Riemann Hypothesis and its Applications
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Lecture – 16

So, this is where we were last time we were trying to evaluate this integral which is not too difficult to fix now.

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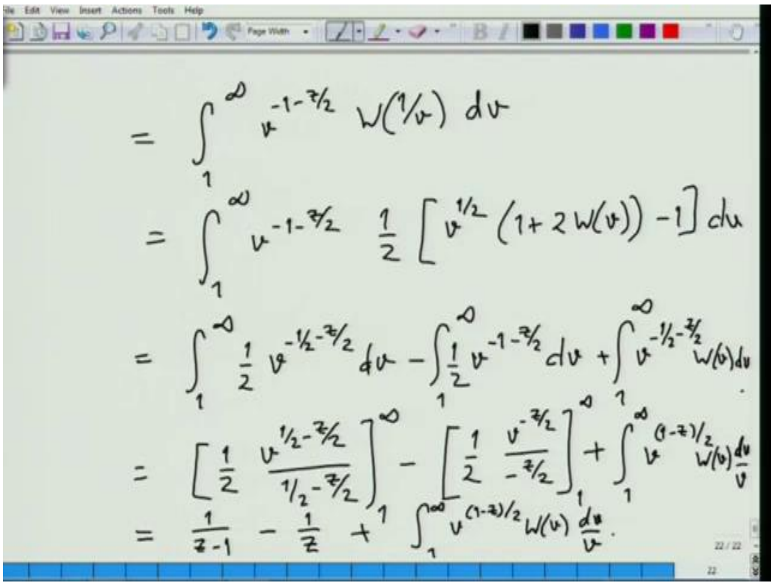

So, let us continue here see what happens to first integral that is basically half of v to the minus half minus z by 2. So, this would be v to the half minus z by 2 divide by half minus z by 2, 1 to infinity. So, we only what happens is second integral something similar half v to the minus z by 2, 1 to infinity and then here we get 1 to infinity this of course we do not integrate, we will just write it as. So, this is just a slight rewrite of the same thing and, now if we look at the actual values here what do you get well here of course we while evaluating this integral.

So, we have to assume that for example the real part of z is bigger than one because otherwise if real part of z is less than 1, then when you said v to infinity it diverges which you does not want. So, under the assumption the real part of z is bigger than one when you said v to infinity this is 0 when you said v to one then what do you, what do you get is one more z minus 1.

Similarly, here we need to assume that real part of z is bigger than 1, sorry bigger than 0 which is anyway fine because we have already assumed real z is bigger than 1. So, when v equals infinity this goes to 0 and v equals one then this becomes a lots of minuses here this should be right and plus this integral, and now let us switch back I gave several of these before.

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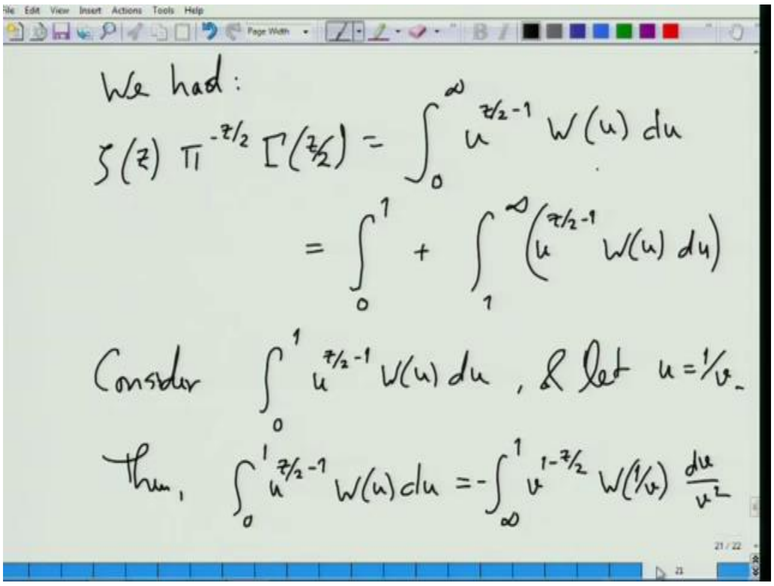

So, this is what the expression for zeta was zeta z pi to the minus z by 2 gamma z by 2 is this integral which we have split into these two parts. So, this is the part that we just managed to rewrite as that long integral this 1, so let us just plot that value in here and see what we get equals.

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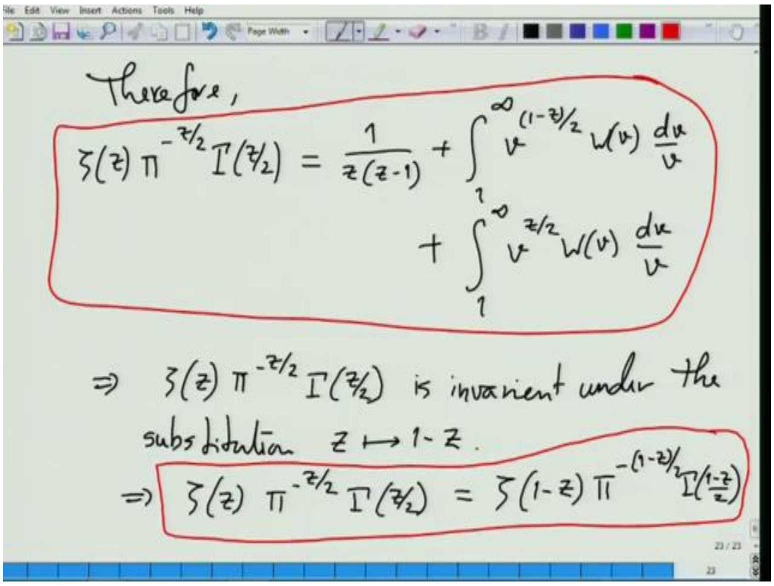

So, stick that in which is 1 over z minus 1, minus 1 over z this is actually also equal to one over z, z minus 1. Then plus this 1 to infinity v to the 1 minus z by 2 w v d v over v and in addition the this part which we have not touched and we will just copy this part as it is which again changing the variable name u to v we get v to z by two w v d v over v. So, this is the result of all this hard work proof for your analysis and what not, but what we get out of this is something very interesting sees the right hand side first thing to notice in the right hand side is not obvious.

But, once you realize it becomes obvious is that the right hand side is invariant under the substitution z by 1 minus z the first term is 1 over z, z minus 1. So, you replace z by 1 minus z, z minus 1 becomes minus z and z becomes 1 minus z, so that is again same, what if you do the same thing in this integral v to the 1 minus z would become v to the z by 2 nothing no z. Here, which and v to the z by 2 becomes v to the 1 minus z by 2, so these two integrals just swap with each other, so this implies that.

So, if right hand side is invariant of course the left hand side is also invariant under the substitution and which means and what does this show us. Well, we know the definition of zeta z for all z with the real part greater than 0 real part greater than 1 it is defined by that infinite sum.

So, we did little bit of playing around with it we defined it also between the strip 0 and 1 and, now the question that we started with what how what definition do we give to zeta

z. So, when z has negative real component well, now let us take a negative component z what do we get same zeta of minus half well pi to the minus, so let us say z is.

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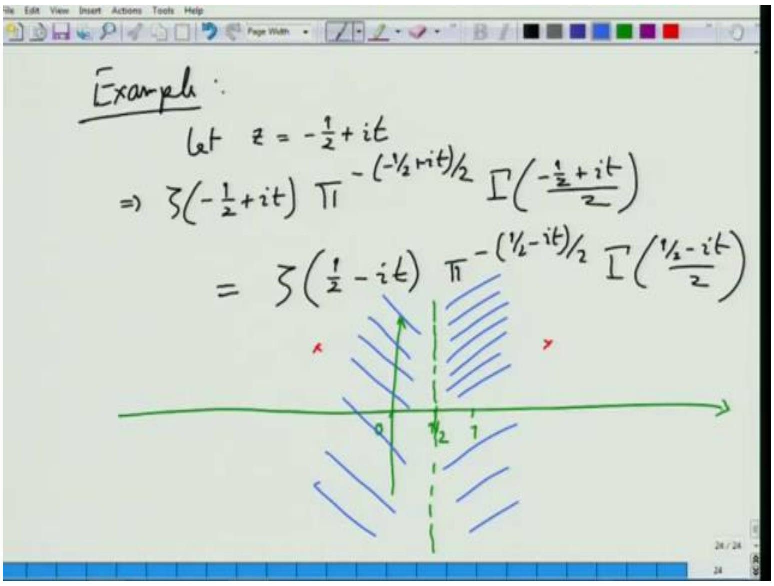

So, let us, let us take an example suppose z is minus half plus i t from complex number with real part minus half then zeta of minus half plus i t pi to the minus half plus i t divided by 2 gamma. So, this equals zeta 1 minus z 1 minus z is what half minus i t, now this is well defined pi to this whatever power is well defined gamma of minus quarter plus i t by 2 is well defined. Now, in fact gamma is well defined on any non integral negative number, so this is well defined gamma of half of minus i t is well defined pi to this minus is well defined and gamma.

So, can gamma quarter minus half i t by 2 is well defined, so that gives me a definition of or a value to gamma minus half plus i t. So, in fact if you just think about it for a moment this is 0 and you take the line that is, that is one this is half. So, this relationship which I just derived this relationship can be used to define the value of zeta at any point, here using the value of zeta at this point which is just a reflection around the line.

So, z equals plus half real z equals plus half because gamma z, sorry zeta z is defined double zeta 1 minus z which is a symmetric quantity with respect to z equals half line. Since, we know what the value of zeta is on this domain we just extend the same thing to get the value zeta on this domain.

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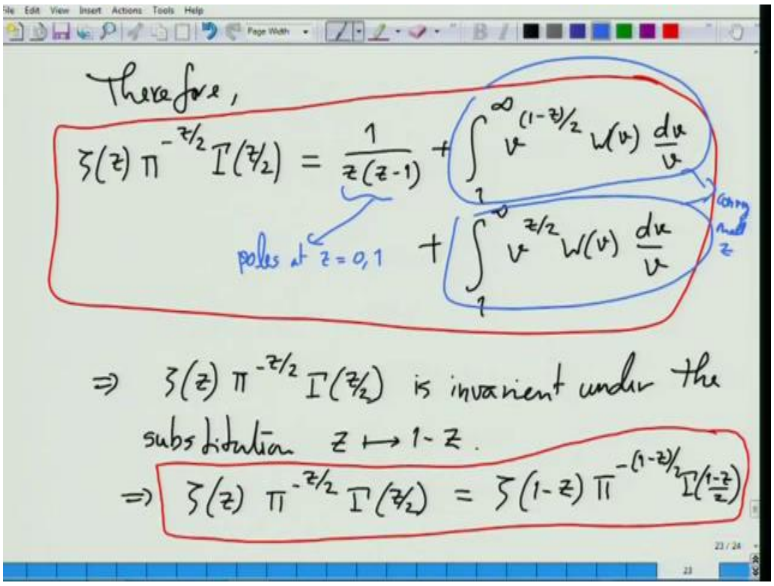

So, that is that is, that is it I mean it although this relationship if you think back that this integral we said when the evaluating this integral and sending this for 0, a real z must be greater than 1 which means that this relationship holds for real z greater than 1. So, there is a question here why when this relationship hold only for real z greater than 1, how can we conclude this equation. So, that is because analytical continuity look at the right hand side it is easy to see that this is defined everywhere the right hand side except for 2 points at z equals 0.

Now, there is a pole at z equals 1 there is a pole because of this, so this gives pole z, z equals 0 and 1 at all other points this term is defined. Now, what about the integrals see if you recall the definition of w v, w v was some n going from 1 to infinity e to the minus pi n square v. So, you this infinite sum converges for every v and in fact it is easy to see that this infinite sum is going to decay very fast something like e to the minus v at least as badly as e to the minus v also because every single term.

So, there is e to the minus first term is e to the minus v second term is e to the minus 4 v e to the minus 9 v e to the minus 16 v when we add all of them up you probably do not get anything better than maybe e to the minus 2 v until we get something even less. So, this decays really fast, sorry not e to the minus 2 v e to the minus v by 2, so this decays really fast and this is a only a polynomial in v the key difference from the earlier case was that. So, when the integral is from 0 to infinity and z real z is bigger than 1, here for example this becomes one over b to the some positive power and when b is 0 this diverges.

But, this integral, now is from 1 to infinity v this part never diverges this behaves at most a polynomial [FL] as in the numerator a polynomial or in the denominator. So, a polynomial which does not matter I mean this is decays so fast that this will convert for every value set. Similarly, for every value of z this also converges, so these both converge on all that, so these two conversion are all that these two are conversion are all that except for two poles at 0 and 1.

So, right hand side is defined for is basically a metamorphic function x with two poles at 0 and 1 and the right hand side agrees with the left hand side on the half plane real z greater than 1. Therefore, by analytic continuity and its uniqueness you can analytically continue the left hand side and make it equal to the right hand side. So, on the entire complex plane which allows me to extend the definition of zeta z all over because I know pi to the z minus z by 2 and gamma z by 2 are defined everywhere.

But, except for some poles, so this equation itself guarantees that zeta z exist on all planes on all points except for some few poles which we will see in a moment. Then we can use this relationship to derive this relationship between zeta z and zeta 1 minus z which gives us a nice way of calculating zeta z on the negative side using zeta values on the positive side.

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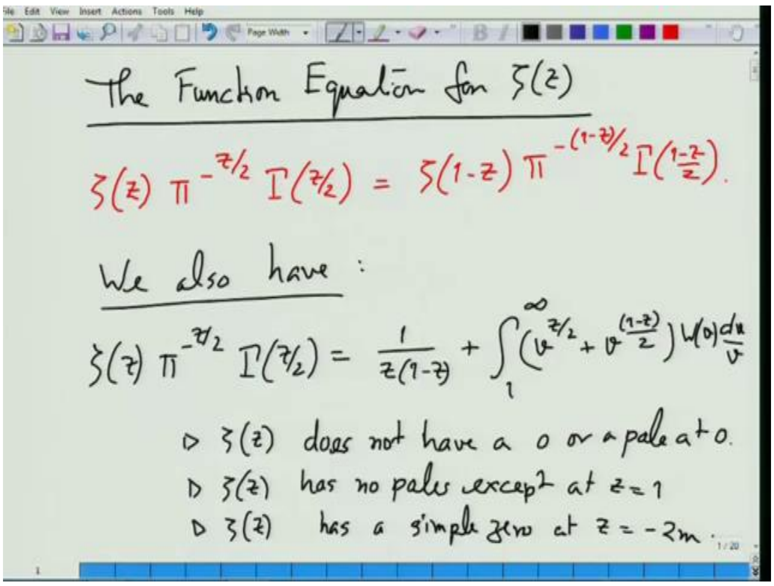

Therefore, have this called the functional equation or zeta and this is as I already said and this is a very important equation which will allow us to talk about zeta values in different contexts. So, we will keep coming back to this equation it also very nicely relates zeta contrary with the gamma function. Now, coming back to this well this same equation or in fact if you do back to the previous one which I will write as why this and, now I already observed that this has exactly two poles at z equals 0 at z equals 1.

So, let us using this, let us find out all the poles of the zeta function over the entire complex plane because this and all needs a definition of zeta function over a complex plane. So, we all know that zeta z has a pole at z equals 1 and in fact that is a simple pole of order 1 and on the right hand side also we see a pole of order one at z equals 1. So, this seems to define you know that gamma z by 2 there is gamma half does not have a pole is not that does not divert.

So, everything is perfect what about z equals 0, the right hand side is a pole at z equals 0 does zeta have a pole at z equals 0, zeta 0 should be zeta 1. But, why that is only for gamma function, yes exactly see gamma has a simple pole at z equals 0 and that matches with a simple pole on the right hand side.

Therefore, zeta does not have a pole at z equals 0, in fact we cannot even have a 0 at z equals 0 because if it had 0 at z equals 0 that will cancel out pole of at gamma on the left hand side. Therefore, you will not have pole on the right hand side, so this how the conclusions from this are zeta z does not have a 0 or a pole at 0. Now, let us move back these are the two poles on the right hand side which cause the poles are 0 then the left hand side.

So, poles are actually on the left hand side the right hand side has no other poles, so which means whatever poles zeta has if at all it has they must cancel out with zeroes of gamma. But, does gamma have a 0 anywhere on the negative complex plane, in fact if you recall that functional equation for gamma. So, gamma z is equal gamma one plus z over z, so if gamma z is 0 then gamma 1 plus z must also be 0 which means gamma 1, 2 plus z be 0, 3 plus z. So, that all points must be 0 all the way up, but at some point the real part of k plus z must become positive.

Then we know that, so that actually tells that the gamma function is never 0 on the entire plane, so gamma function is never 0 on the entire plane. So, it is surely not 0 on the negative complex plane and, therefore zeta has no poles on that negative plane. So, which means that the zeta has exactly one pole at z equals 1, well we can derive observe something more also here because gamma has some poles on the negative side in fact at all negative all negative integers it has a pole.

Student: It also has a pole at 0.

It also has a pole at 0, you are right it also has a pole at 0, but that is fine that pole is already taken care of it has a pole at minus 1 minus 2 minus 3 which means to get that pole in this equation. So, we have to set z equals minus 2 minus 4 minus 6, so all every negative even integer will cause a pole on the left hand side.

But, right hand side has no poles at those points which mean zeta must have a 0 at those points to cancel out those poles. Now, the order of what is the order of those poles of gamma say what is a order of pole of gamma at minus 1, the order is 1. But, in fact all of these poles have order one because see gamma minus one is gamma 0 over minus 1, so it is it narrate, it narrates the order of the pole at gamma 0 and every subsequent 1.

So, all these poles of gamma are of order 1, therefore zeta must have a 0 at every negative even integer and these zeroes must be of order 1. So, it is simple 0 is same as this pole 0 of order 1 minus because there is z by 2 here is there any other 0 of zeta function on the less negative complex plane use.

Student: Constant’s equals to the right hand side

What?

Student: Constant’s equals to the right hand side

Yes, so use the functional equation there if there is a 0 on the say at z which is somewhere on the negative complex plane. Then there will be a 0 corresponding no gamma at this those points is not 0 whether gamma z by 2 or gamma 1 minus z by 2 because we know gamma has no zeroes and no poles also. So, except for 0 minus from minus 2 which we have already taken care of at all other points at any other point zeta z has 0. Then zeta z minus 1 is also 0 if that is negative that means, if there is minus 1 here then this will become zeta 2, that zeta 2 is 0 can zeta 2 be 0, which actually pi square by 6.

Now, in fact zeta z for any value of z with real part greater than 1 will not be a 0 because again that it is a conversion series and numbers is kept added up except 12 there are amplitudes complex amplitudes which can subtract also. But, this series someone can argue a bit to see that it is never 0 on the side of real z greater than 1 which in turn means that it will have 0. So, when real z is less than 0 it is symmetric along real z equals half there is no 0 on real z greater than 1 there is no 0 on real z less than 0.

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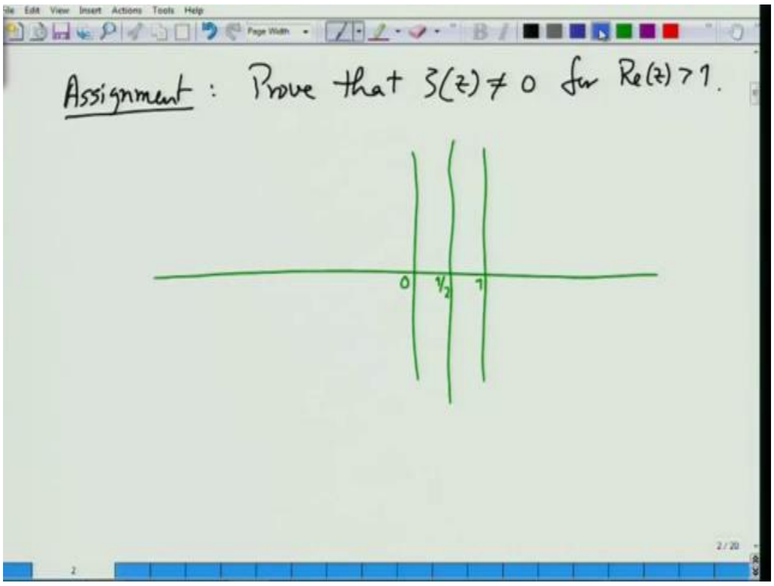

In fact, let me give that as an assignment, so by this symmetry.

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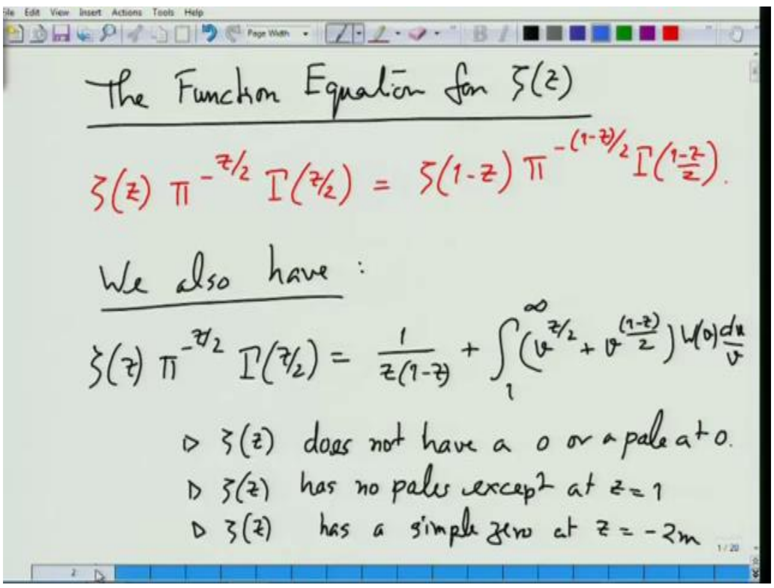

So, symmetry of value certainly holds for the zeroes if zeta z is half then zeta 1 minus z need not be half, but if zeta z is 0 then zeta 1 minus z must be 0 because of the way the equation is setup.

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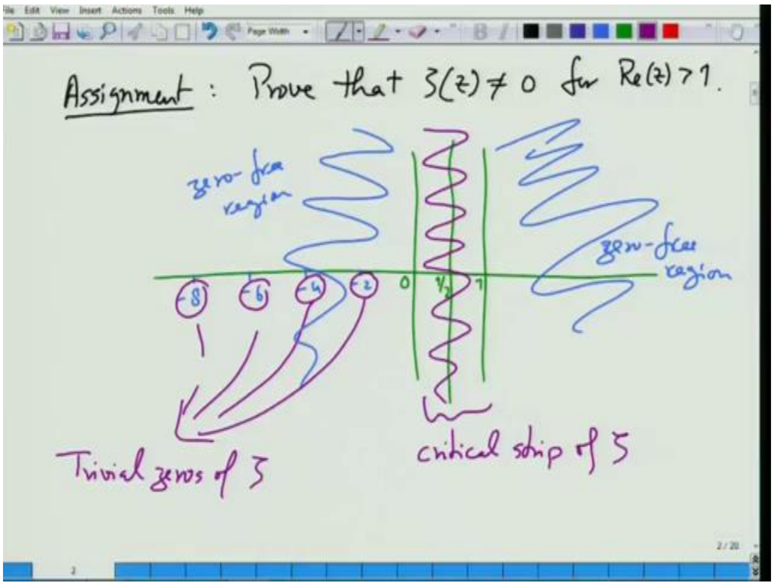

Therefore, since there is no 0 here 0 free region, let us say that this is also 0 free region except of course these minus 2 minus 4 minus 6 minus 8 and so on where the zeroes are caused by the gamma function. So, apart from this zeroes of zeta function minus 2 minus 4 minus 6 minus 8 all other zeroes, if at all they exist must be in this region.

Student: Why is the 0 not reflected on the positive region?

Because gamma has a pole at those points, so 0 and pole cancel at those points on one side and other side, so the remaining zeroes are in this strip between 0 and 1. So, this is called critical strip of zeta function because the behavior of zeta function in this strip is something that we would like to understand why I will explain very soon. But, that is, that is where we have some very interesting behavior of zeta function these zeroes which are at negative even integers. So, these are called trivial zeroes of zeta function trivial because these are basically caused by poles of gamma function.

Now, they are not really of much interest to us well they are of interest, but they do not really change the things in any significant way as we will see it later. So, with all of this we have at least achieved our one of the targets which was to define zeta function over the entire complex planes, so that we can take a contour.

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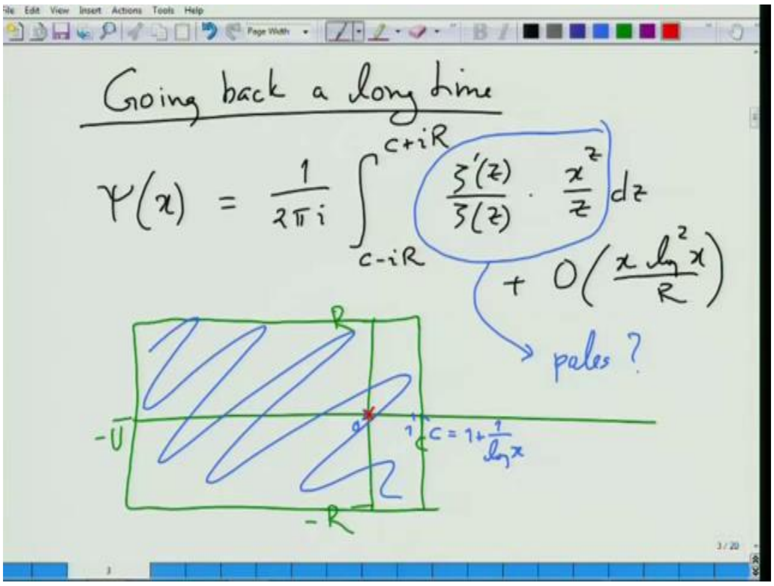

Now, I do not know if, but the origin of all of this v x psi x equals 1 over 2 pi i integral c minus i R to c plus i R zeta prime z over zeta z x to the z by z d z plus order x log square x by R wait and this is how we started. So, we said, let us evaluate this integral, this is the error term to avoid this integral we use this familiar technique of defining a domain and using the residue theorem one cautious integral formula initially. But, in order to use that one thing that became clear is that we cannot go on the right side because if you go the right side.

Since, x is bigger than one x to the z will diverge and will not be able to bound the integral to this site. So, we have to come on the left hand side and look at this region right and c was if you recall 1 plus 1 by log x just little more than 1. Now, to in order to do this domain integral we needed the definition of zeta function in this domain which we have now managed to change when zeta is defined in this domain and zeta is analytic.

Also, by this analytic continuation make sure that zeta is analytic everywhere except for the pole at z equals 1. So, zeta prime z is perfectly well defined everywhere, so this function is defined well defined and we can now go ahead and then do the integral. Now, in order to do the integral we will of course invoke this theorem of Cauchy, the Cauchy integral formula. So, that by the integration around the boundary of a domain equals the residues of the equals the sum of the residues at all the poles right inside the domain.

So, what are the residues of this function this whole function that we have what are these residues in this region or forget the residues poles. So, let us forget, let us find out poles in this region 1 one pole is very clear z equals 0 pole here, but that and that takes care of this no other pole because of this the remaining poles are because of zeta prime. So, zeta what poles are caused by zeta prime or zeta will they have will it have a pole at z equals 1.

Student: Zeta has a pole.

Zeta has a pole, but it is one of zeta prime or zeta.

Student: Zeta should have a pole.

Why, think of this see zeta has a pole at z equals 1 it is a simple pole, so if you look at the loran zian expansion around the one for zeta. So, you will get something like one over z minus one plus high integrator differentiate this what do you get you get minus 1 over z minus 1 whole square plus higher degree terms. But, in fact the higher degree terms will only be constant and higher it is, no term like 1 over z, 1 over z minus and then you look at zeta prime or zeta.

So, you in the numerator you have 1 over 1 minus z whole square in the denominator you have 1 over 1 minus z plus higher degree term you multiply both by 1 minus z. So, the denominator becomes a polynomial or a power series numerator still has 1 over z minus 1 plus a power series, now add z equals 1 what happens.

Student: There is a pole.

Pole same order pole order 1 pole.

Student: z equals to you know zeta at order 2 pole.

At z equals 0.

Student: Order 2, there is one upon z.

No, 1 upon z is not there zeta is not 0 or does not have a pole or a 0 at z equals 0 in fact this character zeta prime zeta is a pretty bad character. So, because whenever there is a pole of zeta or there is a 0 of zeta this becomes a pole one simple way of seeing this is that zeta prime or zeta i sum.

Student: Zeta prime does not share a 0 with zeta 2.

What?

Student: zeta prime should not share a 0.

No, it does let us look at forget about zeta take any function.

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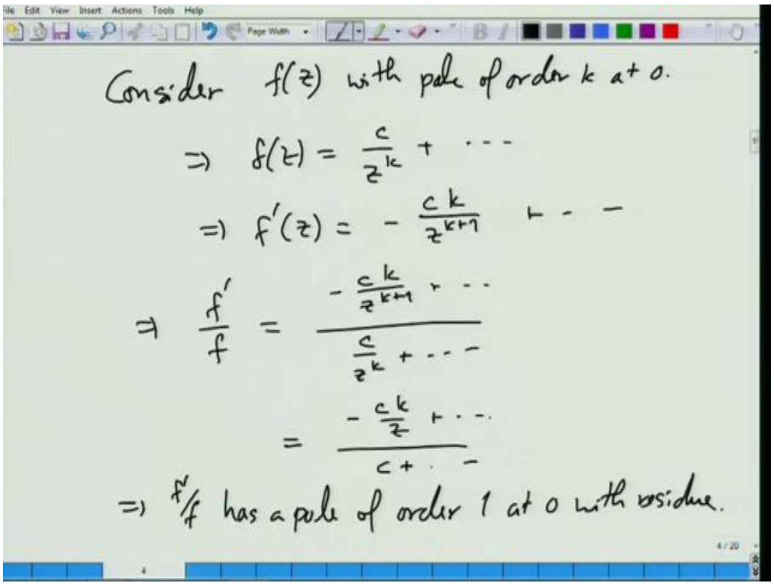

Now, let us consider his is some function f z with pole of order k at 0, so let us we will just assume it has a pole at 0. So, this means f z is 1 over z to the k not 1, but it is some constancy c over z to the k plus higher degree terms what a derivative of f is minus c k over z to the k plus 1 plus higher degree terms.

So, f prime z over f is minus c k over z to the k plus 1 plus higher degree term c over z to the k plus higher degree term multiply and divide by z to the k. Here, we get minus c k by z plus higher degree term c plus higher degree term, now this at z equals 0 is a pole. But, is a pole of order 1 what is the residue of this pole residue is calculated, but multiplying it with z and taking the limit at z goes to 0. So, multiply the whole thing by z take the limit as the z goes to 0, you get minus c k or c, you get residue minus k.

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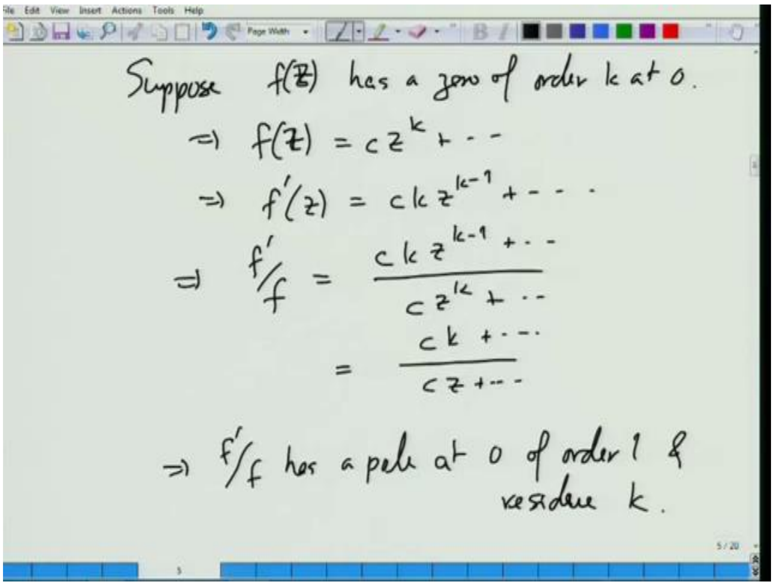

Now, let us instead suppose f k has a 0 of order k then you can write f k as, sorry f z as z to the k some c times z to the k plus higher degree term, so what happens to prime z c k z to the k minus 1 plus higher degree term. So, what happens to f prime over f c k z to the k minus 1 plus higher degree term over c z to the k minus k plus higher degree term z to the k minus 1 can be canceled out from numerator and denominator.

So, you get c k plus higher degree term over c z plus higher degree term, so this has a pole at z equals 0 of order one and what is the residue k. So, f prime over f for any function has this very nice or strange property, if you will that whenever f has a 0 or a pole f prime over f has a pole of order 1 and the residue is precisely the order of the 0 of pole where the order of pole is taken in a negative sense.

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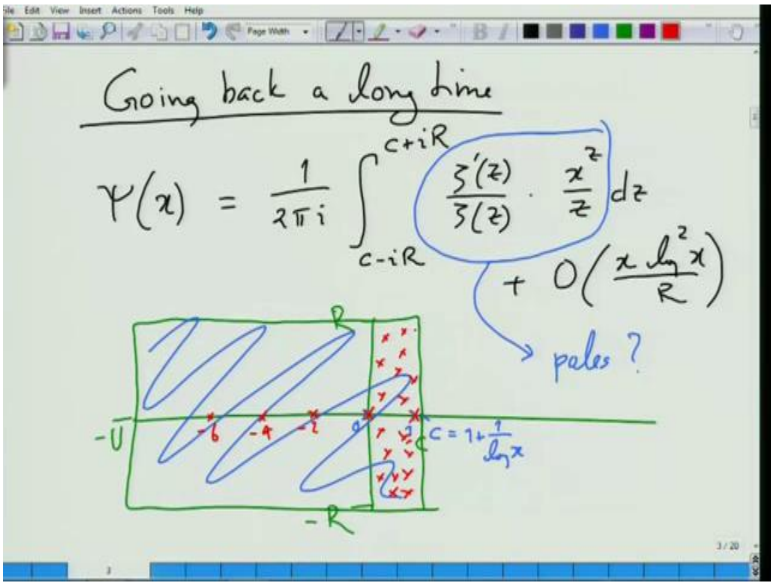

So, that is the knowledge we have, now coming back to zeta prime over zeta which is one instantiation of f prime over f we have to now worry about every single pole and 0 of zeta function. So,, well surely we know here is the pole at z equals 1, so that is going to be a pole of zeta prime over zeta, but not only that minus 2 I have minus 4. So, I have minus 2 minus 4 minus 6, these are all zeroes of zeta function, therefore pole of zeta prime over zeta.

But, not only that all the zeroes wherever they are in this critical strip and we have no idea of how many where they are and how many there are they are also poles of sets. So, in order to be able to estimate what this integral is we will need to do a lot of work, now this part is easy minus 2 minus 4 minus 6 very clean we know it is we know the residue. Now, residue is because these are poles or these are zeroes of order 1, therefore residue at each one of them is one simple this is again one also we know.

So, pole of order 1, therefore residue is minus these are the characters which we really need to understand these zeroes. Here, we need to get a handle on where they are how many of them are there, so that we can calculate the residues. So, the right number of residues appropriately the residues we can only, we can calculate easily by just knowing how many of them are there.

But, there is another there is a reason why we need to know exactly where these guys are also and the reason is the following and as you take the integral. So, see the strategy is going to be that you take this contour integral anti clockwise and we will argue that these integrals are very small. Therefore, can be absorbed in the error trunk and, so this contour integral is roughly same as the integral this one this integral. Therefore, we get x by x not to argue that these contour integrals are very small or these path integrals are very small we will have to be careful as we move along say from here to here.

So, we may be coming close to a 0 of zeta function that is where it will be because here we know exactly where are the zeroes? So, we can avoid them by going far away from them and also come and come down of course we cannot avoid them by too much, but we can certainly speed them right in the middle. So, distance one this side distance one this side, so reasonably far apart, but here we have to have some control overtaking this path.

Otherwise, if this comes too close to a 0 what happens this person diverges or it becomes very large as you move the integral becomes large and then we cannot argue that the integral along this line is small. So, that is why it is important to know the locations of these zeroes also we can avoid the zeroes as we move from right to left. So, we need to know the numbers we need to know the locations and then we can say something useful hopefully about the contour integral. Therefore, we can conclude something useful about psi function, so that is the task ahead of us any questions

Student: Why is the function diagonal?

Zeta prime over zeta has a pole at that point, so as if you this path goes very close to a pole or a 0 of zeta function which is a pole of zeta prime over zeta the integral will shoot down.

Student: Very close as in…

Very close, yes as we have to define what very close is it becomes essentially we want to avoid a situation where when this function takes a very high value along some point on the path integral path right other takes very high value. So, of course we cannot argue that the whole integral is small, so that is a situation we have to avoid, so we cannot just blindly take a.

So, you know take a point here and just move to the left we might even be hitting a pole if we assume pole over then the whole integral is undefined, so that is as I said is a task ahead of us and we are now sort of tells us. So, that we have to analyze this distribution of zeroes in order to get a handle, but let us assume wherever those zeroes are, so let us say we can at least write down this.

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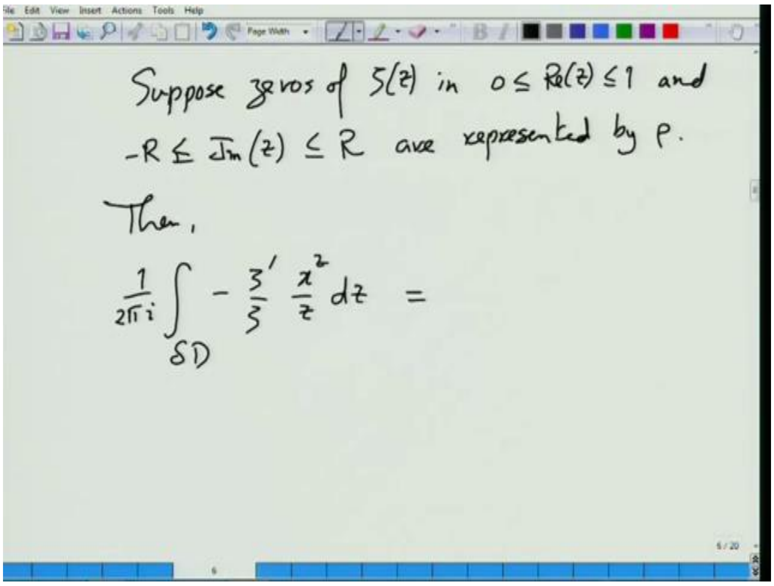

So, suppose, so let us use a symbol rho which I will uniformly use to represent zeroes of zeta function in that critical strip. So, then what is this contour integral if we recall this is domain d oh there is a negative sign also, we write there is a negative sign. Here, what is the value of this integral by residue calculus it equals to the sum of the residues at all the poles, so what are the poles here?

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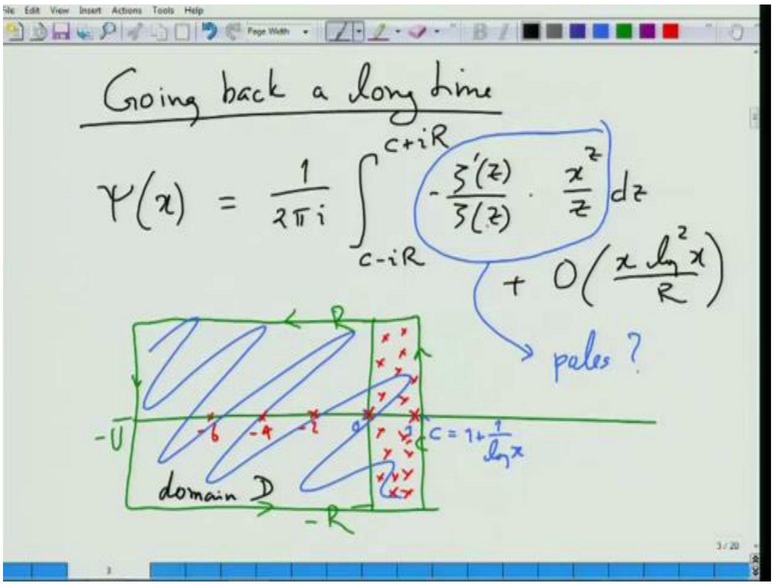

So, poles are of course at 1.

Student: Did you say anything about the order of 0 in this strip

Order of…

Student: 0 in the critical strip.

Order of zeroes, I am not sure I think we can say something, but that is not very important for us anyway because zeta prime or zeta will always have a order one pole. So, only there is value of the residue will change value of residue will change, yes you are right, but what we can say is just let rho run over all zeroes if their order is k. Then let you can sort of pretend that they are k zeroes sitting at the same point, so as rho runs over all the it runs over k times all that. So, this equals the sum of residues in all the poles, so first let us take care of the poles that we know well.

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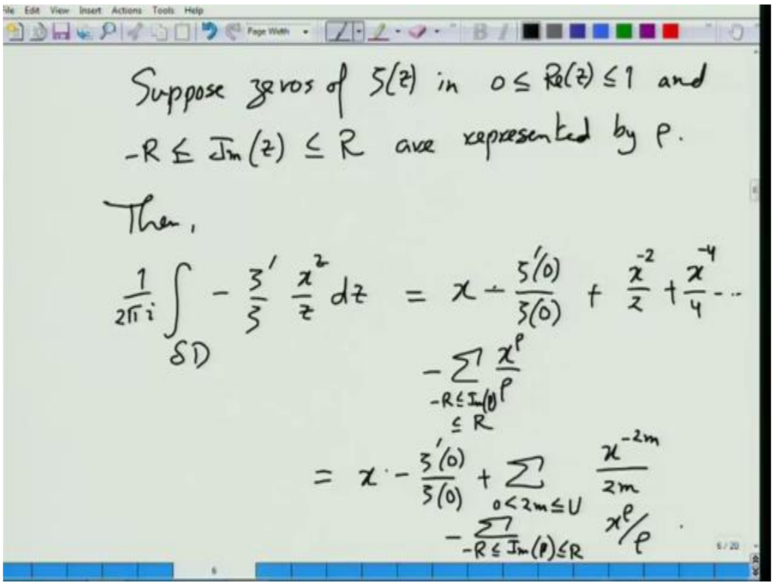

So, what is the residue as the pole when z equals 1 zeta prime over zeta takes value minus 1, so minus zeta prime over zeta will take the value plus 1. But, notice that the residues of the whole function not only this and it is this entire function we are considering at its poles. So, what is the residue of this at z equals 1 x then moving back z equals 0 is a pole of this, now the residue of this, therefore is minus 1 because of this being a pole. Now, this will there then be zeta prime 0 over zeta 0 a simple residue then moving further back at minus 2 minus 4 minus 6 what is going to be the residue this will give always.

Student: Should not there be a minus over there.

Where?

Student: Zeta prime over zeta.

Yes, minus why would there be a minus the pole is pole will give a residue of minus one no, no sorry yeah you are right, this will be minus 1. Now, going back to minus 2 for example minus 2 is a 0 of zeta, so it is residue will be plus 1 zeta prime over zeta. Since, there is a minus here, there will be a minus 1 here and this would be minus x square by 2 minus x to the 4 by 4 and so on.

So, these are the residues corresponding to the zeroes at the trivial zeroes and then there had be residues corresponding to non trivial 0 in the critical strip and what about there will be 0. So, it is residue will be plus 1 here minus 1 here and this would be x to the rho over rho, so this is right this is.

Student: Shouldn't that be x to the power minus two.

What?

Student: Should not that be x to the power minus 2 upon 2 because that means x equals to.

You are right minus 2, yes that is right.

Student: And the minus will come along.

And there will be a minus here.

Student: And zeta of z to the power.

Thank you, so that is the formula we get we will do it next time.