Riemann Hypothesis and its Applications
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Lecture – 15

So, this is the lemma we are going to prove today amongst other things, of course, and I might have to come back and modify the statement of this lemma; this square root u could be 1 over square root u that we will see.

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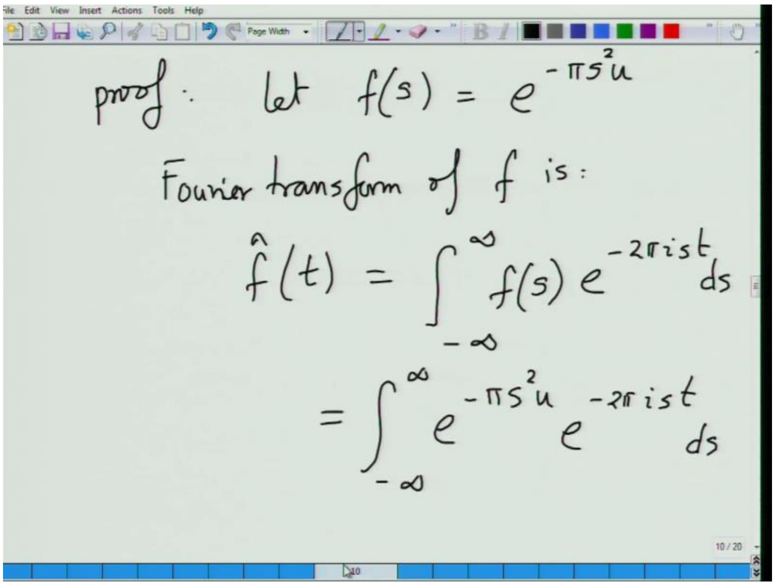

Okay, good. So, the Fourier what is Fourier transform; let us start with the definition. So, let us define the function f n as f s square. So, all I have done is extend the definition of function n from integers to all numbers. Now Fourier transform can be defined for just about any function as long as there as it satisfies some reasonable convergence properties, but there are two distinctions depending on what the function is like. The function is periodic with the certain period, then you only consider that period the interval of the period in order to define the Fourier transform because after that it is all the same.

On the other hand if the function is aperiodic, you just take that the period is infinity and consider it as the whole function. So, in this particular case, this is not periodic; there is clearly no period here. So, we just define the Fourier transform to be t integral from minus infinity to plus infinity; you just take the entire span for the function f of s t to the minus 2 by pi s t ds. Multiply the function by e to the minus 2 pi i st and integrate it from minus infinity to plus infinity, okay. And of course, one has to see whether this is well defined, and for that, one can or one should show that the absolute value of this integral converges incase of the function we are looking at certainly task converges. This is really very, very rapidly decaying function. So, that is the problem, okay.

Now question is what is this? Well, we know it is minus infinity to plus infinity e to the minus pi s square u e to the minus 2 pi s t ds. And what I want to show is that this is actually same as the original function, right; that is the target or lot quite the same up to a multiplication factor by the square root. Now for that, we will use a little trick; just observe here, there is a pi here, there is pi here. So, pi can be taken out in common. So, let us actually just write this in a slightly more simple form.

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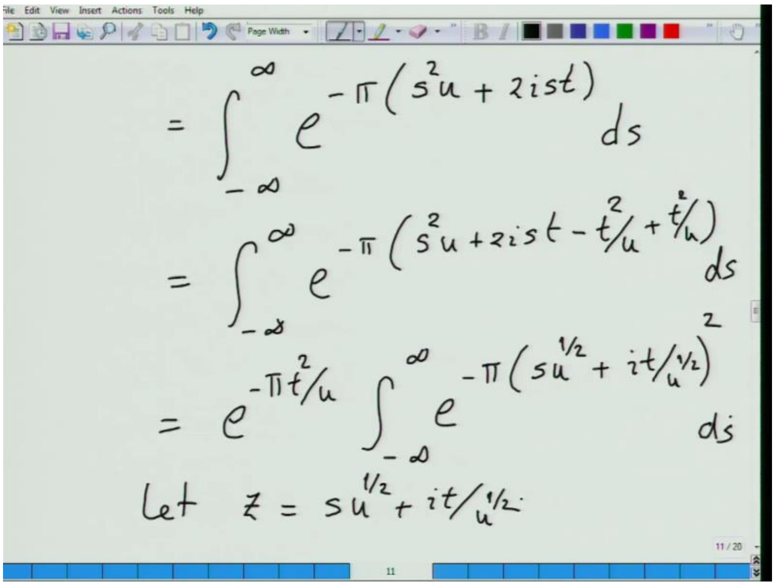

Minus pi, then u have s square u plus 2 i s t s square u plus 2 i s t. So, this almost suggests that we should try to complete the square and see what we get out of this. What do you need to complete this square minus t square by u, and of course, plus t square by u d s. Now this last has nothing to do with s. So, we can just pull it out, okay. This is basically this expression is s square root u plus i t by square root. Now all then we need to do is now we just let z be s square root u plus I t by square root u. So, far s u t u s u t were all real's. Now we move into the complex number by letting z be this and express this integral as in terms of z.

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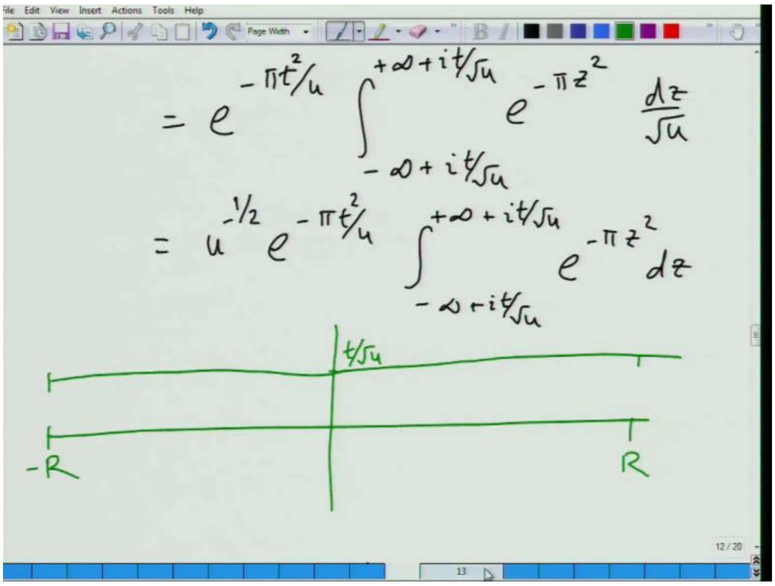

So, where does z goes from? When s goes from minus infinity to plus infinity, what happens to z? Yes, so it just goes from minus infinity plus i t by square root u plus infinity plus i t by square root u e to the minus pi z square; that is what we get here, and what happens to d s? So, u t is fixed; they do not vary. So, let us take d z is d s root u. So, d z by root u, good; so we have not developed this form which is somewhat simpler except the noise a complex integral.

Now we know some ways of handling complex integrals. So, let us try to see what happens with this. If we want to integrate it from minus infinity plus this to plus infinity this in quantity; that is like where does this go from t by square root u which it is by which somewhere here. And the integral is along this line minus infinity to plus infinity. So, we can use the standard tricks that is integrated from minus r to plus r and then we will send r to infinity.

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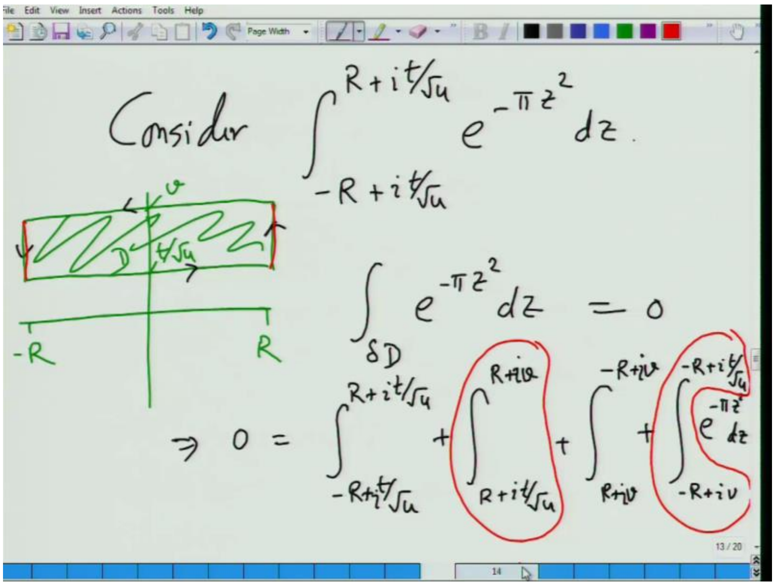

So, we consider minus r plus r and this integral again we will write as this is go here once again from here to here. So, let us define like a contour whichever or else take a rectangle unless this is with the domain d. So, what is the integral of this e to the minus pi z square dz along the boundary of this domain d this zero, is it clear? There is no pole; the integrant has no pole anywhere. So, there is zero, okay. So, this implies that zero equals now if you see that you traversing it counterclockwise minus r plus t by square root u r plus i t by square root u plus r plus i t by square root u to r plus v, okay.

Now if you consider out of these the second and fourth integrals, these two are more or less same similar to each other except that you have one is minus r, one is plus r. It is going from here to here, and another is going from here to here, and you have the integrant in e to the minus pi z square. If you look at the absolute value again using the similar ideas that we have been doing, what is the absolute value of this e to the minus pi z square as we move along these two vertical red line. E to the minus pi is r square. So, it will at least e to the minus pi r square, right, and since it is in the denominator, we can say that its bounded; at least the integrant is bounded by q over let me just write this.

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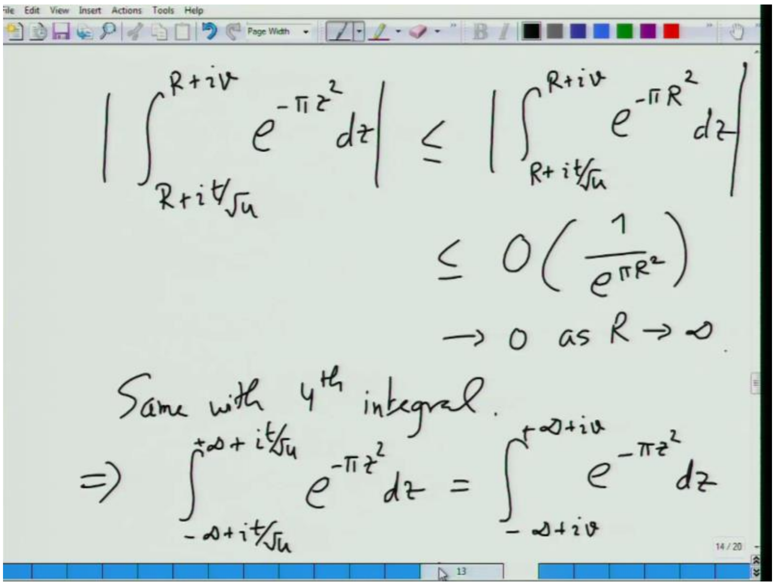

Let us just pick one of them, say, r plus i v. This is less than equal to, okay, and then this is, of course, bounded by order assuming t is n v's t u v to be constants of external values, okay. And as r goes to infinity, this goes to zero and the same here opens with the fourth integral, and this gives that the first and third integrals are equal when we take the limit. Oh, this has nothing to do with integral; this comes out, and you are integrating d z. It is not even r, sorry; this is actually v minus t by square root u. So, that is just one.

Assuming that v t u, these are all fixed numbers and its r there I am sending to infinity. So, it is just that, okay. So, that basically says that the second integral and the fourth integral vanish as r goes to infinity which means that sum of first and third integral is zero. So, third integral is in the opposite direction minus to plus. So, if we start this, we basically get integrating this along this line is same as integrating along this line which, therefore, tells us that if we look at this, we were integrating along this line.

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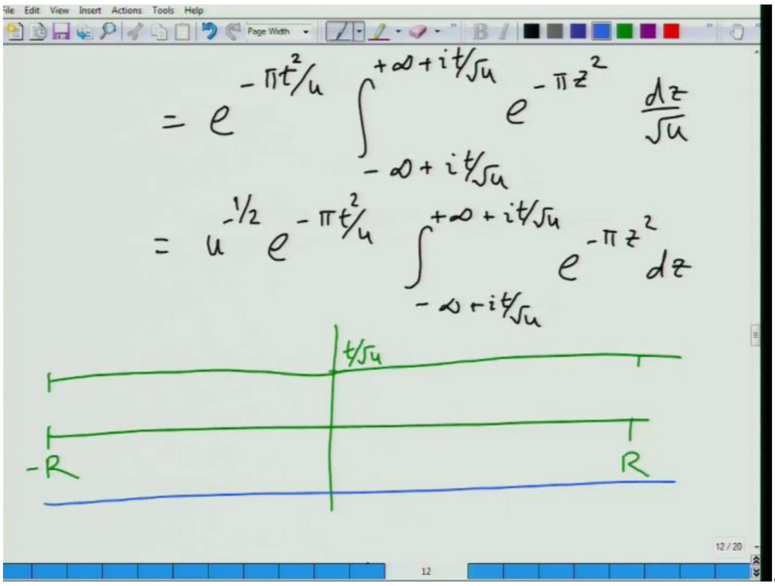

Instead of that if we integrated along any other line horizontal line from minus infinity to plus infinity, the result will be the same, okay. So, that is the first step of our evaluation of integral.

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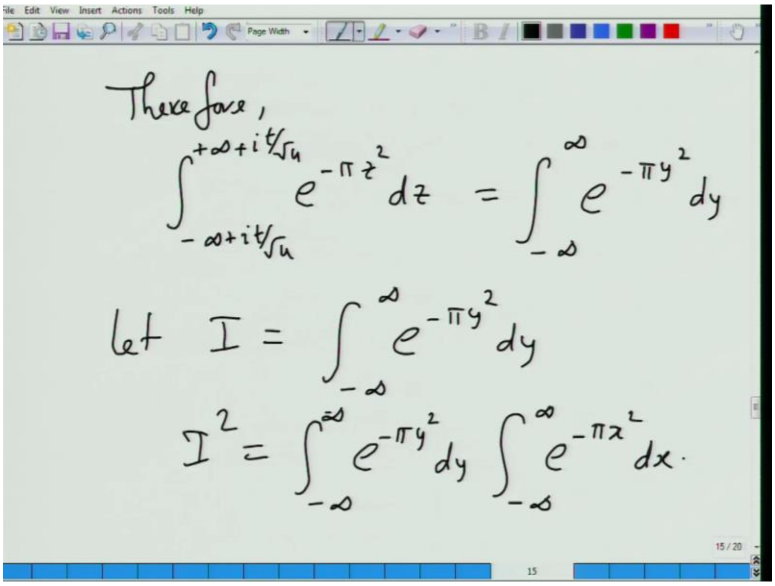

So, what we can say is, therefore, in particular where this equals, let us forget about the complex part; just integrate along the x axis the real axis pi y square d y, okay. So, this is the further simplification. We are now had this simple integral over the complex which is a complex integral; we have now got it as a real integral. Although, it is sometimes easier to evaluate real integral through complex integrals, but in this case, it is easier to just evaluate the real integral directly, and this is integrated by very neat trick.

Actually the integral of this is 1, and to show that, let us give it a name minus infinity to plus infinity e to the minus pi y square dy. This is the normal distribution; it is basically the and you are just saying that this probability mass as 1 which is all this, can you prove this that this integral will be 1. It is very simple actually; just look at I square is again nice trigger as is said, alright.

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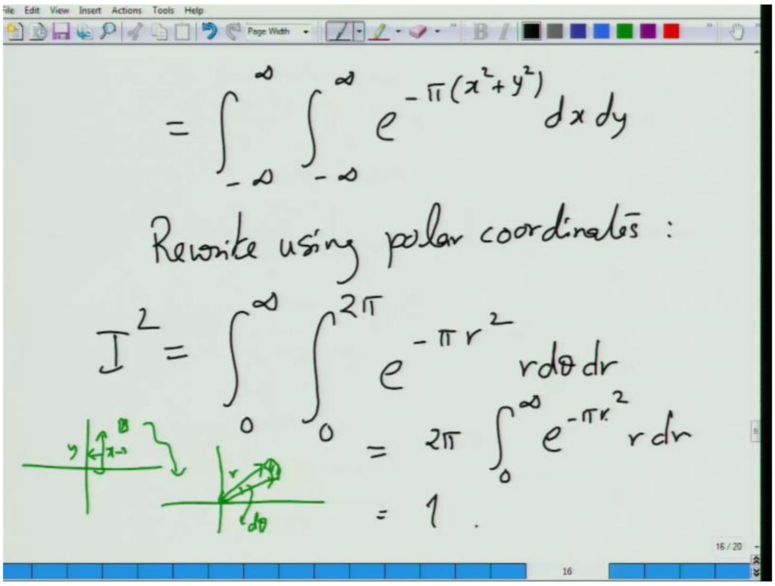

So, just multiply i with itself. So, you get double integrals, and this is now an area integral over a plane were you take the dx dy as a rectangle and integrate along value along that and just add up all. So, it is integral over the entire plane two dimensional plane not a complex plane but normal r 2, right. Now this integral I can rewrite using polar coordinates, okay. So, polar coordinates will be r and theta, r will be going from zero to infinity and theta will be going from 0 to 2 pi; what happens to the integrant? E to the minus pi x square plus y square x square plus y square is r square. So, that is in minus pi r square.

What happens to dx dy? Dx dy is a tiny rectangle; that I am going to change with there is dx dy. This I will place within the polar coordinates. So, this is x, this is y; the polar coordinate I will replace with r and this is d theta with this piece. So, it is r and this d r and this is d. This area is r d theta d r; it is pretty standard but I thought I will just mention it anyway, okay. And now you see that this is oh there is no d theta. So, you get 2 pi here 0 to infinity e to the minus pi r square r d r and this you integrate it with one. Integrating this is pretty straightforward; just do a k equal's r square and let us integrate, okay. So, now going back to where we started from, what did we get? We showed now we have shown that this integral is 1.

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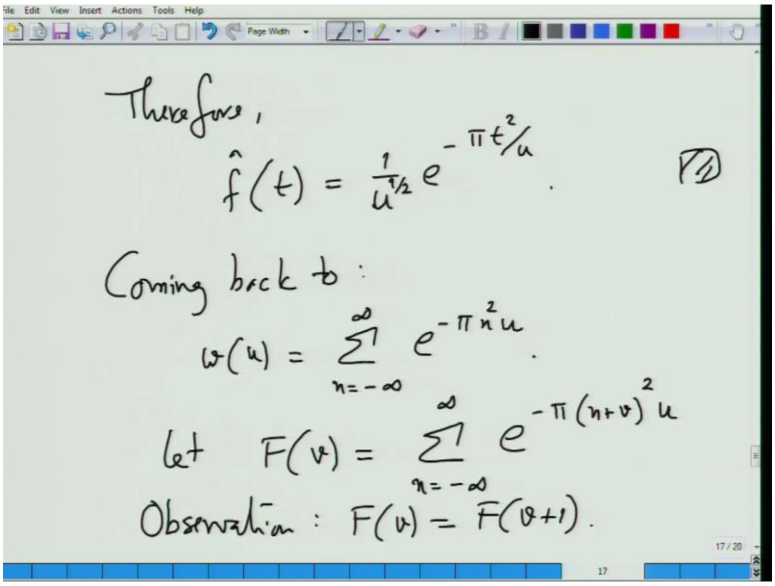

And therefore, my Fourier transform f hat of t is equal to 1 over square root u e to the minus pi t square by u, okay. So, I do need to change the statement; this is minus. So, that completes the proof for the lemma, okay. So, of course, what I have shown is much more general thing, and this Fourier transform holds at every point not only at integral points, but all I need to use it for is integral points, alright. So, that now if you recall, we started with this infinite sum and each term of the infinite sum was at e to the minus pi n square u. And we took one term and then calculated its Fourier transform, okay, but it is really this infinite sum we are interested in.

So, let us now get back to the infinite sum w, what was it; w u plus look at this particular function little carefully. Let me define another function associated with let us say capital f of, okay. Let us define f of v as again is generalized version of w where again u is something I fix, and then we have this infinite sum minus infinity to plus infinity e to the minus pi n plus u the whole square. Now for this, we make this observation F v is f u plus 1 or is this obvious? This is pretty obvious, right, because we are placing v by v plus

1 we will just get n plus 1 plus v whole square, n going from minus infinity to plus infinity its over; n plus 1 will also go from minus infinity to plus infinity. So, this is a periodic function with period one, okay.

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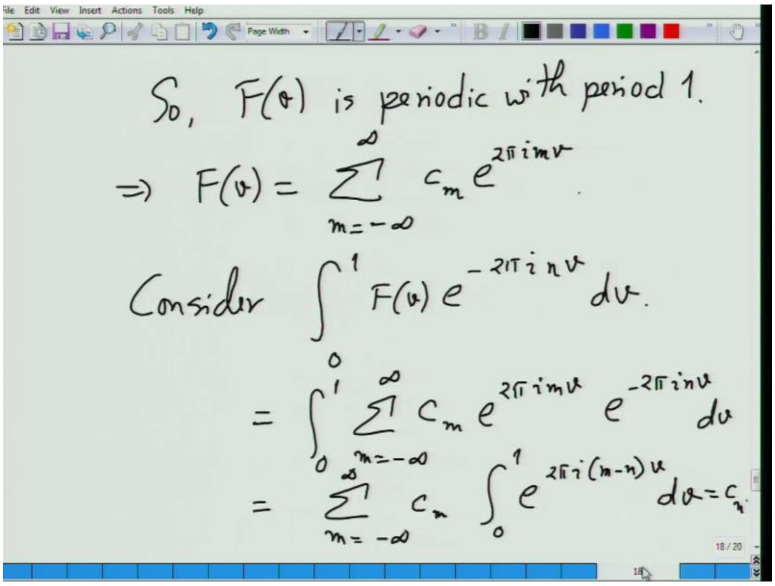

And now again I will oak Fourier analysis on this function which is a different one than the previous; this is a periodic function. So, over periodic function the Fourier analysis shows which I will again show you in two minutes that f v can be written as this infinite sum. So, it says infinite sum of e to the 2 pi i m v these are all each one of this is periodic function with period one, right, and you get this each one of them multiplied by c m which we call the Fourier quotient corresponding to this particular; as you see this is really standard transformation of the function or representation of function as a sum of sign waves.

How do we get this? Well, that is pretty simple as just defined c m out of this. C m would be so this or this, okay, let us see how do we identify for extract out seen. So, this gives now let us consider. So, this is equal to, of course, zero to one, while you substitute this; make this n and used to separate this out. Now again using any form convergence of things, we can swap the sums with integral, and this integral is easy to evaluate. This is going to be one if and only if m equals n; otherwise, it is going to be zero. And so this is equal to c n.

What is that c n, yes; see this is this integral is nonzero exactly when m equals n. So, all the terms in this infinite sum will vanish except when m equals n and then it in that case this integral takes the value one and so you get exactly c n. So, that is why you can express this function which is periodic in terms of this in front power series, okay. Now what do I want to show you? F u was this infinite sum, okay.

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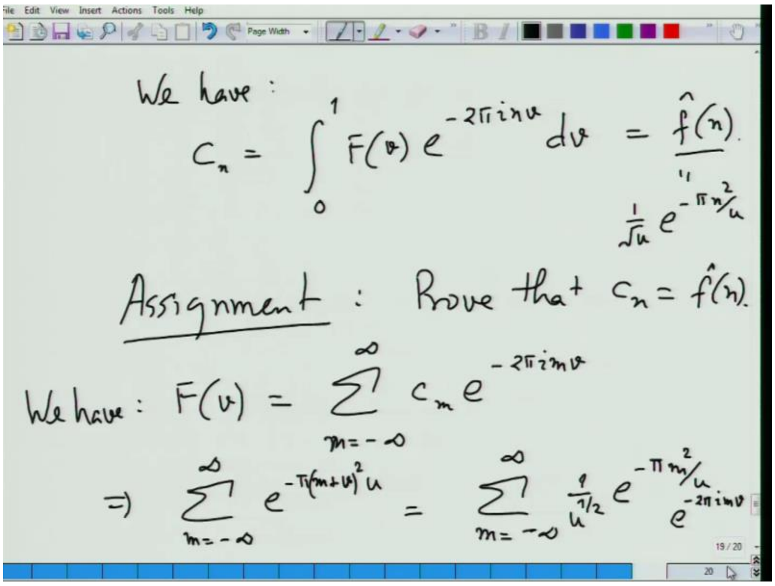

Let us look at this c n in a slightly different way. See this also can be shown easily that c n actually is equals this integral. I will not show I leave it for you to work this out. So, this is equal to zero to one if we substitute for this; for f v I am going from minus infinity to plus infinity and f v was e to this. Now this integral with this infinite sum, we can merge as an integral going from minus infinity to plus infinity, and what happens to these guys? See v was going from zero to one. Now I will make v go from by minus infinity to plus infinity; what happens to this? I am not making the answer; what do the Fourier transform define us.

See what I want to show is that this is equal to and actually the exact form of a factor does not matter. This is a general result that Fourier coefficient of this periodic function which is an infinite sum. This capital F was defined as this infinite sum, and this is a periodic function with period 1, right. For such a function the nth Fourier coefficient is actually equal to the Fourier transform of its n s term; can you see how to prove this? So, this is c n; that is how you start with, and if we plug this in, obviously, if that is worth let me give you we are spending too much time on this.

Let me give you this as an assignment; anything that I cannot prove I am going to give as an assignment; this require some many place in all this integral. Now let us get back; we are now close to what we want to prove. See if we recall f v is this infinite sum c n e to the minus 2 pi i m v, right; f v by definition gives me what I am going from minus infinity to plus infinity of e to the minus pi m square u m plus v square u and this is equal to m going from minus infinity to plus infinity; c m is this f at n. So, f at n is 1 you already know 1 over square root u e to the minus pi m square by u right times, of course, e to the minus 2 by i m v. So, we get this relationship. Now in this relationship again we have proved much stronger relationship then we need just plug v equals zero; what do we get?

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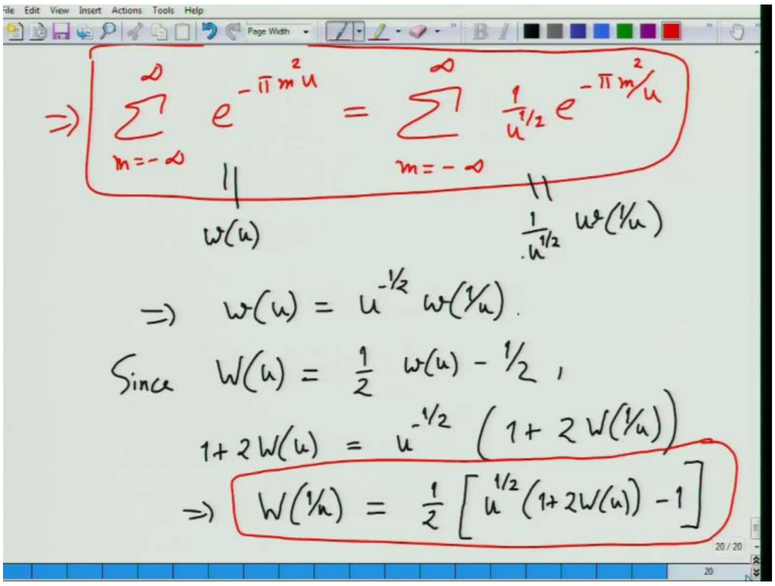

We get m going from minus infinity to plus infinity, plug v equals zero. So, e to the minus pi m square u equals 1 over square root of u. This multiplier is all 1 when v is zero. This is exactly what we needed because this is w u, right, and this is 1 over u square root u w 1 by u, okay. So, we can conclude that w u is u to the minus half w 1 by and the whole point of this exercise was to derive this equation. This relationship between w u and w o1 by u, and let us go all the way back where we started from that same relate w with w 1 by w u with w 1 by u.

And why did w u arise out; where did this come from? We had this capital w u which is related to small w by this formula, and this capital w u actually occurred here in this relationship of zeta function and gamma function inside this integral. So, let me just lift all of this and write a fresh here. But let us first establish capital w u was half of small w u minus half, right this is what it was. Capital w u is half of small w u minus half. So, we can rewrite this relationship in terms of capital w, what do we get? W u is, therefore, 1 plus 2 capital w u is equal to u to the minus half 1 plus 2 capital w 1 by u.

With this case, we have the w 1 by u and this is what we eventually interested in is half of, okay, and this is the relationship we should remember this or take away from this whole. Now go back to that Riemann equation which is not Riemann equation, but the zeta function equation and the let us create a new page which is page twenty one.

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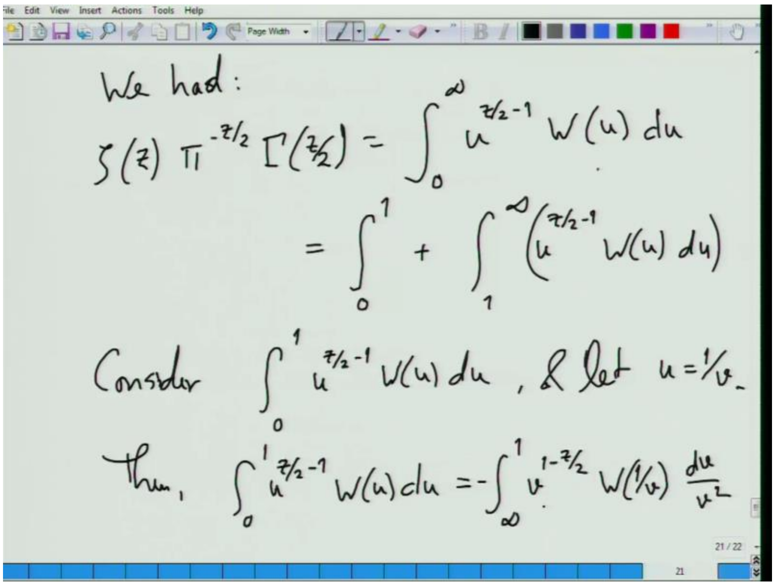

What did we have? Zeta z pi to the minus z by 2 gamma z by 2, is it gamma z by 2, yeah, equals this integral which says 0 to infinity u to the z by 2 minus 1 times w u, okay, and let me refresh your memory that we had a problem in the convergence here when the d l z is less than 2, because then this part sought of diverges as u come close to infinite zero. So, the problem really is when u is close to zero, then this may not converge, and that is what we want to get along get away from. So, let me split it in two integrals; this is a bad integrals 0 to 1.

This is where bad things happen plus 1 to infinity. There is no problem in 1 to infinity; in 1 to infinity, this integrant converges absolutely. So, this integral converges to something is sensible amount; no matter what the value z is, right, because in this one range 1 to infinity w u which is e to the minus u r are worse or that is even faster decaying one. That will dominate due to this z by 2. Let us consider the bad part 0 to 1. Let us do a change in variable; replace u by 1 over v, what happens to this? When u goes from 0 to 1, what happens to v?

It comes from infinity to 1. Due to the z over 2 it becomes v to the 1 minus z by 2. W u becomes w 1 over v; what happens to d u? It becomes d v over v square with a negative sign, of course, okay; are you with me so far, okay? So, since there is a negative sign outside and the limits are also reverse.

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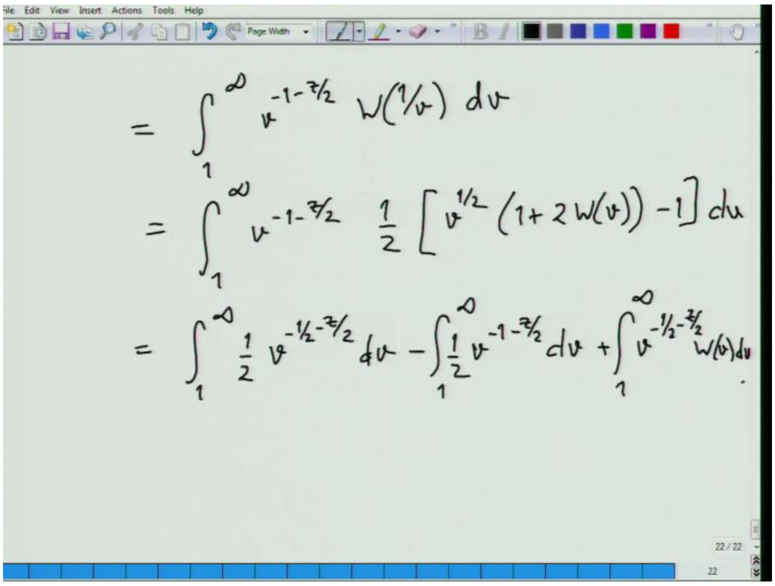

We can flip that both we get from 1 to infinity v to the minus 1 minus z by two because there is v square dividing this w 1 over v d v. So, everything is sensible here except w 1 over v, but we know how to handle w 1 over v; that is 1 to infinity v to the minus 1 minus z by 2. W 1 over v we just go back and stick this is, therefore, half u to the v to the half 1 plus 2 w v minus 1 d, fine. So, I will leave it as a workout at home.