Riemann Hypothesis and its Applications
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Lecture - 14

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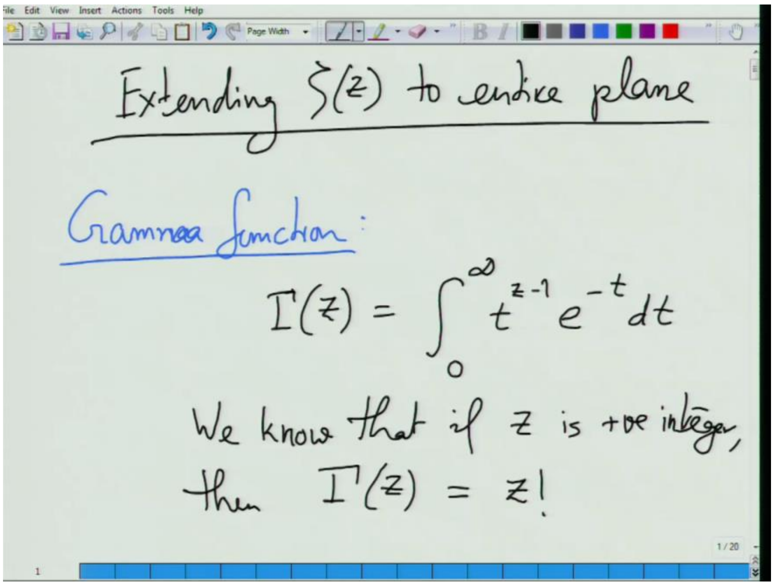

So, today we look at this how to extend this zeta function to entire complex plane. And in order to do that, we will take a dittour short dittour, but still we have to take it dittor, and introduce a familiar function to you which is a the gamma function, you know gamma function with it is right. Let us, this is the definition of gamma function of course this is generalize to complex numbers.

This is this was originally defined for real numbers as with the zeta function, and what we know about this function is if z is a positive integer, then gamma z equals z factorial. So, it generalizes the factorial function, and that is easy to say write this integrate this by parts now. So, do you get z factorial or z minus 1 factorial? Let see, do you get z factorial?

Student: ((Refer Time: 02:57)) z minus 1 factorial.

You integrate t to the z minus 1 that is right. So, then that is you integer z by z good. So, now if you look at look at this functions, and think of z now as a complex number. So, what can we say? So, firstly we have to talk about this guy being first it is define for positive integers very clearly according to this, but when z is not a positive integer, it does not have such a nice expression for itself. So, we have to consider or analyze the behavior of this function on non integral values. So, first question that one ask is does it converge everywhere?

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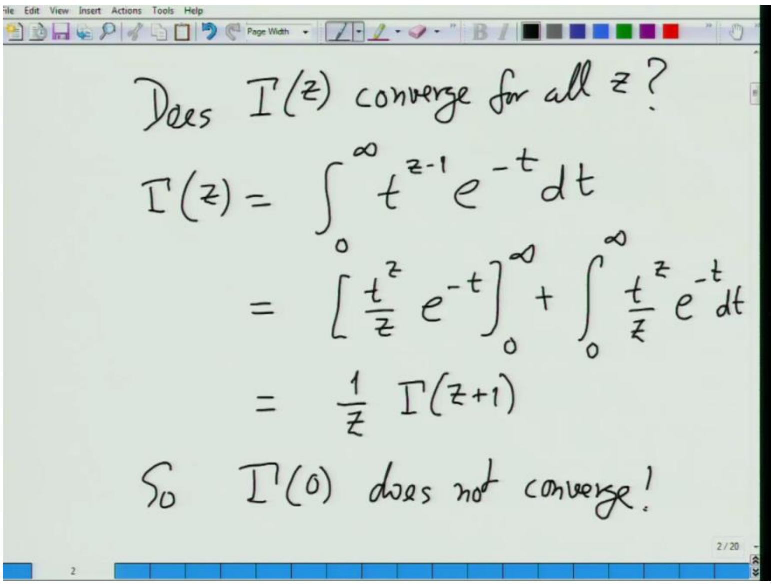

So, what is your guess? Does it look like converging for all z? Actually does not and it is easy to see that. So, so the restless get back to this. So, this definition is of course valid for any z that is how it is defined, and let us integrate by parts. So, we get t to the z, then minus 0 to infinity differentiate e to the minus t which makes this plus, and you get t to the this is as 40 equals 0 this is 0, 40 equals infinity also this is 0, so that first term is vanishes away, and we get 1 over z gamma z plus 1. So, actually this tells us, so that gamma z is z minus 1 factorial.

So this, but this relationship gamma z equals 1 over z times gamma z plus 1 holds for all z. Now, when z is 0 now gamma 1 is 1 that is straight away by the fact that is n minus 1 factorial. So, gamma 0 is actually infinite. So, it does not converge at z equals 0, so that thus imply that excuse me analyze in these function is can be little tricky. So, gamma 0 does not converge.

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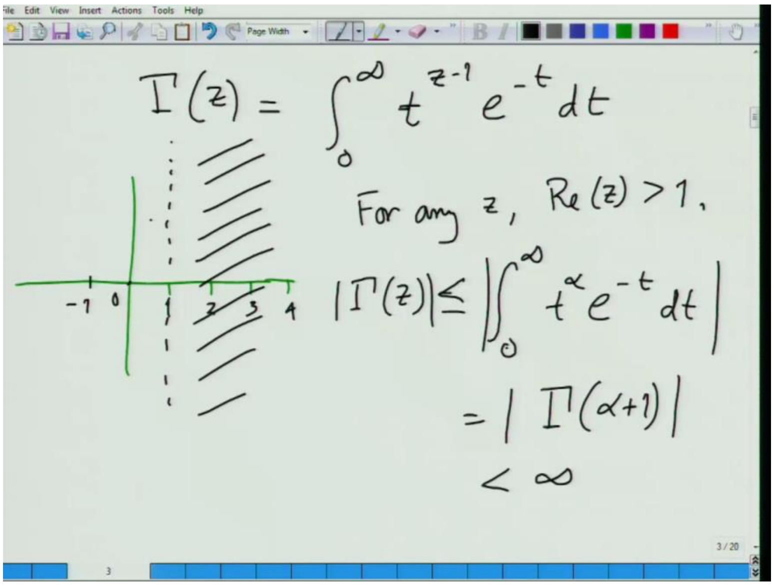

So, let us carefully analyze this, let write gamma z again, and the reason why we are suddenly spending time on gamma z will become a parent in the short way. So, z now takes any value in the complex plane. So, let us first draw this the complex plane never here. So, what we know z for every integral point it is well defined, and as it nice leave defined value. Now, for any z, so z absolute value of z is greater than 1 not absolute value when I have make mistake here real z is greater than 1, what can you say about the absolute value of gamma z?

This is going to a bounded by t to the absolute value t to the z minus 1, which because there is there in the exponent and real z is the complex part of z will vanish away anyway in absolute term, and this so this will be some t to the alpha, where alpha is greater than 1 greater than 0, e to the minus t is for c to the minus t is always positive d t the absolute value this. And this if you see will converge, there is the value this is less than infinity, why? We will firstly alpha is positive, so near 0 this is close to 0.

As you go away from 0 the value increases, but then e to the as t increases e to the minus t really starts dominating t to the alpha, and it very rapidly takes it to 0. So, this is a sort of informal argument, but one can make it more formal by just expand e to the minus t as a e to the minus t is power series not necessarily, because this is going as still going to be with

Student: ((Refer Time: 10:29))

Sorry

Student: alpha is ((Refer Time: 10:33))

Actually, basically we show that beyond a certain t, this more precise argument. Anyone if you have more precise argument, what is this why should this converge?

Student: Now, integral is gamma alpha plus 1 if you look into the real part only integers.

Only integers yes sure.

Student: The absolute value is gamma alpha plus 1.

Gamma alpha plus 1 yes that is true, but what do what can be say what gamma alpha plus 1, sure I can say that this is absolute value of gamma alpha plus 1, but why is this bounded.

Student: gamma alpha plus 1 is always ((Refer Time: 11:57)) no matter how been it is ((Refer Time: 11:58)) not infinity

Why.

Student: Alpha is a integer.

No alpha is not a integer.

Student: ((Refer Time: 12:05)) even…

But, can we say probably yes.

Student: I can ((Refer Time: 12:16)) to find for real value other than this ((Refer Time: 12:20))

Is define through this definition, and but I think you can say that, because if you see as alpha increases the value of gamma will gamma alpha will increase, because you the curve will always be above the curve for smaller alpha. We are multiplying this y t to the alpha which as long as alpha is of course greater than 0. Then it will always be above the curve, and so the area under the curve will always more than the area under the smaller alpha and therefore.

Student: only for t greater than 1.

Only for…

Student: t greater than one…

Only for t greater than 1 fair enough, but between 0 and 1 this is anyway bounded. So, then you do not had to worry about it, so that together will imply that this is not. So that know mix the lie somewhat better that at least it is well defined on the entire real line not only not only entire a line, but actually on the this half complex plane which is real z greater than 1. Now, what about the rest, how the complex plane, so this like again we are with is the situation like the zeta function there is a half plane on which it is well defined, and actually analytic this is also analytic.

It is analytic function of z, because again integrant is an analytic function of z, and the second variable for every value of t is analytic function of z, and the integral is fine now take you may key with analytic. So, it is analytic on this real z greater than 1, and we again raise with question and view the reason why I am want to define it for the entire complex plane will become a parent very soon. So, let us define it to the left of the real z equals 1.

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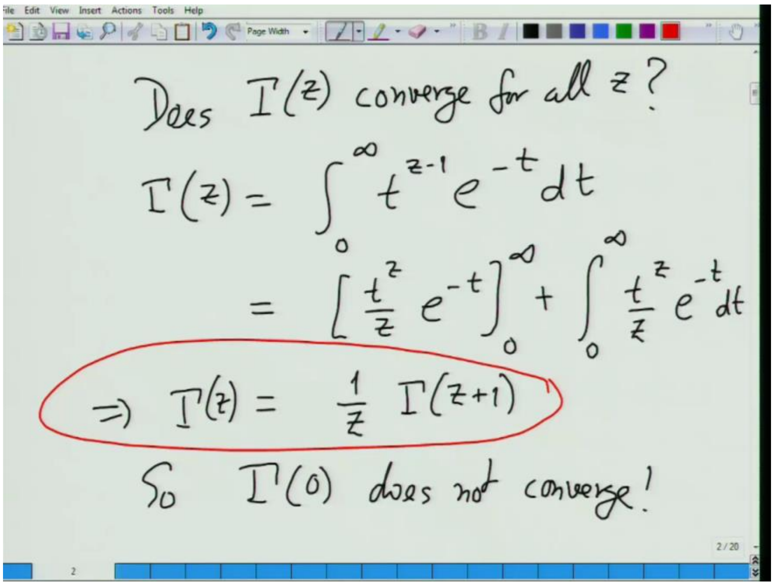

Now, here unlike the zeta function, the life is or the process of defining this is much just use this equation, this is value for all z. So, which means if you want for example, gamma half that is gamma 3 by 2 divide by half. So, you got the definition of gamma at this point. In fact, this is true for all points you can comes, so there is take this strip between 0 and 1 here, you can define the gamma value in this strip by looking at the next strip between 1 and 2 and the gamma value is there, and doing the appropriate translation. Once you have the gamma value is here, you can define the gamma value on the previous strip between minus 1 and 0 just copying there and keep going back, and this way you can extended it all the way.

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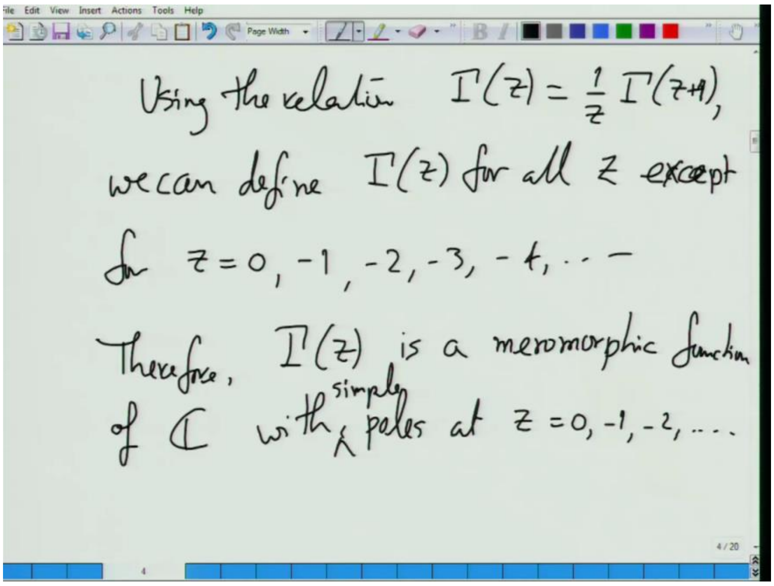

Not all for all z except when of course it diverges. So, when does it diverge? Does it diverge any wear on the right side of this plane real z equals 1, it does not it is value at converges everywhere we are just saying. Does it diverge on any point between the 0 1 strip no, because it is just pulls up the value from the previous one, and it is you it uses as this relationship, and the only way it will diverge is when z equal 0. So, the first value on which diverges starting from the right infinity is at z equals 0, except for z equal 0 what is the gamma is minus 1, gamma minus 1 is gamma 0 divided by minus 1, and minus 0 diverges.

So, minus 1 is also infinity, minus 2, minus 3, minus 4 essentially all negative integers including 0 is very diverges, it does not diverge anywhere else. Therefore, gamma z is a meromorphic function on the entire complex plane with poles at whom by the why does it, why I am calling it pole at what is the nature of gamma 0; it is gamma 1 by z. So, around therefore has a simple pole, pole of order 1 at z equal to 0 at z gamma minus 1 that is gamma 0 divided by minus 1.

So, it is really again the same simple pole will get pulled in here, and so and they will all poles. So, though higher order poles occur anywhere, so each of the negative integer point we get simple poles, so good. So, now we understand the gamma function in the entire complex plane. Now, let us connected with the zeta function and that is very simple, but it is the connection is very simple, but the idea that yes there is such a connection is non real.

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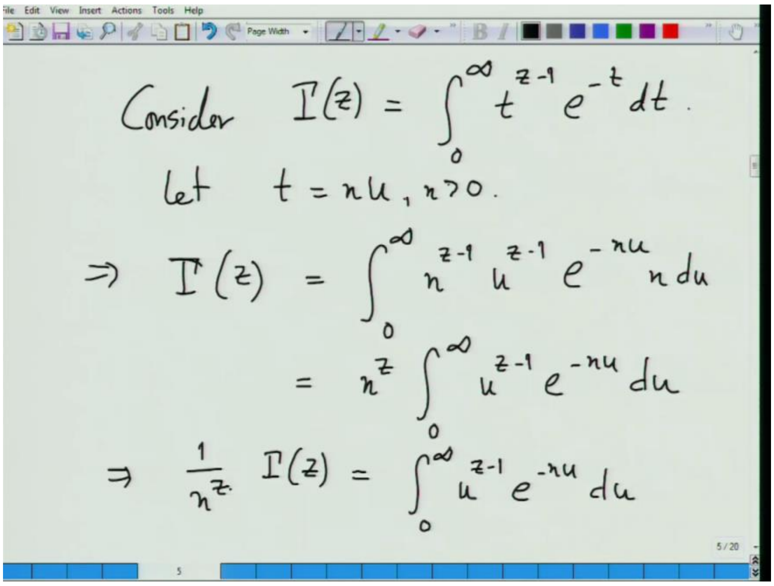

So, let us again consider the definition of gamma function no then, and let us do a change in variable no the other way wrong t is n u. So, for t everywhere replace n u, so then what you did, gamma z is what happens to n greater than 0. So, what happens to the limits this stay 0 to infinity times n to the z minus 1 times, u to the z minus 1, e to the minus n u and d t is n d u, no d u, so n to the z of course is comes out of integral. Now, these guys familiar 1 over n to the z, this is one term in the zeta function summation. So, let us summit over. So, this equation hold for every n in particular for every positive integers this equation holds.

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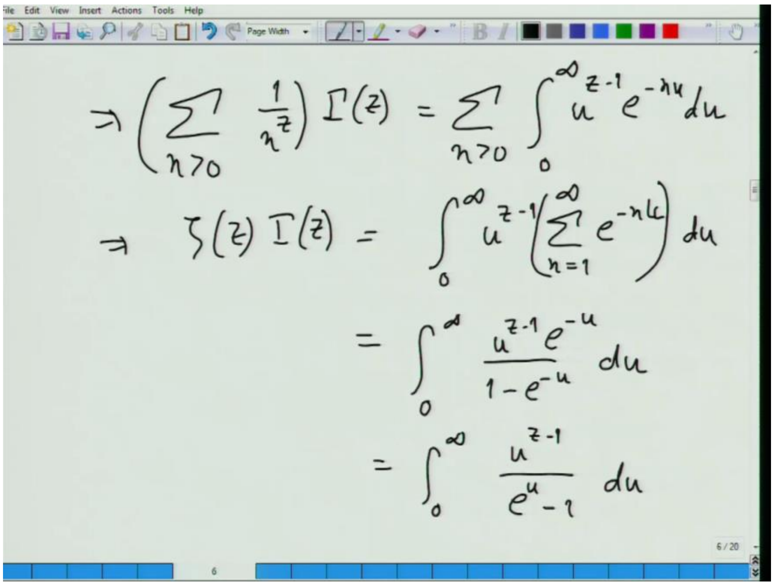

So, you sum over n greater than 1 no not in greater than 1, sum over n greater than 0 1 over n to the z gamma z, this gives you gamma z this of gamma, gamma of z this is zeta function times gamma z equals this sum. Now, what I am now going to do is interchange the integral with sum, again I have to be careful here these are infinite sums and integrals, but because of the uniform convergence on there of the thing inside the integral inside I can do this. So, I will go ahead into this, and what is this? This is a familiar series, this geometric series with multiplier being e to the minus u, what?

Student: ((Refer Time: 23:46))

There should be on e to the minus u on the top you are right. So, now we get this relationship between the zeta function and the gamma function, zeta times gamma is this integral on the right hand side. We know that gamma is defined over the entire complex plane. So, if we can show that the integral on the right is also defined over the entire complex plane, then we are done we got the definition of zeta functions as well over the entire complex plane.

So, let us look at the integral on the right hand side, is it defined over the entire complex virtue means it does not converge first of all over the entire complex plane. It is unlikely, because it just half for the gamma function integral that when real z is less than 1, then z minus 1 the real part is less than 0, and then the converges does not quite hold actually it does diverge. So, we cannot directly use this relationship to conclude that the integral on the right hand side does converge. Unfortunately this is slightly different from the gamma function integral which allowed this beautiful relationship between gamma z and gamma z plus 1 which allowed us to extend it, which no longer holds for this kind of integral, so that is also a problem.

So, just by looking at this relationship it is not clear how to extend it, but one can actually x use such a relationship to extend easily, but for that one has to do little more work. Now, there are two ways that little more work can be done, and both these ways were given by Riemann in this paper. One way is to start with this integral itself, and do something to this integral on the right hand side. And then transform it in a way that eventually it takes a form which we conclude it is define to over the entire complex plane, and then therefore conclude that the zeta function is defined over the entire complex plane.

The second way is a more interesting, which in that he it is start with the slightly different relationship between the gamma function and zeta function, and that is one I will show you, because that gives something even more better then the definition of zeta function over the entire complex plane. It also gives a relationship between different values of zeta function just like there is a relationship between different values of gamma function, and that relationship will be very useful as later on. So, let us adopt the second way.

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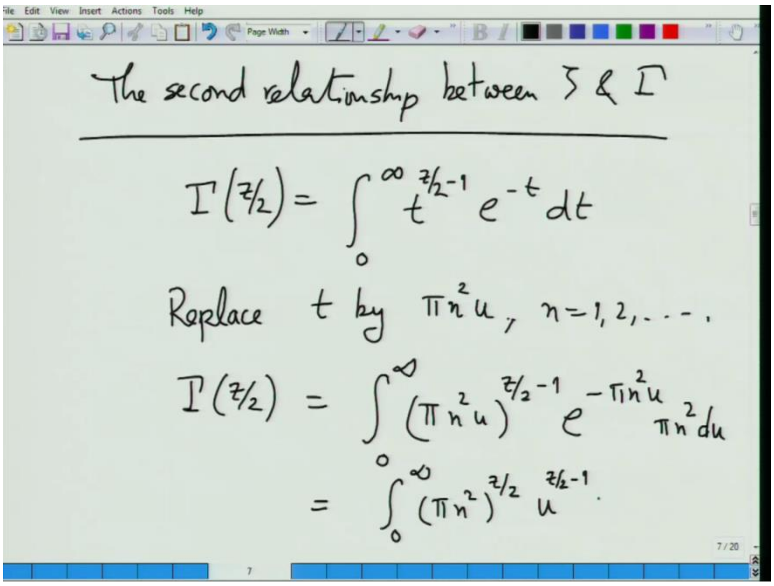

So, the second relationship, so again we start with gamma function, and this time we do a change of variable again, but not in the way that who. So, I should have done this write here. So, I will not consider gamma z, but gamma z by 2. Now, replace t by I think it is pi n square u. So, this is a move fun a transformation as remarkable when his the earth thinks that you just wonder how did it with come by, by magic or something, but it will you see that later on it will fit in perfectly with our analysis.

So, what happens then, of course limits remain the same of course n is of course, 1, 2 these are all n is always positive integer, pi n square u to the power z by 2 minus 1, e to the minus pi n square u and d t becomes pi n square d u. So, this gives us 0 to infinity pi n square to the power z by 2, pi n square z by 2 minus 1 and then 1 pi n square coming from here u to the power z by 2 minus 1, e to the minus pi.

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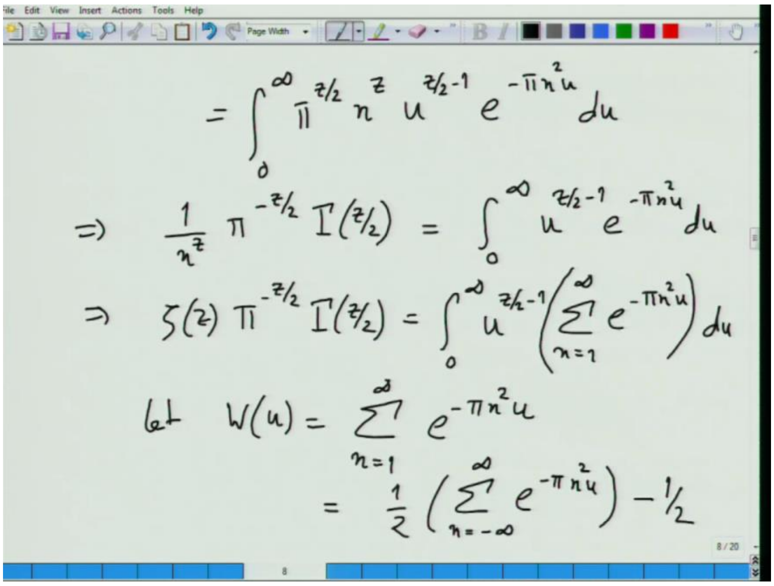

Now, this was expand it out pi to the z by 2 n square to the z by 2 which is n to the z which is exactly the form we like, and again do the sums use that inform convergence to push in the sum inside we get zeta z. Now, this sum which is insight the lies as the previous sum one there is, but do it is advance even faster than the previous. And pi where is the reason why will we choose this, because of the there are some point where it you previous integral was diverging; we choose a much faster converging sum here, so that that divergence can be eliminated, so that is a philosophy.

Of course, the question is exactly why this form, because this is something which we can still analyze not as well as the previous geometric series, but still it is amenable to analysis. So, let look at this, let us give it a name W (u) one thing there you can immediately observe here is that this is symmetric minus 1 by 2 I think see for whether n is positive or negative this takes a same value, because there is n square there, and for n equal 0 this is 1. So, basically all that is taken care of it.

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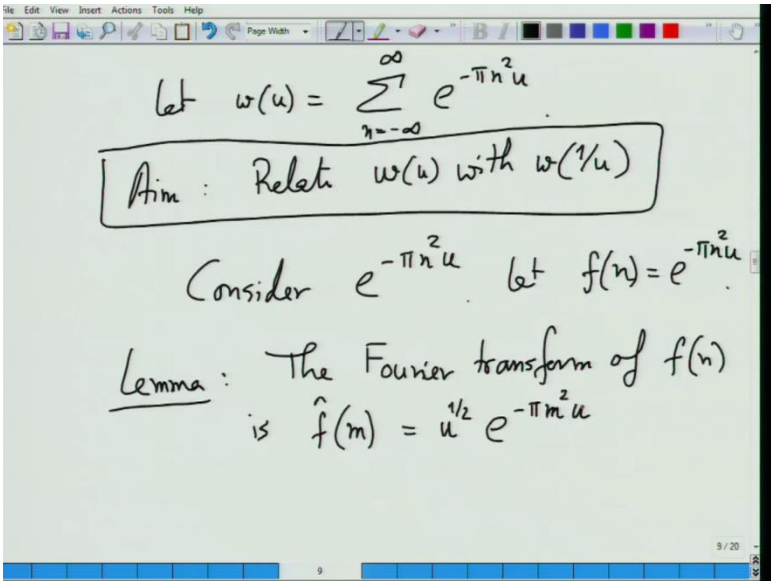

Let us give small w (u) is called as, now this sum is it fairly nice sum those of you have tabled with some bit of Fourier analysis, no does not matter. We will do a bit of Fourier analysis here. So, it is really a (( )) and again I will again warned you that there are parts that I am ask me why somebody could think all this I do not know, but the fact is at it has been discover and behavior. Of course, to demystify it little bit, I do not think anybody sat down and 5 minutes, and in 5 minutes came up with all these analysis, it required a lot of hard work I am sure a lot of different experimentation with different sums, and finally identifying what is the right one.

So, now you want to analyze this, and in particular what I want to do is w (u) with w 1 by u. And we let just define a function, how are you use this symbol f have you not, and consider the Fourier transform of this. So, let us mish prove the following lemma, just read out this, I should consider this as a function of n not as a function of u. So, this is defined for different values of n, and then the Fourier transform of this is f at m or we can still write f at n all is written. There is something here I think it is square root of u or 1 by square root of u. So, do not take my word for this when we derive this will forget this out.

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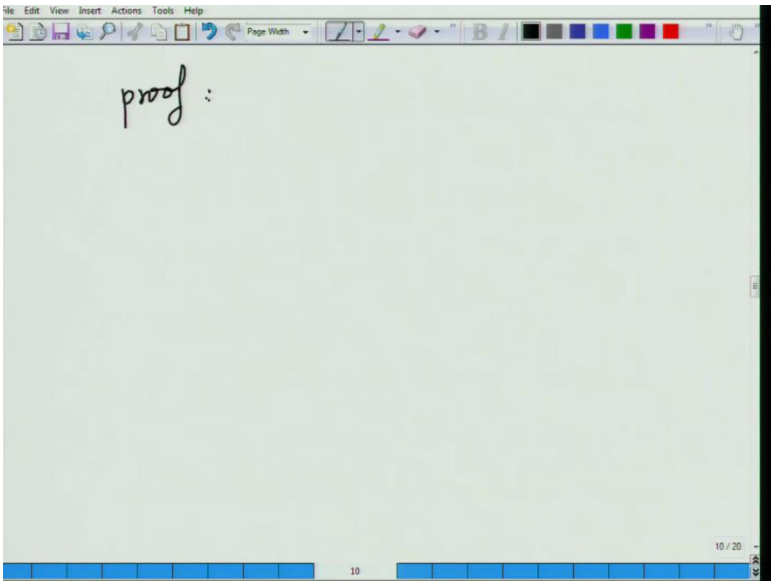

So, what is the Fourier transform of this function, anybody remember definition of Fourier transform of function?