Riemann Hypothesis and its Applications
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Lecture – 12

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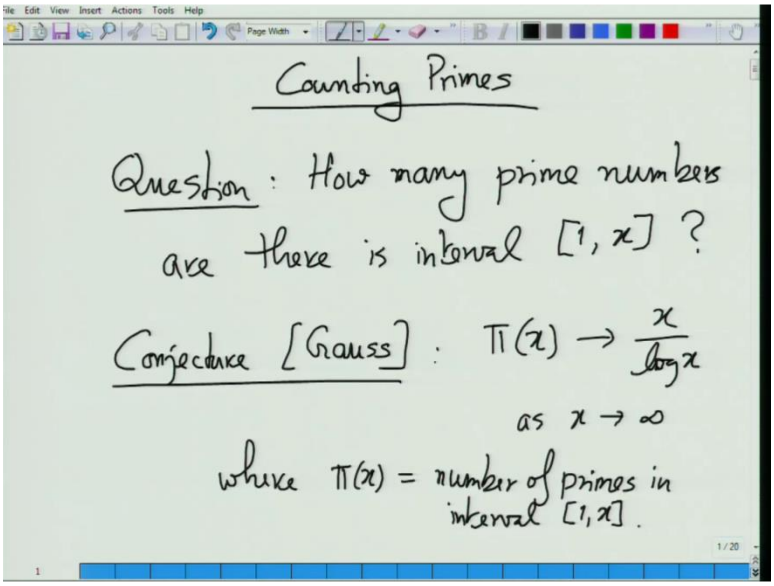

So as primes less look at the problem of counting prime numbers, and this is the main topic all these goes any way. So, let us see where we go from here I am will slowly bring in zeta function, and third setting its properties.

So, the motivation for this is simply the following question. So, given an X how prime numbers are there, and the interval one to X. This has been around for a long time sort of fascinated lot of mathematicians who try to come up with some formula capturing this number of prime then pretty quickly people realize that, it does not exist any nice formula for this. Because prime numbers seem to be have very random numbers.

And so, any sort of nice formula expressing exact number of prime numbers in the interval is not likely to exist at all; however, it was again observed to, and particularly by cos who did huge amount of calculations about number of prime numbers in interval that, there is definitely some pattern here. In fact, what you observed was that the number of prime numbers are very close to X by log X they are not exactly equal to X by log X, but are within a small distance away from X by log X and this was pure numerical from hand calculation on that time there is no computer. So, use to in all is paired time it is said used to compute tables of prime numbers, and then you based on those calculations e.

Made a conjecture that this is conjecture well it was made in some different form of others also, but it was very specifically highlighted by gauss. If we denote by pi X the number of primes in the interval one to X and pi X tends towards X by log X as X goes infinity this is not equal to but a tends towards which is another way of saying that limit X goes to infinity pi X over X by log X tends towards zero not zero tends towards one.

So, what is a conjecture? In that conjecture motivated to the lot of interest around that time which was early nineteen century, and Riemann who was the student of gauss; spend quite a bit of time studying this prime numbers not the density of the prime numbers as it is called. And it was he has name we have be looking thinking seeing is there it was he that came up with this connection of the density of prime with this complex analysis, and zeta function, and everything else how lot of things other things may he wrote a very small paper about eleven twelve pages long.

On which was counting prime numbers or on density of prime numbers, in which he laid out this whole program define the zeta function it was still around earlier also, but this the complex very end of this zeta function it is pi then study the properties of way made the connection with density of primes made the Riemann hypothesis, and derive the conclusion was remind happened it just about everything. It is at in small paper, and it is there avail at least translation that paper are there on the web and this very interesting read all of you should go and read it.

So let us trace back the thought process of Riemann from we will not follow completely that, but more or less. And at many points in this sequence that I am going to tell you a sequence of step that I am going to tell you. You will realize that several of this steps seems to be like pulling a rabbit out of hat there is no a prime justification s two y are be doing this, or how did this idea come about? And that was genius of Riemann that he just came up with so many new ideas. In that paper that is will be very hard for me to give you a intuition or justification, why this idea is being used?

We will realize the use of that idea only when we apply it and do the calculations see yes it does work. And that is the nature of all great mathematician mathematical ideas they are all very simple, and they all seem like pulling a rabbit out of hat. So, and I will try to point out those idea at least from my prospective things are I do not understand I will try to point out I do not know why this how this came above, but it has come out.

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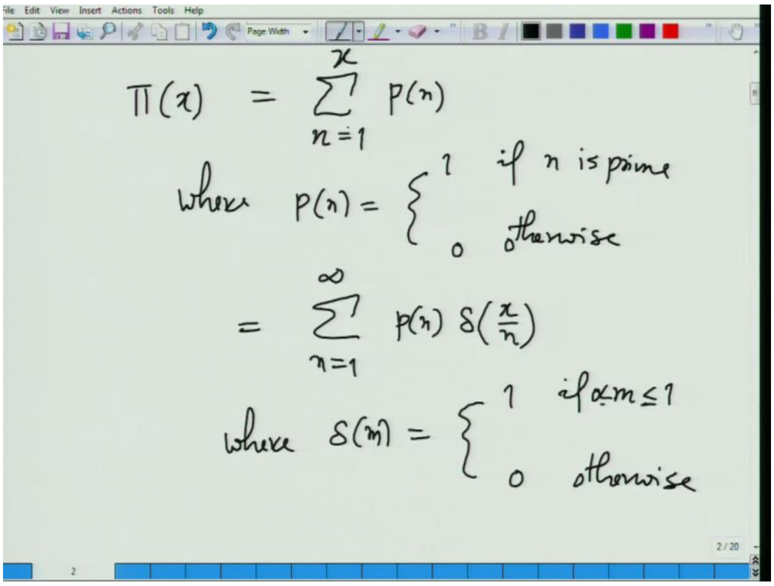

So let us go back, and we are interested in this prime counting function which is pi X. So, let us spend some time on this, the pi X is all primes in the interval 1 to X. Let us write it more mathematically, and that can be done by writing something like this that N going from 1 to X, and let us write here a notation which would mean P sub where P of N is 1 if N is prime 0.

Now, this is a sum going from 1 to X which is a variable quantity. So, it is a finite sum not even terminating at a fix point. And analyzing this sum is especially given the quantity inside which is also not very nice in sort of is should be 1 and 0 at the primes are sort sum of randomly distributed it is 1 and 0 with some random probability. Then it becomes easier to analyze if we can transform this finite sum to an infinite sum. Because then we at least we do not have to worry about to up to what point we have to sum up we just sum up all the way. One problem is taking care the problem of P N is still there, but it at least the sum in problem is taking care off. So, how do we translate this from to a sum from up to X this is sum up to X how do we translate this to the sum up to infinity is a very straight forward idea.

That I can write it as N equals 1 to infinity P of N times delta of X by N. where delta of M equals 1 if M is between 0, and 1 0 otherwise. This take a delta function you have the infinite sum here just to feel careful I should be saying that this is between M is between 0 to less than equal to 1. Because X can be integer, and want to sum up that point. So, that is nice that instead of finite sum nor infinite sum both the limits are well known at, but that finality I have paid is I have introduce another funny function delta here multiples multiplied the P N which was to begin with not. So, easy function that to analyze. But then just recall last lecture this delta we have a nice for handle on.

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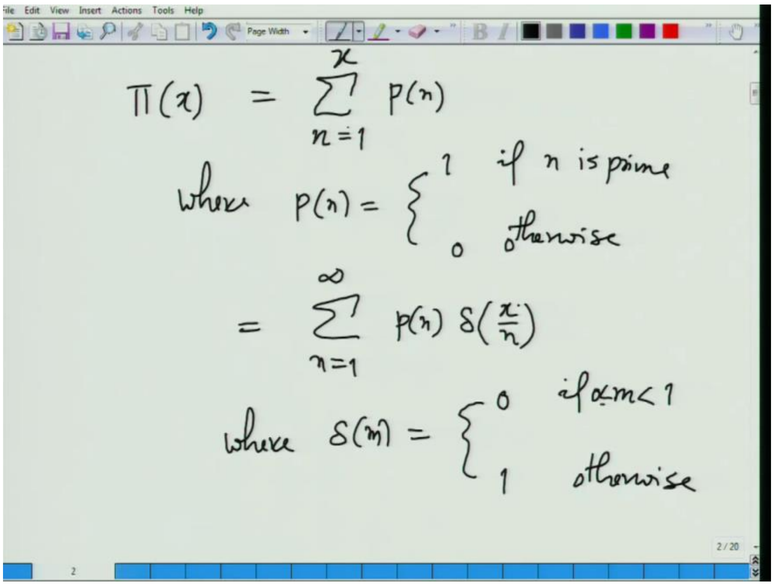

We can stick in place of delta, where are we? P N, and what can we write for delta X for N? If we recall what is that function it was 0 between 0 and 1 and then it would step up to one all the way. The delta function there are I want is should be 1 between 0 and 1 and then go down to zero. So, that is the N by delta hat N by X N by X if X is bigger than N. I think there is something wrong here.

I think we already had it correct. The delta N here should be 0 and here is 1 this delta X by N. And here if N is less than X then you want to be 1 as long as N is less than equal to X, which means as long as X by N is greater than equal to 1. You want it 1 as soon as N becomes bigger than X, you want to become zero and therefore, that so, we already have had it correct.

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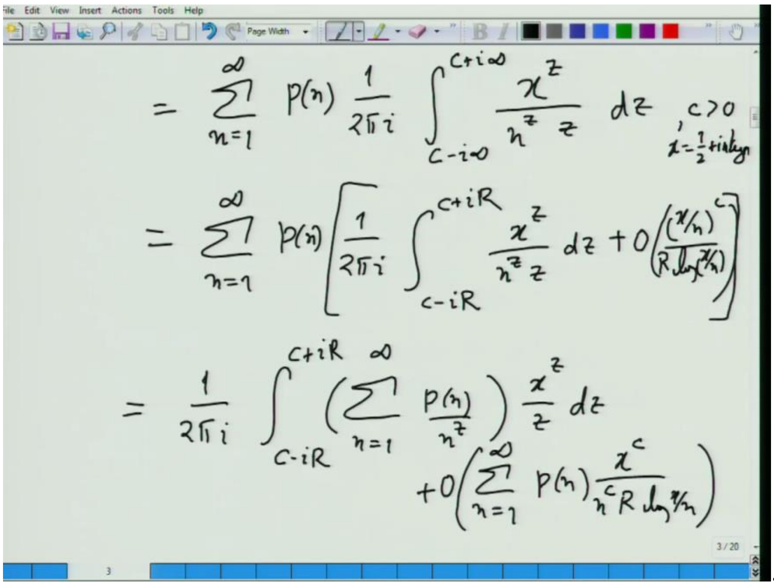

So, now let us invoke that previous theorem, and we can write this is now, who remembers that integral it was 1 over 2 pi I integral C minus I infinity to C plus I infinity X is now X by N. X to the s by N to the s and let us now that we are treating with it write as Z because we have been typically writing Z for complex numbers, and C greater than zero. So, I have to be careful here so at exactly 1 this does not quite fit N.

And when do when this take values exactly by when X equals N. N is always an integer. So, as long as I say that X is lets a half plus integer, which is pi I can always count number of primes between 1 and X, where X is half plus an integer that number. If I can do that I know the number of primes everywhere that is not a problem. So, this is just a simple technical point which I can get around by fixing X always to be half plus integer and that is what we will consider now, and forever x is half plus integer.

Now, I did turns out that this is fine I can use this, but this is not quite what we would like. Because see, why are we trying to do this? We if I put it in this formulation now, I have a complex integral. I will try to see our, what is our goal is to estimate the value here right now, that I have written in terms of complex integral. I will use all my complex analysis information to evaluate that complex integral, and we saw some examples last time. And the on all of them the way we did was we want to evaluate the integral on some curve or line, we fix a domain whose one of the boundary is this say line then put other lines there.

And then use a coil this residue calculus formula to see evaluate the integral along that entire boundary of the domain. Then we say that, on the entire boundary that we are interested in that we leave a side and all other boundaries we say that this goes to 0. So, we can ignore that, and therefore, we get this integral on the kind we want. So, that was strategy in that the strategy we were going to use here, because as seems like a natural want to do.

But if I put my line on which I mean integrating as going from minus infinity to plus infinity, then I do not have a any definition of domain for defining a domain in a proper way. So, that whose boundary is well defined I need to limit this integral to a finite called it cannot be an infinite line.

So, that is fine instead of this I can write it as we already. In fact, we did evaluate this integral C minus IR to C plus IR, X to be Z divided by N to the Z, Z DZ, and then there was an error domain associated with, how much was that? X to the C by R log x. By N, and here also you get X by N. So, I should replace this by this now, let us continue with our simplification of this, the integrant inside is an analytic function. Moreover, as if we look at this let us me write it down then I will justify this.

I am writing this one second part is clear I mean I just the sum is separate it take there, but here what I done is the swap this summation integral. And the reason why I can do it is because, this sum is uniformly convergent for absolute value of Z greater than 1. Why is this uniformly? first of all if this is uniformly convergent then, sure you can start summation integral we have seen that. Why is this uniformly convergent?

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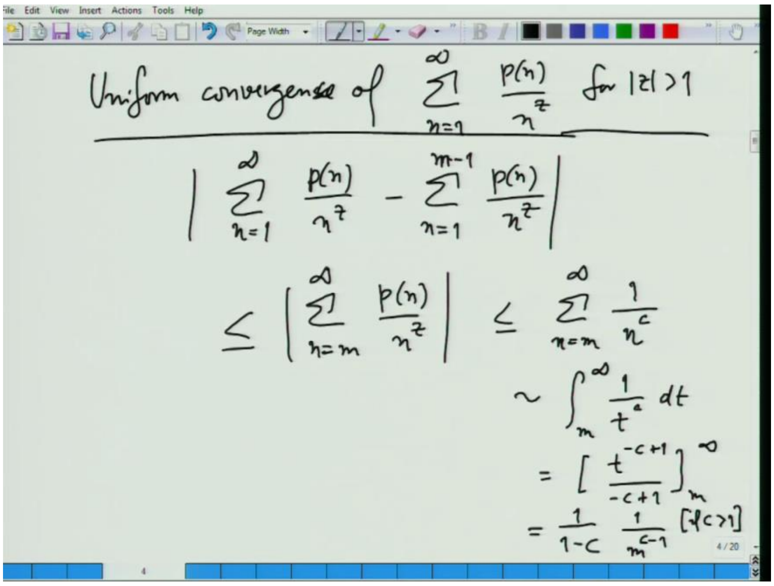

What do we need to show that this is uniformly convergent.

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Even if PN is there PN is always either 0 1. So, as assume PN is all once. So, what you get is there is zeta function right there is a zeta function, and we although we not seen that zeta function uniformly convergent. So, let us just write this by, what is the condition from uniformly convergence? You take a finite any finite sum, and subtract it from the infinite sum take the absolute value that the difference between the two is, if value which goes to towards 0 as the length of the finite sum increases. So, let us say if the length you take N equals 1 to infinity P N by N to the Z and subtract it from N equals 1 to M minus 1 P N over N to the Z look at the absolute value this is less than equal to absolute value N going from M to infinity. P N over N to the Z this is less than equal to summation N equals M to infinity.

P N is at most 1 always. So, it is 1 by absolute value N to the Z, which is N to the C CZ, and this integral is going from C minus IR to C plus IR. So, we only look at the real bigger absolute value you only look at the real part if it is in exponent so, it is C. That also tells me let us first do that sum, and we will come back this N to the C, C is greater than 1 well not we do not know, but if C is greater than 1 then, what is this sum? This sum you can approximately there is a integral M going to infinity 1 over T to the C, DT.

Within some constant factor of this is t to the -C +1 by -C +1 M going M to infinity, and this is the place where we use the fact that C is greater than 1. So, the second part of this is 0, and the this is therefore, 1 over 1minus C, 1 over M to the C minus 1. This is the C is greater than 1, and now, you see that this is uniformly convergent because, as M tends towards infinity. And there which is the as the approximation of you would sum increases this tends towards 0. Irrespective of the value of Z that you take so, that is settles the fact.

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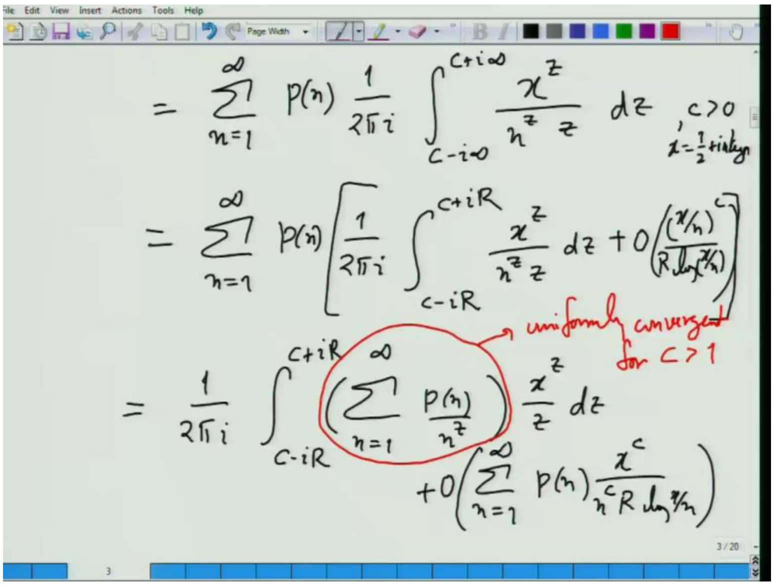

This sum is uniformly convergent. But we need C greater than 1 remember original sum for the delta function was valid for any c greater than 0, but this uniform convergence is not true for any C greater than 0 we have to assume C greater than 1. So, from hence forth we will assume that C is greater than otherwise you cannot see otherwise you assume, how does they help? Is your question that y is how does uniform convergence help in switching? Is what you are asking all to switch because you can show that the result without the switch, and result after the switch they are both equal. And the way to show that is that your question. So, when the way to show that is I think I have briefly mention earlier is to look at.

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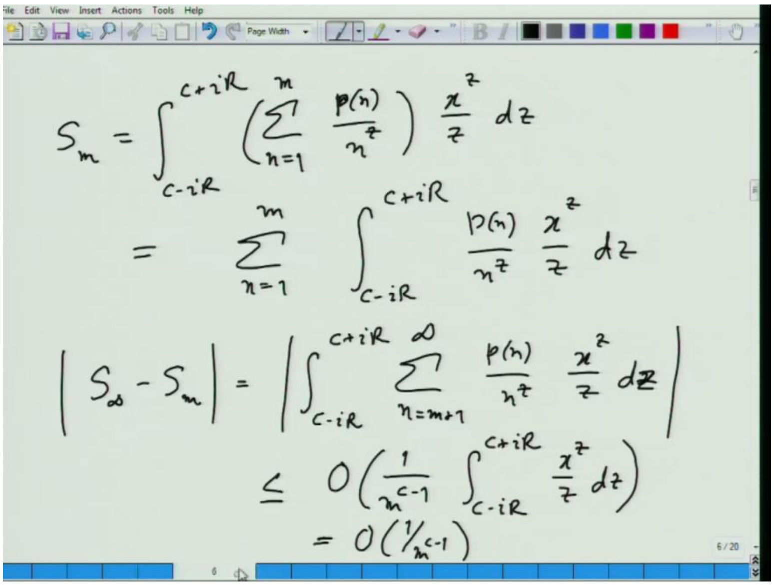

Let us take this basic case. So, that you look at this integral C minus IR to C plus IR. And one to N P N by N to the Z, X to Z by Z DZ. This is the finite sum inside this let us call this S M. Now, what we know is that this surely is equal to N equals 1 to N integral C minus IR C plus IR PN over N to the Z. Because is a finite sum right say it is integral of a finite number of terms which by we know by standard integral theory that this is the sum of the integrals. So, this is fine.

Now, let us look at s infinity minus SM, S equal to. And this is less than equal to this by uniform convergence is 1 by order 1 by M to the C minus 1 we just derived. So, this is order 1 by N to the C minus 1 times this integral C minus IR to C plus IR X to the Z by Z DZ. And this integral is equal to what you know what this integral equals, it depends on whether X is less than 1 or greater than 1 depending on there it is either 0 or 1 so, at most 1. So, this is really order 1 by N to the C minus 1.

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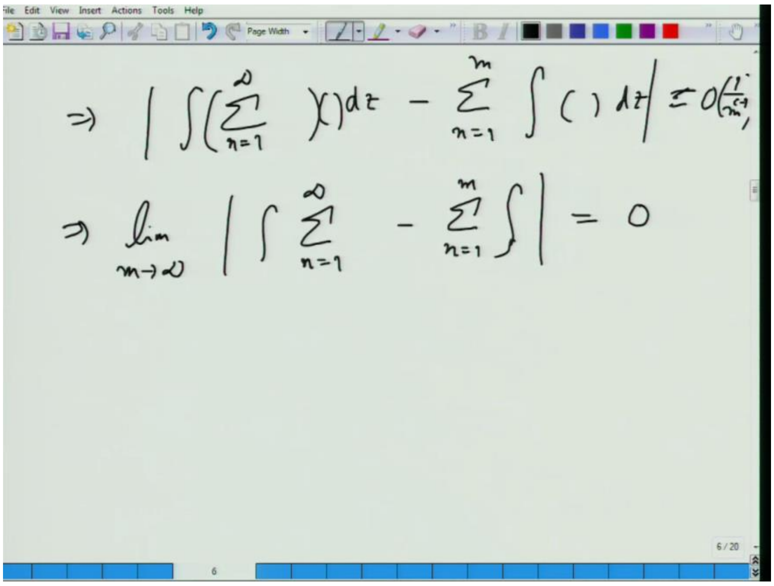

Now, use the fact s in, what is the s infinity? If the integral inside the sum nice sorry the other way the sum inside the integral, I am not writing this is stop here that is as infinity minus SM was that finite sum inside the integral, but we know that is equal to that sum N going from one to M of integral. This is equal to order 1 by M to the C minus 1 now, take the limit as M goes to infinity the right hand side goes to 0 what happens to the left hand side, well this is fixed there is no change to this.

This tends towards that infinite sum followed by integral, and that absolute difference is zero. So, this says at integral of sum is equal to sum of integral. So, the uniform convergence is critical here we do not have uniform convergence we do not get. So, is does that convince you the uniform convergence?

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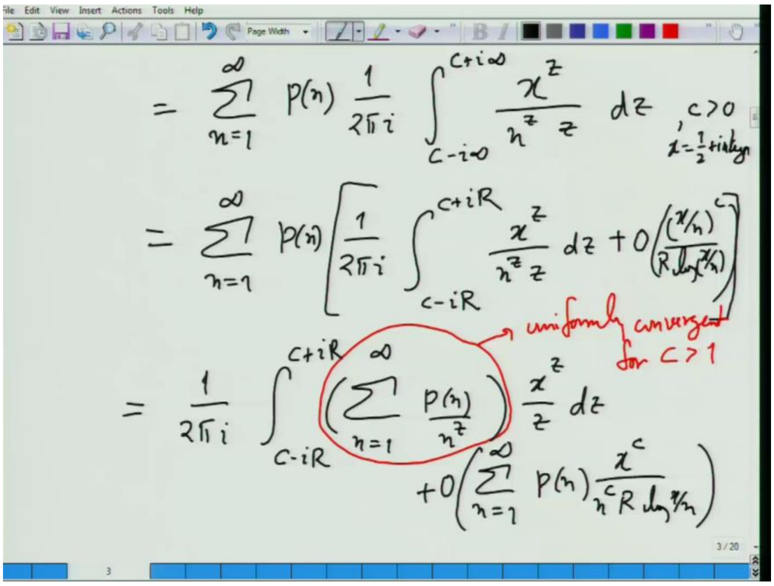

So, now coming back to this we have this uniformly convergent what is C greater than 1, and this is alloying we to swap the integral with sum we get this inside the sum, and now, if we look at this, is well not quite zeta function zeta function had PN equals 1 everywhere, but some close cousin of the zeta function.

So, this gives you an idea that here is a integral complex integral with something like zeta function sitting inside times X to the Z by Z which is was the integrant of for the delta function. And you integrate this on this line minus IR to plus IR at C greater than 1 this integral is equal to the number of zero from one to N up to this error term. So, now, we have two jobs avail more than two jobs let us, but two immediate jobs are first see, what cousin of zeta function this is?

Right at because I want to get some early because this PN being sometimes zero sometimes 1, and do not even know exactly where it is 0 where it is one that is not a very good function to analyze cannot really. If I ask start ask you know querying about the property of this we will probably get start. We want some nice of function one very important requirement. The second important requirement is to the estimate this error term because if this is error term is to bake then again we will not be able to say much.

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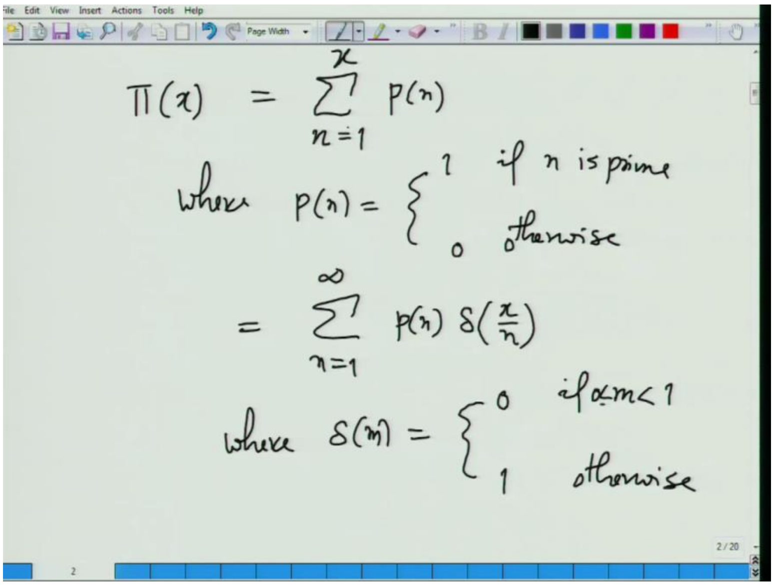

Now, as it turns out so, by the way here already is this beautiful idea of translating this finite sum to the infinite sum with the delta function multiplying. And which is written as just complex integral. So, this connection between the complex integral and the prime count established right here, and this is beautiful idea, and which is not at all; obviously, just a incredible inside which let to this connection.

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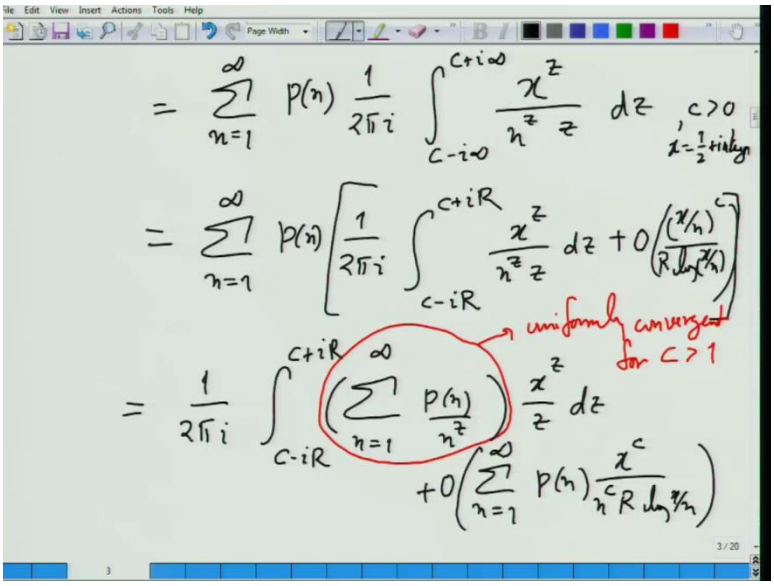

Now, as it turns out that this particular sum P N over N to the Z is not amenable in expression as a nice enough function. So, although there is goals for very loadable we started pi X and did manage to establish this connection, we cannot really make too much progress from this. So, we have restart it does not work. What the problem that pi X is messy very messy function to handle. So, what we, and then we are trying to connect pi X to sum cousin of zeta function which is a does not seem infeasible. So, we do the next first thing we start with the cousin of pi X, and try to connect with it some cousin of zeta function. So, at least two levels of interaction and then it works.

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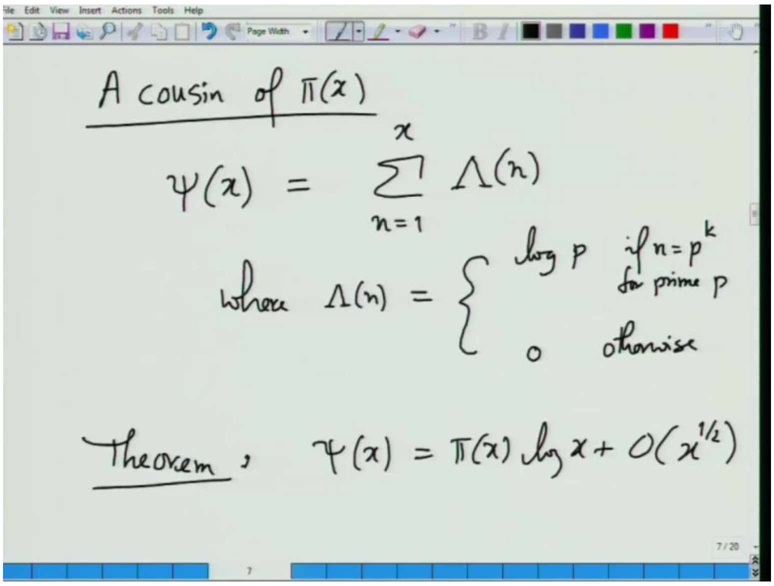

Here is a cousin of pi X. And is called psi X is this sum of one N going from one to X of lambda N where lambda N equals log P, if N is P to the k for prime P now, pi X had P of N which was one precisely when that is prime otherwise it was zero this is slightly more not general it is a function which is non zero at more points. At every prime power it is non zero, and it is non zero with value log P with P is the prime of which we have go forward. Zero and so, you can see this is a close cousin of pi, and as it turns out these two are very tightly related.

And fact there is a theorem which I am going to prove not today because; toady is the task is to connect this with complex integral with this theorem is interesting. And it was proven by Riemann this paper as well that the psi X is pi X times log X which is sort of you expect because, at every prime instead of one there is log prime that way. In fact, every prime where you say. So, this kind of a relationship one can suspect that psi X would be that actually pi X times log X, but what is somewhat unexpected is that the error in that relationship is very small.

This error is at most order square root of X. Is this point is clear. So, we this tells us that if I can get a good estimate on psi X, we got a good estimate on pi X whatever, is the estimate for psi X divide by log X that gives a very good estimate of pi X as well. We restart the whole thing with psi X in place of pi x.

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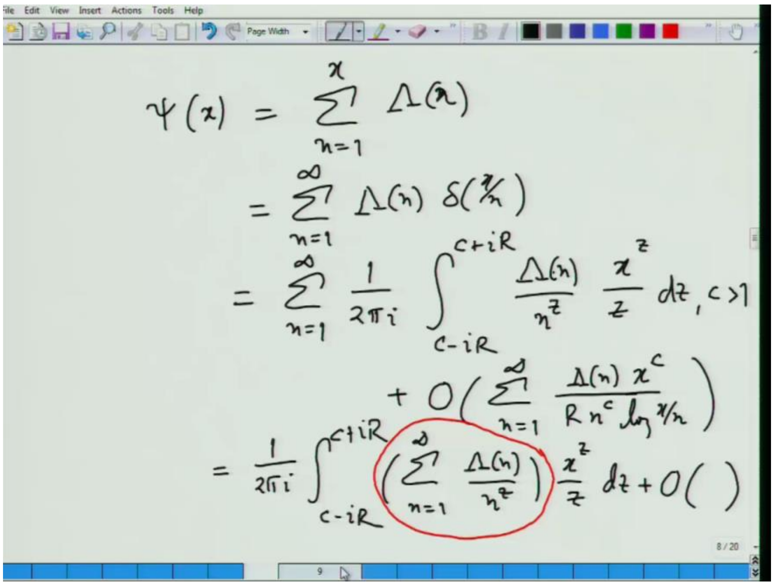

Now, psi X is summation N going one to X lambda N, this is same as summation N going from 1to infinity lambda N delta X by N this is summation N going to N infinity. I think this expression I will write, and again I can interchange this is for C greater than 1 this to make think work write on the beginning. And that is not worry about error terms that is look at the main term. After extending the integral with sum we get this. And now, this is the cousin or they assumed cousin of zeta function is this is only difference of course, is an instead of P N we have lambda N here. So how this get related to zeta function this is how.

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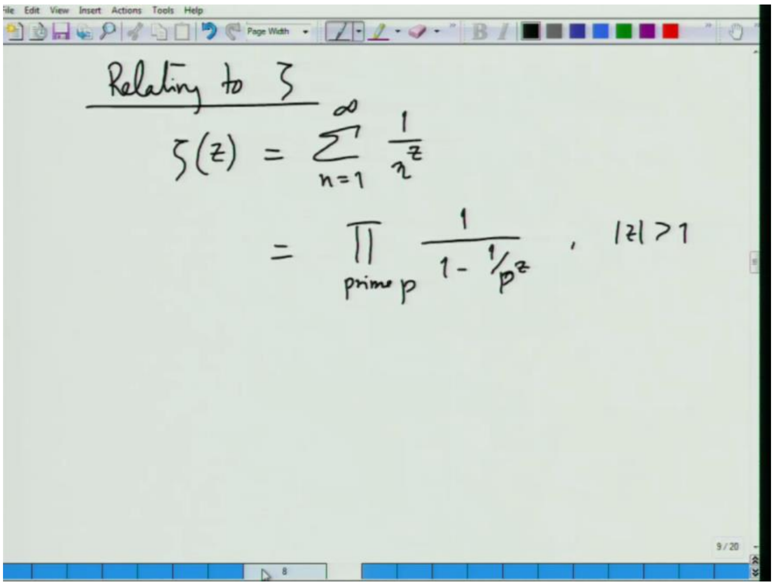

What is zeta function? Zeta of Z is this. And we know that we can write this as product over all prime P we did this long time ago. And 1 over 1 minus 1 over P 2 because, this express is as a geometric series, and then we can get it ((Refer Time: 43:28)). But now, that we have started worry about convergence is we have to see that this is also convergent this product, and well requires a bit of an argument, but you can see in the not too difficult that as long as the absolute value Z is greater than 1 this convergence nicely this equality holds. Let us takes this condition that absolute value of Z is greater than one. So, that is zeta function now.

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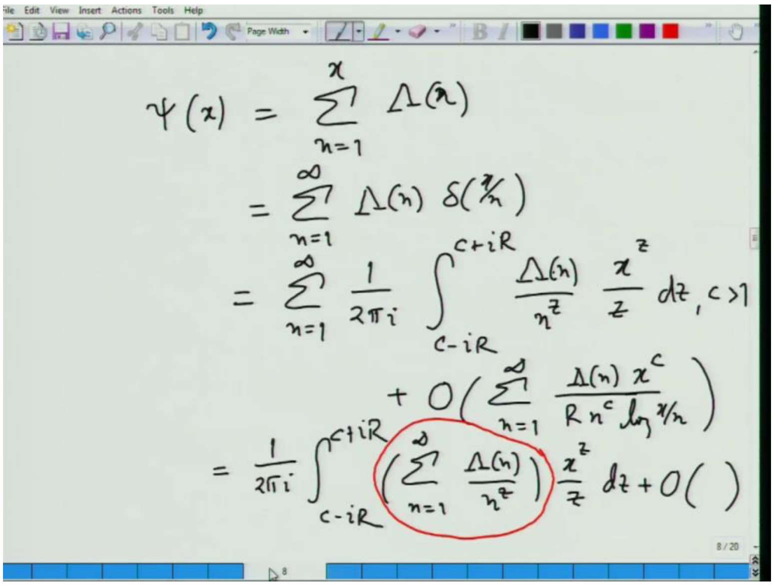

If you look back at this lambda N the way is defined it is zero whenever N is not a prime bound. So, this sum is actually only a sum over prime powers. In fact, same was the case with summation N P N over N to that was only sum over primes.

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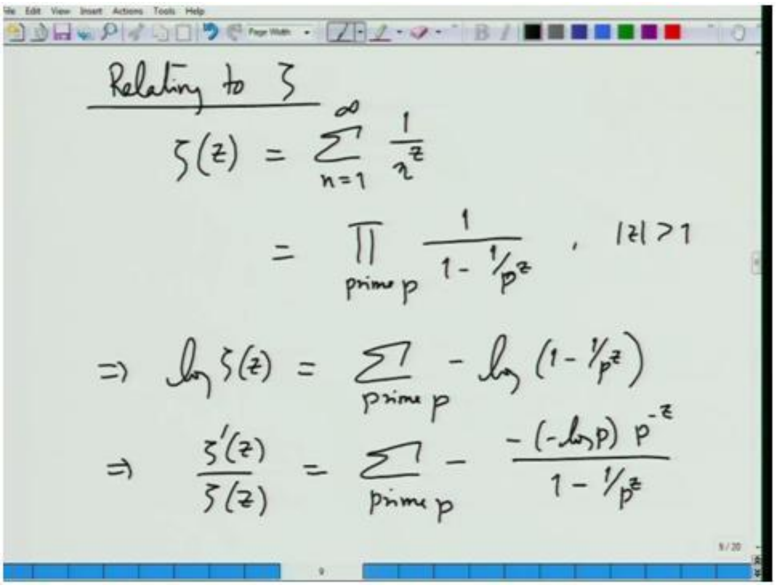

Where is in zeta function this sum over all X. So, to convert zeta function as a sum over primes, this gives a clue zeta is a product over all primes. How do you convert a product into a sum in standard way? you take log, but now, the sum as all these funny creatures logarithms send again you want to get return then, how do you get greater them again standard take derivatives differential left hand side what you get zeta prime Z over zeta Z. Differentiate this side, what you get? What a derivative of this? 1 minus 1 over P to the Z.

so this is well one is of course, goes away then there is a minus here, and then differentiating one over P to the Z that is P to the minus Z log P we differentiate that you get minus log P out. So, minus log P, and the rest you say the same P to the minus Z divide by one minus one by P to the Z. What is this you do not follow? What you get? Let us go step by step.

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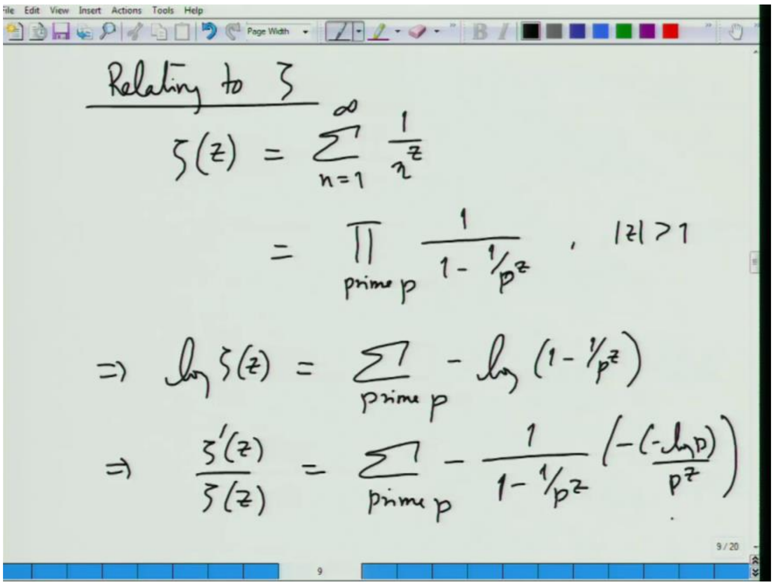

Differentiate log you get 1 minus 1 by P to the Z now, you differentiate the N sides the insides well one goes away. So, you get minus then you differentiate 1 by P to the Z that is e to the Z log P by e to the minus Z log P actually. So, when you take derivative of that respect to Z you get minus log P popping out and e to the minus Z log P remains e to the minus Z log P it is which is same as e to the Z.

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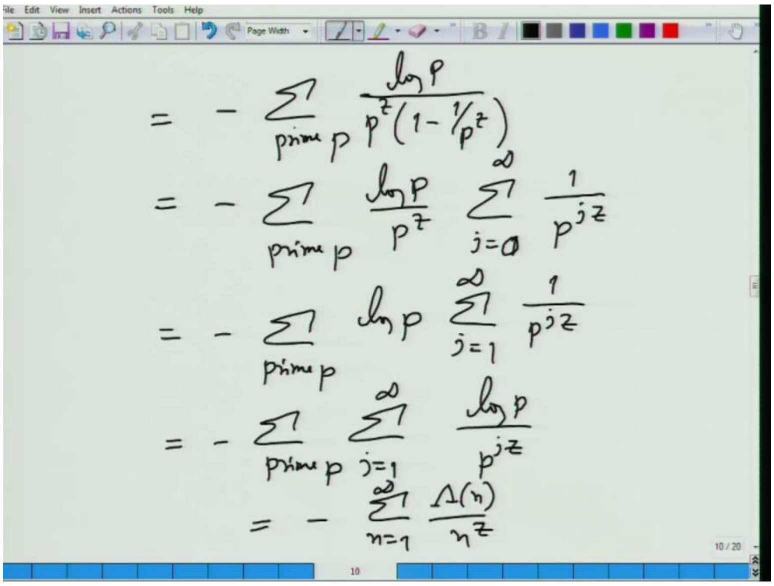

So, that for you get sum minus sum I am taking minus outside sum over prime P. And then you get log P at the top because that minus minus becomes a plus, and here you get e to the Z here so, we get, I write it as P to the Z 1 minus 1 by P. Now I express 1 minus 1 by P to the Z there is a geometric series, sum over prime P log P times by P to the Z of course. And then this geometric series which is j going from 1 to infinity 1 by P to the j Z. Z 1 or this goes from 0 to infinity, and there is a P to the Z sticking outside take it inside all that happens is that j is start going from 1 to infinity.

We get this now, what is this sum? Look at this is running over all primes P running over all j going from 1 to infinity. And the number in side that you will look at the split region. So, this is essentially running over all prime powers sum over all prime powers, and the inside is log t divide by that time power therefore, this is precise. That is precisely cousin of pi X that we define this also inspired idea. So, we will considered psi X instead of pi X, and just fits in perfectly. So, let us go back to this and write.

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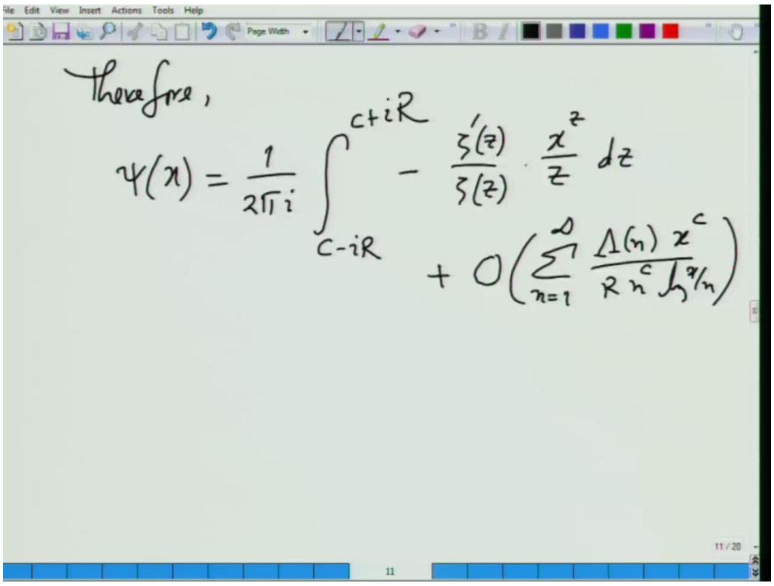

Therefore, psi X is 1 over to pi i C minus IR, C plus IR, and inside the integral the integrant is this we know what exactly what this is. This is zeta prime over zeta is minus of this. So, minus zeta prime over zeta then X to the Z by Z DZ plus this is random. So, this looks very good this is now in a very nice shape the complex integral. we can handle the error term, and could it nice get a nice bound on the error term then we are right on track to estimating psi X. Because that all we need to do is to estimate the integral, and we know tools estimate this integral this is on a line we will choose, and appropriate domain integrate over the boundary or domain use cautious theorems to get the integral value. And then try to make the other boundaries of domain go to 0 or become very small if not 0 because we are willing to tolerate some error in the estimation anyway.

There is already somewhere here in this part this is already somewhere. I hope what happened this is unexpected. So, there is already some error here in this part. Even if there is an error estimating the integral that is fine alright clear to everyone. So, we have established this that is enough today and next time our first task will be to get a handle on this error term. Exactly how lye this is naturally, it will be a function of r and that we want to get at to make it as smallest possible alright any questions. And once you of course, we get a handle on this then we get down to integrating this function inside.