Riemann Hypothesis and its Applications
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Lecture – 11

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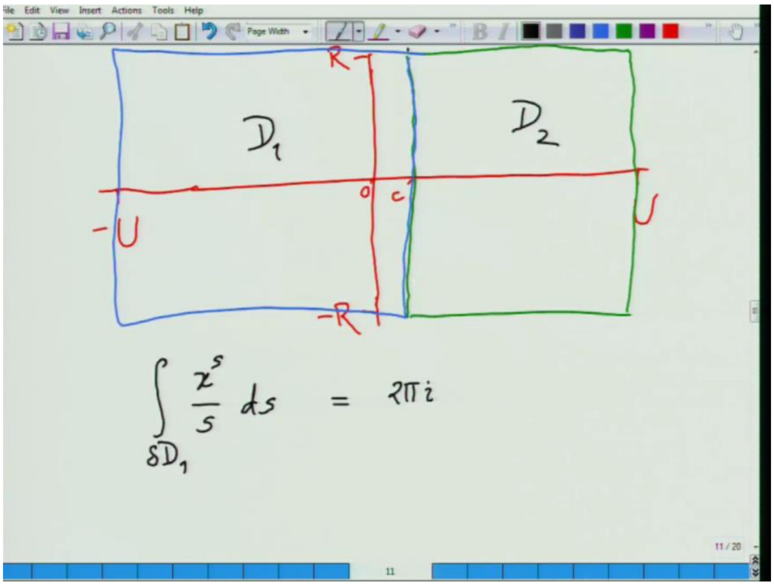

This is where we were last time and we want to prove this theorem. So, let us see how do we do this, what I will do is look at that line which is just defined, and so what I am going to do is, to take define two domains, they are both rectangles they are the common part of these rectangles is this vertical line at c; and going up to plus R up there minus R down there, and this extending on the this particular domain is rectangle extending up to plus U here and this domain D 1 is going up to minus a.

And, the reason I have defining two domains is because, I want to show a certain behavior of this function which is very different, depending on the value of x. So, below one it is one behavior, above one is enough completely different, so let us see, so if you recall the integral that we are looking for is x to the s by s d s. So, let first integrate it over or around the boundary of D 1, what is this, this should be easy the integral 0 why, in D 1 this is 0, thought an analytic at s equals 0, it has pole attach equal 0, in fact this is only pole for this, there is no other pole here.

So, inside, and so, 0 inside D 1, so that is pole at D 1, what is residue at this pole, you multiplied this function x d s divided by s by s and take the limit, s is goes to 0, and hence it is 1. So therefore, this is, this integral is 2 pi i, so that is what we know directly from the residue theorem. Now we can, now split this integral as before into because, it is over a rectangle, so let us split it into four parts.

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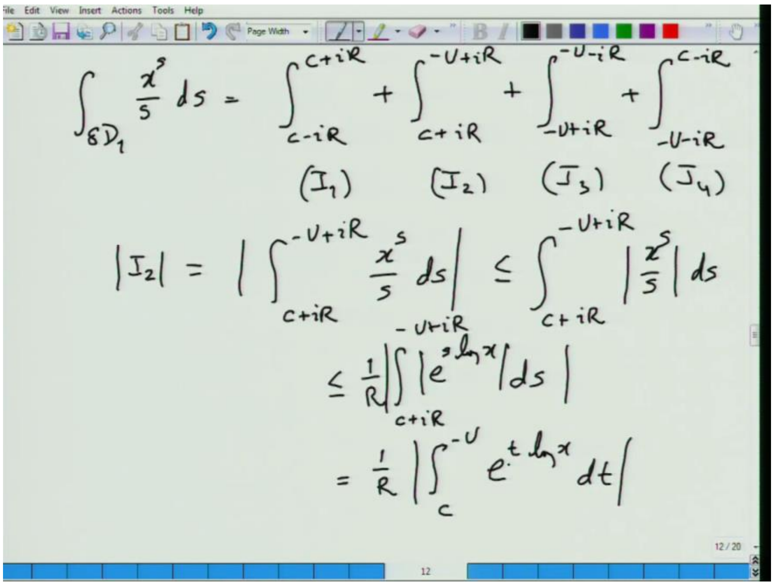

The first part goes from c minus i R to c plus i R, the second part goes from minus U plus i R c plus i R. So, I am traversing the domain in counter clock wise passion, c plus i R to minus U plus i R, plus minus U plus i R to minus U minus i R and finally, minus U minus i R to c minus i R. Let us call this I 2 I 3 I 4, and the strategy will be same as before, this is the integral of interest, that we want to show something for, and so, we want to get read of the remaining one.

Let us try to do it, and the same replies same strategy let us look at the absolute value of these integrals. So, what is the absolute value of I 2, that is absolute value of this integral c plus i R to minus U plus i R, x to the s by s d s this is less than equal to integral of absolute value, what is absolute value of s, as s goes from c plus i R to minus U plus i R. So, s is going from here to here, what is the absolute value of s, it varies surely, but it is at least R, and that is all we need, because we dividing by absolute value of s is at least R.

So, this is at least, and what about x to the s, x to the s is e to the, let me write e to the s log x d s c plus i R minus U plus i R, and note here that although s is a complex number the only part that varies in x, s is the real one. So, I can rewrite this same integral as by the way this is absent taking the absolute value.

So, s I can write as real part plus the complex part, when we take the absolute value the complex part any way goes away, because that is absolute value is 1. So, only thing I left with here is 1 over R c to minus U e to the t log x d t, and there should be somewhere outside, I should just put absolute value. I do not want any negative values anyway, and this can be integral will you say, what is the integral of this e to the t log x d t.

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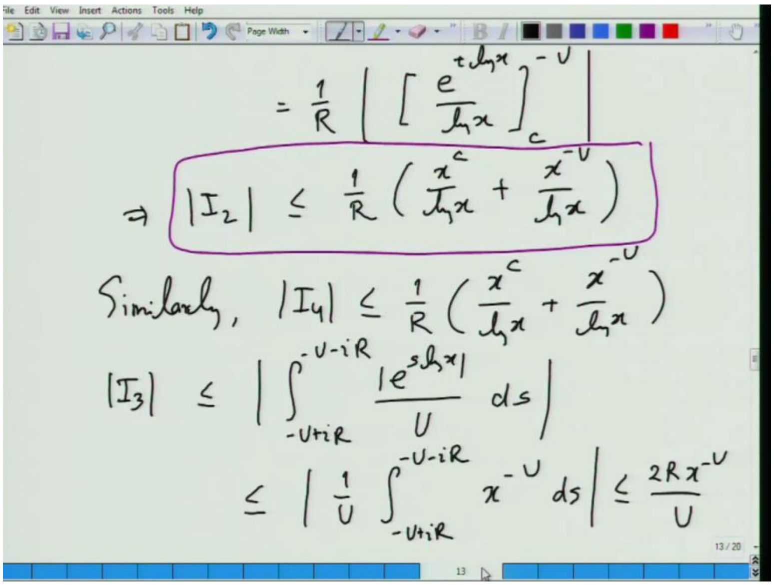

E to the t log x divided by log x going from c to minus U and this is less than equal to, well the bigger part would be one t equals c that is x to the c by log x minus x to the minus U by log x, and by less put a plus also it does not matter, just to be on the same side. So, that is the estimation for I 2, and if you look at this one, I 4 is the same thing because that the integral going up and this is same along the same line just reflecting down in the imaginary side, all the along there will be excess. So, it is the same the absolute value, why it would be the same.

So we can write bound this also, that leaves I 3, so let us do the same exercise for I 3, that is from minus U plus i R to minus U minus i R, what is the absolute value of s ((Refer Time: 09:25)). So, here it at least U, so we can just use that at least U, the absolute value d s and now again s breaks into real and imaginary part, when we take the absolute value the imaginary part vanishes. What about the real part, how big is the real part, it is exactly minus U and so, this is less than equal to 1 by U minus U plus i R to, this becomes x to the minus U d s.

So, there is really no integral to be done and this is tribal integrals and therefore, and this integral is at most of length to R. So, and in fact, when you take the absolute value the other it will becomes at most 2R, so that you get basically 2R x to the minus U divided by U.

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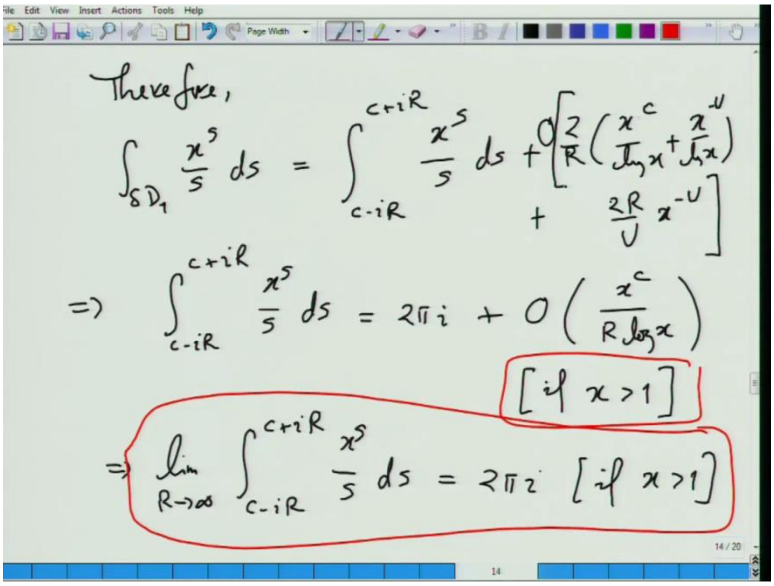

So, plus this two times this plus this, so that is equal to, I should stick it, ordered here, what is actually it, plus x to the minus U by log x. Now, let us take the limits and the first thing we do is we send U to infinity, this is of course 2 pi i, and as you sent U to infinity, what do you get, because U does not occur in, on the here and anyway.

So, I can easily send U to infinity, what comes or what remains in the error two, two of course, I can hide in the constant, this goes to infinity, this go to infinity and now, if I sent R to how variable I should vary very, very it should be carefully. So, when I send U to infinity, what happens this will happen only if x was greater than 1, so this is a very important class to be remember.

As long as the x is greater than 1, this is what happens, if x is less than 1 then this will go to infinity that will be pointless. So, assuming x is greater than 1, I get this approximation and now, send R to infinity, x is greater than 1, ((Refer Time: 13:58)) so that is in result of a integrating on this domain.

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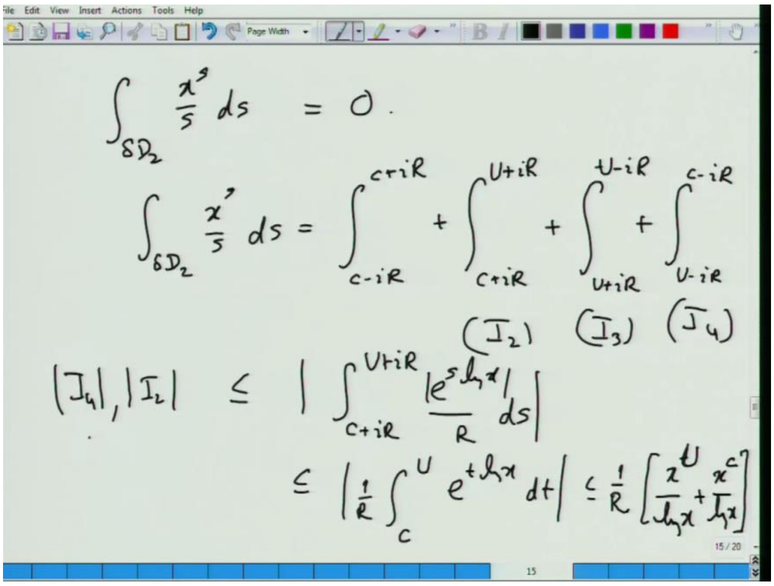

Now, let us look at the result of integrating on D 2, it will be pretty much similar, first of all, by residence around what is the residence, if zero is completely analytic on this domain, so this is 0. You see that is the reason we chose these two domains, ((Refer Time: 14:28)) here because of the pole being here, we get residue which is 2 pi i or divide by 2 pi i you get 1 here it is completely analytic, so it is goes to 0.

But, also the important thing is it to sort of coincide this 0 or 1 with the value of x, what I did was to look at this specific integral ((Refer Time: 14:48)). So, this says that the integral thus converge but, this only when x is greater than 1, now let us look at these, this one and same we will express this as sum of four integrals, again we let this to be I 2 I 3 and I 4, I 2 and I 4 are again similar integrals in absolute value.

So, their absolute value is going to be less than, again using the same ideas x to the s is e to the s log x and now absolute value for s is going to be how much, at least R is going from R or we write.

And now, n absolute value here again the imaginary part vanishes and then we can this we can convert to a real integral, which goes from c to U e to the t log x 1 by R comes out here d t and this is less than equal to x to the U by log x plus x to the c by log x, and same with actually I 4, the absolute value I 4 also has the same property.

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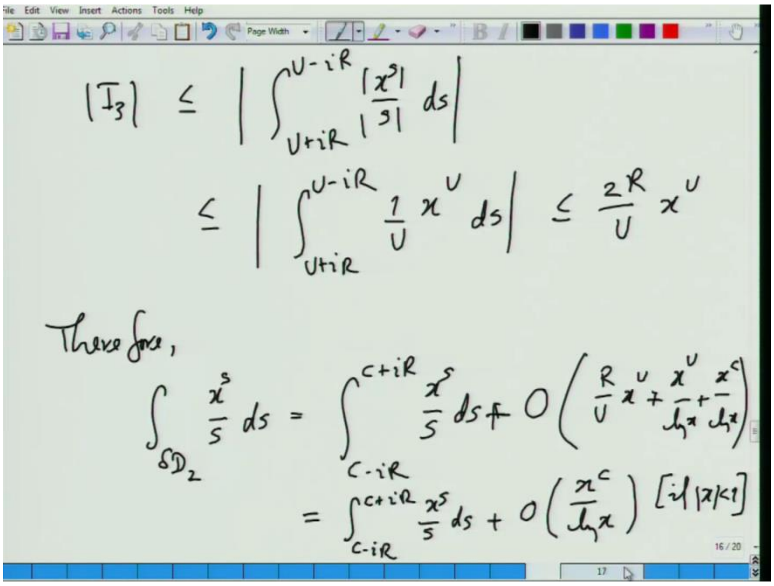

And what about I 3, I 3 is U plus i R to U minus i R, well s is going to be absolute value phase is at least U and absolute value x to the s is going to be x for U d s and this again is 2 by 2 R by U x to the U. Now send U to infinity, what happens to the first term, well depending on x, if x is less than 1 this goes to 0, if x is bigger than 1 this diverges. So, we focus on x less than 1, and we get that this is equal to same thing with the second term, in fact this is the absolute value of x must be 1.

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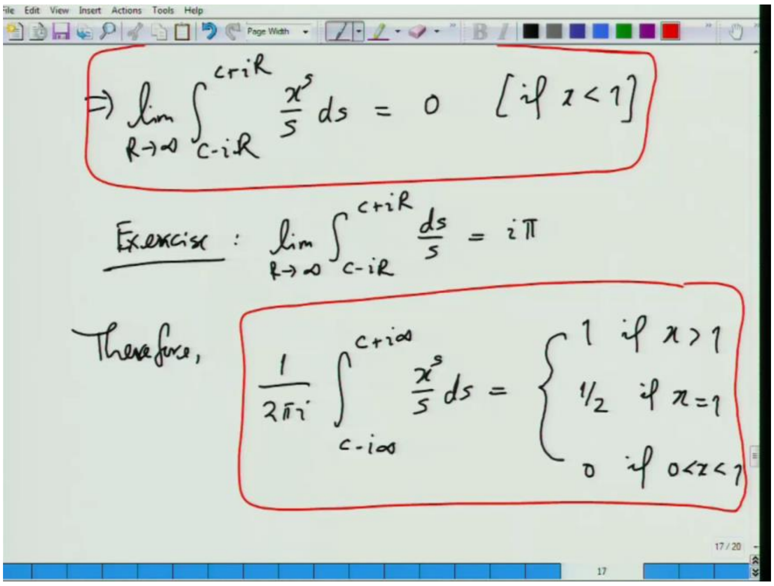

And now, if you send R to infinity we get ((Refer Time: 19:51)), what we get, this is 0 here is something missing this is 1 by R for outside, for I 2 and I 4. So, this also you justified, this gives me therefore, the function I will write slightly change, because this is when x is between 0 and 1, this is 0 and then it steps up, and then it stays one forever. What you wanted was, 1 between 0 and one there is step down, but there is trigger to change, so slip, so it replace x by 1 by x and you get the other option.

So, this is going to be it is very interesting function to begin with, and is going to very useful for us later on, also it demonstrates the power of this contour integrals, that you can use the knowledge about the poles inside a contour, actually evaluate the integral very easily. And x equal to 1, we have x equal to 1, this what you can, let us you what happen if x equal to 1,if x equal to 1 you get d s by s, d s by s is log s, log s integrated from c minus i R to c plus i R, what is that? Well, you can, then let us see, firstly, we have to think about what is a interpretation of log s we are taking we are already discussed in last class.

The nice thing about here is because of the line is on the positive side is not cutting on any of those funny things on the left hand side. So, it is only on the one single sheet, and it is going from c minus i R to c plus i R and so, we can just use the principle, the main value of our log s which is, so log of c plus i R minus log of c minus i R. And then that it becomes log of c plus i R divided by c minus i R, and as you send R to infinity, what you get, you get i divided by minus i, it will minus 1, you get log of minus 1, what is log of minus 1?

Student: ((Refer Time: 22:50))

1 hyper, log of minus 1 is i pi, then this is in secondary, now that is, that is about.

So, this is let me just say exercise that, and therefore, 1 over 2 pi i c minus i infinity to c plus i infinity x to the s by s d s this equals 0, if x is between 0 and 1, half if x equals 1, 1 if x greater than 1. So, this is the step function for us, any questions? Good, so I am pretty much done with introduction to complex analysis, and now I would like to start on the real content of the course. Now, in this we have 15 minutes, so I would like to start it from the next class.

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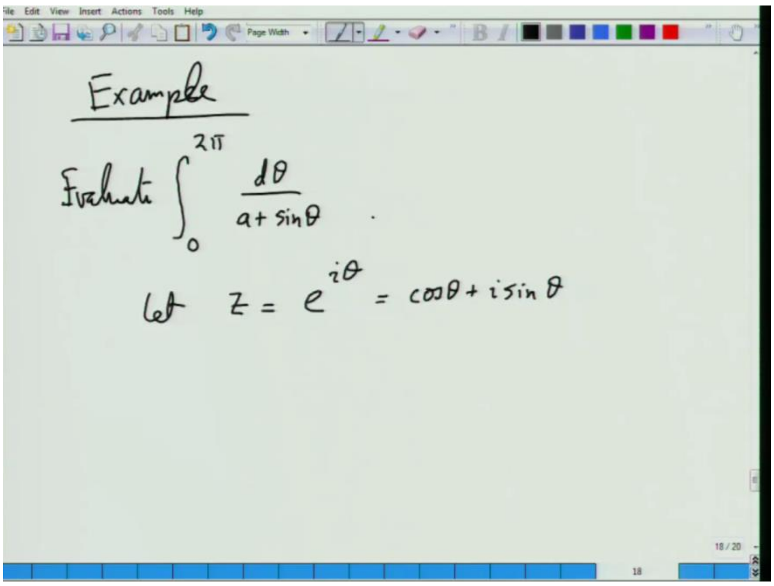

So, to fill up the time, let us do one more integral, this time, let us do something different, now how can integrate this, do you know this integral classically, how do you solve this?

Student: ((Refer Time: 25:08))

Define a as sin alpha that, will require a to be between minus 1 to plus 1.

Student: ((Refer Time: 25:21))

Then do something is it then, you can sin alpha, suppose sin alpha plus sin theta.

Student: ((Refer Time: 25:38))

And then do this. So, that is one way of doing it good, so what message here is a much simpler way we have to doing it, same strategy think of this integral as happening over complex plane. So trans, firstly, I want to translate everything over the complex numbers, so here theta is varying 0 to 2 pi, let us imagine that we are integrate, that is we are integrating over complex plane, same theta 0 to 2 pi, but now instead of theta, we can actually view theta as a all the points we are integrating over r complex numbers.

So, let us write this as, first write this as to substitution, so let z be e to the i theta which is therefore, cosine theta plus i sin theta. So, what is sin theta in terms of that, what?

Student: ((Refer Time: 27:05))

Simpler z bar no, we do not want to get z bar, z bar is by the way derive talk about is, that the function z, f z equals z bar is it analytic or not, is it analytic, f z equals z bar, how do you prove it we are adding,

Student: ((Refer Time: 27:37))

Sure. So, continuity is R as zero, assume that is simple that is satisfy partially.

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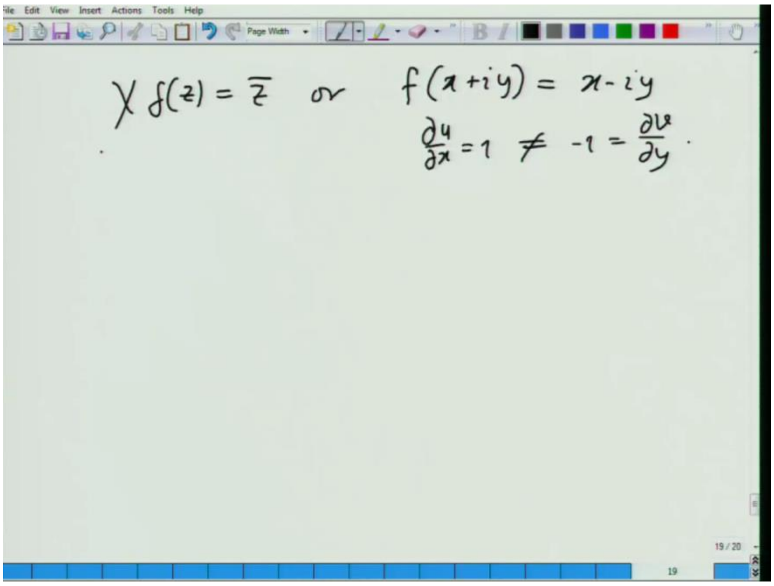

It has, let us pull out an intermediate page or I can write f of x plus i y equals x minus i y, so what is quasi remark, this is U this is minus y is being. So, delta U by delta x is delta v by delta y minus fails, delta v by delta x must be equal to delta v by delta y, so not analytic. So, this function is not analytic, so which means that we cannot use z bar anywhere in an analytic function essentially, so coming back to this z bar is no go, but that idea is good, but you want to get cosine theta minus i sin theta somewhere. So, that you can subtract and get just sin theta, how do you get cosine theta minus sin theta, minus i sin theta without using z bar.

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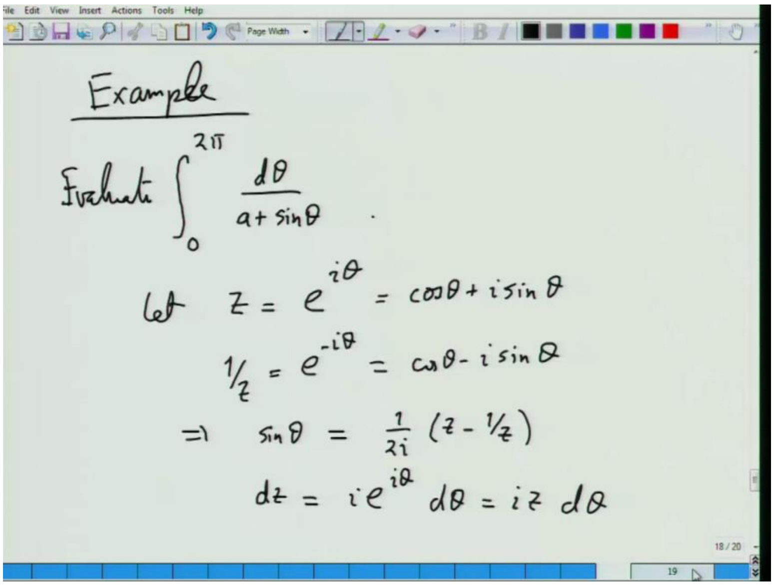

Student: ((Refer Time: 29:21))

e power e to the power minus i sin theta exactly. So, which is, so e to the minus i theta is cosine theta minus i sin theta which is very good, but what is e to the minus i theta in terms of z is 1 by z is pretty fine. So now, you subtract and so, we get sin theta is therefore, 1 over 2 i z minus 1 by z.

Student: ((Refer Time: 30:01))

We are choosing that, we are choosing z to be e to the i theta, making sure that modular is 1, because we are integrating over the circle and we can choose any radius of the circle, because 1, radius 1 is the best for us good. So, that is sin theta and what is d z, d z is i e to the i theta d theta which means i z d theta, so now, we got everything to do the substitution.

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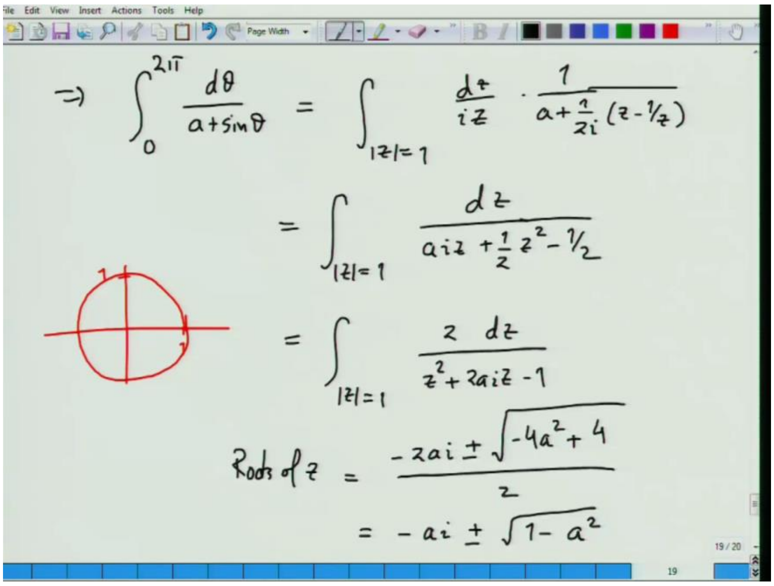

And then, I can write d theta is d z by i z and sin theta is 1 over 2 i z minus 1 over z, and this is equal to d z over a i z plus 1 over 2 z square minus 1 over 2. So, that is what the integral looks like, and this is over a circle of radius, so at the value, what is the value of this integral, it depends on whether there is a pole inside, that you need desk or not.

So, and that is easy to determine, it is a quadratic in z, what are the two roots of z, they are minus 2 a i plus minus square root of minus 4 a square plus 4, divided by 2 and therefore, we get minus a i plus minus 1 minus a square. So, where are these roots, first of all it is real and imaginary parts are very clear, real is either plus or minus square root of 1 minus a square and the imaginary part is minus a i.

Student: ((Refer Time: 33:32))

It depends on a, yes absolutely it does depend on a. So, what happens if a is greater than 1, then there is no pole inside and then the integral would be 0 and what happens if the a is between minus 1 and plus 1. Then it will have, will you have both roots in, we will have both roots in same side, wait, is that necessary this square plus this square is 1, if a greater than this, probably, this is probably then, it is not even does not even make sense. When this, this process will fail if actually a is bigger than 1, so then you or be let, if a is greater than 1, then this actually becomes complex, which is a just makes things massive than before.

Student: ((Refer Time: 34:56))

No less than is not necessary if it will be outside if a is bigger than 1, then this would be i times, something i times a real number. So, then you get either add or subtract that real number to a, so you might get inside or outside the circle. In fact, if a is large one of them, there is certainly be inside the circle because, that real number would be very close to a square root of a square minus 1, if a is large as going to very close to a. So, when you both get added that falls out, if the one gets subtract from the other that would be inside, and if a is between minus 1 and plus 1 then this is both are inside, good.

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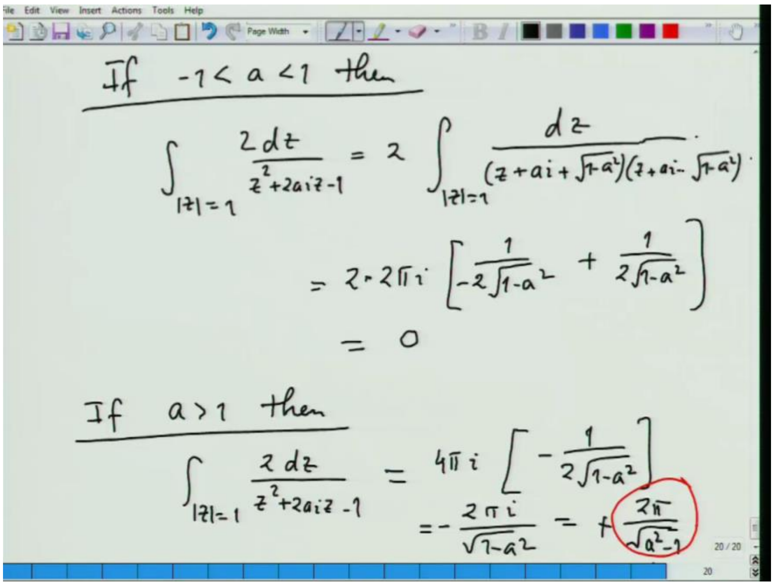

So, we split this in therefore two, so let us analyze the first, if minus 1 is less than a is less than 1 then what happens, so then the integral would be what, what is that, I have write this instead of first doing it less factor this, this is a factorization and therefore, this is equal to at this residue is what is. So, basically we eliminate this and we substitute for z minus a i minus square root of a square which will be 1 by 2 minus to square root of 1 minus a square plus, when we substitute this what is going to be the residue, this correct and this is 0, this is cancel, so this.

And, if a is bigger than 1 then, going back ((Refer Slide Time: 38:21)) one of them is the residue, so when this is plus that is only residue that survives. So, the one that is survives this one, that is the pole inside and though you make this substitution, and then you get minus 1 over which is of course, wrong where is this wrong, it should be a real number, this is a complex some.

Student: ((Refer Time: 39:19))

1 over a square is complex, got excellent, so this is basically 2 pi by square root of a square shown, so there be a minus, now there is a minus already here, and this when you take this or they should minus 1 square root is find, which gets cancel there should be a or it could have been plus 1 and minus 1, depending on whether square root of minus 1 we choose to be plus i or minus i. Probably it is minus i it will be this, this one has to look at then see, what, what is the right value may be going back to the original integral (Refer Slide Time: 40:13)

Student: ((Refer Time: 40:18))

A is greater than 1, that is right and sin theta where is between minus 1 and 1, so everything is positive, so this must be positive. So, this should be 1, so there you are gets a solution of a integral, then very straight forward when no clever substitution needed, so apply the blindly the residue theorem, and you have the residue. And, not even well of course this is just, now that we have playing around.

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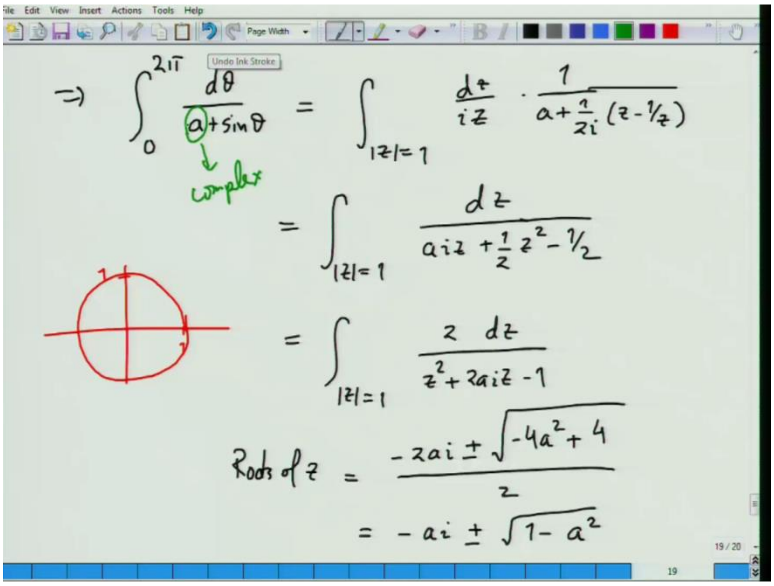

Let me just take another sort of obvious thing also, that if this a was a complex number instead of a real number greater than 1, a complex number with absolute value greater than 1, then what would happened?

Student: ((Refer Time: 41:31))

So, this is an analytic function of a, it does not have there is no pole and because, of absolute value a is greater than 1, so there is no pole of this, and we saw that one another theorem we proved was that in, if it is there is a analytic function it is integral is also analytic. This is an analytic function n a, which on the real line, positive real line a greater than 1 agrees with this, therefore, it must agree with it over the entire complex that half plane absolute value a greater than 1. So, you get that result hopefully of course, this is also analytic for absolute value a greater than 1, so it both are analytic both are agree on a line and therefore, they agree everywhere, so that is it for today.