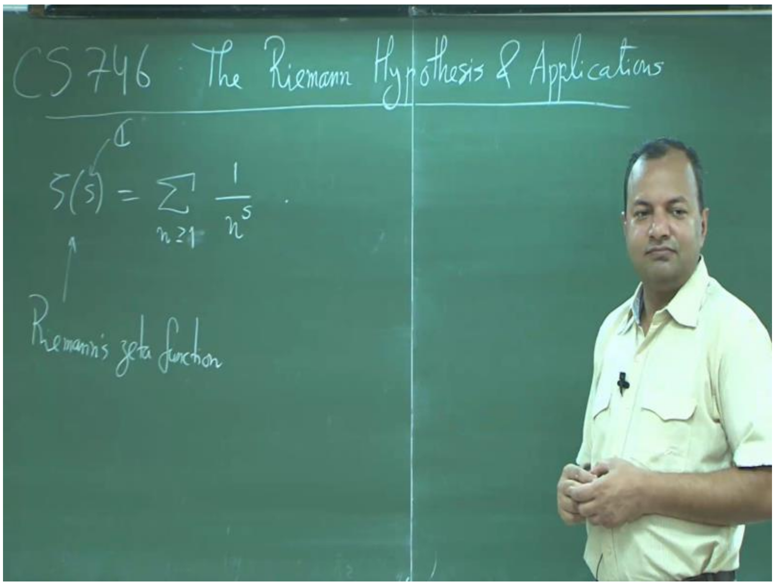
Riemann Hypothesis and its Applications
Prof. Manindra Agrawal
Department of Computer Science and Engineering
Indian Institute of Technology, Kanpur

Lecture – 1

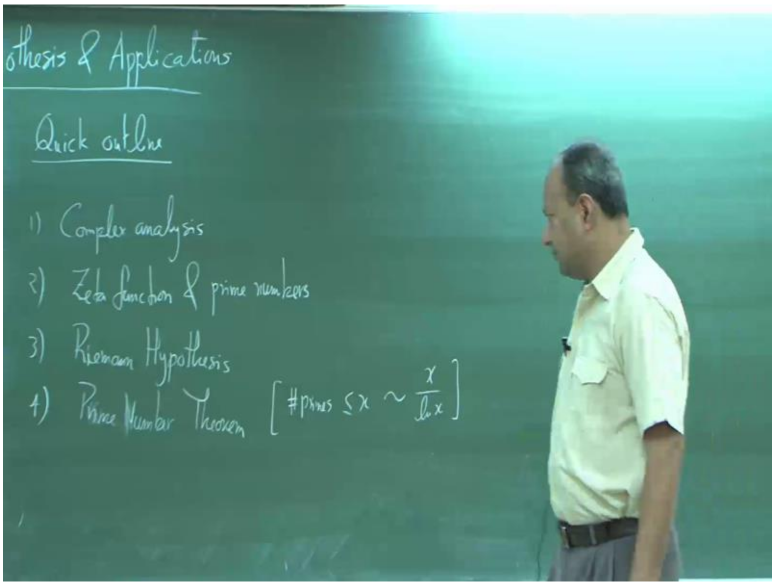
So, that is the course all about Riemann hypothesis, we will mostly be looking at one particular function.

(Refer Slide Time: 00:29)



This is the Riemann zeta function, the number s is the complex number and this is an infinite sum, which has many interesting properties and the course as I said will be devoted to understanding some of those. You will also look at once we have a good grasp of this function and now, its connection with many interesting objects like number of primes up to a certain range. Then we will look at generalization of this function in different domains and see what they say about those different domains. And finally, we will also look at the titles says about some applications of the zeta functions. Or I should not say zeta function application of Riemann hypothesis, I have not talked about Riemann hypothesis is which I all do latter. So, the hypothesis pertains to some properties of zeta functions and the hypothesis also has many consequences in various parts of mathematics and computer science. So, we will look at some of those consequences.

(Refer Slide Time: 02:26)

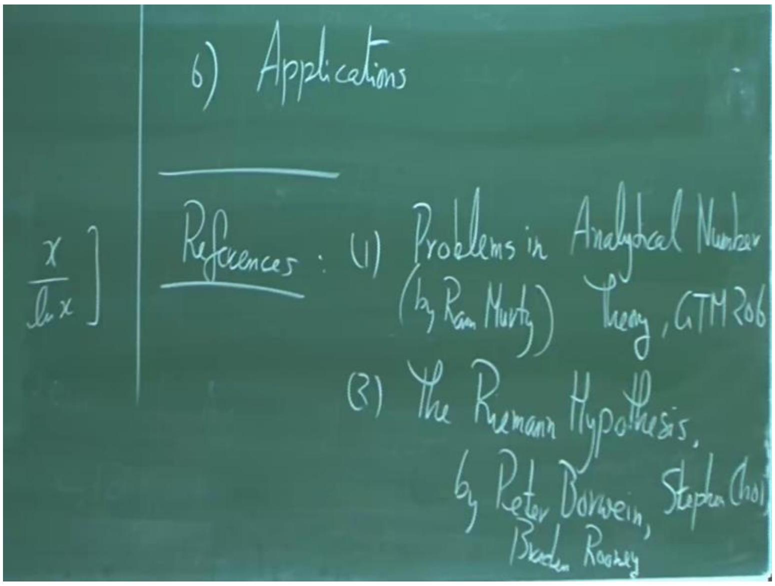


A very quick out line, some of you who have read the pdf file, which I have attached, would know this, will start with complex analysis. I am sure all of you have done some course on complex analysis, is that a correct assumption? And many of you have done it long time ago right. So, I can guess that many of you will not remember much beyond the definition of complex analysis, is that a right assumption? No, you know more? That is good. Then I will have spend less time here, what I plan to do is, to start with definition of complex number and develop the necessary tools, not necessarily all with the proofs, but give you a feeling of those tools in complex analysis, that we will be using in our analysis of this function.

Fortunately it turns out that we do not really need too many fancy tools. So, it is going to be not too bad, but it is going take off some time of this course and that time very it contingent upon on how much you know or you remember about complex analysis. Once we have basic background will look at, well I have already defined zeta function, but zeta function and prime numbers. So, that is very critical connection that is how zeta function started its life when Riemann was trying to understand the distribution of prime numbers. So, we will show exactly what Riemann discovered this was a quite a remarkable piece of work by Riemann, I will get into details of that latter. So, let us stop it here.

And then so Riemann also made a conjecture about the nature of zeta function when he did this study and that conjecture has still unproven after more than 150 Years. And there are number of mathematicians who have try to prove it, but failed and it is considered to be greatest unsolved problem in mathematics, there also happens to be one million dollar prize for anybody who proves this hypothesis. Then will prove prime number theorem, this is essentially follows from a weaker version of Riemann hypothesis that one can prove and from that follows the prime number theorem. Let us see how many of knew what prime number theorems? Number of primes exactly, number of primes this theorem says number of primes, less than equal to x is sintotacally x over log x.

(Refer Slide Time: 06:49)



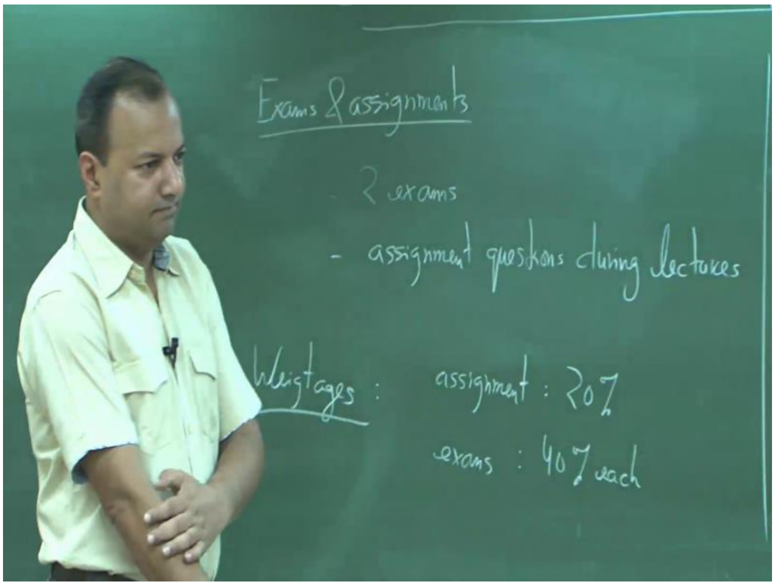
So, this we can prove, in fact, this was proven more than 100 years ago, building on works of Riemann. Once we have done with all these studies we will move on to the generalization of Riemann hypothesis, which really come fall out of generalization of zeta function to other domains. And just as Riemann formulated his hypothesis for zeta function for those generalized zeta function one can formulate a similar hypothesis, which are called generalized Riemann hypothesis for extended Riemann hypothesis depending on which domain it is being generalized to.

And we will look at some those variants there are huge number of variants, which will not have time to go into details of all. So, we look at some of the important ones. And in particular we look at generaliz ation to higher character, which we do not need to understand right now, elliptic curves, which again do not need to understand right now, if you do not know.

And may be one more, but I will leave it at this point. And finally, we will look at some applications and since this course has a cs title, I will justify that by looking at some computer science applications here. So, that is the basic outline, any question? No. There is no text book, because it turns out that of course, there are large numbers of books written about Riemann hypothesis, zeta function connection with prime numbers, but it turns out that mathematicians have a way of writing or expressing their ideas, which I cannot quite understand. So, therefore, I will have sort of gone through, many of those put everything in my own way and which is what I will present.

So, therefore, there is no book, I will be following chapter by chapter, but there are reference books, which basically says that everything that I cover is more or less in these books. Now, what are those? I did give couple of names in the let me see, if I can pull it out, this is a GTM series book two zero six problems in analytical number theory by Ram Murthy, Ram Murthy is one of leading number theorist in world. So, his he has written couple of really nice books on number theory. Then there is second book is the Riemann hypothesis, this is available on the web. So, it fairly easily access, there is some URL, which you can just Google up for this, this is by Peter Borwein, Stephen covey and Braden Rooney.

(Refer Slide Time: 12:31)

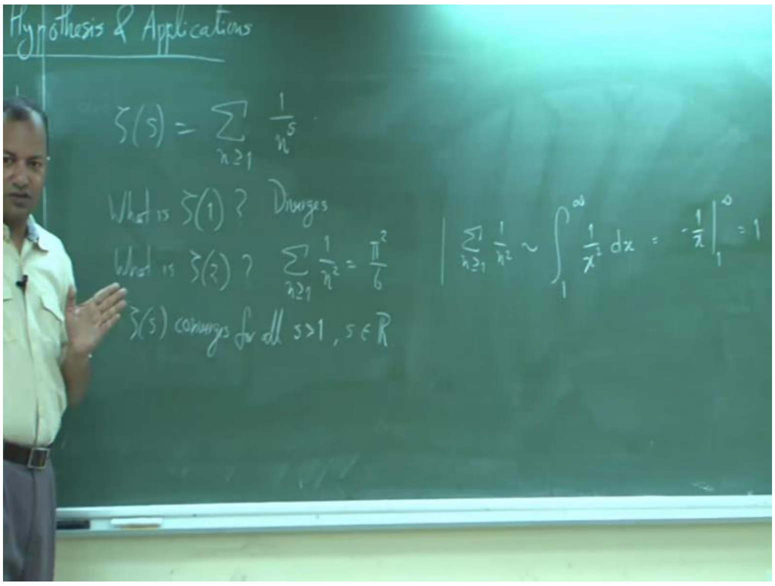


Exams; this is the first time, I am teaching this course. So, I really do not know how the assignments or exams will evolve, because it is very easy to make questions in this subject, which are very hard to solve. And if try to reduce the difficulty it very quickly becomes very easy to solve. And for examination one needs questions, which are sort of intermediate difficulty. So, I will make an attempt in formulating those questions, but it might turn out that there either too easy of too difficult, you will have to just suffer. There is really no other way out of this.

Two exams, mid sem and sem, one of them will be take home well will discuss with you which one do you want to be take home. Assignments again, assignments can be done very easily here, because there are many things which, I will not go into details, which I will just leave out the assignment problems. So, assignments will be more of dynamic in nature that during the lecture as I ok, assignment problem number one, assignment problem number two, as we go on.

Weight ages, assignment how much, if you take to want assignment, twenty percent, forty percentage, is that ok. That finishes of the standard beginning of every course, any questions on this? So, let us now began our study of zeta function, I did say that, I will first talk about complex analysis, but before jumping into complex analysis, let us spend some time in trying to see what this function is about.

(Refer Slide Time: 16:00)



So, this lets write it down again. Now, this function of course, has a parameter s an is an infinite sum, quite clearly if there are some values of parameters for which it does not converge. For example, what is zeta 1? What, l n 2? This diverges. In fact, you sum up you take s to be 1 and then you sum up n up to x; you will get about log of x. So, as x goes to infinity this diverges.

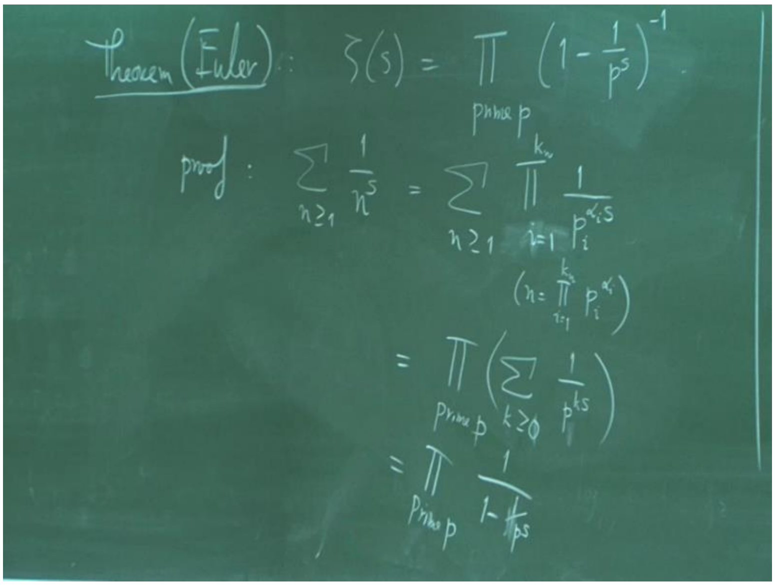
What about zeta 2? That actually does converge and we will see latter on that it converges to as z pi square by 6, can anybody prove that it converges to pi square by 6. You know the proof, ya, Euler was the first mathematician to prove it. It has Euler proved it actually without complex analysis and when we look at complex analysis will find some really nice proof of this equation. Fine anyway it does converge that is whole point and this anybody can see it converges or can is that simple to see convergence of the series.

Why does it converge? So, we are going even before complex analysis to math one o one, infinite series is converges, that is one simple way of doing it yes, basically this is approximately this integral. In fact, by setting it up properly you can show that, this is less than equal to this and this integral is easy to handle. And this is just minus 1 over x, 1 to infinite, which is 1. So, zeta 1 diverges, where as zeta 2 converges. In fact, for all the same analysis will show that zeta s converges for all s greater than 1, where s is the real number not talking about complex number right now, the same proof works.

So, that gives you some idea of what this function looks like. And of course, for s less than equal to 1, this is going to diverge again talking about the real line. So, on the real line the nature is quite clear at top level that for s greater than 1 it converges, for s less than equal to 1 it diverges fine. But the fundamental question is fine this is an infinite series of some form why should it be interesting? Why should we bother studying this series?

As I been saying already that this as some intimate connection with prime numbers. So, this connection at least one of those connection was discovered by Euler.

(Refer Slide Time: 21:17)



And it is a very simple observation. So, let us call it a theorem proved by Euler, this is says that zeta s is equal to product over all prime numbers p of this quantity, 1 minus 1 by p to the S to the minus 1, that is 1 over this quantity.

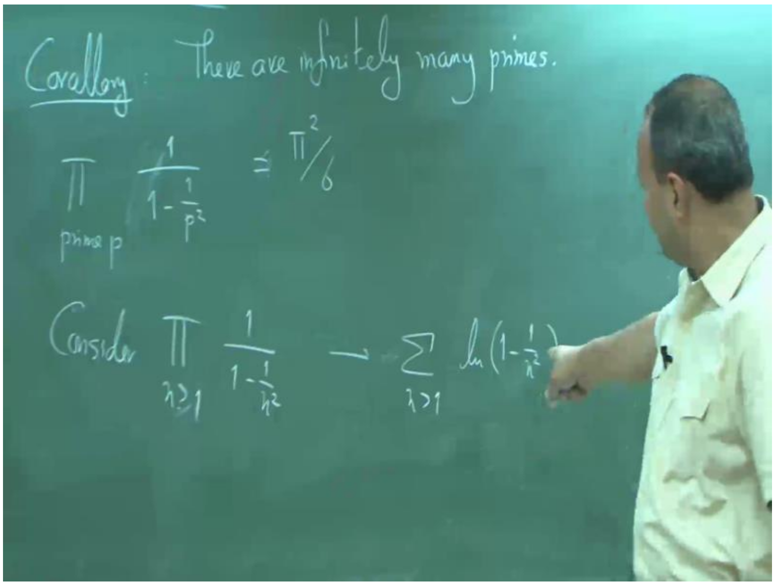
Proof is fairly straight forward, all you need to do is to observe look at start from there. And let us, write it in terms of prime numbers and to do that let us notice that this sum runs all over integer sign. And by the fundamental theorem of arithmetic, this goes even before that right. I am going to high school and I am the sure all of you remember fundamental theorem of arithmetic, that is every number can be uniquely written as product or prime number.

And so if you write that, we get n greater or equal to 1 product prime P, product over P to the, what should I write it? I going from 1 to lets say k n 1 by P i to the alpha i, where n is. So, n is written as product of this prime, so we just rewriting this structure again. And now, let us exchange the product and sum by observing that in this sequence take a particular prime say P 1, what all, in what all way does it occur, P 1 occurs in every n, which is a multiple of P 1. And more over, if P 1 to the k divides n and P 1 to the k plus 1 does not divide n. Then it will occur as P 1 to the ks. So, basically for every prime P an every power k of the prime P, P to the k occurs in all multiples of, exact multiples of P to the k, that is P to the k divides n, but P to the k plus 1 does not divide n. So, we can write this is as the product over all prime P and then sum over all k greater than or equal to 1 or zero. So, notice what I have done, this is product over all primes P sum over all exponents k of P and 1 over p to the ks. So, you take this sum and multiply it over all primes P and other way of seeing same thing is that you take any set of primes here, any finite set of primes in this product.

Pick any for each prime pick any exponent k and you multiply out those primes with those exponents, those chosen finitely many primes with corresponding exponent, you get in number and that is a unique number. Right that is by fundamental theorem of arithmetic that, this number you get, because you have picked up a unique set of primes and unique set of its corresponding exponents you get a unique number and that number will be present here. And every such choice will create a unique number and every number n will be created by one such choice. So, that is the equalance, so this there is one more thing one is to do when we are doing manipulating infinite series is convergence. We have to be very careful about convergence before we do this.

So, that is how this is convergence series, again for S greater than 1 as long as s is greater than 1 this each one of this is absolutely convergence. So, remember absolutely convergence, ok, good. And therefore, we can do this exchange. And now, it is pretty straight forward, this is product over prime P and this is familiar geometric series, which is equals 1 over 1 minus P 1 by P to the s. So, that expression connects the zeta function with some form of product over prime numbers. In fact, this tells us some very simple facts about prime numbers almost immediately.

(Refer Slide Time: 28:51)



For example, there are infinitely many primes, how do you derive from this theorem? Exactly set s equal to 1. We know zeta 1 is divergence and for s equal to 1 this product on the right hand side is 1 minus 1 by p to the minus 1, if there are only finite in the primes. Then this product will not diverge with each term is a finite number and there are finitely many products here. So, it does not diverge, but we have to be little careful, this equality only held when s is greater than 1. So, we cannot use this equality for s equal to 1. So, although this seem like simple and direct proof, but this not quite correct proof. We have to be little more careful here, the same idea works, but what one is to say realize or do here is that we know that, this equality hold for any s greater than 1. So, set s to a number bigger than 1 and send it towards 1. As it tends towards 1, the left hand side is getting bigger and bigger and bigger, keep going towards infinity .The right hand side does not, if there are only finitely primes. In fact, you can place an absolute upper bound on the right hand side, depending on what is the largest prime number. And therefore, there exists only prime numbers.

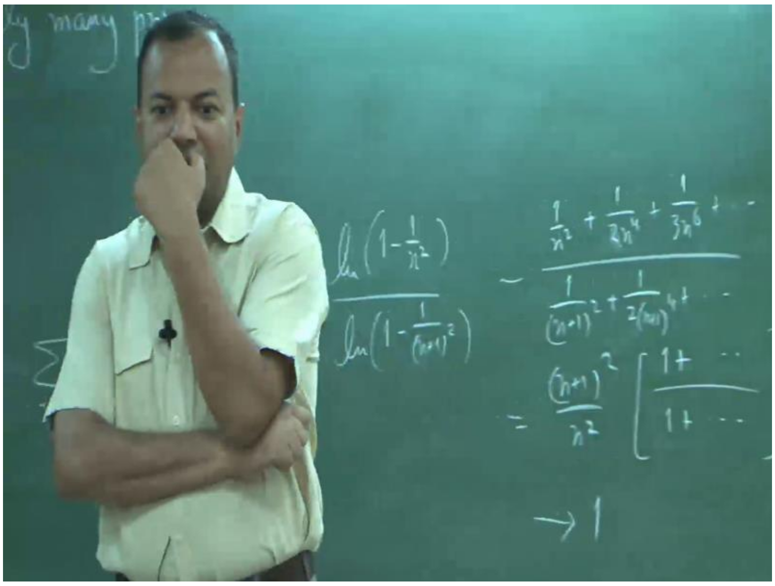
Let us see what more can we get out of this. Yes, currently it is all real s yes, we are not, that is right, solve for this. So, should I have mention that, when Euler started the studying of this function, him and there were previous mathematicians also, who always thought of this as a function over real numbers. The complex where not considered at all until Riemann and that was one of the great insight of Riemann to extend the domain from real to complex. And we will see during as we go along this course, the remarkable effect the simple change in perception had on the nature of this function on everything else ok. So, what more can we derive of this, can we say more about prime number? Well let us see, zeta 1 for s equal to 1, told us that there are infinitely many primes, let us just try out zeta 2. So, we know zeta 2 is bounded certainly, but is actually equal to pi square by 6, which is less than 2, some small number, that all we need to do. This is pi square by 6 again the exact number is not important here, only the fact right now will be using is this is bounded number. So, right hand side is bounded clearly left hand is also side is bounded there is no problem of equality here, because we are using s greater than 1.

How can we relook at this product in some way? Let me put it this way, suppose or let us consider this product n greater than 1, does this converge. So, how do we decide about convergence of series? What? That is one way of doing it yes. This, ya, this is of course, first two terms of cosine series, but that may not be that useful cosine is always this is an infinite product and one has to do error estimate and start clear, how bad that will be. One is to take convert into sum, take log right. So, from here go to and look at this convergence of this series, and this convergence of this will tell us whether this is convergent or not. Now, is this convergent?

There are some simple criteria is to for checking convergence of series. One is to check the ratio of two successive terms. Yes, this, will this be less than or equal summation of 1 by s square. The product of infinitely 1 is always still 1, but this is also product of all terms greater than 1 would they all they are, one has to see whether these terms are also tending towards 1.

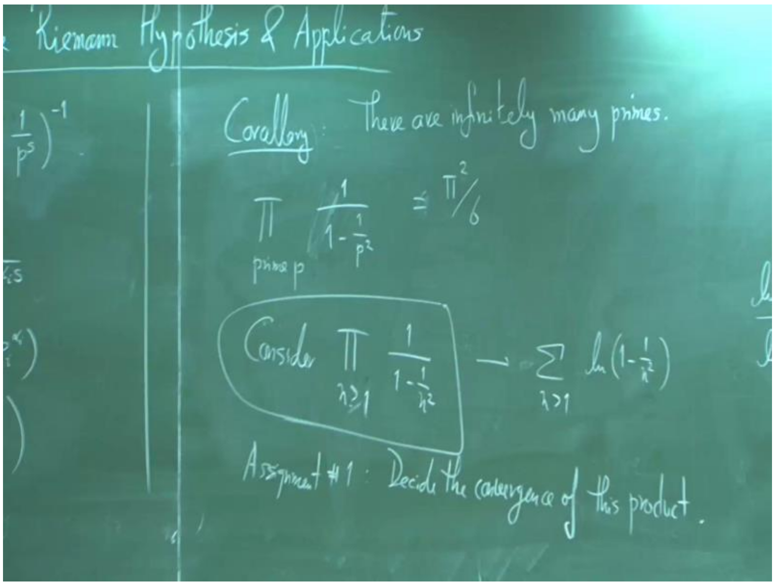
So, whether in the limit the product goes up or may be stays close to less than equal to a boundary. You saying compare it with summation 1 by n square, so this bounded by 1 by n square, what is log 1 minus x? That is x plus x square by 2 plus x cube by 3 and so on, is bigger than 1 by n square. Sequence these are all again tending towards 0, the successive terms get closer and closer to 0 as n goes bigger this inside gets closer and closer to 1 and log of that is close to 0.

(Refer Slide Time: 37:41)



So, let us look at ratio of two terms, I do not know if we can sort out, does not look very exciting. Let us try this out expand this, what is log 1 minus x is? 1 by n square there all pluses here. So, is it 1 by x square by 2 or x square by 2 factorial? x square by 2, so this is n to the 2 n to the 4 plus 1 by 3 n to the 6 and so on. And this is 1 by n plus 1 whole squared plus and so on. So, let us multiply out, this with n square, and this with n plus 1 whole squared, so that we get, n plus 1 whole square by n square times 1 plus n times to infinite this ratio goes to 1 and this ratio also goes to 1 right. And what happens if the ratio of successive term is going to 1, does this series converge or diverge? Converge? What? Ratio has to be less than 1, then it converges, if it is 1 then it diverges.

(Refer Slide Time: 40:38)



So, this product actually diverges. Sorry, yes. At 1 you cannot say, so that is bad. And then I have to do more work, assignment problem number 1: decide the convergence of this product. And hopefully by tomorrow you will have decided and will continue with our work.