Computers Algorithms-2
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Lecture - 9
Flow Networks

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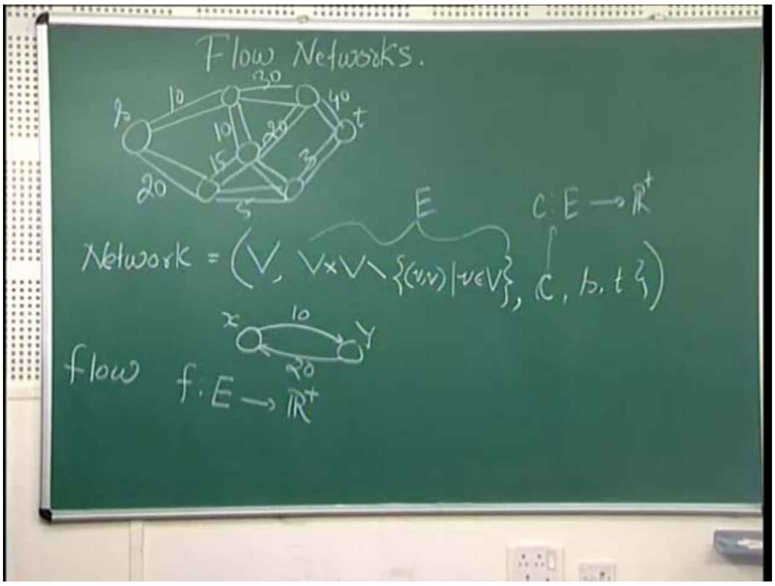

Today we will start with a new topic namely, Flow Networks. To motivate our self, let us take a simple situation where, we have a water source, which supplies water and there is a network of pipes. These are some junctions where, more than 2 pipes meet and so on, this is sort of a situation we want to deal with. This is a source and this is the sink, we want to transfer water from here to here and we have pipes of different capacities so, they can have a various capacities like may be take an example, may this is better or may be this is 5, this is 15, this is 20, this is 3, this is 40.

And we want to transport maximum water from s to t node and the issue is, how much water should be transported through various pipes subject to of course, their capacities, Now, normally of course in nature, water finds it is pressure and decides, where to divert itself but, we have to now compute in a discreet set up. So, we will first define the flow network itself and gives some other definitions necessary for the discussion. So, let us start with the definition of flow network, which is a couple of set V of vertices or the nodes.

The pipes will be now represented by directed agents so we may have a little bit more general situation, we have a directed pipe so, pipe which can carry water in a specific direction. So, we will assume, that there is a connection between every pair of node but, not including of course, loops so, we will remove. Through this, denotes a complete graph, a directed graph where, there is an edge from each node to every other node. In addition to that, we have a function C, C is a function from edge set.

So, let us just call this E to non negative reals, and finally we will have 2 special nodes s and t, these are elements of V so, this is a the algebraic setup, that we want to study. Now, this allows for having different capacities in a edges between the same pair of vertices. So, let suppose, this is x and this is y, I can have a capacity 10 here and 20 here so, this is how, it is slightly generalize this picture.

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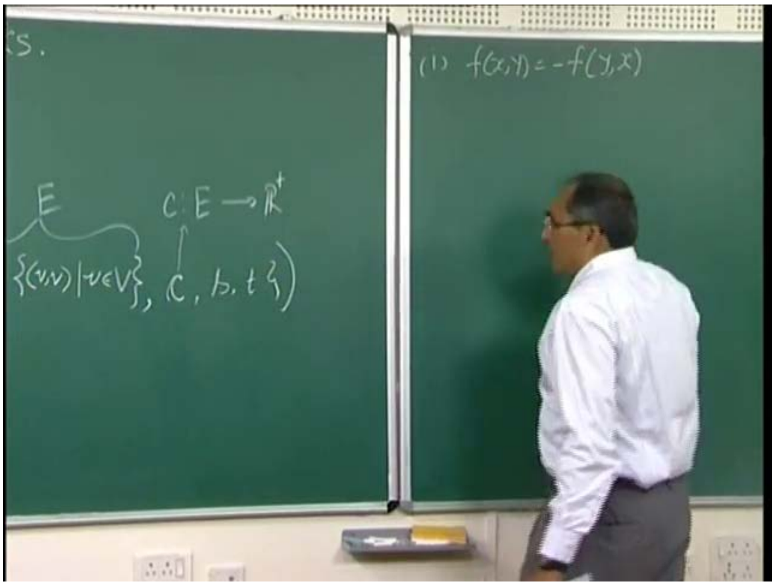

Now, let us try to define the concept of flow. So, flow is a function again, f is also a function from the edge set to non negative reals subject to the following conditions. So, the conditions are that f of x comma y is minus f of y comma x.

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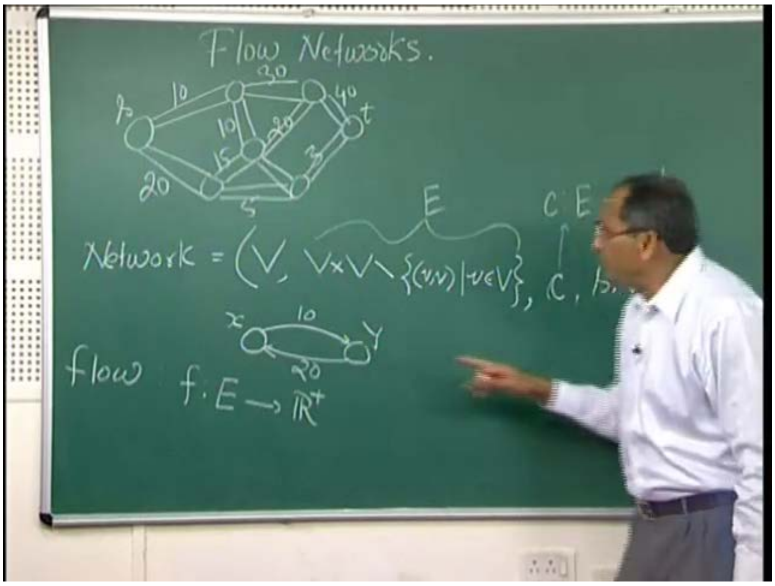

After all, whatever flow is going from x to y directly suppose, some 8 units are flowing in this direction and 2 are flowing in this direction then, effectively it is as good as 6 flowing on this direction and 0 in this direction. So, we will instead of, associating this value to 0 and this to 6, we will a slightly different notation.

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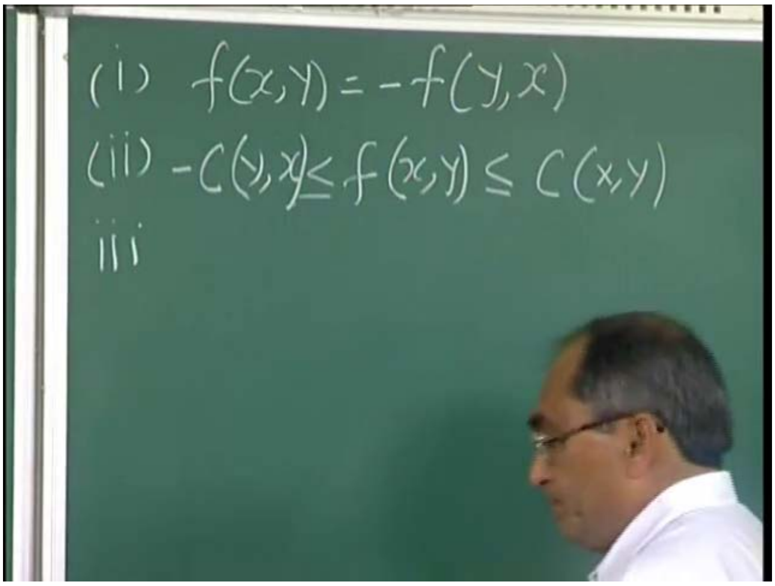

We will denote effectively the net flow so, this is if x to y is 6 then, it is simply is that, y to x is minus 6. Another important condition is, that the flow can never exceed the capacity of the edge or the pipe. So, we have a bound of C x comma y, the edge x comma y has capacity C x y, this has to be an upper bound for the flow from x to y. So, you can never send the flow from more than 10 units from here to here or you can never send more than 20 units from y to x. And because of this condition, this is also implied that, this is greater than or equal to minus y comma x because, this is equal to minus f y x.

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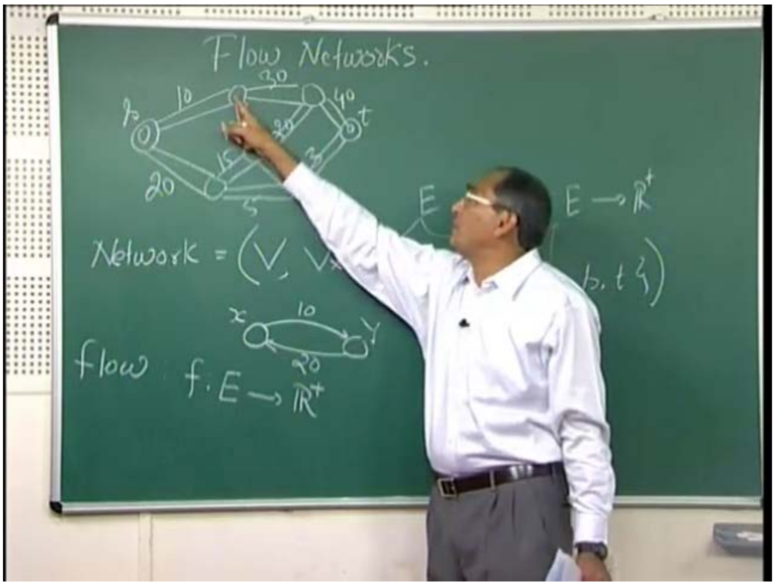

The third condition is let us just again look at this diagram, these two are special nodes, this supplies water and this one consumes it. But apart from this two nodes, if you look at other nodes, there the total flow must remain conserved, whatever comes in, is what should go out. We always talk about stationary state picture so, this we have to capture that, there is no net flow coming from any node or net flow consumed by a node.

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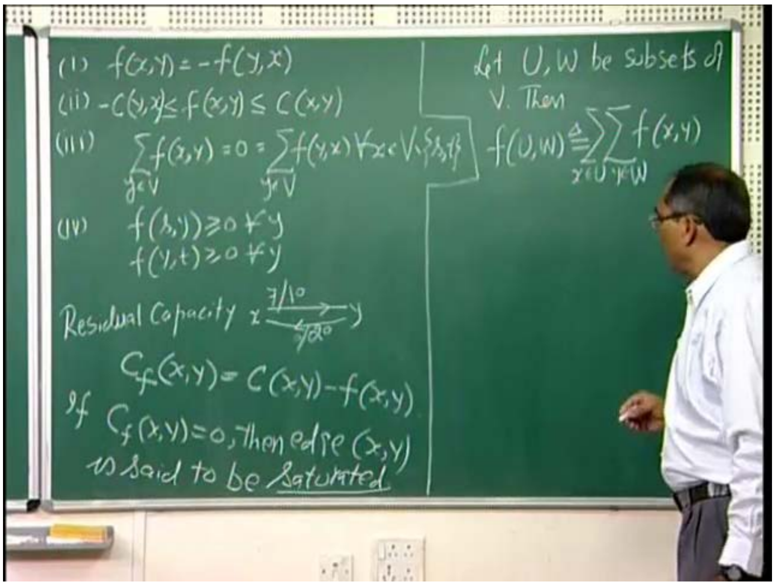

So, we will add this condition namely, some f x comma y, y remember, f x, x is 0 always. So, this is the net flow coming out of node x should be 0 and similarly, the net flow going into x should be 0, this is true for all, for all x in V minus s comma t that is, apart from these two. We also insist that, source is the one that supplies so, f of s comma y should be grater than equal to 0 for all y and f of y comma t should be grater than equal to 0, for all y.

So, these are the conditions for a valid flow in the network now, there is a notion of residual capacity. So, suppose, you have an edge with capacity 10 and there is some seven unit of flow going in this direction, there is an edge between the same pair of nodes with capacity 20 so of course, 0 is going in this direction. So, if you know this, we have capacity to add 3 units of flow on top of it, in the direction from x to y. So, subject to the given flow, the residual capacity from x to y is 3, how about the residual capacity from y to x.

Obviously, we can put 20 units of flow in this direction but whatever 7 units of flow is flowing in this direction can also be neutralized. So, that can also be added in this direction and in all, there is a residual capacity of 27 units from y to x. So, we define residual capacity as C subject to a given flow f, from x to y as C of x to y minus f of x to y. So, that defines a residual capacity, which will be useful later on. If C f x y is 0 then, edge x y is said to be saturated so, no further flow is possible in the given direction x to y.

Now, one notation which comes in handy, I want to define so, let U and W be subsets of the vertex at V and indeed, U and W may overlapped. They are not necessarily disjoined then, f U comma W denotes the sum total of the flow f x comma y where, let us write down two sums x belongs to U and y belongs to W. So, this is just a compact rotation for this notation so now, let us try to give a graph based representation of flow network and a flow in it.

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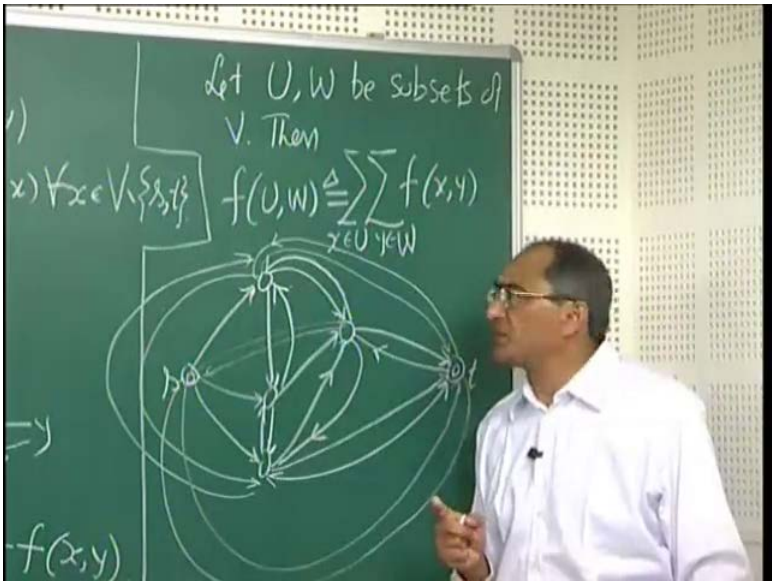

So, let us say, we have following network with this node, source node as sink node t and may be there are few nodes here, there are of course, edges in both the direction. So, let us just complete this first, except on these nodes there are no incoming edges into s because of our condition, that all the flow through s will be always emerging out of s. Similarly here, every edge is actually converging into the sink, all other edges are in both direction so, one can think of such a notation.

Now, of course, this is a complete graph so, there are a edges of between all pairs of nodes so, may be I will add these edges and there is of course, here as well and so on. So, it is not complete, just add that because, you will also have edges here and you will have edges from s to t directly but look at this, it looks pretty dirty, not very useful way to represent it. So, what we will do is, we will if I have to show of flow, I will only show those edges on which there is a flow and if the flow is empty, I will withdraw those edges. So, let us just simplify and imagine a simple picture.

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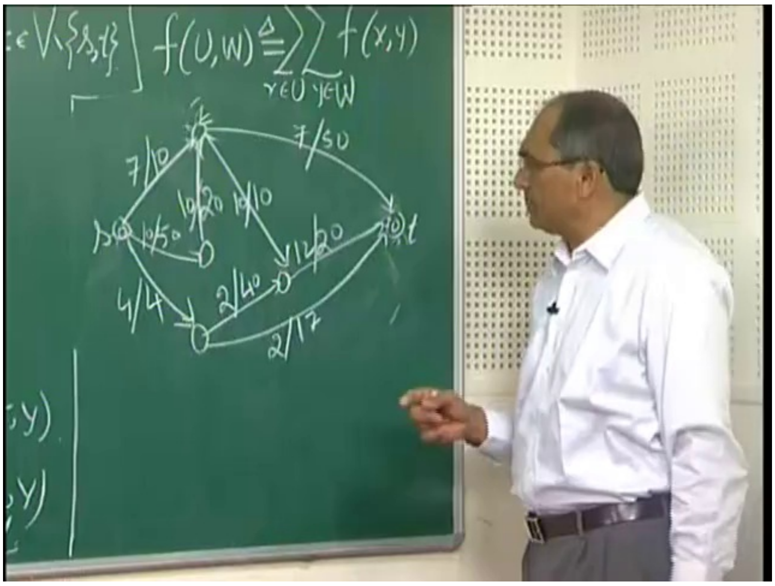

So, let suppose, you have only flows, may be there was a vertex here and let suppose, there is flow over here into this as well and there is a flow going in there, on going there, over here, here and here. So, we normally show the capacities on the edges like that and we will show the flow, actual flow passing through it so, may be it looks like this and may be 10 out of 50, 4 out of 4, etcetera. I just want to construct one example of a valid flow so, I will just contracts some numbers, here I am showing 10 units passing into this, 10 going from here to here.

So, this seems to be conserving the flow, 4 it is coming here say, 2 going into this, 2 going into this, 17 is coming in here, may be the region edge over here, 10 going through this, 7 going into this and so on may be 12 here. So, I hope this is correct but, in case it is not, one has to just check but, anyway the whole purpose is, that one can actually represent a particular flow in a network like this by just removing those I just which are currently not being used by the flow.

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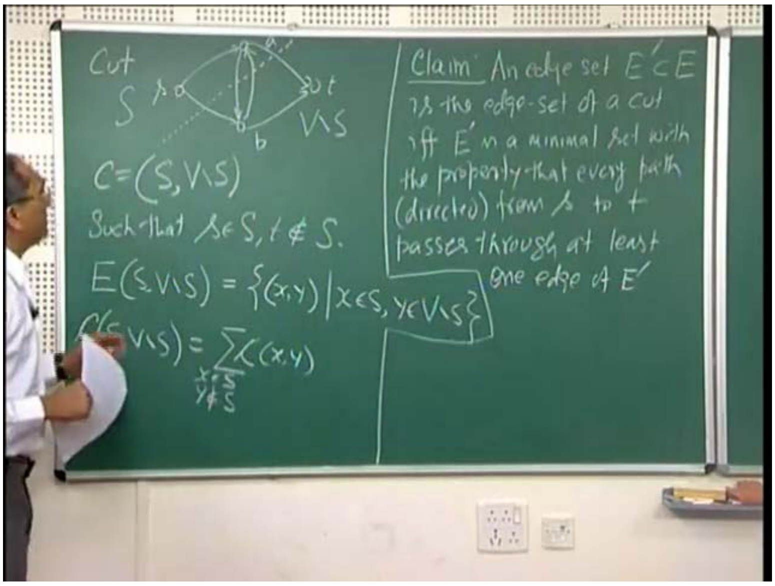

So, now, let us define a notion of a cut, a cut of a flow network now, take a small example, very, very simple example s, t, a and b. So, cut is actually partitioning of the vertices into two sub sets such that, s belongs to one of the subsets in t, to the other. So, we define, a cuts C as topple S comma V minus S. So, this is S side and this is V minus S side such that, s belongs S and t does not belongs to S.

hen, we define the notion of the capacity of the or rather first of all, let us talk about the edge set, the edge set of a cut E, of a cut S comma V minus S. So, this is the set of the all the edges x comma y such that, x belongs to S and y belongs to V minus S. So, we are here representing those edges, which are pointing from a vertex one S side towards a vertex on the complimentary subset. And capacity, capacity of a cut S comma V minus S is the cumulative capacities of these edges so, we will add up C x comma y, x in S, y not in S.

So, this is the total sum of the capacities of the edges crossing this so, here, we will consider this edge, we will not consider this edge in the wrong direction, this edge and this edge, these three edges constitutes the edge set and their capacities sum is the capacity of the cut. So, now here, I am going to claim a very a simple fact but, useful so, the claim is, that an edge set E prime of the complete edge set E, is the edge set of a cut if and only if, the following.

That every path starting at, remember this is a directed graph so, paths are all directed, every directed path from s to t passes through at least one edge of E prime. And E prime is the minimal set with this property if and only if, E prime is a minimal set with the property, that every path which is directed from s to t passes through atleast one edge of E prime. So, this right inside is actually a characterization of the edges of a cut or the edge set of a cut. What this is saying is, that every path passes through at least one edge of E prime and this property break down if I withdraw even one edge of E prime from the collection, then that sense it is minimal.

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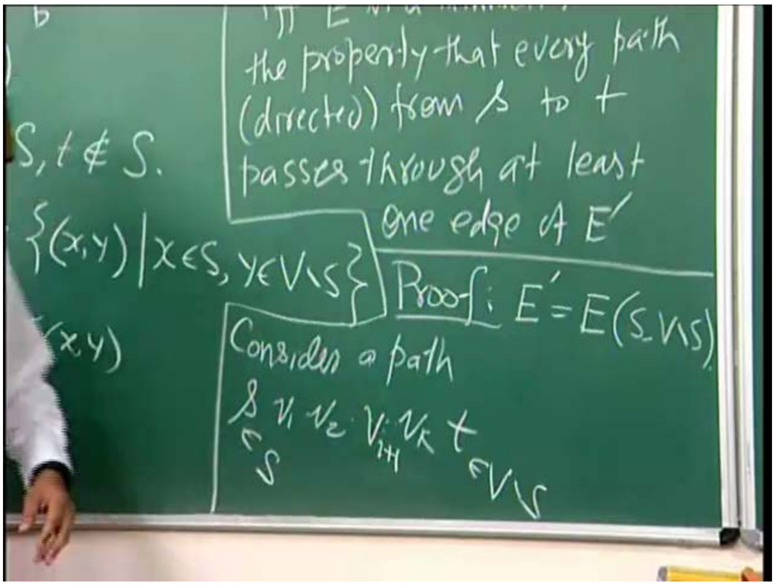

So, let us try to prove this claim, first of all let us say, we are given that E prime is the edge set of the cut, some cut. To see that, every path should pass through to at least one edge of E prime let us take some arbitrary path from s to t. Consider a path s, v 1, v 2, v k, t so, there is an edge of course, s to v 1, v 1 to v 2 and this is a complete graph so, there are edges and it start at s ends at t. This belongs to S, this belongs to V minus S so, as you proceed from left, there has to be some first vertex, which belongs to V minus S. So, let suppose, that vertex is v i plus 1 which means, that upto v I, all the vertices are in S and here, the first vertex occurs which belongs to V minus S. Obviously, v i plus 1, v i plus 1 could be t so, there is no problem about existence of this hence, v i belongs to S and the edge v I, vi plus 1.

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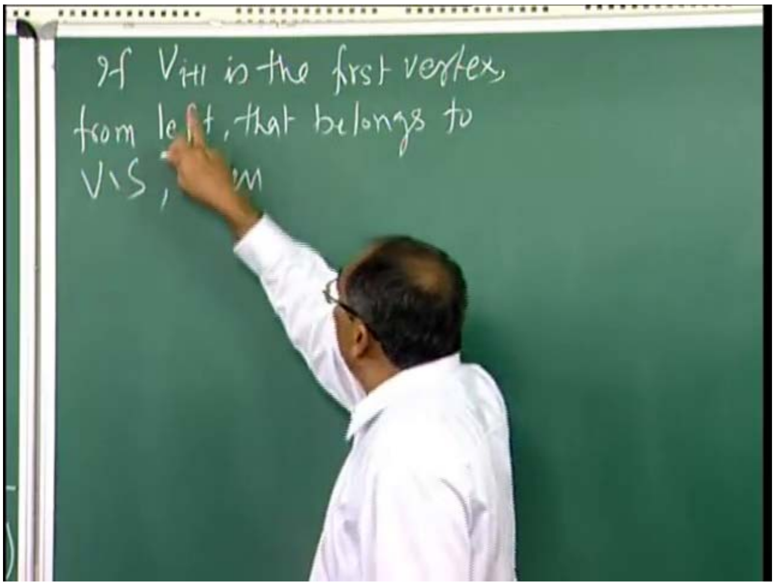

If v i plus 1 is the first vertex from left, that belongs to V minus S.

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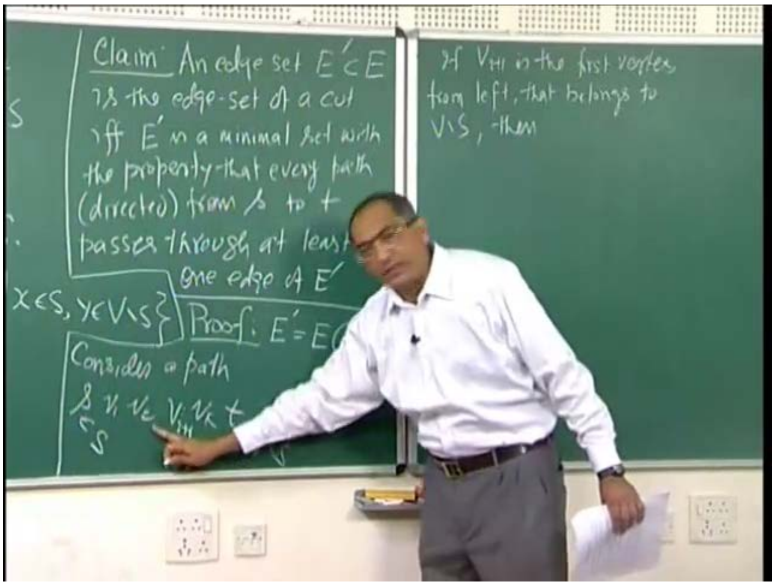

Then, now note that, v i plus 1 cannot be S, so there is always a vertex before it.

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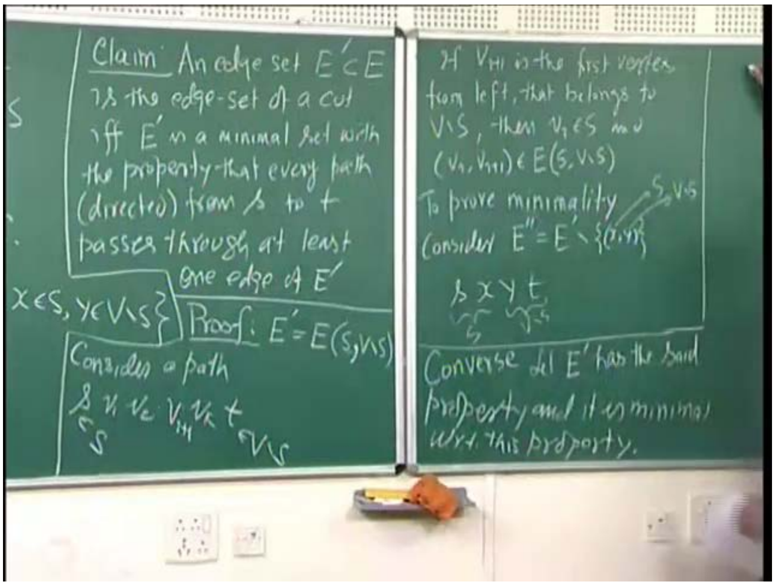

So, that why, I am just writing this as v i plus 1, in that case v i belongs to S and edge v i v i plus 1 belongs to the edge set of S comma V minus S. So, we have established that no matter what path you give me, there is always at least one edge in the path belonging to this set. Now, to prove that, this is a minimal such set so, to prove minimality, to prove let me take off an edge. So, consider E double prime equal to E prime minus a some edge x comma y. Clearly, this is in S and this is out side S so, this is one edge we have taken off.

Now, I can actually form a path namely, s x y t consider the simple path of three edges, as s and x both belong to S and these two belong to V minus S. This edge does not belong to the edge set of the cut, this edge does belong to the edge set of the cut and now, x y also does not belongs to the set E double prime. So, E double prime fails to have that property, that we have described here hence, E prime is a minimal set having the property, that every s t path passes through it.

So, this is the one side of the story, let us prove the converse let suppose, there is a set, which has this property and it is minimal. Let E prime has the said property and it is minimal with respect to this property so, we want to prove that, this has to be equal to be edge set of some cut.

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So, let us define a set S, we define S to be those vertices which are such that, x is reachable from s, small s without passing through E prime edges. What it says is, to consider paths from s, which do not pass through any edge of E prime, wherever it can reach those vertices are 14 S which alternatively means, there is a path from s to x without any E prime has on it. All such vertices are collectively put into S prime now, consider the cut, the cut S, V minus S, we will prove now, that the edge set of the cut is contained in E prime.

So, let us define the sub claim that, E S comma V minus S is contained in E prime so, to prove this, we go by assuming the contrary and prove an absurdity. So, assume contrary, let edge x y belong to E, S comma V minus S but, does not belong to E prime, it does not belong to E prime but, it belongs to the edge set. So, by the definition of this edge set, x belongs to S and y belongs to V minus S, this is the definition. Now so, there has to be a path from s to x because, by definition S is the set of vertices, reachable from s through directed paths without passing any edge of E prime.

So, we must have had a path s to vertex x now, that we have also assumed that, this as a x y does not belong to E prime so that, path can be extended to y because, this also does not belong to E prime and this is one of those valid paths. So, this implies that, y belongs to S as well because, y is also reachable from s without passing through any edge of E prime. But, what we notice here is, that y belongs to V minus S, which is a contradiction so, we conclude that, this set must be empty.

In other words, E prime contains every has the cut set of it, let us just write down this complete argument. Now, since x belongs to S, there is a path from s to x without passing through E prime edges. Now, since x y does not belong to E prime, this path can be extended to y hence, by definition of a S prime is y also belongs to S but, as given here, y belongs to V minus S so, this a contradiction. So, we have seen that, a every edge of the cut belongs to E prime but there is another assumption namely, that E prime is a minimal set with this property.

We have already shown that, the edge set of every cut has this property which means, that property is satisfied by E as well. So, if E prime is strictly larger than this set then, this cannot be a minimal set with the property. So finally, we can say, since E of S comma V minus S has the property, what property I am talking about, I am saying the property that, every path from s to t passes through the given set, has the property and E prime is a minimal set with the property. Hence, E prime better be equal to so, the better be equal so now, let us talk about one more interesting fact about the networks.

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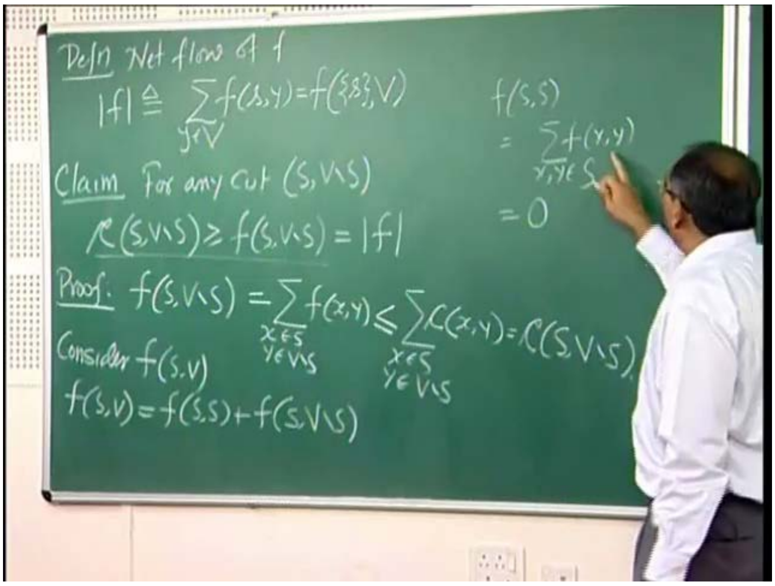

So, the second result that I want to state, needs this definition, definition of net flow so, given a flow, flow has we have defined is a function, which gives values associated with edges. Say, f net flow of the function f, we denote that by modulo f, this actually defines the total amount of fluid that enters into the network. So, this can be stated as, f s comma y where, y belongs to V, this is the total quantity of fluid that emerges from the source and enters into the network.

And in our notation that we have defined earlier, this is same as f s comma V so, here is a claim so this is, that for any cut, the capacity of the cut is greater than equal to f S comma V minus S, which is equal to the net flow as defined here. What this says is that, the total capacity of the edges jumping from S to V minus S side is always greater than or equal to this quantity, which actually is nothing but, the net flow passing from this set to this set of nodes, the net flow.

So, let us try to prove this, f S comma V minus S, this we can express as from the definition f x comma y, x belonging to S and y to V minus S, this was the definition. And from the one of the conditions namely, that the flow can never exceed the capacity, we can bound this and say that, each value f x y is bounded above by C x y where, x belongs to S and y to V minus S. This is precisely C of S comma V minus S that is, this in equality.

So now, let us try to show that this two are equal, consider f of S comma V, f S comma V is f of S comma S plus f of S comma V minus S. If you just expand this, you split that sum into two parts, you get f S comma S plus f of S V minus S. So now, the question is, what is this expression f of S comma S, let us just look at it, f of S comma S is sum f of x y where, x and y both belonging to the side S so, this is actually 0. The reason is, for every pair x y, we also have pair y x and their value is equal and opposite so, they cancel out each other and this term reduces to 0. So, the f S V is same as f S V minus S.

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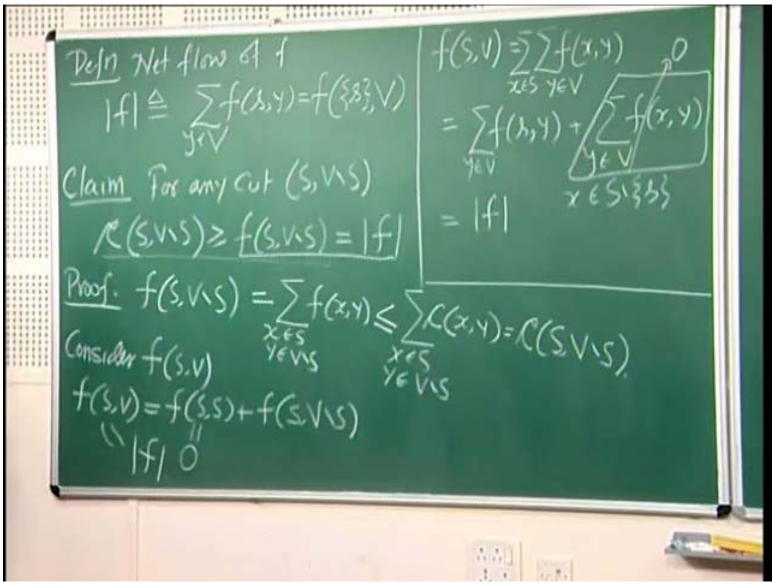

Now, let us expand f S comma V and that is, sum f x comma y, x belonging to S and y which runs over entire set V. So, let us split this into two sums, one is f s comma y, y in V plus summations x belonging to S minus small s, f x comma y, y belongs to entire set V. Or may be, I will write down this slightly differently y belongs to the whole set V where, x runs over S minus small s. This is the same expression as this so, let us just take a look at this sum now, that this x is neither equal to s nor equal to t because, t belongs to the complimentary set of S.

Hence, the conservation law says that, this is equal to 0 so, this second term vanishes and the first time is by definition, the net flow remember that is how, we have defined it. So, this is nothing but, net flow so, this we have shown is equal to mod of f. Hence, the net flow is equal to this, that is the second equality. What does that mean so, let us take a look at a simple picture.

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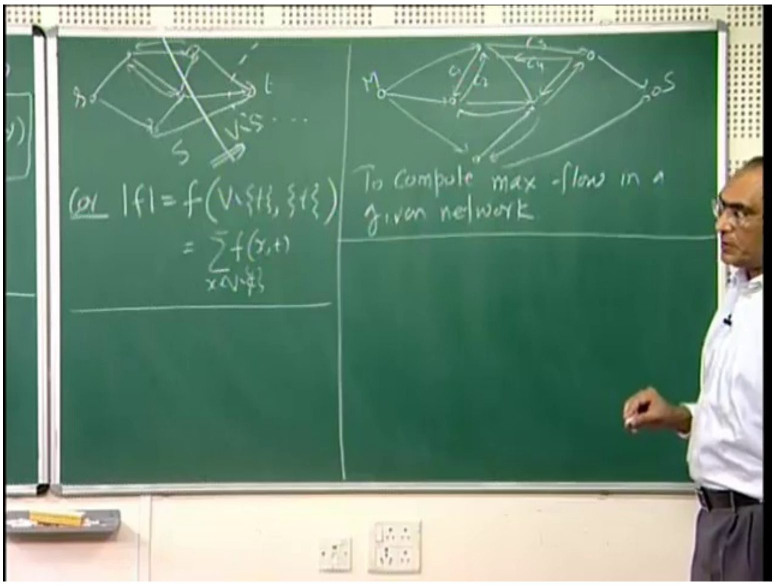

I have suppose, we cut the network here, this result states that, whatever flow that has emerged out of the source is exactly equal to the net flow passing from side S to side V minus S in this step. And that is precisely, what we should have due to the conservation rules hence as a corollary, the net flow should also be equal to f V minus t and t, this is a special cut namely, this cut. This is also equal to sum f x comma t, x belongs to V minus t, which is the net flow going into the sink t, which is again as expected, whatever emerges from here eventually enters into the sink.

So now, we have more or less defined all the basics and towards the end, I want to define a problem. So, let us just imagine a situation where, you have a mining company N, which has a side somewhere or there is a steel mill somewhere quite far off. And there are several towns in between shown that, mining company can generate enough coal and steel mill is also ready to consume as much as is available. We have some transport companies, which provide connectivity which can carries coal between towns so, may be this is how, it looks like and so on.

Our objective, one more thing about these transport companies is that, they have certain limited capacity, in a given unit time, they can carry atmost so much coal. So, they have been, you know there is some capacity c 1, c 2, c 3, c 4, etcetera associated with each of this lines in the transportation. We want to compute what is the maximum coal that can be transported from M to S and how. So, we would like to know, what is the maximum amount that can be transported and which edge must carry how much so that, we can realize that maximum flow. And that is precisely the problem, that we want to address next in the next lecture that is, to compute max flow in a given network. Before I close the last point, is about multiple sources and sinks.

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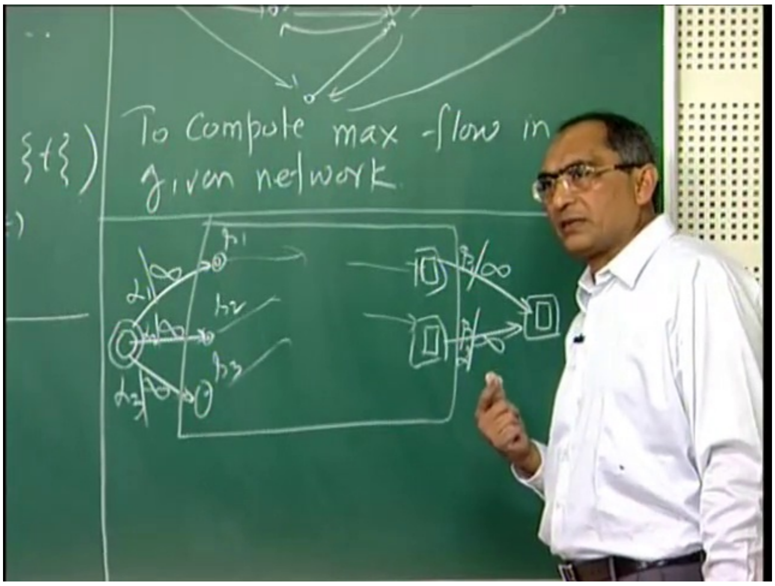

Suppose, we have several sources as well as several sinks in the network, how does this problem change in that case or how do we handle this. And the answer is actually very, very simple what we do is, we define two artificial nodes. So, these are new source and sink nodes and these are reduced to regular ordinary nodes so, they have whatever edges we have. On the top of it we add the new edges in the direction from new source to the original sources and from original sinks to the new sink and associate infinite capacities to each of these edges.

Now, this becomes a regular source one sink problem and the thing that I want to argue now is that, every solution of this is a solution of the original problem and every original problem solution is essentially a solution of this and the reason is the following. Suppose, we are given a solution of the original problem, that would generate certain flow outs, out of each of these sources. So, let us call them s 1, s 2, s 3 whatever the flow emerges out of it let us say, some alpha 1.

So, we can send alpha 1 flow from this alpha 2, from this alpha 3, from this similarly, whatever is sinking into this namely, beta. We will send that beta 1 into this, beta 2 into this and so on so, that becomes a valid flow of the extended or enhanced flow network because, now the net flow here is 0 similarly, net flow here is also 0. So, every solution of the original problem is the solution of this, that the converse is also true.

If you once you solve it, you remove this portion completely and now, you have got a solution of the multi source, multi sink problem. Hence, one has to compute only a max flow solution for this enhance problem and from that, we will get the next flow solution of the multi source, multi sink problem. We will continue with more of this, in the next lecture.