Computers Algorithms-2
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Lecture - 27
Linear programming I

Hello. Today we will discuss an optimization problem for linear programming, which finds an application in many industrial problems. It is also used in designing approximation algorithm. One of the algorithms which are to the popular to solve this problem is called simplex algorithm, which is what we are going to discuss the problem is specified as follows.

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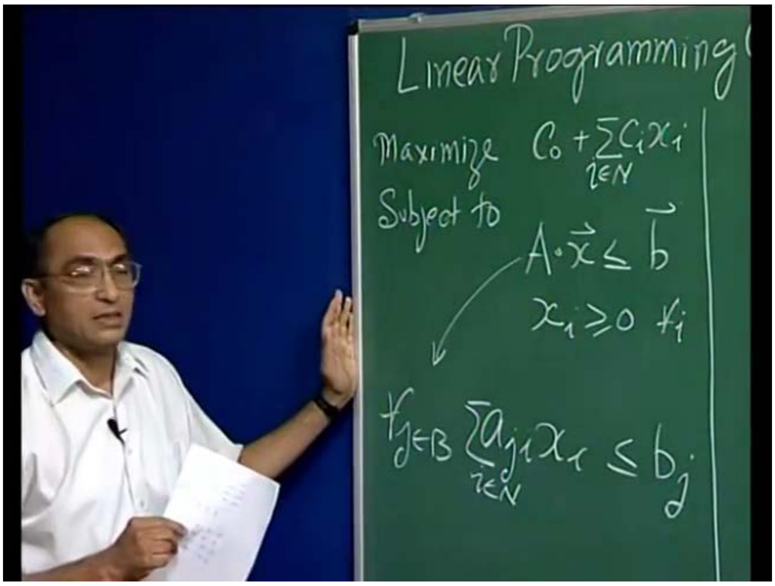

Maximize c 0 plus c i x i subject to we have a matrix. We have a variable vector x and a constant vector b. These are the conditions, which are equivalently stated. As for all j in B, a j i x i, i belonging to say N is less than or equal to b j. So, N denotes the indices of the variable. j denotes the indices of the equation. So, this stimulatory stated as its matrix a of constraints, coefficient.

Each of the elements of this vector is less than equal to corresponding element b j. The second condition is that each variable is non negative. Subject to these two conditions, we want to maximize this linear expression where c 0, c 1, c 2 etcetera. Again, i belongs to N. These are given. First, I am going to describe a couple of applications to the problems that we have discussed in these lectures.

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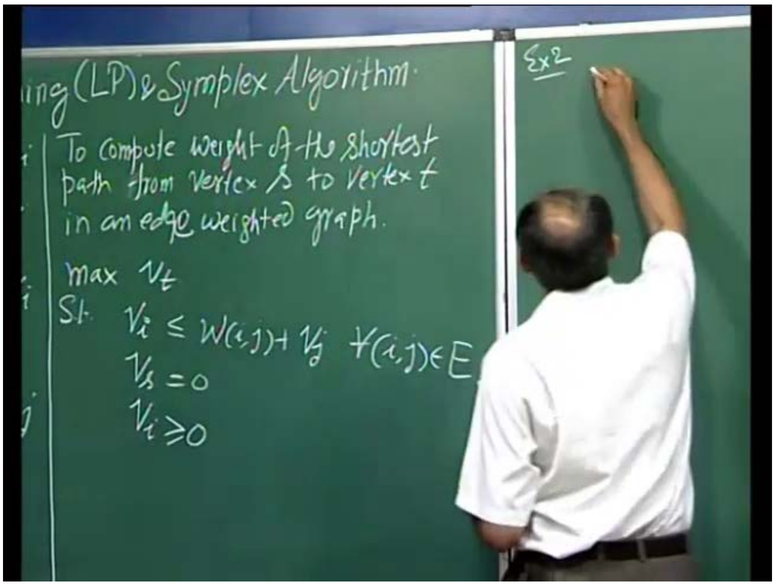

So, first one is to compute distance or weight of the shortest path from vertex s to vertex t in an edge weighted graph. We assume the weights are non negative. We have discussed this earlier in these lectures, how to compute the shortest distance from a fixed vertex to all the vertices. In our case, we are and the distance of fixed vertex. So, a linear program can be designed for this in the following fashion. We say that there is a variable let us say, V i for ith vertex. For each vertex, I am designing, defining a variable V i. We say that V i is less than equal to W i, j plus v j.

This is for all i, j edge in the edge set. We also said that V of s 0 and subject to these conditions. We want to maximize V p. This is subject to these conditions. We have as many conditions as the number of edges in V. In addition to that non negativity conditions for the variable, one can verify that the solution of my mistake. Yes and the non negativity condition, I did not mentioned is actually this. One can verify that the solution of this program, linear program. In that, the value of V p will be the shortest distance, the weight of the shortest path from s to t. The other example, example number two is for the flow problem.

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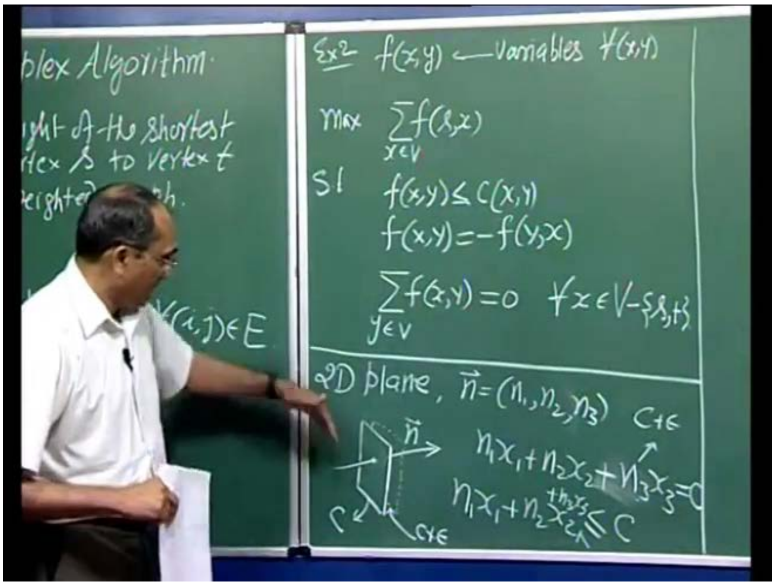

In a flow network we define variables. f x, y are the variables for all x y pairs. In this case, we are going to maximize sum f s, x for every x in V. Notice that, this is precisely the value of a flow if f x, y denotes the flow value on the edge from x to y. Subject to these constraints, which are f of x, y is less than equal to the capacity of the edge C x y. Further, we want to have f of x, y equal to minus of f of y, x. Thirdly, we will have a Cut chef law which says f of x, y for y all y in V equal to 0 for all x s in V minus source. So, these are the standard conditions that of course, satisfy subject to that we want to maximize the net flow.

Notice that our requirements are linear conditions inequalities or equality. Now, in this formulation, we express our conditions through inequality. So, one would want to know how do I express an equation? Suppose I would like to express that some linear expression should be equal to this as we just saw in our in the second example. So, if I want to express that some alpha should be equal to b, then, we can splits this into two inequalities. Alpha should be less than equal to b. Minus alpha should be less than equal to minus b. That will ensure that alpha will be said to be equal to b in any solution of this in if this linear program.

Now, in case, we want to avoid this condition for a variable. We are not interested in the positive value of a variable. Then, we can define for that variable x i to new variables x i prime minus x i double prime. If we substitute for x i this expression and then, which we insist both of these should be positive as a linear program refigured; that will allow me to have both the negative positive and values of x i. So, this is a very versatile formulation. Many problems can be expressed in this form. Now, let us try to give a geometric prospective to this part. Let us say you have a two dimensional plane; a 2D plane and a vector n which is n 1, n 2. So, you have a vector n 1, n 2.

Then, the equation of this plane which is perpendicular to this n is precisely, n 1 x 1 plus n 2 x 2 equal to some constant C. Now, if the position of the plane is moved forward or backward, that would be effectively saying that the constant is changed. So, if I ask what exactly is n 1 x 1 plus n 2 x 2 less than or equal to C. n 1 x 1 plus n 2 x 2 is equal to C plus some positive epsilon. Suppose, I replace this to C plus epsilon; then, I know the plane will be parallel to this. But, it might move let us say, on this side; one of the two sides. We do not know which one let us say, it is in this side away in this direction. This indicates that every point, which for this expression as value will greater than C will be on the on that side of this plane.

This is the plane corresponding to C. This one is corresponding to C plus epsilon. So, everything greater than equal to C will be on the other side of the plane. Less than equal to Cc will be on this side of the plane. In other words, this plane divides the entire space. So, well my drawing is in 3D. So, it should have been actually 3D. But, does not matter; so, the plane actually partitions. So, to be accurate, let me put the third dimension here. This could be plus n 3 x 3 equal to C. Similarly, we will have over here plus n 3 x 3.

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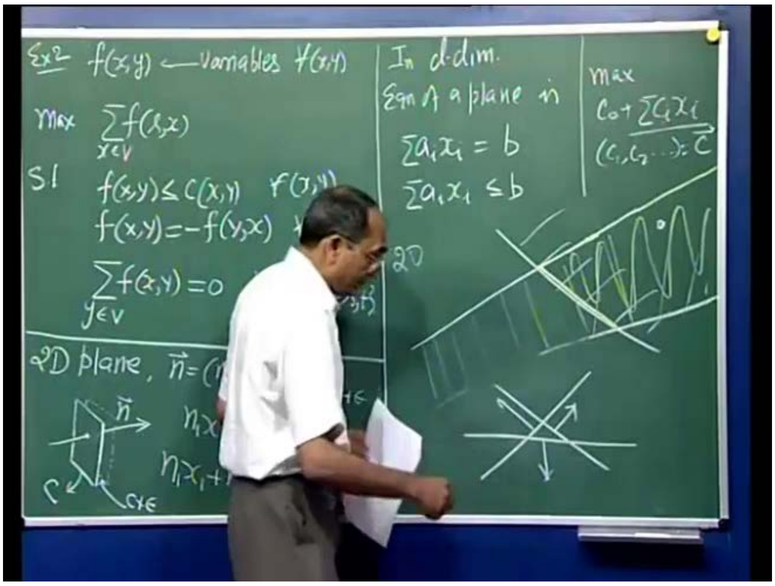

In d dimension, equation of a plane is exactly the same way. Sum a i x i equal to say a constant b. Let us put b here. Then, an equation such as a i x i less than equal to b indicates the half space. One side of the entire space, one side of the plane and that is known as a half space. So, the collection of these conditions as we have here, we have several conditions here. This one for example, says for all x, y. This is for all x, y. In 2D case, if I have let us say three such conditions, I may have three half plane.

In 2D, a half plane, a plane is a line and half plane is one side of the line. So, suppose the condition says that I must be on this side of this line. I must be on this side of this line. Similarly, I must be on this side of this line. In that case, the region in which I am supposed to stay is this region. Any point in this region will satisfy all the conditions of in inequality of this form. Now, suppose we have a region defined by these lines, which is completely bounded. So, it says that we should be on this side of the line or in general planes.

Let suppose c as a vector. c is the subject condition that we are maximizing. The expression is c 0 plus some c i x i. Let this c 1, c 2 be the vector c. Let this be vector c. Then, this expression is nothing but the dot product of the position vector of that point, which indicates the solution and the vector c. So, if i have a coordinate system, if I have a point in this space; this space is known as feasible space. This is the feasible space of the constraints.

I have this position vector. If I take the vector c from the origin and I will project. So, let us just take and I project my vector x. This is the position vector x. I project it. Then, this distance is nothing but c dot x. This is c dot x. Our goal is to maximize this expression. Remember, this is just a constant. Hence, what this is indicating is that it shows me where I need to find a point, which is as far as possible in this direction.

In this case, this is that point. This is the point, which is still in the space. Hence, it satisfies the constraints and the value in the sense that x dot c will be maximum. That will be the answer to our program. In case we have a situation whether feasible space looks like and my c vector is in this direction. Then, this will be the solution even though the feasible space is unbounded. In another situation, in this very case, if c vector is in this way then, as I proceed, I will move along this direction. I will find that the c dot x continue to increase.

As a result, we can never find the maximum possible value of this expression. This situation is indicated as unbounded solution. Although there is a feasible region, there is no point which gives you the maximum value of our expression. Finally, the third situation can be that we have let us say these three planes. This as an example and which indicates that I want to be on this side of this plane, on this side of this plane and on this side of the third plane. In this situation, there is no region which is consistent with all the three conditions.

In this case the feasible region is empty. Therefore, there is no question of any solution because nothing satisfies our conditions. There is nothing to maximize. So, this is the geometric interpretation of the problem. This format of the problem is known as standard format. Now, I am going to restate the same problem. That is known as a slack form of the problem.

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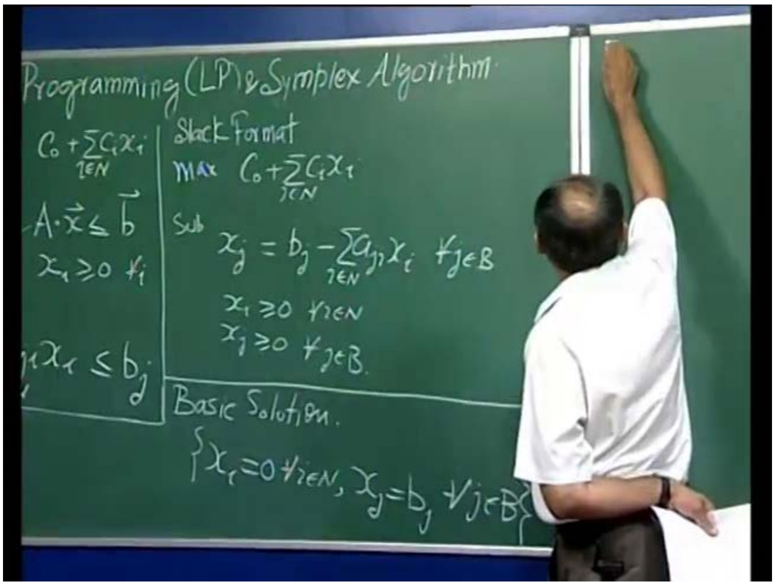

Slack format, what we will do is we will take these conditions, each of these. Each of these wants us to ensure that the difference this minus this must be non negative. So, we are going to introduce one variable for each of these equations. Now, we have max c 0 plus c i x i; i belonging to N subject to x j equal to b j minus sum a j i, i belonging to N; x i for all j belonging to b. B is the index of the equation. We have introduced a new variable called slack variable as the one for each of the equation.

Now, this condition is equivalent to saying that x j should non negative. So, we now add conditions that x i should be non negative for every i in N. x j should also be non negative for every j in B. Now, we have two sets of variables. One set sometime these are called basic variables. These are called non basic variables. Now, we will describe the potential solution called basic solution. A basic solution is nothing but all x i being equal to 0 for all i in N and all x j are equal to b j for all j in B.

This will constitute a solution if each of these bjs are non negative because we want to guaranty that these variables are also non negative. We do have all these. Sorry, these are also non negative. These conditions are satisfied and that is about all. So, this is a possible point in the feasible space. We are not claiming that this is going to be maximum solution; solution which maximizes the object function.

So, in case all bjs are non negative then, we can always have one solution for basic solution. Now, initially we are going to consider the problem with the assumption that for the given problem, the basic solution is a feasible solution. Later on, we will discuss the problem when this is not feasible. Now, let us look at the situation in which basic solution is optimum solution as well.

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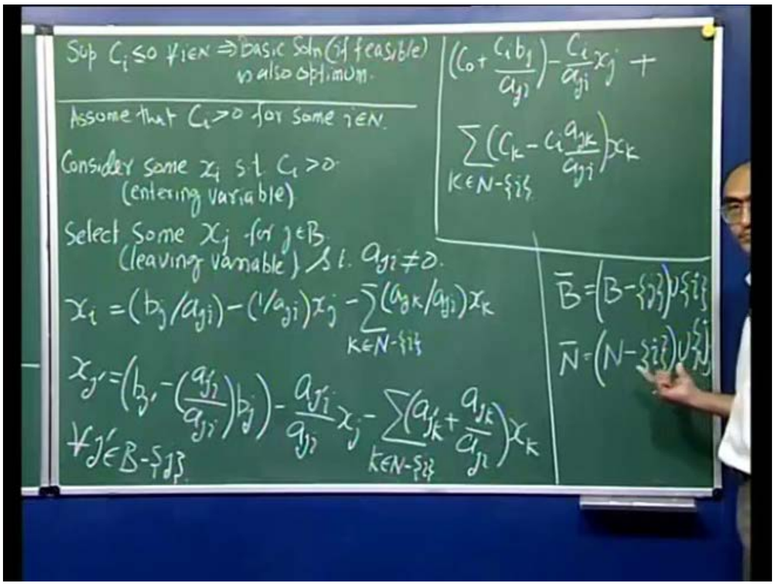

Suppose, each of the c, I suppose c i are non positive for all i in N. In that situation, raising the value of any x i will decrease the value of this. Hence, the best bet for maximizing this would be setting exercise to 0. That is precisely what we do in the basic solution. So, the basic solution is also optimum in this case. Basic solution if feasible is also optimum. So, in that case we do not have to do any work. Therefore, we are going to consider situation when not all of them are negative.

Some of them are positive. So, let us form; let us perform certain transformation in this formulation of the problem. Now consider, so here, assume that c i is positive for some i in N. now, consider some x i such that c i is greater than 0. We will also call it. Later on, we will see why and entering variable. Let us also select some x j for j in B. so, we all stable. Now, what we will do is that we consider that equation which corresponds to this x j.

In case a j i is non zero then, we can state that equation in the following fashion as x i equal to b j divided by a j i minus 1 for a j i x j minus some k belonging to n minus i a j k divided by a j i x k. Now, this is exactly the same equation that we have here. Only thing is assuming that a j i, this particular i is non zero. Then, we can take the term on the other side. Divide by a j i the entire equation. Take this term back on this side.

That is this one and we get this equation. So here, we are putting a condition such that a j i is not 0. Subject to this, this is a valid transformation. The remaining equations from this set will look like as follows x j prime. So, j prime are the remaining indices in b other than j. That will look like b j prime minus a j prime i divided by a j i b j minus a j prime i by a j i x j minus sum of a j prime k plus a j k by k j i x k k in N minus i. All we do this is a fourth power all j prime belonging to B minus j.

All we have done is taken those equations. In case there was an occurrence of x i this i then, we substitute that for that this whole expression. We end up with this equation for the remaining equations of this set. Then, the third item in the program is the object function. So, our object function transforms to c naught to c i b j over a j i minus c i over a j i x j plus sum c k minus c i a j k by a j i x k for k in N minus i. Now, this is also because here if we have x i we can plug in the new value the new expression for x i. They are going to see the distance form to this expression. Now, we are going to see what are the implications of this new formulation of the same problem.

Notice that, as long as a j i is non zero, there is no change in the information. These are exactly the same equations only written slightly different. But, for our purpose now, we will say that these are new equations where the variable set or the basic set. The basic set is if I want to denote the new basic set as b bar is nothing but b minus j union i. The non basic set of variables is N minus i union j. The reason is we have an equation for x i. Now, this corresponds to a variable corresponding to an equation. Hence, this index now goes to set b; that is b bar and among the variables. Now, we have x j. Now, x j goes to this set N bar. All right. Now here, we have said that a j i should be non zero.

Now, suppose it turns out that a j i is either 0 or negative for every possible j. So, here we have set of equation. We have take, chosen a specific x i. It turns out that for every equation, the coefficient a j i is either 0 or negative. In that case, if we set all the variables other than x i to 0 and set x i to be as large as we guess the term here; will either be 0. The coefficient is 0 or will become positive, because we have a minus sign here. a j i is negative. So, this will be positive. Hence, this will remain positive. So, these constraints will remain the satisfied. x i we have a for all x is other all x is prime other than x i are 0; x I, we are setting, taking it to infinity.

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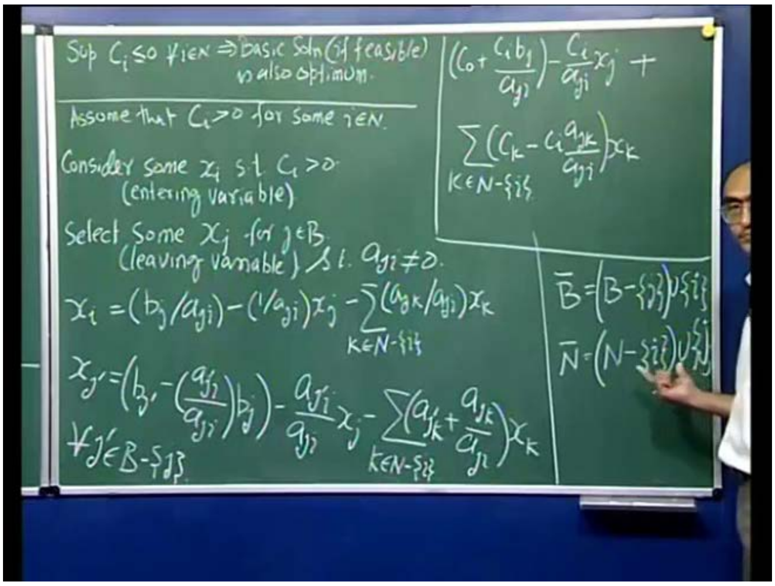

So, if a j i is non positive for all j in B then, x 1 equal to 0, x 2 equal to 0, x i equal to infinity, x i plus 1 equal to 0 and so on. This will be a feasible solution. But, the consequence of this is that in this solution, the value of object function will also be infinite. So, what we notice is that the solution will be unbounded. This will correspond to the situation we had shown in geometric picture of the problem. If it is an unbounded feasible space and my c goes in this direction; so as I continue to move forward the solution; the solution remains the solution. But, the object function continues to grow.

That is what this indicates. So, as long as we find the situation such as this for any x I, we know that the solution is unbounded. We can stop. Now, let suppose we have found a j for which a j is positive. It is not supposed to be negative. It is not supposed to be 0. Then, we have now. Now, suppose a j i is positive. Now, under this condition what is the information we can extract from these equation. So, the one thing that we notice is if we look at this term, which corresponds to the b value, the constant value in this equation. Now, this value says that b j prime minus a j prime i divided by a j i b j.

Now, this can be noticed that we are interested in those bs, which are positive or 0. But, not negative in case we want the basic solution to be feasible. So, we look at this equation. If we want to make sure that this is positive, this is indicating that b j prime divided by a j prime. i should be greater than equal to b j divided by a j i. If these are positive and if this is the smallest such value then, we guaranty that this will be non negative. So, while looking for a j; for a given i let us go back to this. Here, we choose a i such that c i is positive. Now, if i chose an x j such that a j is positive and for every other a j prime i b j prime by a j prime i is greater than equal to b j by a j. If this in equality also holds then, these terms for all the equations will be non negative.

Further, because a j i chosen here is positive, this will also 0; be positive if we had b j positive. Remember, we assume that b j is positive. We assumed our basic solution is feasible. So, this will also be feasible. So, subject to this condition that this holds for the choice of j. Then, in the new set of equation, the basic solution will also be feasible; because these terms are non negative.

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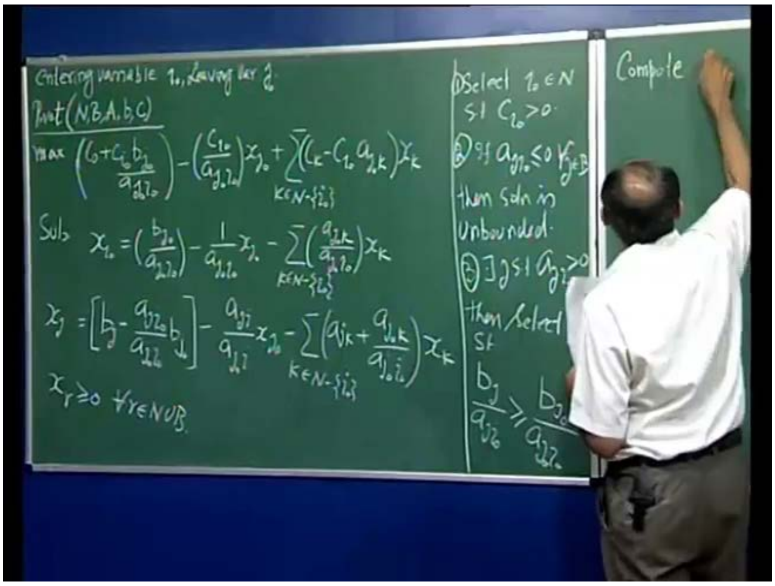

Simplest algorithm is nothing but the series of this transformation that we have discussed. I have now rewritten this. Let us say i 0 is an entering variable from set N. a j 0 is a lining variable directed from set B, which corresponds to one of the equation. Then, I will just re express the object function and the equation. Here, of course, we still have all variables; all variable with non negative for all r belonging to N union B. Now, how do I select my variable for the entering and labeling variables? What we seen is the criteria for selection will be that select i naught from N such that c i naught is positive.

That is the first criteria. Second if a j i naught is negative or 0 for all j in B then, solution is unbounded.

In that case, we have to start. There is nothing to be done. In case, there exist j such that a j i naught is positive, at least one j such that a j positive then, select j naught such that b j a j i is garter than or equal to b j prime. No, sorry, b j naught divided by a j naught for a j i naught being positive to all those equations j, in which a j i naught is positive. Then, among them we chose that j naught for which this is the least. For that, what we do is we compute delta j as follows.

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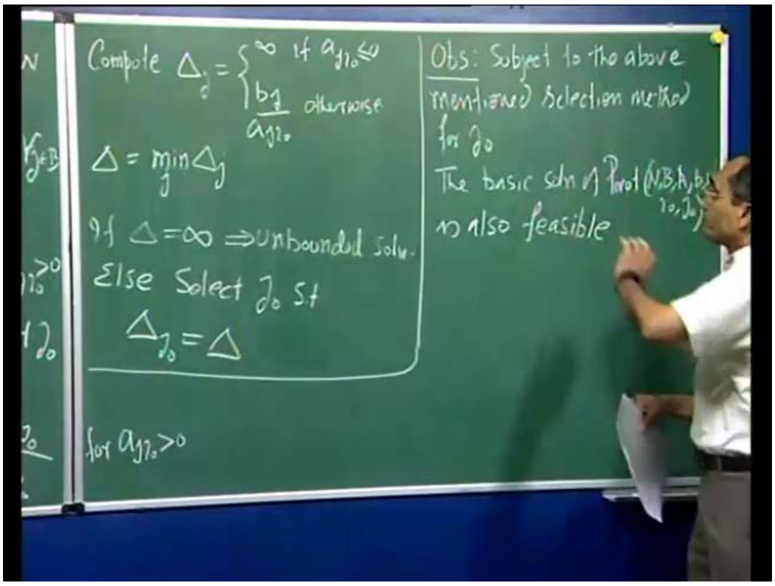

It is infinity if a j i naught is less than or equal to 0 otherwise, we give the value b j divided by a j i naught. We define delta to be minimum over all j among all j. Now, if delta is infinity that indicates that all our a j i naught are non positive. That means unbounded solution else select j naught such that delta j naught is equal to delta. That guarantees this inequality. Once we guaranty this, what we find is; notice one more thing that if this quantity is negative. Then, we know that chosen our j naught to be such that a j naught i naught is positive. This quantity is always positive.

But, by choosing with respect to those js for which a j a naught is positive, we have chosen j naught in such a way. Hence, this expression is always positive or 0 but never negative. This expression is obviously positive. So, what we notice is that subject to these selections, conditions; we will notice that subject to the above mentioned selection method. For j naught, the basic solution of the new problem or new version of the problem, which we are denoting as pivot N, B, A, B, C. Solution of pivot N, B, A, B, C.

Let us just notice that this linear program is denoted by pivot. This thing and we can specifically add i naught j naught. They are transformed subject to these in this. So, with this choice, maybe I should add that here i naught, j naught. The basic solution of this is also feasible. Recall the only condition for the linear program. The basic solution of the linear program is feasible is that those b s must be; all of them must be non negative. We have guaranty that they are non negative if we make our choice of j naught as described here.

Hence, after the transformation, we get another linear program for the same problem whose basic solution remains feasible. Remember, we started out with a feasible linear program, in which the basic solution was feasible after this transformation remains feasible. One more thing that we need to observe that if a j naught i naught is positive.

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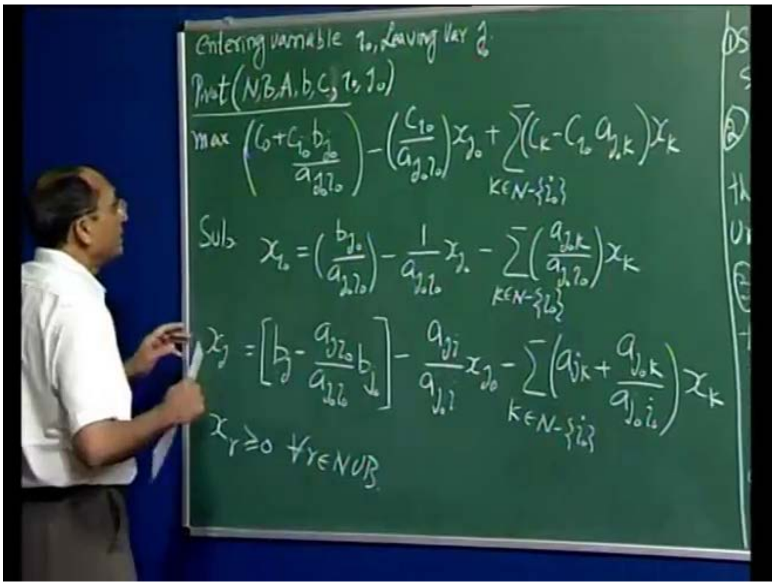

As we have chosen, we already said that c i naught is positive, in case b j naught, which we know is not negative because, our basic solution of the original problem was feasible. So, this was 0 or positive. In case, this is not 0, that is say, if this is positive then, all the three quantities being positive; the new c naught prime, the new value of this constant will increase. Now, what is the interpretation of this term? Notice that in our basic solution, these variables are set to 0. The B variables or the N variables are said to be the equation variable; are set to these values. If these values are non negative, we know that our basic solution is feasible.

In that case, the value of the object function is this expression. The value of the object function in the basic solution of the previous problem was just c naught, which has increased to this value. Of course, b j naught is positive. So, what we notice is, this transformation also enhances the value of the object function in the basic solution, when b j naught is objective. Later on, we will see what happens. When b j naught is not positive, what are the consequences?

But, we now notice that this transformation actually appears to be pushing us towards the solution. Let us now notice that it is time to combine the all these observation. Here, I am just going to describe what we are going to do is, we will first check whether the c i are all negative or not. If they are all negative then, our basic solution is the solution. If at least one of the c i is positive then, we pick that as a new entering variable. We check whether a j i 0s are all non positive or not. If that is the case then, we say our solution is unbounded.

In case at least one of them is positive then, we consider this expression b j divided by a j i naught for all a j i naught, which are positive. Pick the one, which that j for which this is minimum. So, this is the choice. We chose our j naught. Once we have chosen our j naught, we compute the pivot with respect to pivot of the old l p with respect to these variables. We write these equations. This equation, we have observed will have still at feasible basic solution. The object function will have a greater value if b j naught was positive.

One last comment in the next lecture, I will start with this algorithm. But, one last comment, I want to make in this context is that every solution of the old linear program must be a solution of this program. The reason is all we have done is rewritten those equation. All we have done is moved. Some terms on from left side to right. Other from right side to left side in these equations. Just rewritten the ex, the value of x i naught here in to the object. So, every solution of the old problem is solution of the new problem. Every solution of the new problem is a solution of old problem so that two linear programs are completely equal.

Hence, we can move on from this point onwards and continue in this direction. So, in the next lecture, I am going to write down the complete algorithm. Then, we will analyze what are the final the terminating conditions for this program.