Computer Algorithms-2

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Lecture -10

Ford Fulkerson Method

Hello, so let us resume our discussion on fluid networks today. First I am going to just review the main results of last lecture recall that a flow network is a portable.

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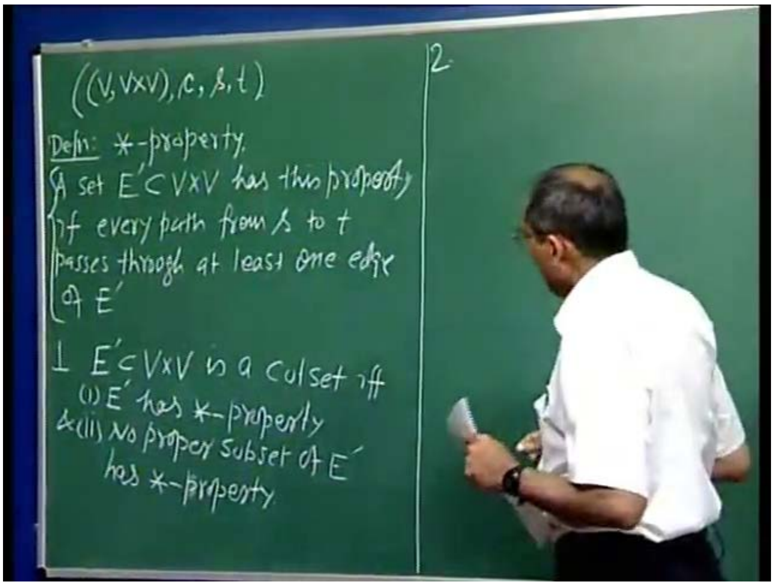

The first one is a complete directed graph, the capacity function a source vertex and a sync vertex. So, let's first define a star property, set E prime of the edges has this property, if every path from s to t. Now, naturally we are talking about directed path from s to t passes through, at least one edge of E prime. So, if E prime satisfies this condition, we say it has star property.

Then just let me review the results one was that E prime is subset of V cross V is a cut set is the cut set or the headset of a cut set if and only a E prime has star property and two no proper subset of E prime has star property. Last time what we mention the same thing was mentioned as E prime is the minimal set having star prime. The second result that we had shown in the last lecture goes that.

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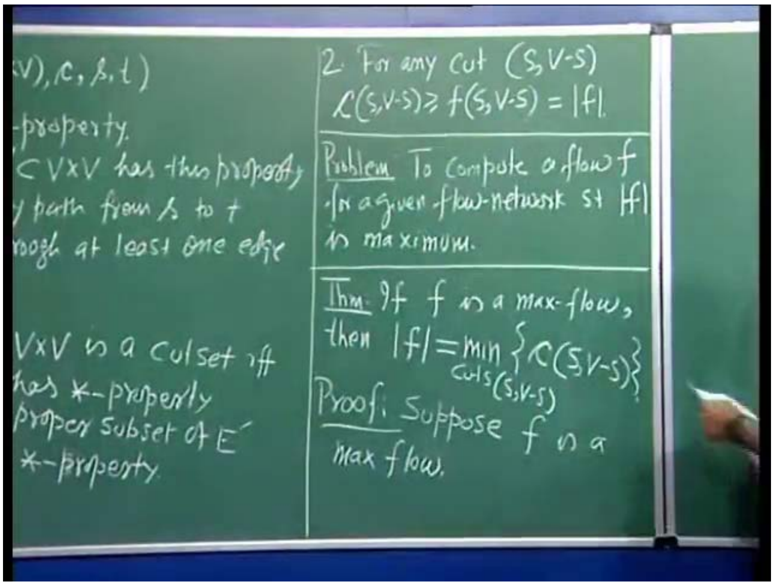

For any cut S comma V minus S, the cut capacity is greater than equal to flow from S to V minus S which is equal to the magnitude of the flow. And we had stated our main problem, a problem was to compute a flow for a given flow network, flow f such that this is maximum. Now, in order to compute such a flow what we need to characterize, what property of flow which is decidable in a local competition, says that the f is maximum. So, we will state a very well known result in this domain, which says the following which says that if f is a max flow, which means that mode of S mode of f is maximum possible flow.

So, if f is a max flow for given network, then the value of the flow is equal to the smallest cut capacity. So, overall the cuts S comma V minus S, the capacity of that cut here what we have notice this that the magnitude of any flow is less than equal to capacity of any cut, any cut. And this theorems is that if this if the flow is maximum then this will be equal to the smallest cut capacity. So, let us try to prove this, suppose f is a max flow. So, this f is one of the possible flows where the magnitude is maximum. Now, what we will do we will defined a set.

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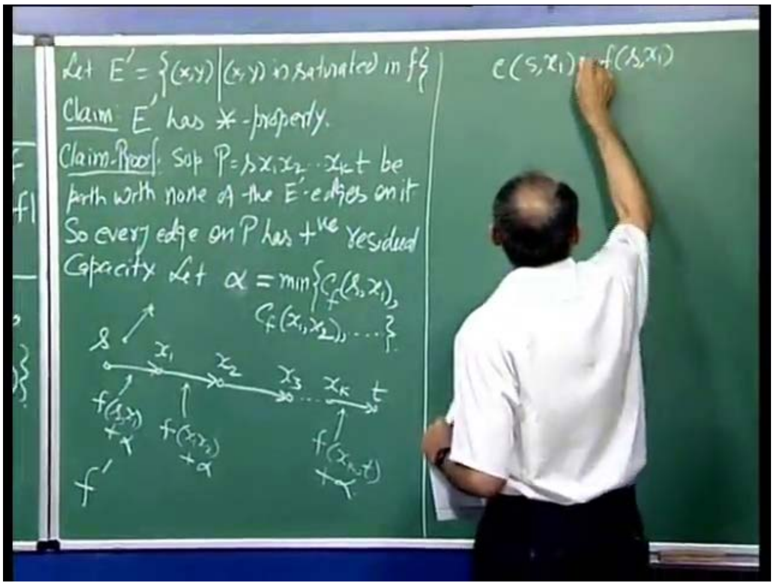

Let E prime with the set of edges x comma y such that x comma y is saturated in f. Now, recall that by saturated we meant that the flow in this edge that is f x y is equal to the capacity of this edge c x y or equivalently the residual capacity of this edge for this flow is 0. So, I have collected every single edge which is saturated in this set E prime and now first claim I am make is that E prime has star property.

Star property is that every path from S to t must pass through some E prime S. Now, to prove the claim, suppose we assume the contrary let say there is a path which does not pass through any of the E prime. So, suppose P equal to s x 1, x 2, x k t be a path with none of the E prime edges on it in that case edge s x 1 must have flow strictly less than the capacity s x 1. Similarly the flow on x 1, x 2 must be strictly less than the capacity of x 1 x 2 and so on. So, every edge on p has positive residual capacity positive residual capacity is the capacity minus the flow. And since none of them are equal they are all going to be positive differences. Let x be equal to the minimum of all this residual capacity. So, let us just say minimum of we denote residual capacity as C f of s x 1 comma c f of x 1 x 2.

Now, you can visualize this has here is s we have x 1 x 2 x 3 and so on. Finally, it reaches t, this is x k may we should use some other symbol for this, let say this is some alpha. Now, suppose we add the flow so we define new flow on this as f of s x 1 plus alpha. Similarly, here s x 1 x 2 plus alpha and similarly, original flow x k t plus alpha. In the rest of the graph, we leave the flow in that we know do not change anything. So, the first question is that the new flow that we have defined, let say the new flow is denoted by f prime this is a function assigning values on each of that, is this a valid flow?

So, notice that in any edge other than this paths, the flow remain intact, if the two end vertices of this edge are different from this. Then the cuts of flow on this vertices remains valid because we are not changing any of the flow going into it or leaving it, the capacity constraint. Namely the flow should not exceed the capacity is also un-effective. Now, let us talk about these this edge had capacity c s x 1.

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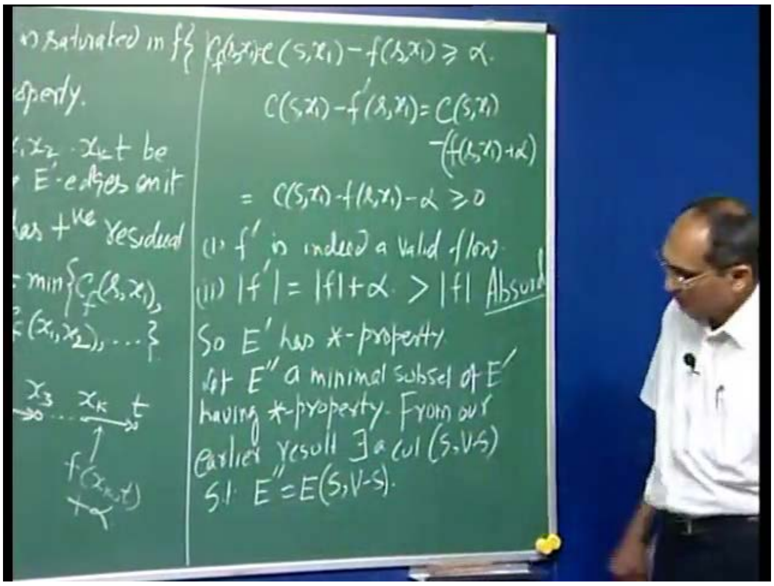

c s x 1 flow was s x 1 and the difference was greater than equal to alpha that was the choices alpha was the smallest of this, this is our c f of s x 1. So, when we append the flow we have c s comma x 1 minus f prime of s x 1, which is equal to c s x 1 minus. Now, f prime is f of s x 1 plus alpha which is c s comma x 1 minus f of s comma x 1 minus alpha from this inequality this is still non negative. Hence we do not violate the capacity condition on this x and the same goes for everyone of this edges. So, we are not violating an of that.

Now, ((Refer Time: 14:51)) the law of conversation of the fluid is concerned at this vertex, the earlier sum was 0, whatever was entering was living and now we have added alpha entering and added living. So, once again the net flow at each of this vertices is 0 a certainly is more for this and more for this. So, what we notice is first we observe is that f prime is indeed a flow of valid flow. And second observation we have made is that the magnitude of this flow is the reason is we had shown that this is equal to the net flow emerging from the star vertex.

The only thing that has change from f to f prime is that now added alpha flow emerges along this edge. Hence we have mode f plus alpha, which is strictly greater than f, what we have found is that we manage to improve the flow which should be absurd, why? Because we chose f to b one of those maximum flow nothing should improve upon this. Hence we have establish that E prime has star property so E prime has star property.

Now, let E double prime be a minimal subset of E prime so I am picking some subset of E prime, which is as small as possible and having star problem. So, I still want this two have this property, well certainly if double prime exist because if every subset of e prime fails to have some property than E double prime will E prime. So, there is no problem in establishing the existence of E double prime. Now, we have shown that whenever set of edges has this minimal a minimal set E double prime has star property, it is actually the h set of a cut set.

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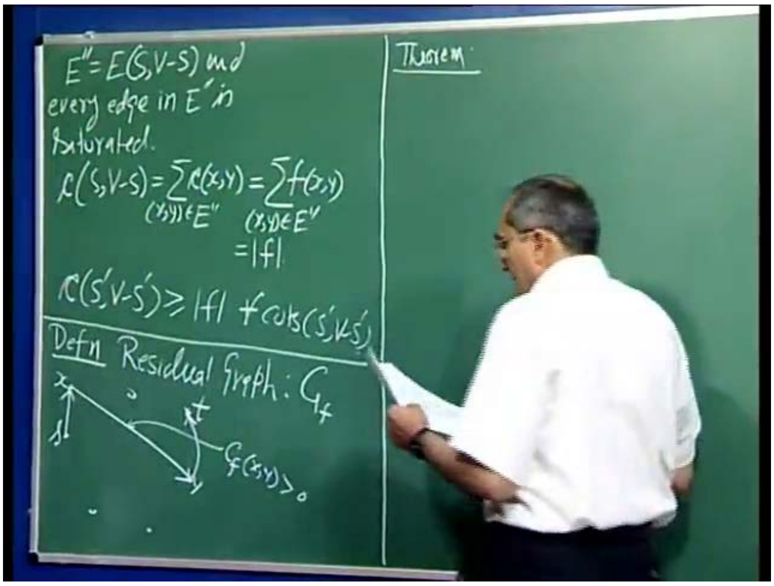

So, from or from our earlier result, their exist a cut S comma V minus S such that E double prime is the edge set of this type. Now, we know set of edges which is the cut set which is the h set of a cut and what is more is that each edge of E double prime is saturated because that is how we had constructed E prime and E double prime is it subset. Now, what we have is we got to now show that the capacity of this cut matches with the magnitude of the flow f.

So, now we will take the cut that we just figured out and we know that its S set is E double prime, and every edge in E double prime is saturated. Then let us ask what the capacity of this particular cut, so the c of S comma V minus S is sum c of x comma y where x comma y belongs to E double prime the h set of this cut. Since, every one of this edges is saturated then the flow through the edges is equal to the capacity.

So, we can also say that this is equal f of x comma y for every edges E double prime, this from our result that we mentioned at the start is equal to f. So, what we have found is that the max flow is equal to the capacity of this particular cut. Now, we had earlier seen that c of S prime comma V minus S prime is greater than equal to f for all cuts. Hence this as to be the minimum of all the cuts and that establishes our theorem, that if f is a max flow then the magnitude of that flow is equal to the smallest cut capacity, this hints towards the fact that where up to what point we can actually go. This tells us that as long as there is no cut, which matches whose capacity matches with there, there is room for improve in the flow. It can always increase.

Now, in order to develop an algorithm to compute max flow I need to define something called a residual graph. So, in our network graph we have say V set of vertices these is are the vertices then I am going to put an edge a directed edge in the residual graph, I will call it G f if the residual capacity of this edge in the flow is positive. That is if c f of x comma y is strictly positive and we will not put any of the edges in this graph, where the residual capacity is 0. All the edges which are saturated are drop from this graph and now we have built a graph based on the given network and given flow f.

Now, what is the property of this graph, so we have some s and we have some t in this. Suppose, there is a path from s to in this graph let us suppose we have a path like this, then on this path there is a minimum positive capacity residual capacity because there is some positive residual capacity alpha one, alpha two, alpha three. And say alpha is the minimum of all those capacity then as we see in the proof what we can do is that we compute alpha and in the original flow we append that much flow alpha along these edges. And if we do that we have see that our net flow will improve by alpha. Hence the problem of finding such a path with positive residual capacity is same as finding a path in residual graph that is the purpose of defining.

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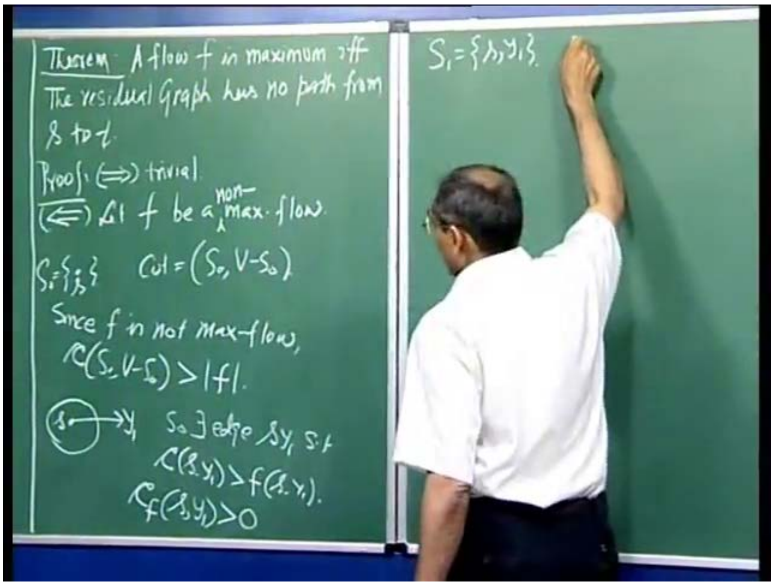

Now, we have a theorem says that a flow f is maximum if and only if the residual graph has no path from s to t. We just saw that in case there is a path then I can always improve the flow, I can increase the flow. So, it is clear that if there is a path then f is not maximum. So, in so this implication is trivial. So, let us try to prove the converse this time I want to show that if indeed f is not a maximum flow, then there always exists a path from s to t in the residual.

So, let f be a max flow sorry a non max flow because we want to show that there will be a path in this case in the residual. So, how do we proceed, let's start with a set s naught equal to just vertex s I have just the vertex s in this set, then we can define a cut s naught comma v minus s naught this is a cut. In our previous theorem what we have seen is that the maximum flow meets the minimum cut capacity. Since f is not maximum the cut capacity of this cut has to be strictly greater than the value of the flow.

So, the capacity c of S naught comma V minus S naught as to be strictly larger than the value of a flow this comes from our previous theorem, what does that mean? It implies that there is at least one edge in that h set of the cut which is not saturated there has to be some edge. So, here is our set containing a single vertex this is set S naught, the edges are like this and there is one such edge which is not saturated, let us call that s y 1. So, there exist edge s y 1 such that capacity of s y 1 is strictly larger than flow on it. In other words the residual capacity c f of the edge s y 1 is positive.

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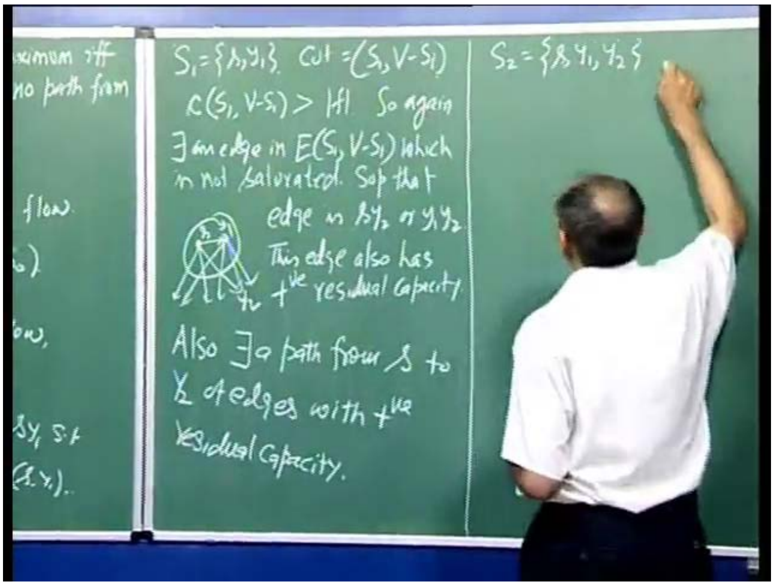

Now, let us go and define set s 1 as s comma y 1 define the cut to be s 1 v minus s 1. Once again I am claiming the capacity of s 1 comma V minus S 1 is strictly greater than the flow. And hence so again their exist an edge in the edge set of s 1 comma v minus s 1 with which is not saturated. Now, you have set s 1 containing two vertices s and y 1. So, there are edges coming out of this and at least one of this is not saturated, now that edge may be s y 2 or y 1 y 2 it does not matter where it emerges from, but say ends up in some vertex y 2.

Then let say suppose that edge is s y 2 are y 1 y 2, then this edge has positive this edge also has positive residual capacity. What is more I also know also their exist a path from s to y 2 of edges with positive residual capacity, well in case the edge was s y 2 then that is the path you have a path from s to y 2. And since it is the only edge, we already know that its residual capacity’s path on the other hand, suppose the edge was y 1 y 2 then the path is the edge that we had picked upon earlier was s y 1 followed by y 1 y 2. Both of them had positive residual capacity now we still satisfy this requirement of existence of a path from s to y 2.

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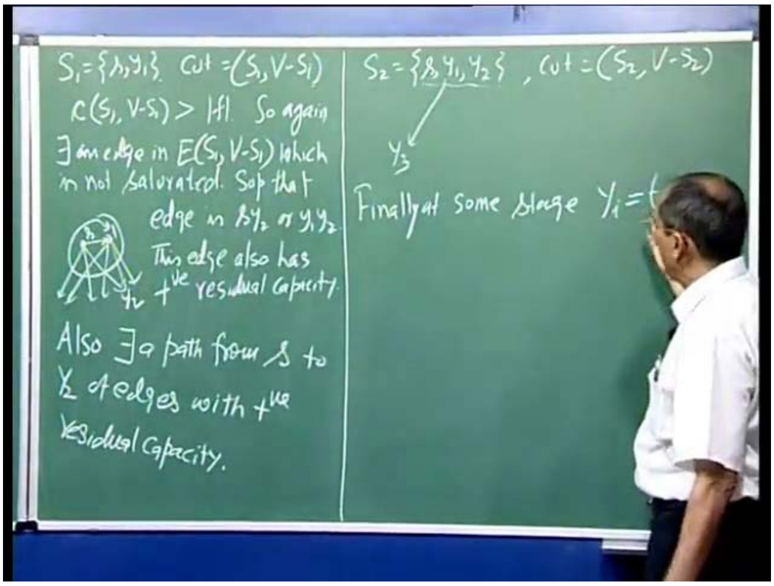

Now, once again the proceed in this fashion now we take s 2 to be s comma y 1 comma y 2 and define a cut to be s 2 comma v minus s. Now, what is happened once again we will argue that our capacity of this cut is strictly greater than the flow and hence again there is a edge from this set to somewhere out to some y 3. And the residual capacity is positive once again I will be available to argue that there is a path from s to y 3 with each edge having positive residual capacity because if it starts from S then that is the edge if its starts from y 1 then the path will be s y 1 edge which has positive capacity residual capacity y 1 y 3 has positive.

And similarly, if its starts if the edge is y 2 y 3 then s y 1 followed by y 1 y 2 if that was the previous path followed by y 2 y 3 or s y 2 if that was the previous edge then s y 2 followed y 2 y 3. In all situations I will have a path with positive residual capacity is on its edges from s 2 the latest model this process will continue. Finally, at some stage some y I will be t because of we have a finite graph some at some point t must enter. And once that happens we have shown that there is a path from s to t on which every edge has positive residual capacity that establishes our second path of a theorem. Now, our result leads to an algorithm which is known as Ford Fulkerson algorithm.

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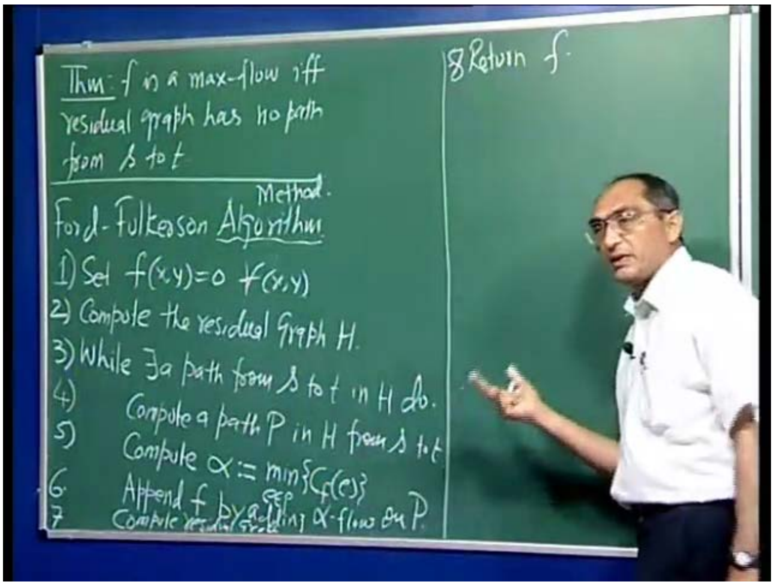

To be precise this is actually not an algorithm, but more like a method because it lacks one important property of an algorithm which will see, what this says is that lets initialize a zero flow in the graph. So, set f(x) comma y to be 0 for all edges x y. So, note that this is a valid flow it satisfies all the conditions of the flow required. Then compute the residual graph, note that all this is in this case is doing is dropping all the edges with zero capacity.

Now, then while their exist a path from s to t in H do compute a path P in H from s to t, compute alpha equal to the minimum of all the edges in P the residual capacity of E. So, all the edges on this path have residual capacity alpha or more, append f by adding alpha flow on P, leave all the other edges in that, but all the edges of P now will have alpha more flow.

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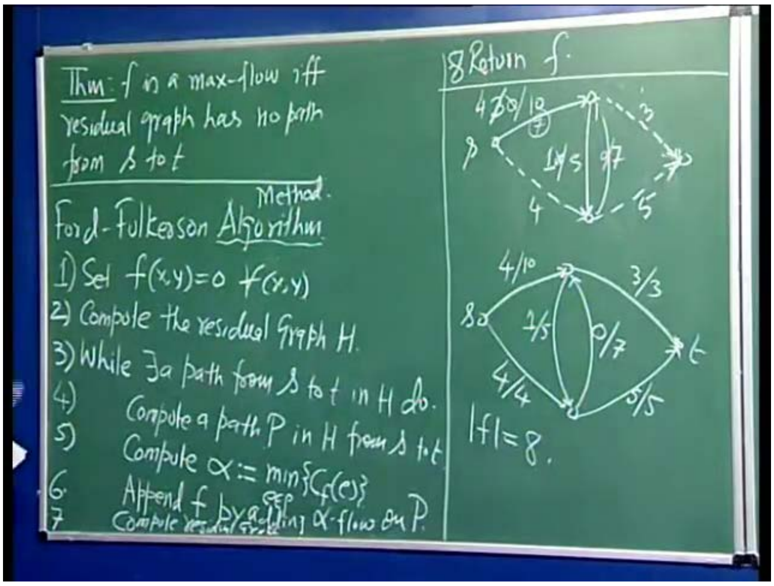

And when this thing is over you return. Well if this is over that means there is no path in the residual graph to be precise, I should say compute residual, compute a residual graph and that one. So, if we come out of this loop, then we known that the residual graph has no path from s to t. And then this theorem tells me that this flow has to be maximum so this correctly returns the some max flow, the problem why I am not going to call this is an algorithm is that there is no guaranty that this process will turn up, it may go on forever.

Although you notice what that if all the capacities where integral, suppose all the capacities are integers, then each increment will be at least of one unit and the minimum capacity a cut capacities say capital C. Then you will need at most capital C iterations in that case it will terminate in fact you can argue the same even when the capacities are rational, but if it is general really number then we may not be able argue that this will terminate. Now, let me run a small example to this lets take.

Let us suppose we have a network, remember I will not show in our representation I will not shown edges with zero capacity for convenient. Initially I will have zero flow in this all the edges this is the inertia. Notice that the residual graph is exactly this so I pick a path, suppose I take a path namely this the residual capacity here is 10 it is 3 here to the smallest number is 3 and I will append that my flow I will make this 3 and this 3. This is a saturated edge so I am going to remove this edge, the broken age is the missing edge from the residual graph of the new flow. And the residual capacity for this is now 7 because we have 10 capacity and 3 is flowing fluid others are in term.

Once again I will look for a path this term say I choose this path and I find that the residual capacity here is 4 and here is 5 minimum is 4. So, let's appends at flow we put four here we put four here and once again this is saturated. So I will drop this from our residual graph and I am just leave it as dotted line, where the 4 denotes the capacity still there is a path which goes like this, this and this. Here the residual capacity is 1, it is 7 here this has 5. So, we have 1, 5 and 7 the minimum being 1 I can append 1 here.

So, this time this will become 4, this will become 1 and this will become 5. Now, the residual capacity is 0, this is a saturated edge so I am going to drop this as well. And now this is one of the edges which had 5 capacity. So, the flow that we have computed of course, the capacity is being 10, 3, 5, 4, 5 here and 7 we are putting 4 units here and 4 units over here 1 unit into this, 5 here because this was saturated 3 here and 0 here.

So, what we have show is that such a flow, where the residual graph has a no path left has to be maximum. And notice that that value of this flow is the sum of the fluid coming out of the source, or what enters into the sync which is 8 this is the maximum flow for this network. Now, to be able give a proper algorithm, we need away to ensure that this process terminate and this is what we will do in the next lecture, will offer in adoption of this method. And that is called Admen Crop algorithm, which suitable tells how to compute the path such that this loop terminate and that will be our next topic.