Discrete Mathematics
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Set Theory
Lecture 2

Welcome to the NPTEL MOOC course on Discrete Mathematics. This is the second lecture on Set Theory.

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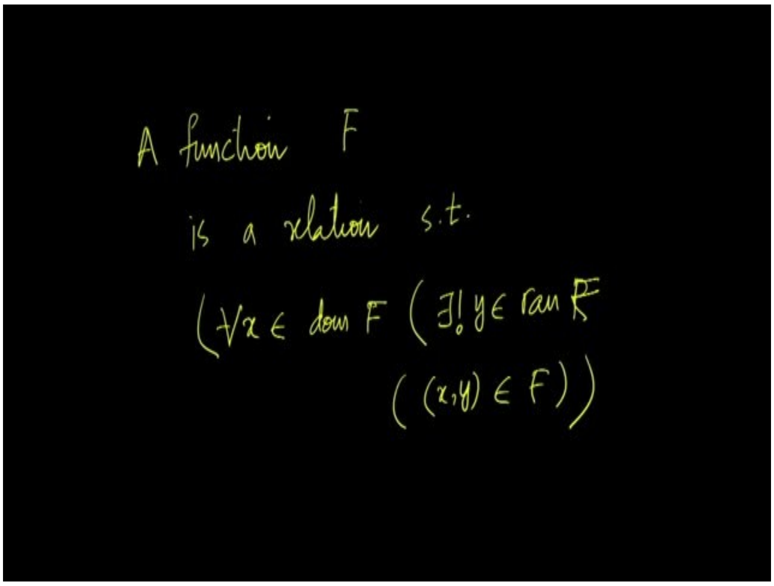

At the end of the previous lecture we were discussing functions and relations. A function we saw is a relation such that for every x belonging to the domain of the function there is a unique y in the range of the relation so that xy belongs of f. This would be the range of f. For every f belonging to the domain of f there is a unique y in the range of f so that xy belongs to f. So that is what a function is.

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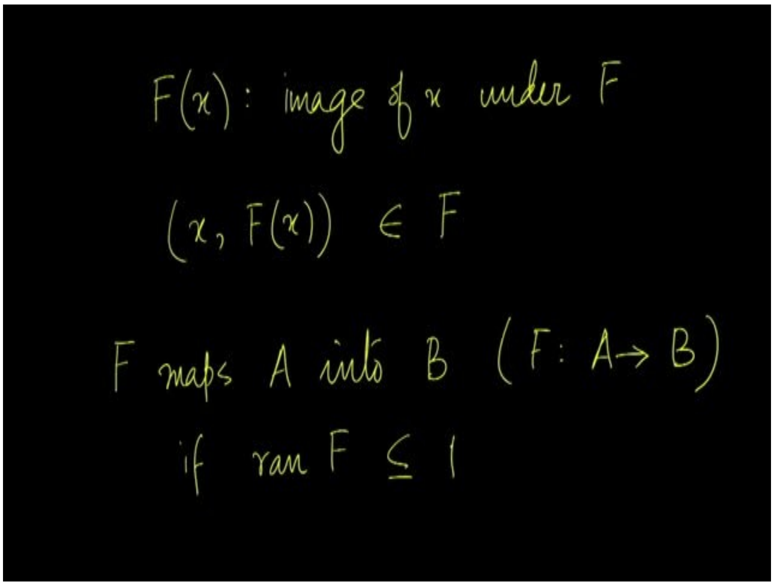

Since an x has unique mapping in the range the image of x can be denoted as f of x this is the image of x under f. It means x f of x is the only ordered pair with x as the first comp1nt and belonging to f. We say that a function maps A into B where a function is from a to b. We said that f maps a into b if the range of f happens to be a subset of b.

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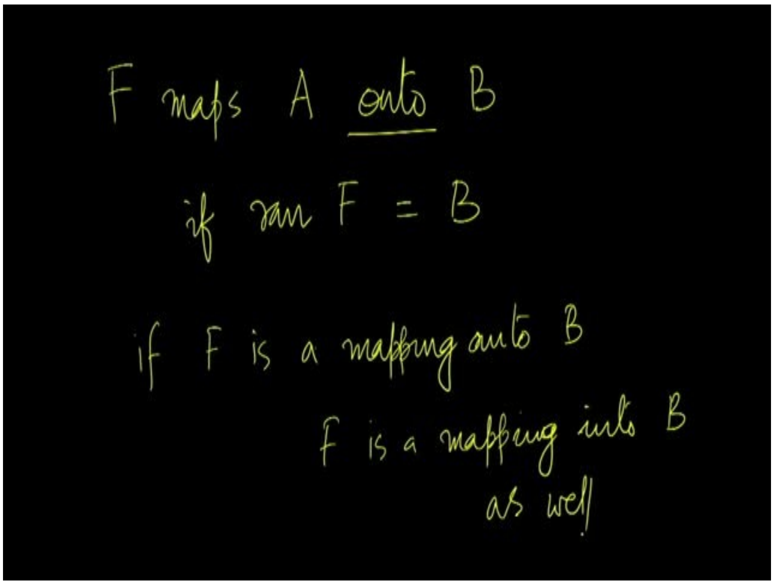

We say that a function f maps a onto b, if the range of f is equal to b. So, if f is a mapping on to b or f is a mapping on to b then f is a mapping into b as well.

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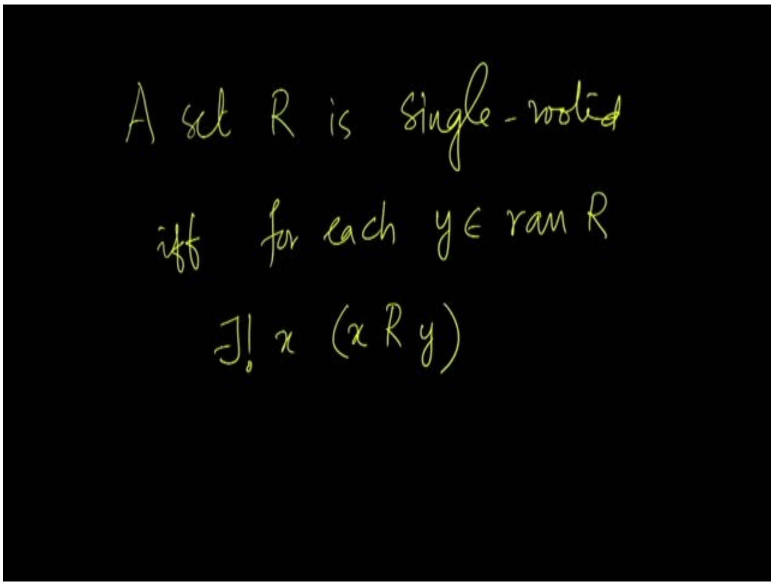

We say that a set R is single rooted if and only if for each y belonging to the range of R there is a unique x so that xRy.

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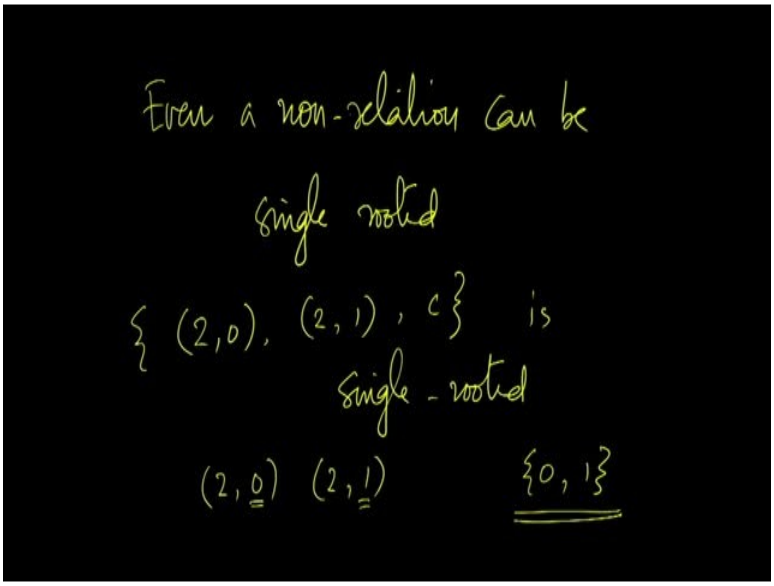

Even in a non-relation can be single rooted. For example, consider this set, this is single rooted. For every ordered pair belonging to this set this we consider 2 0 and 2 1. And consider the second comp1nts, 0 and 1 are the second comp1nts. Each of them has a unique pre image, the only pre image of 0 is 2 and the only pre image of 1 is 2. Therefore, this is a single rooted set. So single rootedness can be a property of a set.

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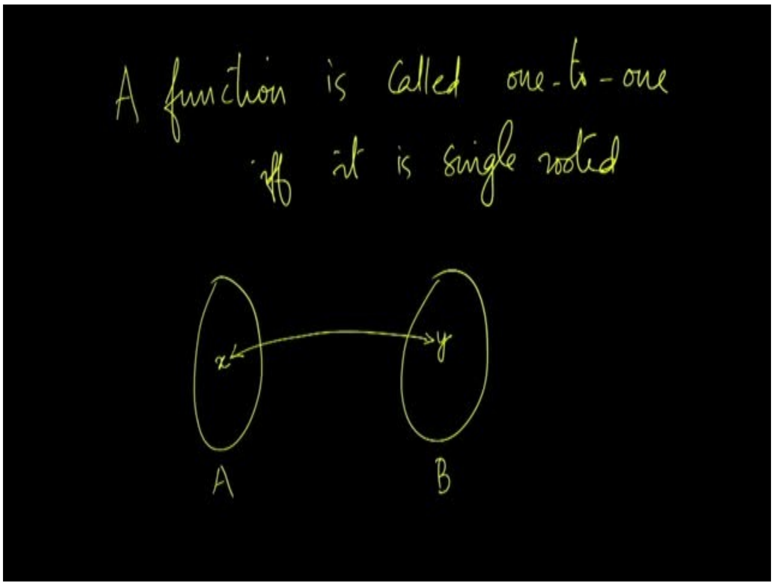

A function is called 1 to 1 if and only if it is single rooted. So if the function is from set a to b for each x belonging to the domain of the function there is a unique y that is because it is a function. Now since it is single rooted then y has unique pre-image x so each x in the domain has unique y in b and each y belonging to be the range of the function has a unique pre-image x and a that is when the function is called 1 to 1. So a 1 to 1 function is single rooted set.

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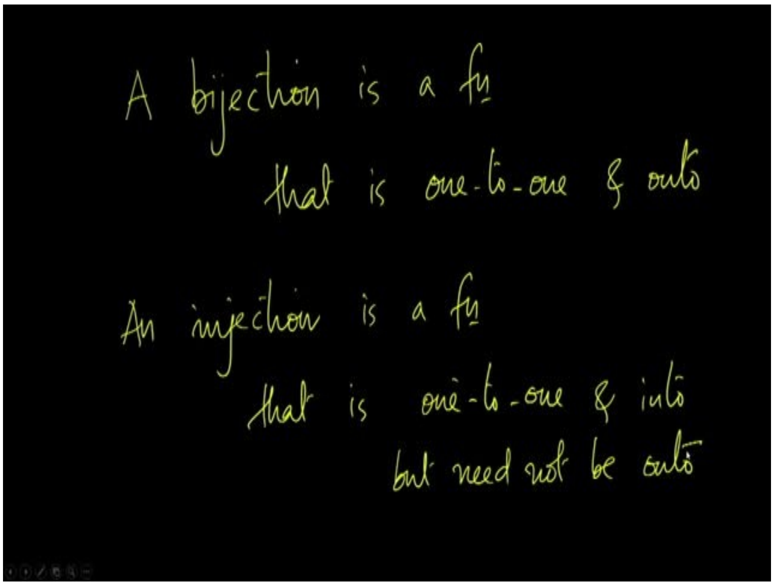

A bijection is a function that is 1 to 1 and onto. And injection is a function that is 1 to 1 that is 1 to 1 and into but need not be onto. So a bijection is also an injection.

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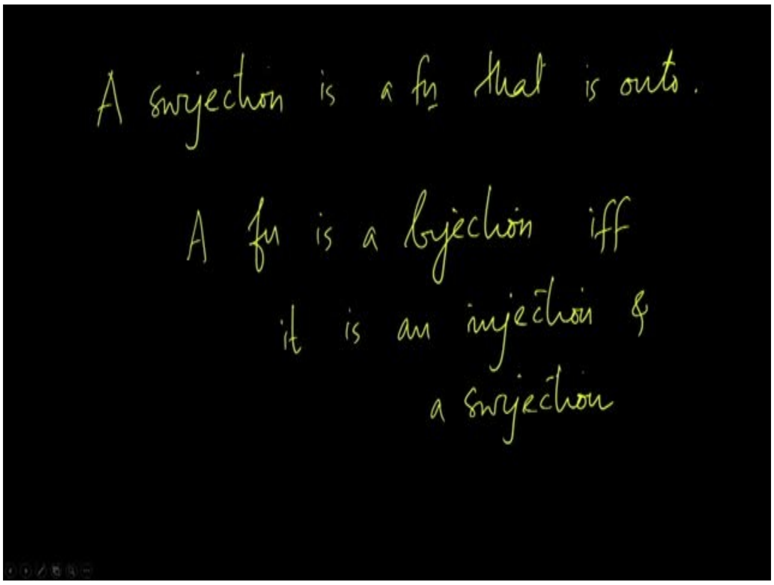

A surjection is a function that is onto. So combining all these definition we say a function is a bijection if and only if it is an injection and the surjection. You must be familiar with these terms, bijection injection and surjection.

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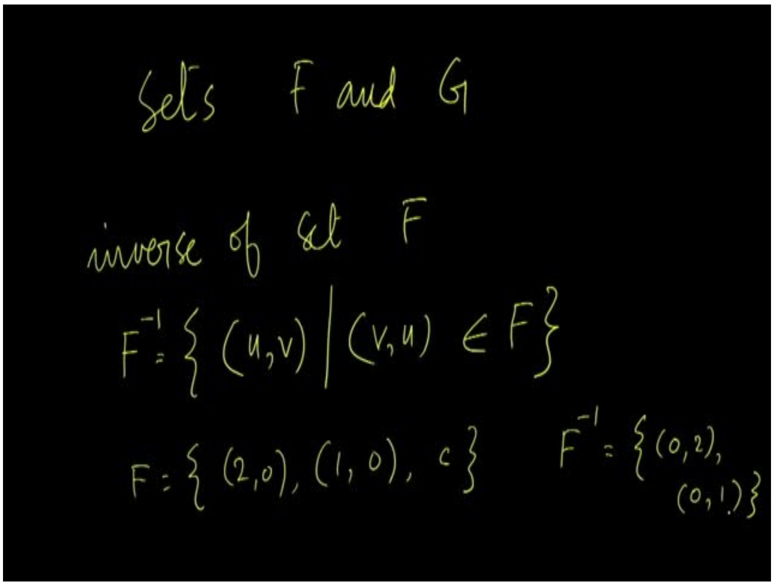

Consider 2 sets f and g. We define the inverse of set f denoted f inverse f to the power of minus 1 this is defined as the set of all ordered pair uv such that, the ordered pair uv pair belongs to f. That is you consider all ordered pair belonging to the set f and then invert the ordered pair make the first comp1nt the second and the second comp1nt the first.

The ordered pairs that you obtain will form f inverse. For example, if f happens to be 2 0 1 0 and c, then f inverse is 0 2 and 0 1. So f need not even be relation for it to have an inverse.

Even for set we can define an inverse. So to find an inverse of a set you take out all the ordered pairs belonging to the set and then invert every single ordered pair.

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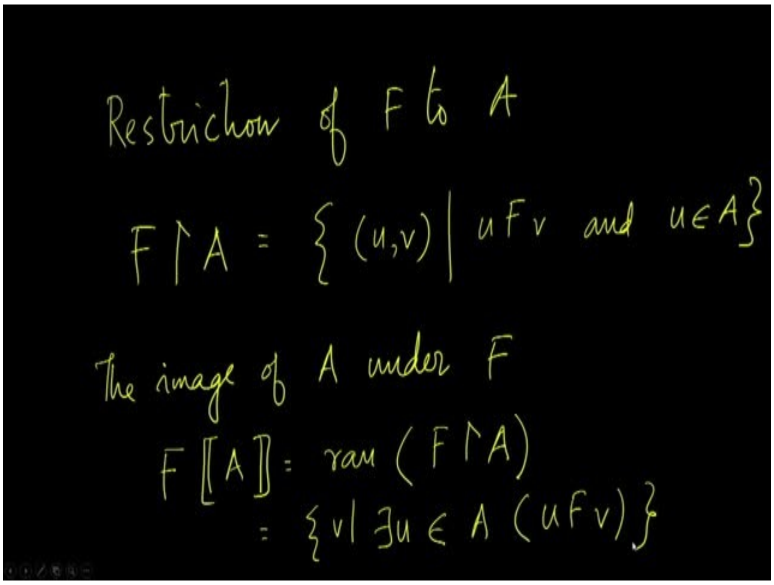

The restriction of a f to a denoted in this manner this is called restriction of f to a this is defined as the set of all ordered pairs u v so that u f v and u belongs to a. The image of a under f which is denoted like this the image of a under f this is defined as the range of the restriction of f to a. This is the set of all v such that there exists you belonging to a so that ufv. The range of the restriction is what the image of a under f.

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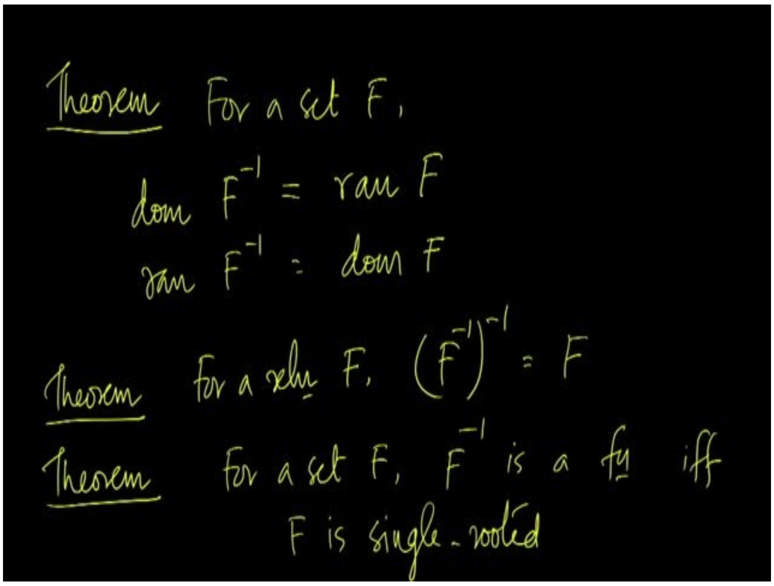

Then we have a number of theorems I will not prove these theorems here. These are trivial enough for you to prove them as exercises. For a set f domain of f inverse is the same as the range of f and the range of f inverse is the same as the domain of f. Another theorem says for a relation f f inverse of inverse the inverse of f inverse is same as f which is trivial enough. For a set f f inverse is a function if and only f is single rooted. That is every element in the range of f has his unique preimage that is when f inverse is a function.

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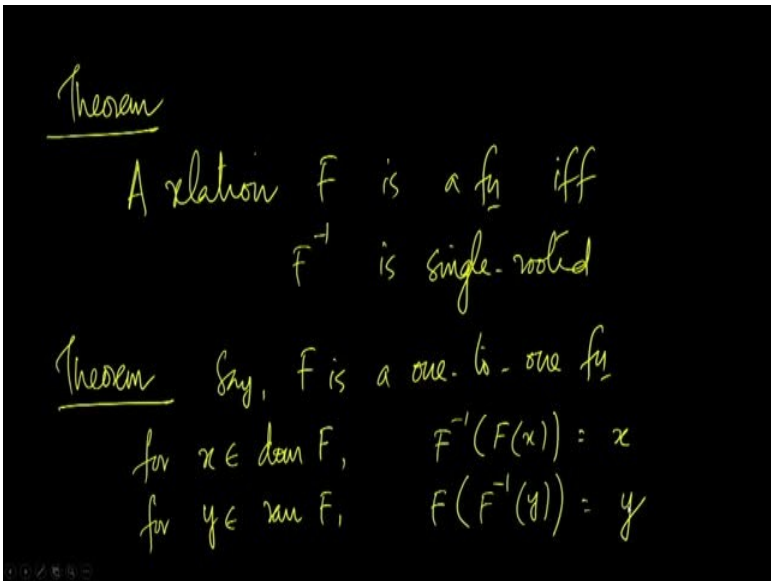

A relation f is a function if and only f f inverse is single rooted. Say f is a 1 to 1 function. For x belonging to the domain of f f inverse of f of x is the same as x. And for y belonging to the range of f f of f inverse of y is the same as y. This is true for a 1 to 1 function.

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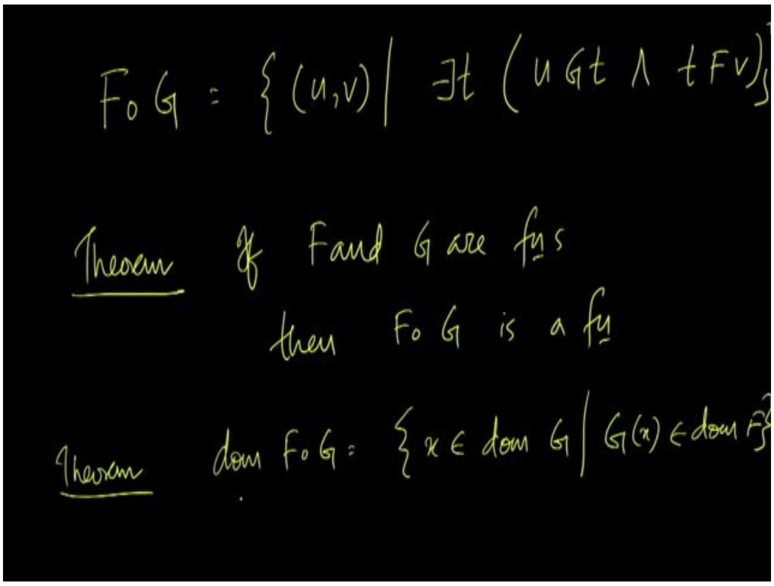

For 2 sets f and g we define f composition G thus. It is a set of all ordered pairs uv such that there exists t so that ugt and tfv. And then theorem about function composition. If f and g are functions then f composition of g is also a function. If g is a function and g is applied on u then it will produce a unique t and if f is a function and f is applied on t it will produce a unique v. Therefore f composition g will also produce a unique v for every you belonging to the domain of it. That is the theorem. The domain of f composition g is all x belonging to the domain of g. Such that g of x belongs to the domain of f.

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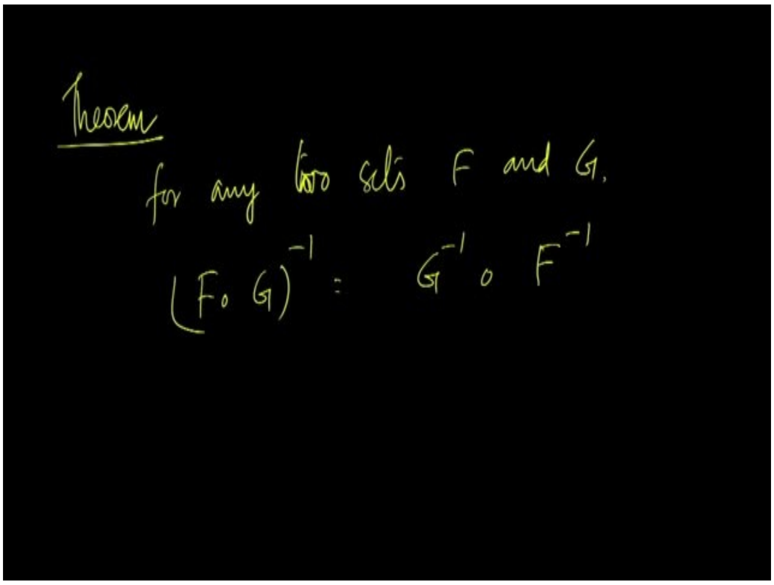

For any 2 sets f and g the inverse of f composition g happens to be the compositions of g inverse and f inverse. So these are theorems which are easy enough to prove as exercises. Do attempt to prove them. So we've seen some basic definitions of set theory.

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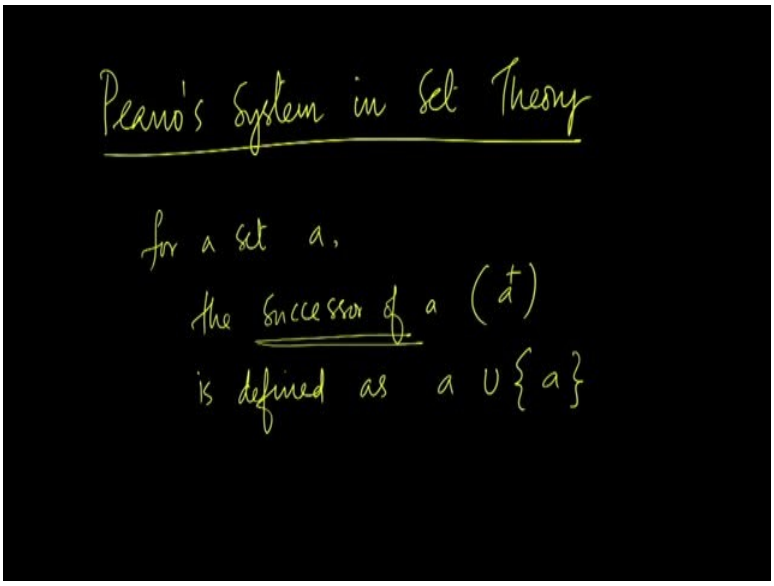

Let us show that Peano system can be embedded within set theory. Russell and Whitehead proved that the whole of classical mathematics can be embedded within set theory. I will give you a peak as to how to do this. So we will begin with peano system. Peano system deals with arithematics, we have already seen peano is axioms and module on logic. So let us see how to embed peano system in set theory.

First of all for a set a the successor of a denoted a superscript plus is defined a union the single turn set containing a. That is when you take a set and add it to itself we get the successor of the set. So this is how the successor of the set is defined.

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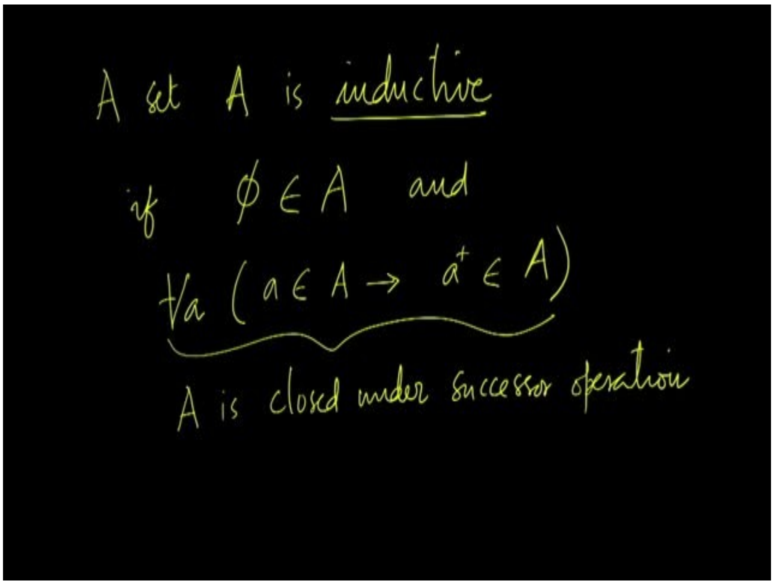

And then we say set a is inductive if the empty set belongs to a and for every a a belongs to a implies that the successor of small a also belongs to a. For every small a small a belongs to capital A implies that A plus also belongs to capital a that is a successor of small a also belongs to capital a. In other words a is closed under the successor operator. So for a set a to be inductive which should contain the empty set and for any set belonging to a you should also contain the successor of that set. That is the set should be closed under the successor operator. That is when we say that the set is inductive.

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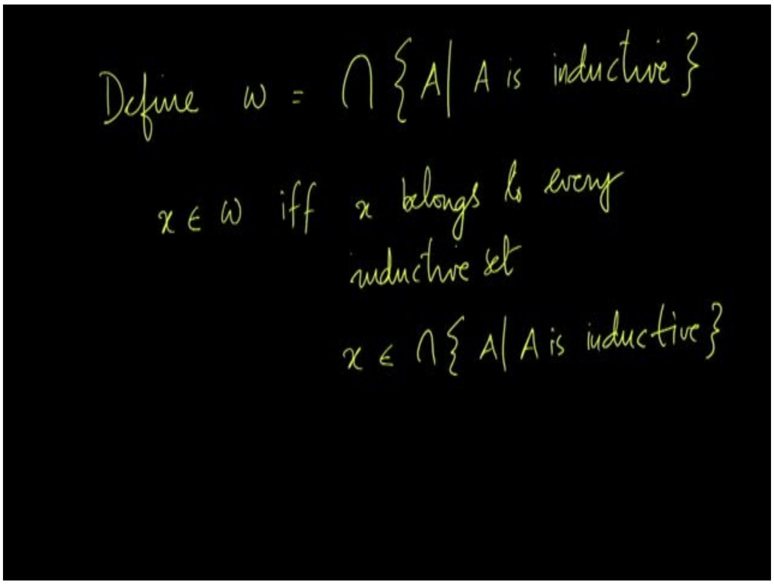

Now let us define omega as the intersection all a such that a is inductive. We define omega as this intersection of all inductive set that is x belongs to omega if and only if x belongs to every inductive set which is precisely when x belongs to intersection of all inductive sets. So the definition says that x belongs omega precisely when x belongs to every single inductive set.

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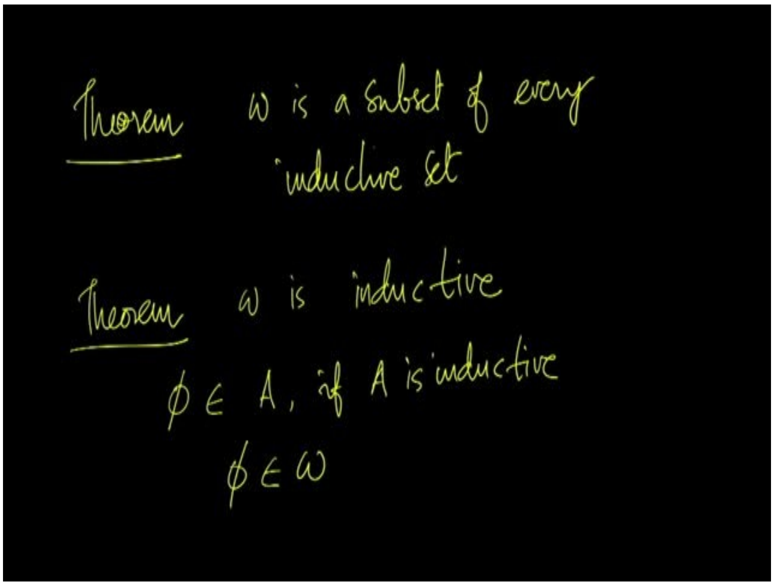

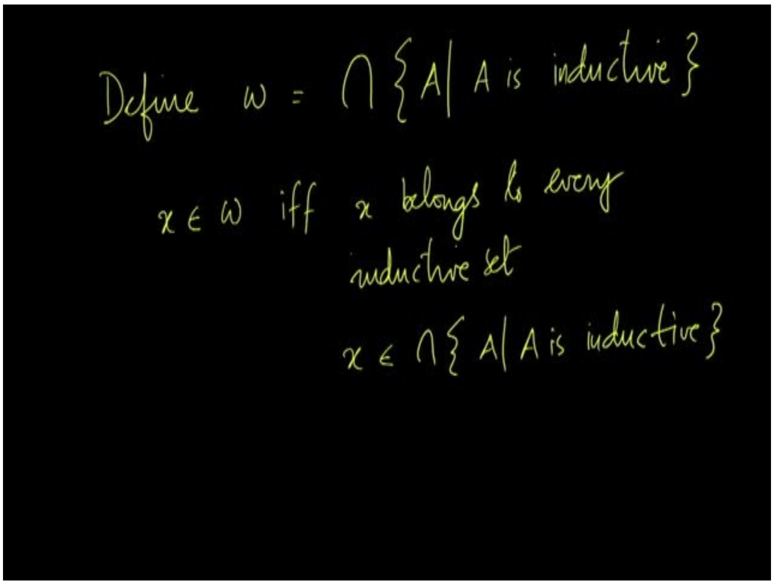

Now let us consider some interesting properties of omega. First of all omega is a subset of every inductive set that should be obvious because in the previous slide we've just seen that if x belongs to omega then x belongs to every single inductive set therefore omega is a subset of every single inductive set. That the members of omega are members of every single inductive set. And then omega is inductive as well. Why should those be so?

This is because phi belongs to a if a is inductive for every a that is inductive phi belongs to a. Therefore phi belongs to omega as well because omega is the intersection of every single inductive set so if phi belongs to every single inductive set it should also belong to omega. So that is the first requirement of omega to be inductive. It should contain the empty set, so it indeed does contain the empty set.

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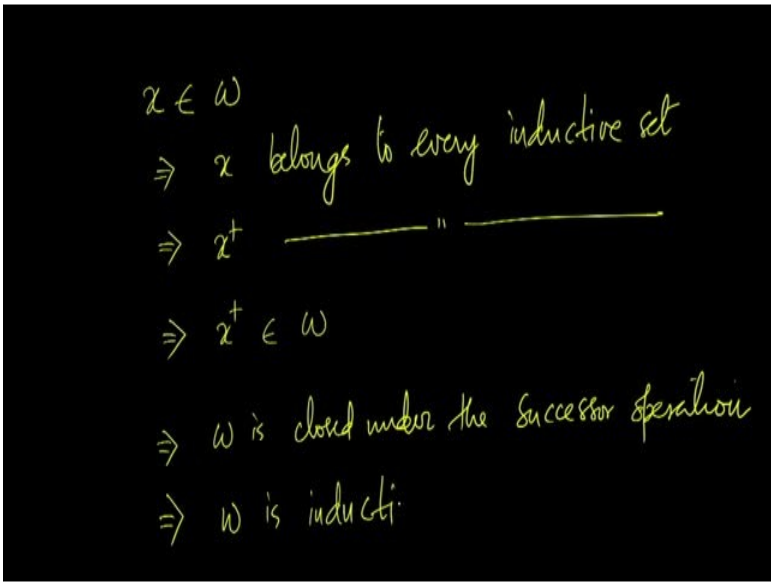

Now consider some x belonging to omega then x belongs to every inductive set. But then all inductive sets are closed under successive operator which means x plus belongs to every inductive set. In order words x plus belongs to the intersection of all inductive sets which is precisely what omega is which means x plus belongs to omega. So what we have is this if x belongs to omega then x plus also belongs to omega.

Which means omega is closed under the successor operator. That is the second requirement for omega to be inductive. The first requirement is that omega should contain the empty set the second requirement is that it should be closed under the successor operator. Since we've established both we know that omega is inductive.

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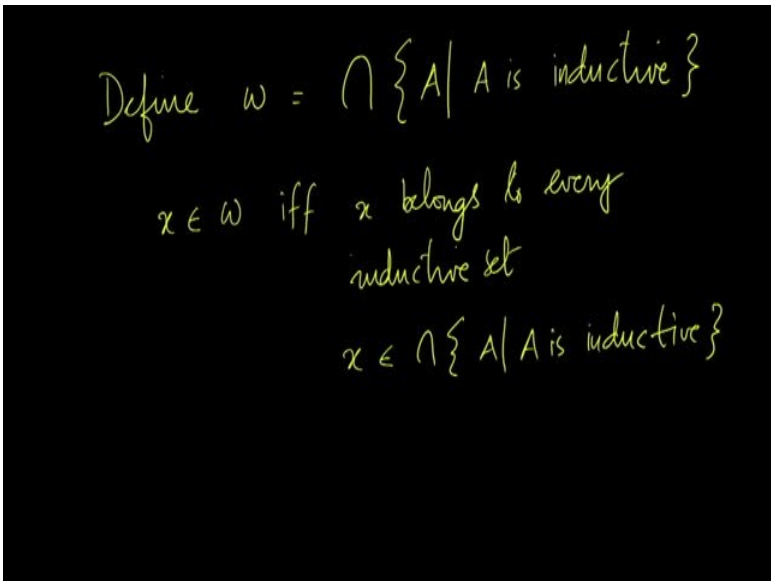

So omega which is defined as the intersection of all inductive sets is itself and inductive set.

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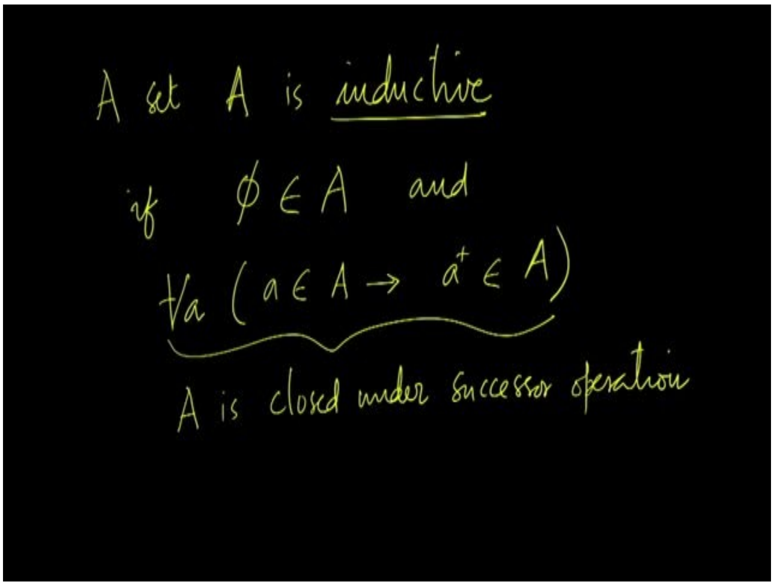

And once again and inductive set is a set which contains an empty set and is closed under a successor operator.

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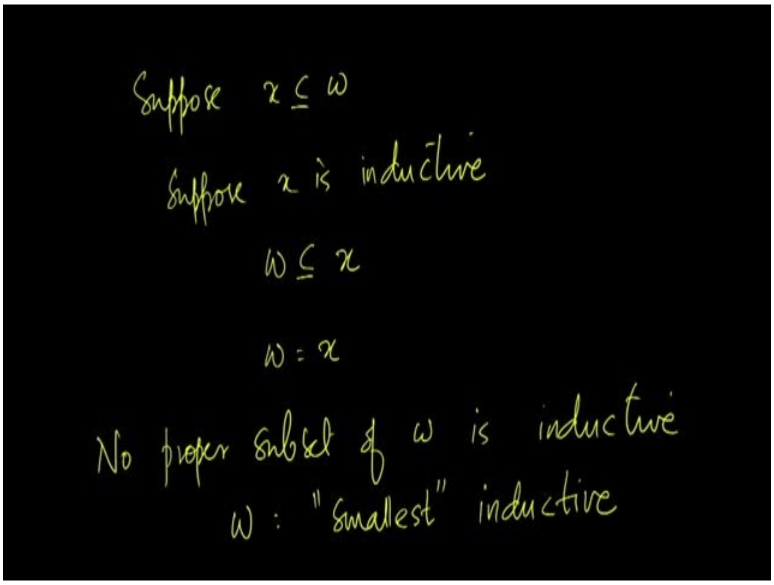

Suppose x is a subset of omega n suppose x is inductive but then omega is a subset of every inductive set therefore omega is a subset of x as well. But when x is a subset of omega this implies that omega equal to x. What it means that if x is a subset of omega and x is inductive then x has to be same as omega. Or in other words no proper subset of omega is inductive. So to combine this with the definition of omega.

Omega is the intersection of every inductive set. And omega is a subset of every inductive set. And no proper subset of omega is inductive so in that sense omega is the smallest inductive set. Every inductive set is a super set of omega.

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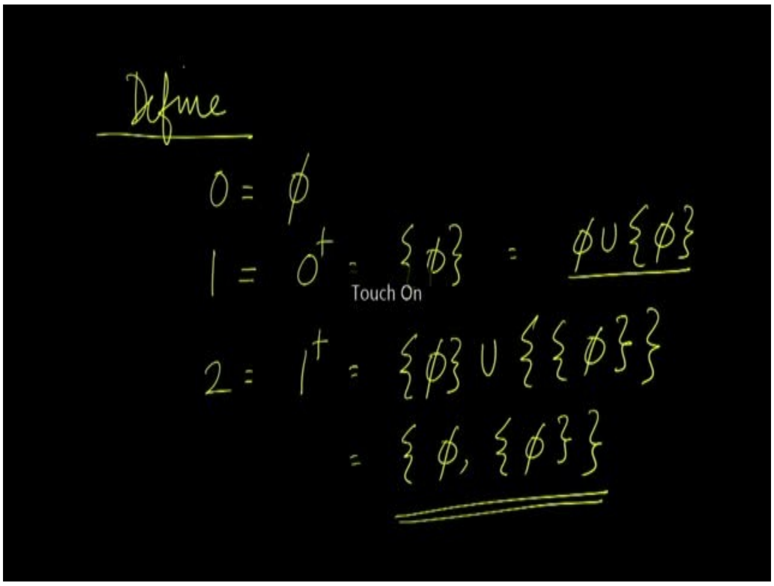

Now you've seen this omega we can embed natural numbers in set theory so what we do is this. we define natural numbers in this manner. We construct natural in this manner. We define 0 as the empty set we define 1 as 0 plus which is the successor of 0 which is 0 added to itself. Which the successor of 0 which would be phi that is because this is phi union the single turn phi.

So 1 is the single turn containing phi and then 2 is the successor of 1. Which would be phi union the single turn which contains the single turn phi. This would be the set containing 2 elements 1 is phi and the other 1 is single turn containing phi. This is what is defined as 2.

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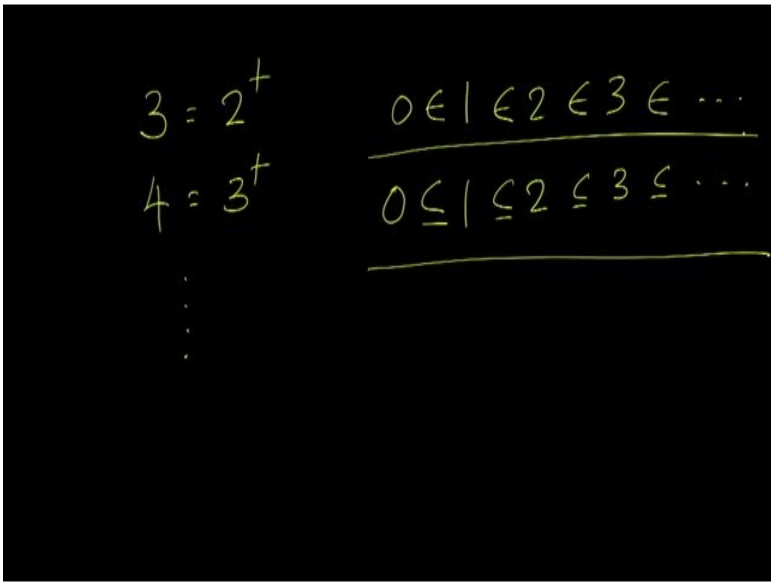

And then the number 3 is defined as successor of 2 the number four is defined as successor of 3 and so on. So in this manner we define all the natural numbers. So you can readily verify that. 0 is a member of 1 which is a member of 2 which is the member of 3 and so on. Not just at 0 is a subset of 1 which is a subset of 2 which is a subset of 3 subset of four and so on. So both these chains of relations hold.

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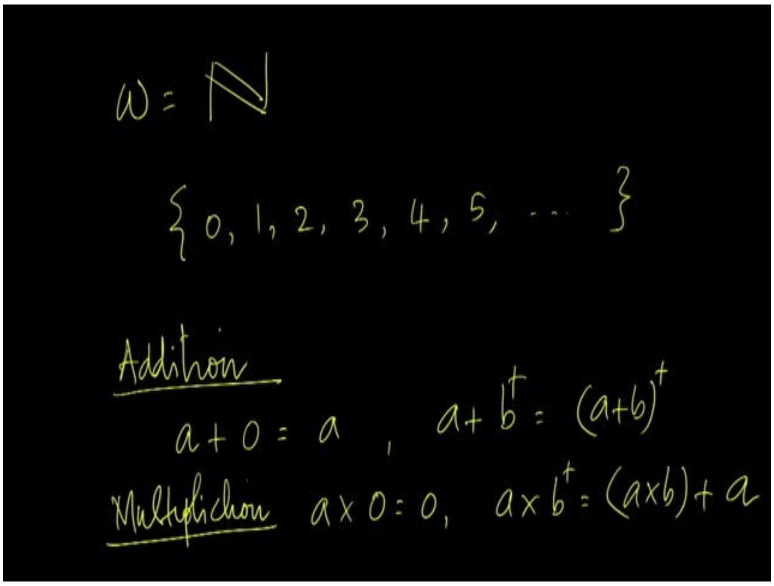

So we have in this sense defined omega as set of all natural numbers omega is the set of all natural numbers. That is because omega contains 0 it contains the successor of 0 which is 1, it contains successor of 1 which is 2 and so on. For every set it also contains a successor of that set. Is closed under the successor operator. So we define omega as the set of all natural numbers. And then you would recall Peano's axioms that we saw in the module on logic.

We define addition in this manner a plus 0 is a and a plus the successor of b is the successor of a plus b. Similarly multiplication is also defined, multiplication is defined as a into 0 is 0 and a into the successor of b is a into b plus a. So multiplication is defined using addition and addition is defined using the successor function. So in this manner we can construct natural numbers and the 2 operations addition and multiplication. You can verify that once we defined numbers in this manner then all of Peano's axioms will be satisfied.

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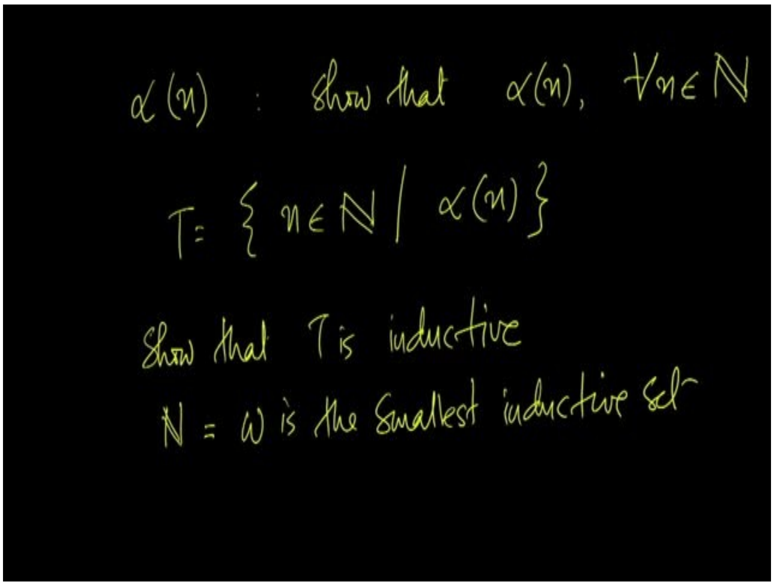

Suppose we have 1 variable predicate alpha of n suppose we want to show that alpha of n is true for every n belonging to the set of natural numbers. We can define t as set of all natural numbers so that alpha and holes. Suppose we prove that t is inductive then we would be d1 because the smallest inductive set is omega. And omega is the same as all natural numbers. So once we show that t is inductive we would be d1.

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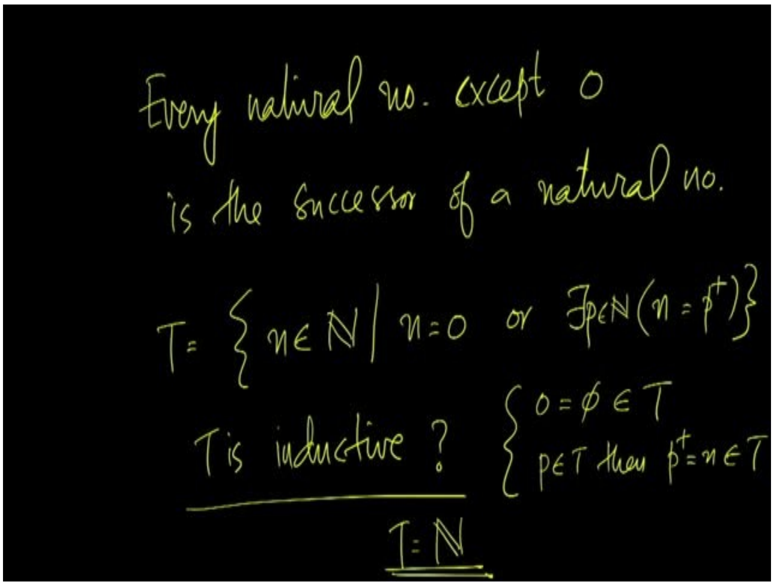

Suppose for example we want to show that every natural number except 0 is the successor of a natural number. Suppose this is what we want to show. Then we can define t as all those natural numbers that are either equal to 0 are successors, suppose this is what t is and then we want to show t is inductive. So this the question we have is t inductive? To prove that t is inductive we have to show 2 things.

First of all we have to show that empty set belongs to t but the empty set is the same as 0 and 0 certainly belongs to t because t contains all those natural numbers that are either or are successors. So 0 is explicitly included here so 0 which is the empty set belongs to t. So t satisfies the first requirement. Then what is the second requirement the second requirement says that if p belongs to t then p plus which n also belongs to t that is also satisfied that t is indeed inductive.

So t is an inductive sub set of the set of natural numbers but we have just seen that there is no inductive proper sub set of the set of all natural numbers therefore t must indeed be the set of all natural numbers. So this is the property that is satisfied by the set of natural numbers. Every natural number except 0 is a successor of a natural number. So this is how we embed the set of natural number in set theory.

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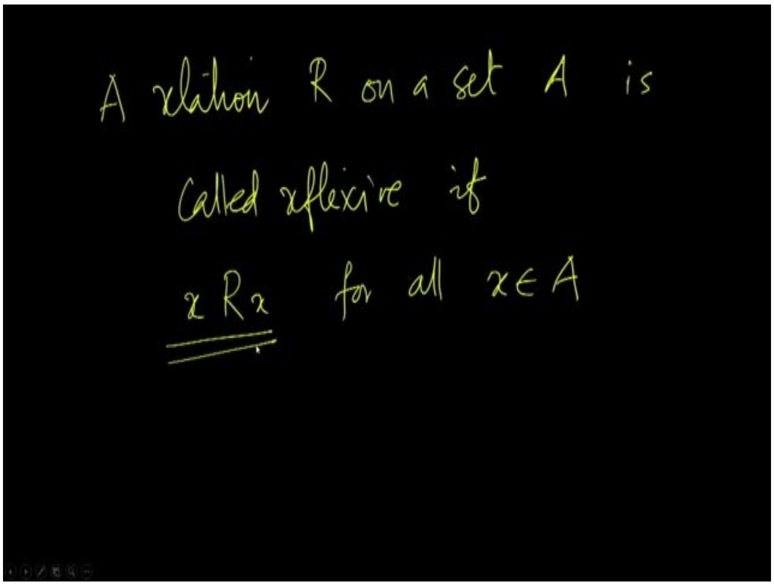

A relation r on a set a is called reflexive if xrx or all x belonging to a. A relation are is reflexive on a set a if xrx is true for every x belonging to a.

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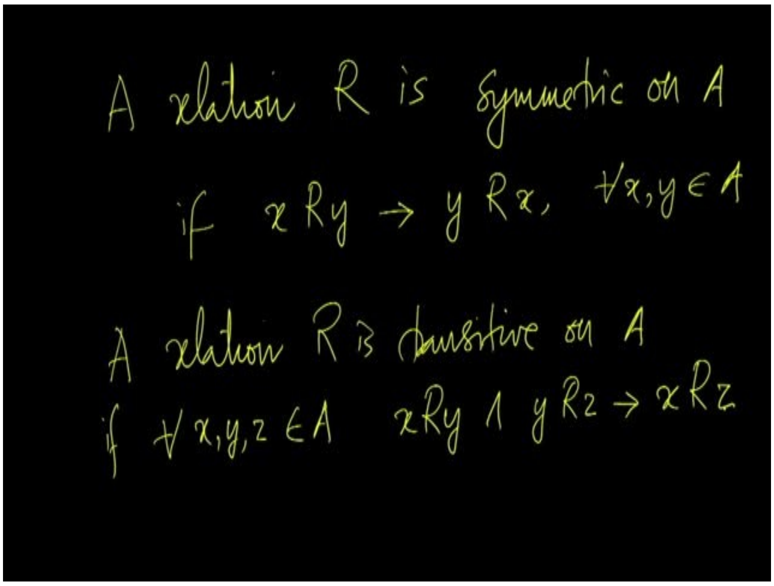

Similarly we say that a relation r is symmetric on a if xry implies yrx for all xy belonging to a. And a relation r is transitive if for all xy belonging to a xry and yrz implies xrz. When a relation satisfies all these 3 properties.

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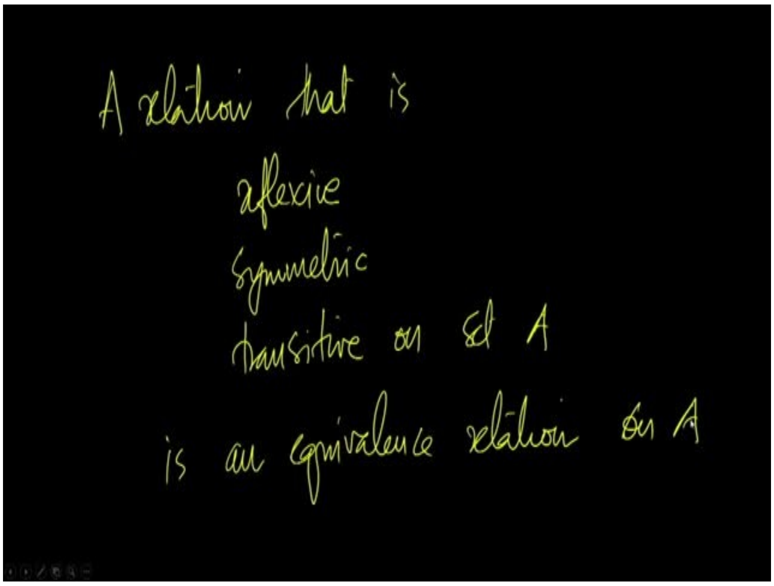

A relation that is reflexive symmetric and transitive on set a is and equivalent relation on a.

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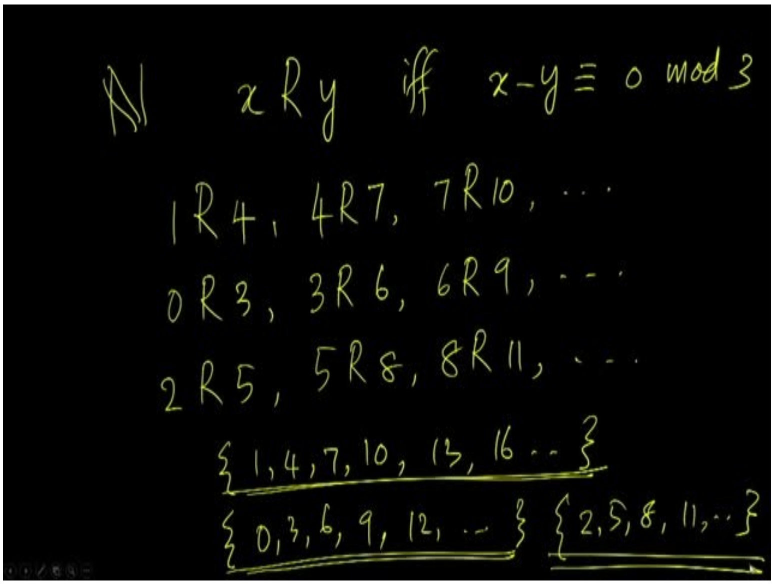

As an example consider set of natural numbers and let us say 2 numbers are equivalent these 2 numbers are in relation are xry if and only if x minus y is 0 mod 3. Which means ir4, 4r7, 7r10 and so on. These are all 1 mod 3 1, four, seven, ten etc. are 1 mod 3. Similarly 0r3, 3r6, 6r9 and so on. So since 0r3 and 3r6 by transitivity 0r6 as well. We also have 2r5, 5r8, 8r11 and so on.

So the equivalence classes in this case would be 1,4,7,10,13,16 etc. this is 1 equivalence class. 0,3,6,9,12 etc. are from the other equivalence class. And the third equivalence class would be 2,5,11 and so on. This corresponds to numbers that are multiples of 3 these are numbers that are 1 mod 3 and these are numbers that are 2 mod 3. So this relation has 3 equivalence classes

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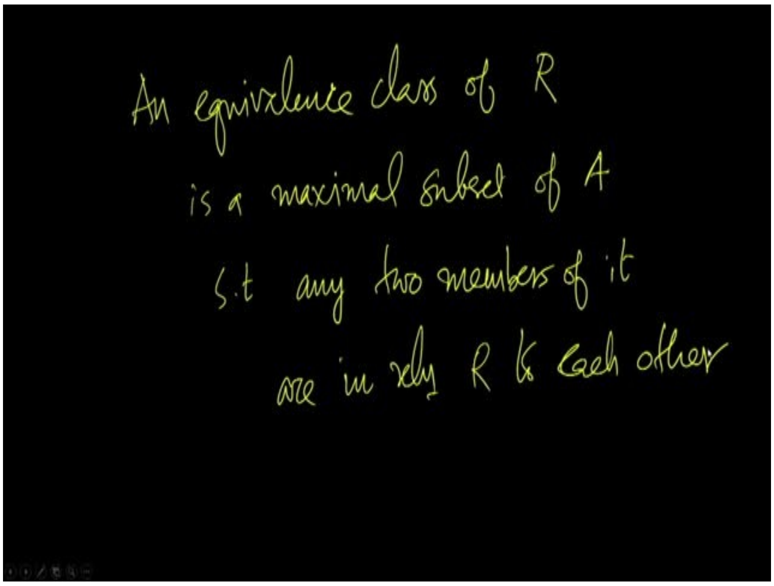

An equivalence class of r is a maximum subset of a such that any 2 members of it are in relation r to each other.

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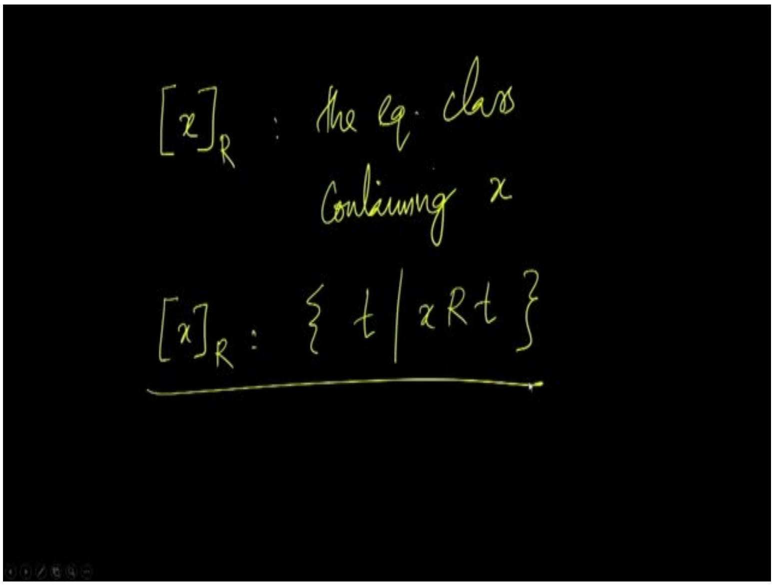

So the equivalence class containing x will be denoted thus formally this is the set of all t such that xr or tr by reflexivity you might as well say trx by symmetric.

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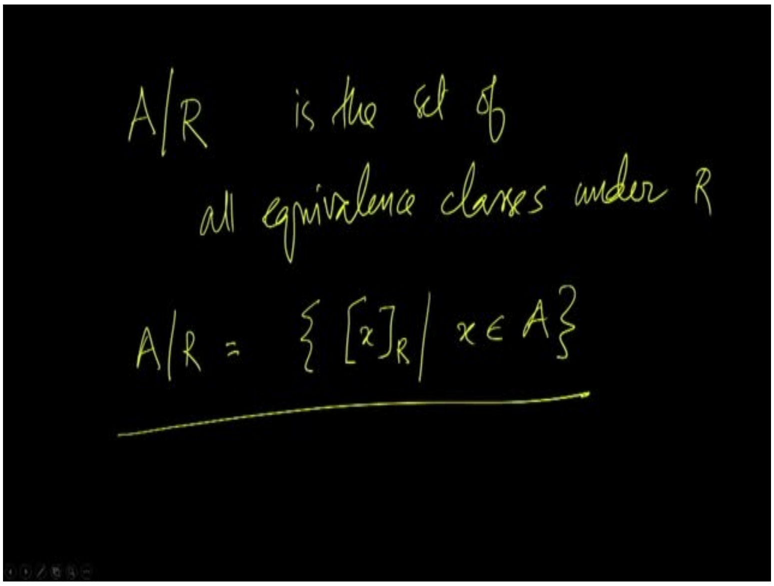

The quotient a with respect to r is the set of all equivalence classes there are formally the quotient of a with respect to are is the set of all xr such that x belongs to a.

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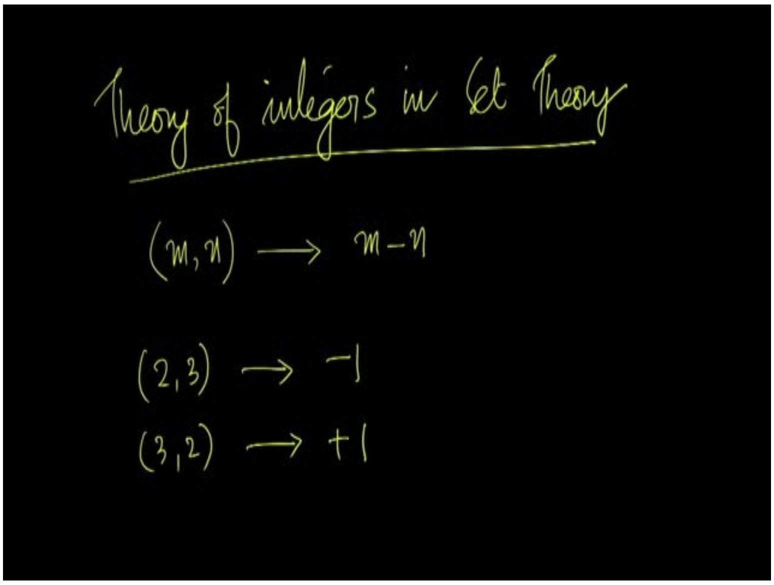

Let us see how the theory of integers can be embedence a theory. Let us consider ordered pair of natural numbers if mn is an ordered pair we would like to associate this ordered pair to the integers m minus n. For example 2 3 can be associated to the integers minus 1 whereas 3 2 will be associated will the integers plus 1. So we would map ordered pair of natural numbers onto integers and the snash.

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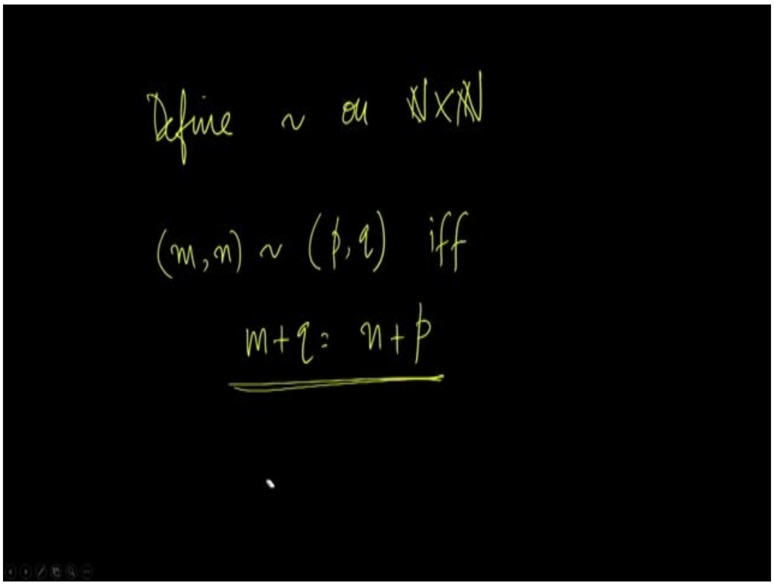

For this purpose what we do is this. We define a relation on natural numbers. This relation is defined in this manner, we say that ordered pair mn and pq are in this relation if and only if m plus q is the same as n plus p. Of course we would have wanted to say that m minus n is equal to p minus q but we can't say this because the set of natural numbers is not closed under subtraction so m minus q may not be a natural number but of course from our arithematics we know that if m minus n is equal to p minus q then m plus q is the same as n plus p. So using additions we're affectively saying the same thing.

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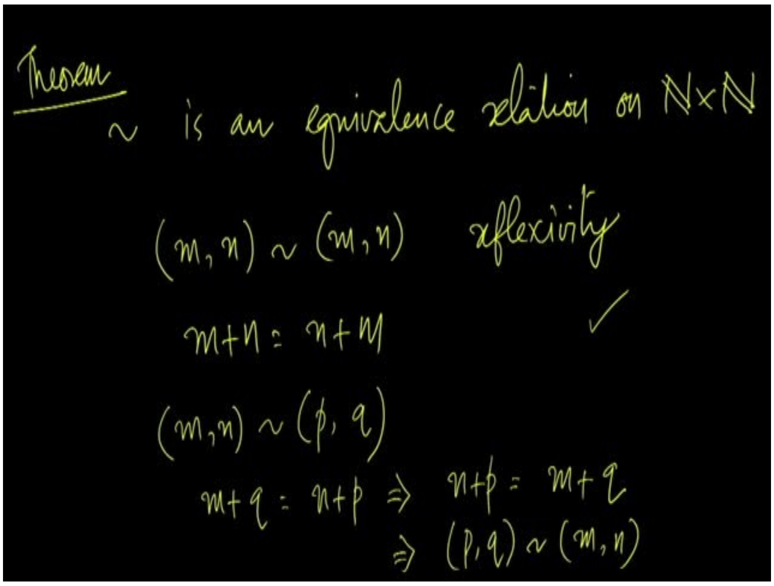

So this relationship is an equivalence relation on n cross n. That is because this holds reflexivity why would this hold that is because m plus n is equal to n plus m by commutativity of addition of natural numbers so reflexivity holds for symmetry we require that if the relation holds between mn and pq then relation should also hold between pq and mn. If the relation holds between pq mn and pq then we have while definition m plus q is equal to n plus p.

Which by commutativity of addition which implies that n plus p is equal to m plus q which would imply that pq is equivalent to mn by commutativity of addition n plus p is the same as p plus n and m plus q is the same as q plus m. Therefore pq is equivalent to mn. That is pq is in this relation till they are under with mn.

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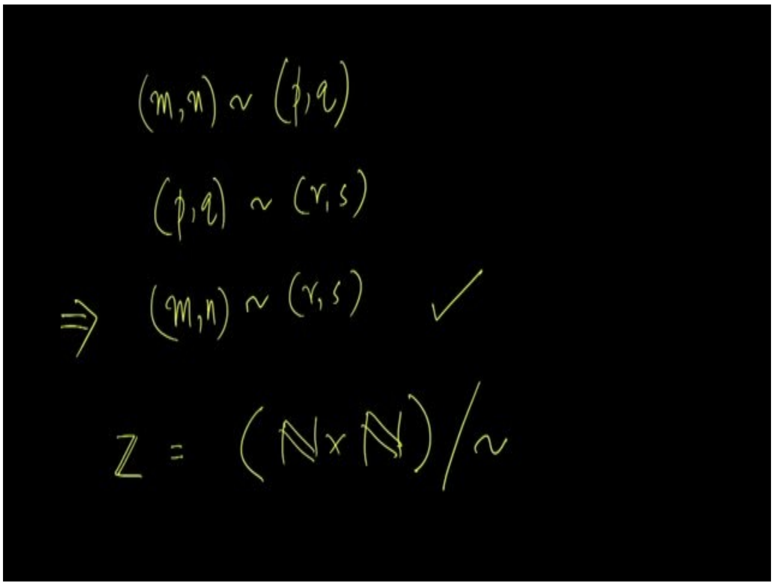

Therefore this relation is symmetric as well. And then you can verify transitivity if mn till the pq and pq till the rs then you can easily verify that mn till the rs. So transitivity also hold therefore this is an equivalence relation. Then do you think those equivalence relation we can define integers in those manner. Let us define a set of integers z as the quotient of n cross n under the till the function.

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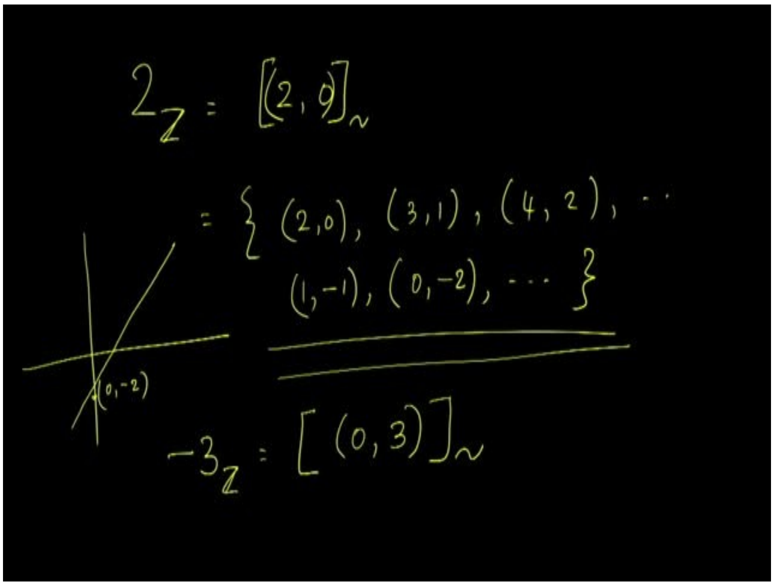

So what exactly do we do here? We define integers 2 as the equivalence class that contain 2, 0 the equivalence class under till day which contains 2 0. This would of course contain 2 0, 3 1, four 2 this is a set. And it would also contain 1 minus 1 and 0 minus 2. So in particular on the xy plain if you consider the integral grids and consider line of slope 1 which passes through 0 minus 2. All the integral grid points that fall on this line will form this set. So this set is what we define as integers 2. Then what would integers minus 3 would be? That would be the set of all pairs which are equivalent to 0 3, under those equivalence class.

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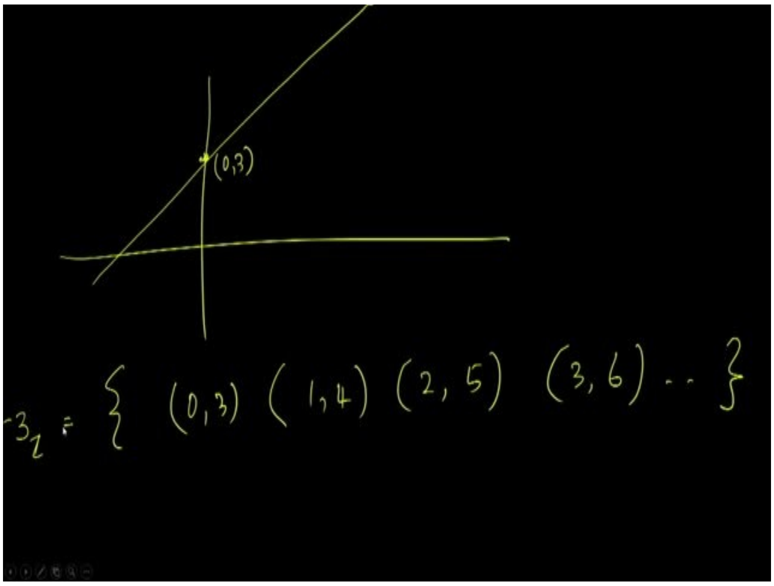

That would be all the integral grid points that fall on 0 3 the line with slope 1 and passes through 0 3 will define the equivalence class. For example 0 3, 1 four, 2 five, 3 six and so on. This is what is defined as integer minus 3. So we can define integers in this manner as equivalence classes under this equivalence relation till day.

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And then we can define the operators on integers addition and multiplication on integers appropriately. So do you think this device what we do is to define each integer as a set and then operations addition and multiplication at work on integers would be defined as operations on these special sets.

If the operators that we define behave in the manner which is exactly mimicking of the addition and multiplication operations on integers then we would be able to say that we have embedded the theory of integers on n-cross on inset theory. So that is precisely what we will attempt to do. The details we will work out in the next class. That is it from this lecture hope to see you in the next. Thank you.