Discrete Mathematics
Professor Sajith Gopalan
Professor Benny George
Department of Computer Science & Engineering
Indian Institute of Technology, Guwahati
Lecture 7
Mathematical Logic

Welcome to the NPTEL MOOC on Discrete Mathematics, this is the seventh lecture on mathematical logic.

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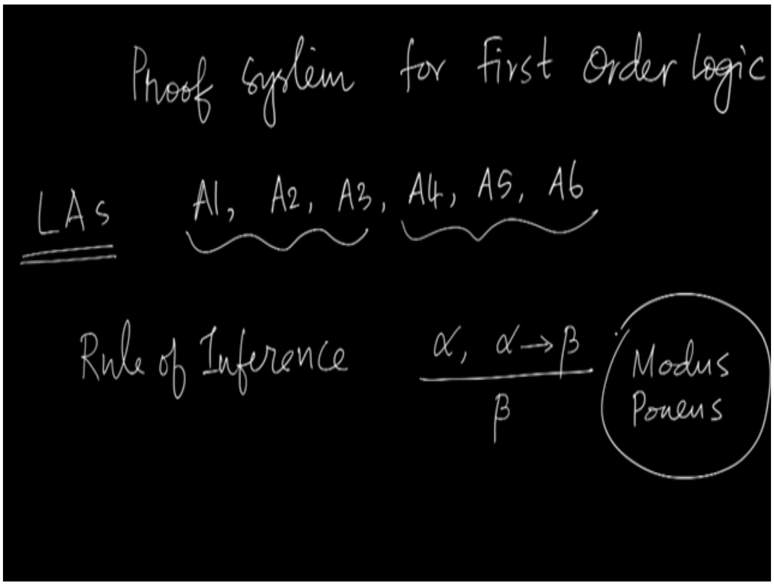

In the last class we were talking about a proof system for first order logic the proof system consists of a set of logical axioms, we saw six templates for forming logical axioms, three of them were identical to the logical axioms of the first order logic, the proportional calculus and then we had three additional logical axioms for first order logic. So these form the logical axioms of first order logic and then we have one rule of inference exactly as in the case of proportional calculus, the rule of inference that we have is modus ponens which says that if we have alpha and alpha implies beta then we can prove beta.

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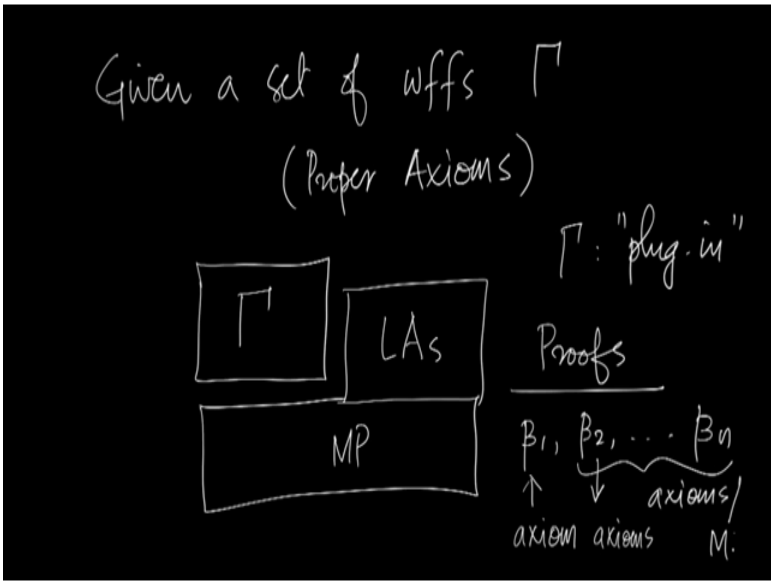

What this means is that given a set of formulae gamma which will form the set of proper axioms we have a system like this we have the logical axioms and we have the rule of inference modus ponens and we have a set of proper axioms gamma. So here gamma is plug in you can change the set of proper axioms that you have when you change gamma the conclusions would change then with this system with gamma the logical axioms and modus ponens you can write what are called proofs.

A proof is a sequence of statements or well form formulae so that the first of the proof is an axiom this could be either logical axioms or a proper axioms and subsequence statements are either axioms or obtainable from the previous statements by modus ponens but of course modus ponens should have two formulae to be applicable. We know that beta 2 also is an axiom so in any proof the first two statements are axioms the remaining statements are all axioms or obtainable by modus ponens.

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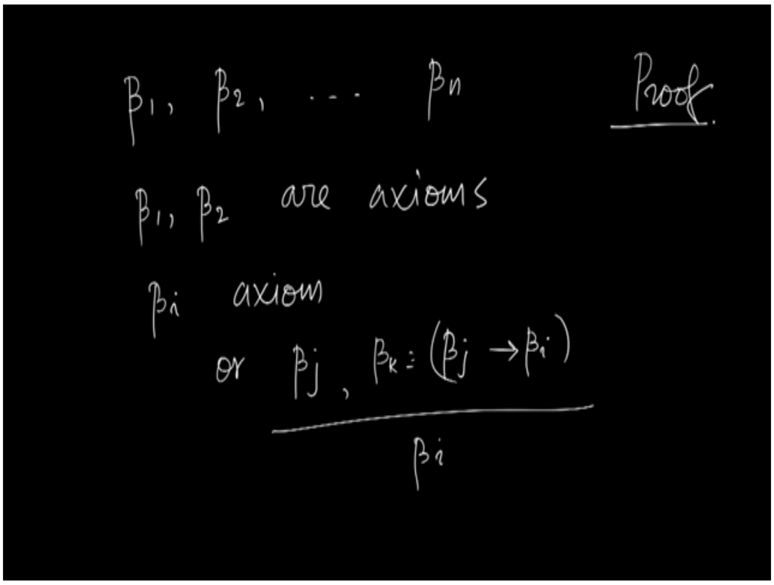

That is we are visualizing sequences of the sought beta 1 beta 2 are axioms any beta I is either an axiom or is obtained by some beta J and beta K which is beta J implies beta I these 2 together will provide beta I. So, every statement is obtainable in this manner, such a sequence of well form formula is called proof.

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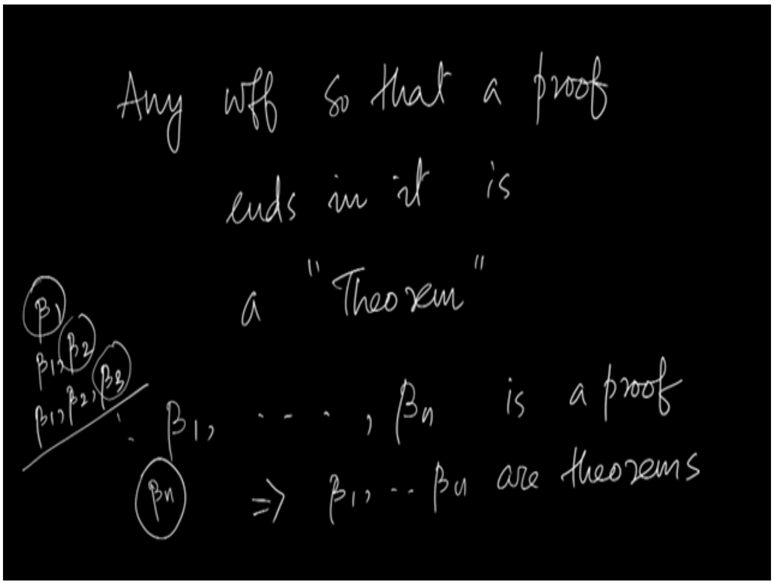

Any statement in a proof any well form formula so that a proof ends in it is a theorem. So if beta went through beta and is approve then all of this statements beta 1 to beta n are theorems that is because you can stop the proof at any point beta 1 is proof beta 1 beta 2 beta 3 is a proof.

In other words any prefix of the sequence is a proof, therefore the statements at which these proofs culminate are all theorems that is beta 1 beta 2 beta 3 etc. and beta and are all theorems. Every statement in a proof is a theorem but usually we attach the theorem to significant conclusion but in the case of logic that is not the case any statement in a proof is a theorem.

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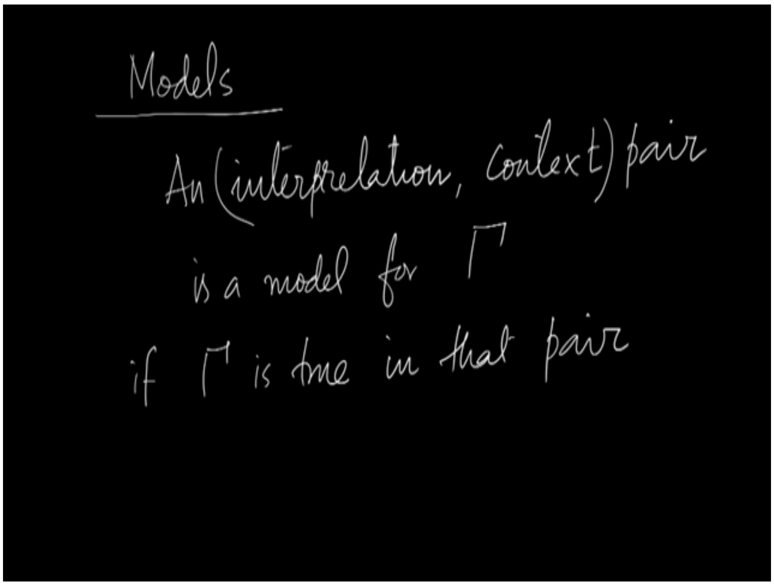

Then we spoke about models. An interpretation, contexts pair which is what breathes life into the syntactic system we have laid out. An interpretation context pair is model for the set of proper axioms gamma, if gamma is true in that pair. Now the words every well form formula in gamma is true under that interpretation context pair in that case we say that this interpretation context pair is a model for gamma.

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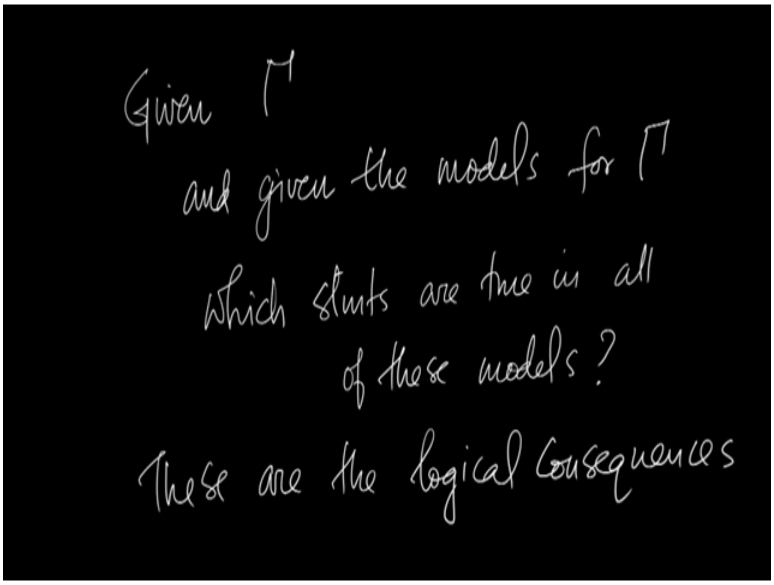

Then the question is this given gamma and given the models for gamma which statements are true which statements or well form formulae are true in all of these models? These are exactly the logical consequences of gamma. What is a logical consequences of formula? Alpha is beta is logical consequence of alpha.

If beta is true whenever alpha is true that is any interpretation context pair which makes alpha true will also make beta true that is when we say that beta is a logical consequences of alpha. Now, when we are given a set of formulae gamma we say that beta is a logical consequences of gamma, if any interpretation context pair which makes every single statement of gamma true will also make alpha true.

So that is precisely what we are saying here a model for gamma is an interpretation context pair which will make every statement of gamma true, if such a model will also make a statement alpha true if every such model will also make a statement alpha true then we say alpha is a logical consequence of gamma.

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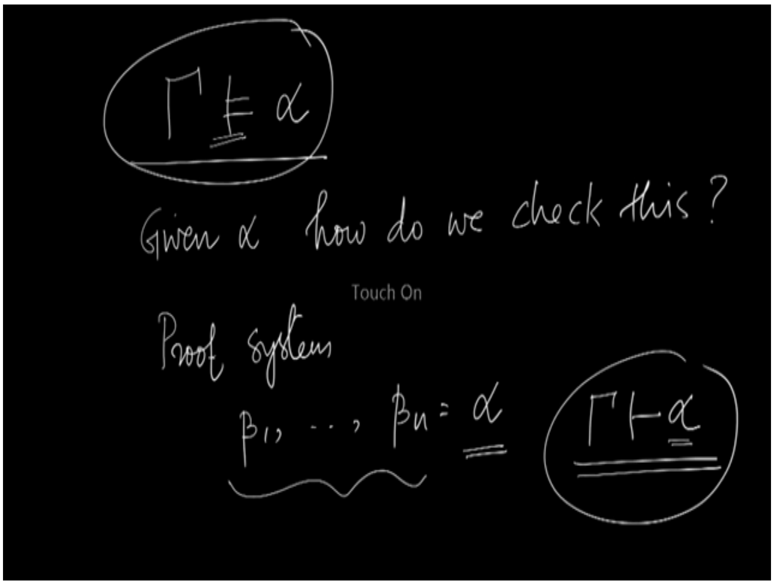

Notionally we write in this manner alpha is a logical consequence of gamma. Now the question is, given alpha how do we check this? This is where the proof system comes handy. When the proof system furnishes us with a proof, where the proof culminates with alpha, we would say that alpha is provable from gamma that is using gamma and the logical axioms we are capable of proving alpha.

Now we would like this to be identical to this that is the semantic notion of logical consequence and the syntactic notion of provability you know proving is strictly syntactic process, we are merely looking at the forms of the statements and rewriting them. So a proof is strictly syntactic process using the semantic process we are arriving at alpha here logical consequences and semantic notion we want the syntactic notion of provability and the semantic notion of logical consequence to be equal. That is when we say that the proof system is sound and complete okay

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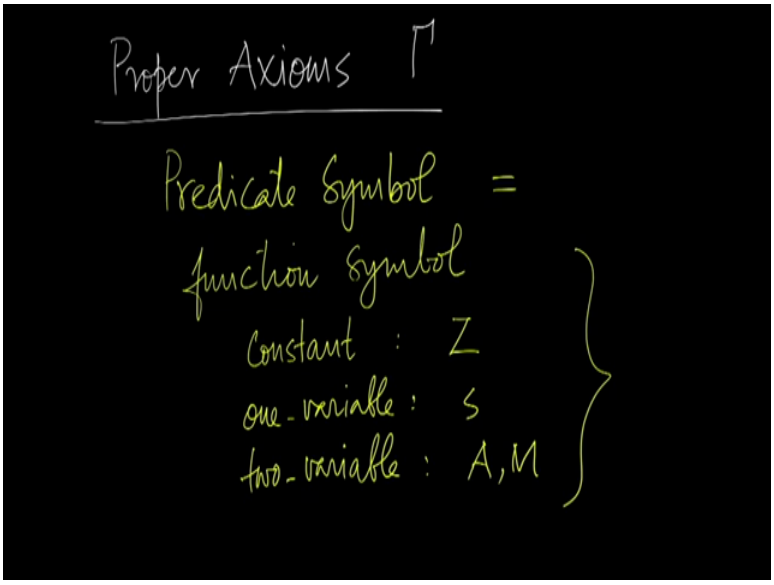

Now, let us consider one concrete example for proper axioms. Let us consider a system of logic in which there is one predicate symbol which is the equality symbol and then there are function symbols there is one constant symbol or 0 argument function symbol which is Z and there is a one variable function symbol which is S, there are 2 variable function symbols which are A and M.

Let us say these are the only symbols that the system has the function symbols and the predicate symbols. So these along with the logical connectives quantifies brackets will form the alphabet of the language

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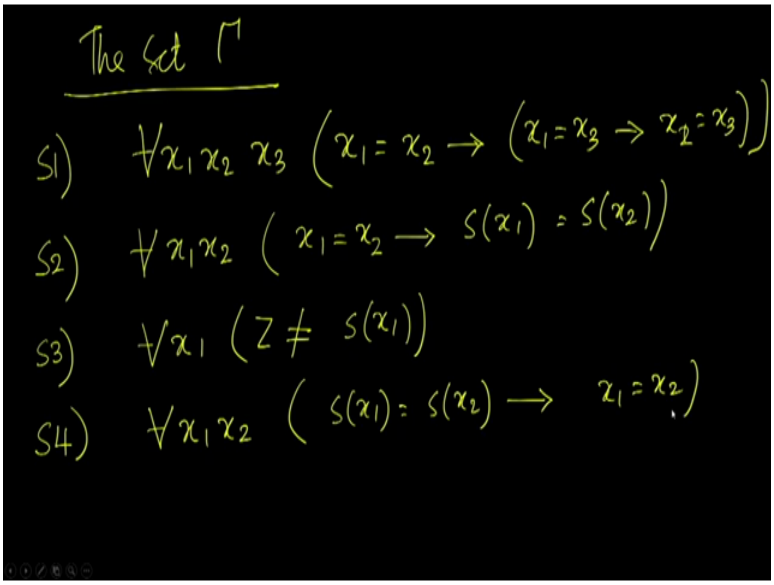

And the set gamma consists of the following axioms, axiom X 1 is this. It is essentially a statement about equality, it is about the transitivity of equality. What it says is that for every X 1 X 2 and X 3 if x 1 equal to x2 and x 1 equal to x3 then x2 will be equal to x3. We will use this as a short form for this so, whenever I write like this you should understand that this is what I mean. The second axiom is also about equality no second axiom is about the S function S function can be thought of as the successor function we would call it the successor function.

So what it says is that for every x 1 and x2 if x1 is equal to x 2 then the successor of x1 is the same as the successor of x2 which means every number has a unique successor and then we say that Z is not the successor of anyone that is what the third axiom is and the fourth axiom asserts for every x1 and x2 if the successor of x1 and the successor of x2 are the same then x1 and x2 are the same. This is the converse of the second axiom.

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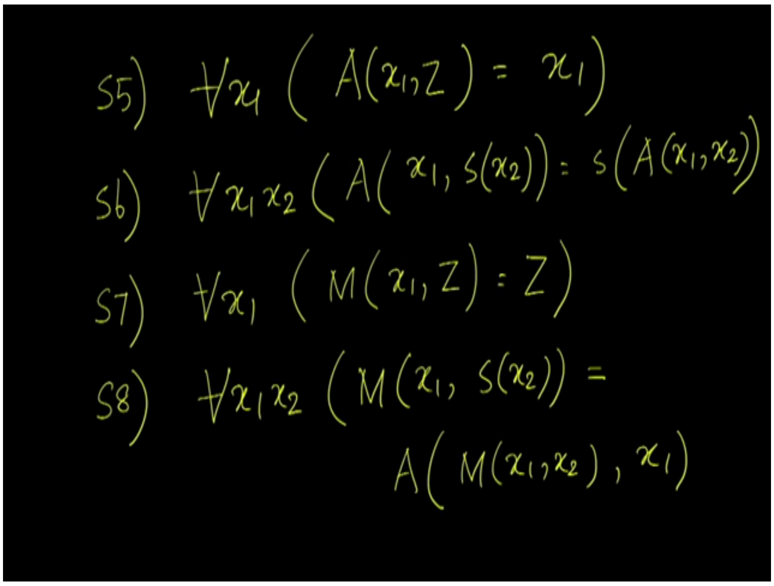

Then the fifth axiom asserts that for every x1 apply a on x 1 and Z we get x1 the fifth axiom asserts that for every x1 a applied on x and Z will give us x 1. The sixth axiom say that x 1 x 2 for every x 1 x 2 a of x1 and the successor of x2 is the same as the successor of a of x1 and x 2. Now, probably you can guess where we are headed, what is the meaning of a in the seventh axiom we assert that for every x1 the m function applied on x1 and Z will gives us Z. The eight axiom says that for every x1 and x 2 M on x1 and the successor of x2 is the same as a applied on M on x1 x2 and x 1.

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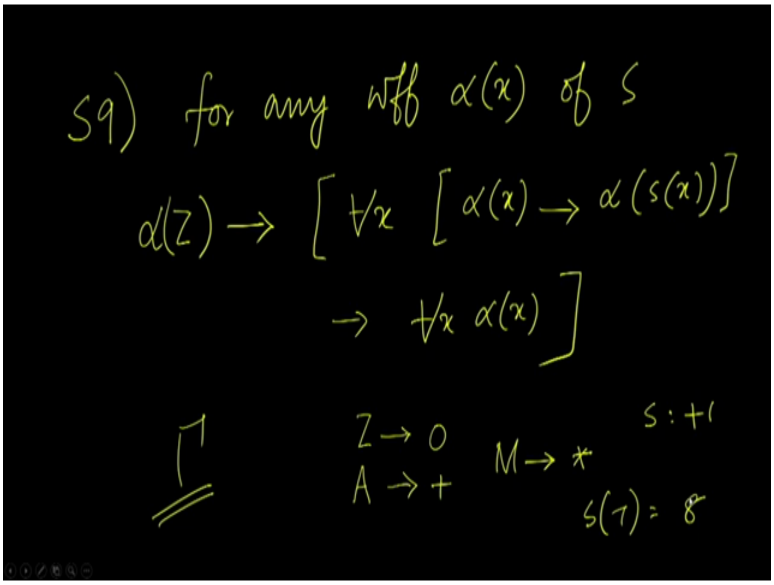

And the final axiom S9 is in fact an axioms gamma it says that for any well form formula alpha with one free variable x alpha of 0 implies that for all x or alpha of Z implies that for all x alpha of x implies on alpha of successor of x implies for all x alpha of x. So, let us say gamma is this set of proper axioms. Now we have written gamma with an intention of a particular interpreting them in a particular way, for example, we would like to interpreted Z as 0 we would like to interpreted A as the addition function and M as the multiplication function and the successor function is the plus one function in the sense that successor of seven is eight

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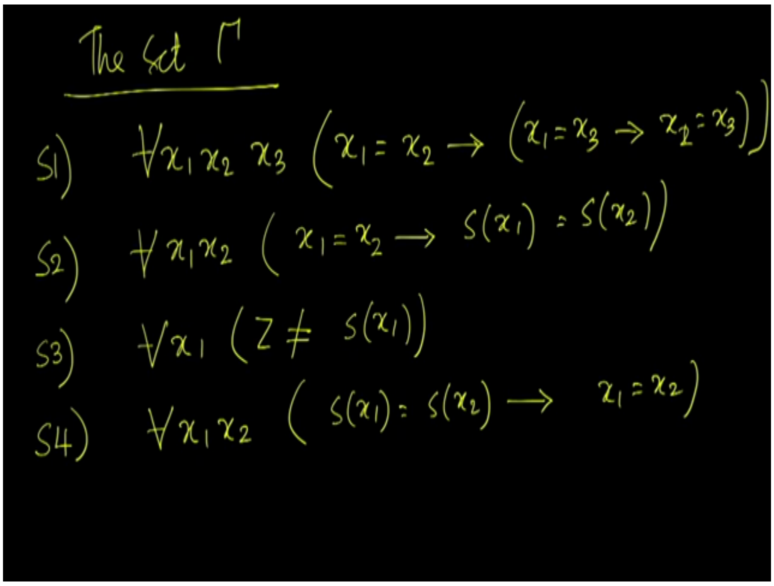

Let us take another look at the axioms with this interpretation in mind so the first axiom say is that equality is transitive the second axiom say that for every x 1 and x 2 if x 1 equal to x 2 then S of x1 equal to S of x2 or another words if x1 is equal to x2 then x1 plus one is the same as x2 plus one. The third axiom says that 0 is not the successor of anyone and the fourth axiom says that if x1 and x2 have the same successor then x1 equal to x2 in other words if x 1 plus equal to x2 plus 2 then you can cancel one from both sides and get x1 equal to x2.

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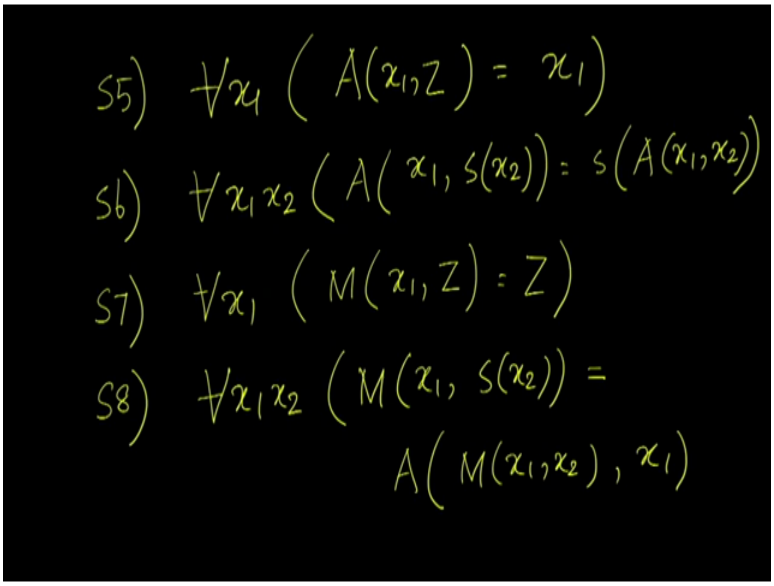

The fifth axiom says that when 0 is added to x1 we get x1 that is 0 is the identity of addition. In the sixth axiom we say that x1 added to the successor of x2 that is x1 plus x2 plus 1 is the same as x1 plus x2 plus 1. So, from this you can essentially derive the associativity of addition. In the seventh axiom we have multiplication with 0 we know that 0 of multiplication that is x1 into Z is Z. In the eight axiom we have distributivity x1 multiplied by x2 plus 1 is the same as x1 x2 plus x1. So from this we can derive the general form of distributivity

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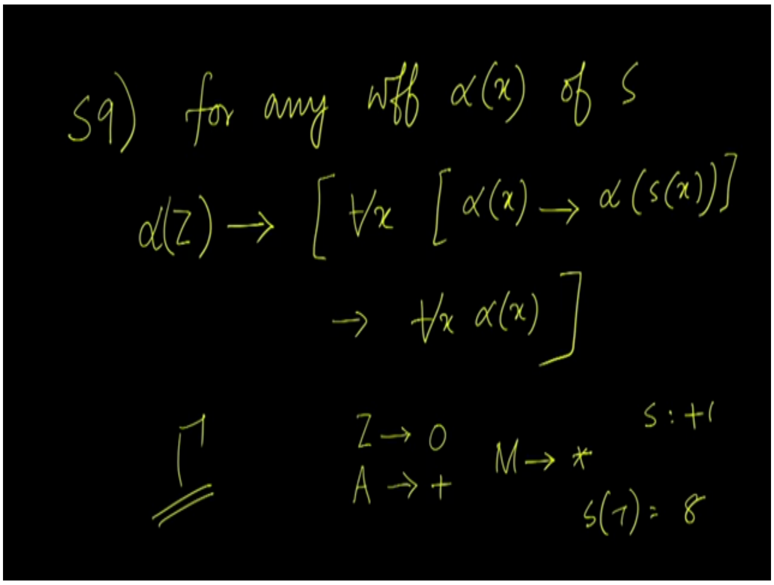

And the ninth axiom is in fact the principle of induction for any well form formula alpha of x if you can show that alpha is true for 0 and also that for any if alpha is true for x then alpha is true for x plus 1. If both of these can be shown then we can argue that alpha is true for every x. so this is the principle of induction. So all this axiom put together form are set gamma.

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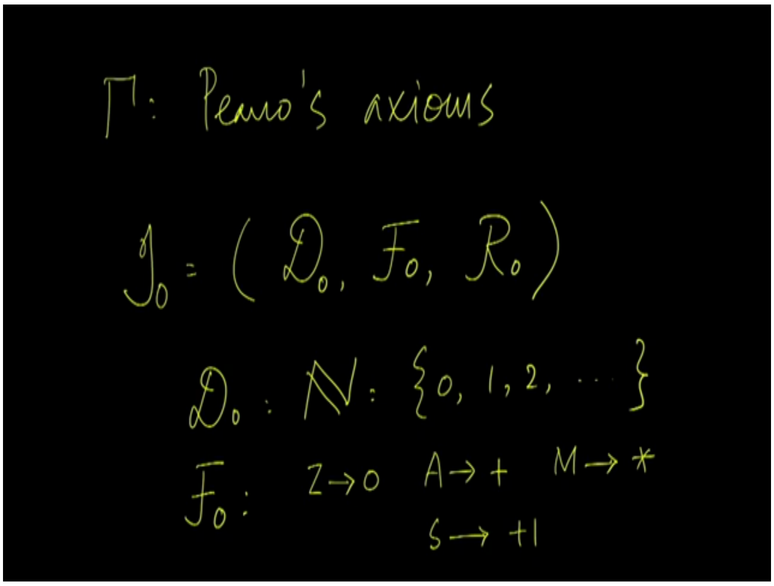

So gamma is called Peano's axioms and forms the basis for theory of natural numbers. So we can think of an interpretation I naught D naught F naught and R naught. Where D naught is the set of natural numbers F not maps the functions symbols it maps Z to 0 A to the addition function M to the multiplication function and one variable successor function to the increment function plus 1. So that is what F not does.

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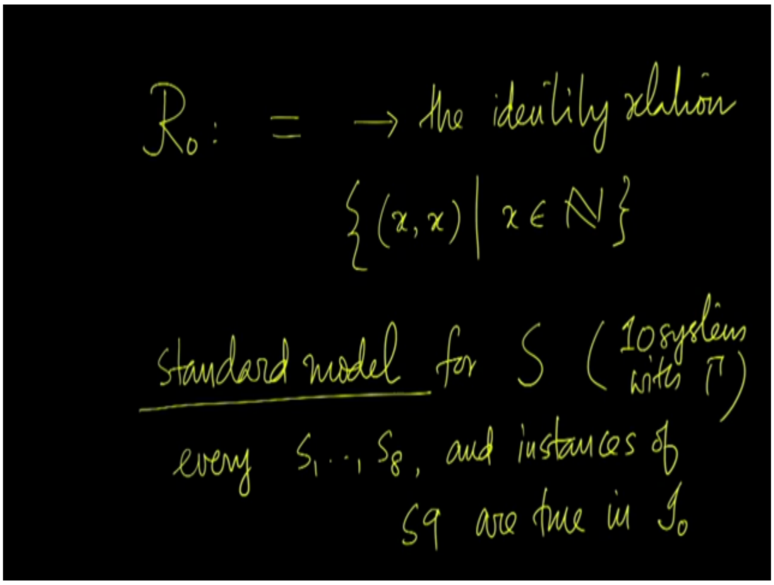

R not maps the equality symbol to the identity function the identity relation which consists of pairs of the form x x. so this way we interpreted all the symbols function symbols and predicated symbols of the language. So this interpretation is called the standard model for our first order system S, S is the first order system with Peano's axioms as the proper axioms that is when within the proof system we plug in gamma the Peano's the system of Peano's axioms what we get is the first order system of S.

This interpretation where D not is defined as the set of all natural numbers and F not and R not are defined in this fashion is called as standard model for S. In fact any interpretation which is equinumerous with this interpretation is also called as standard model. But let us do not bother about the equinomersoity as this point.

So this is what we will call a standard model for S. We call this a model because we can guess we can check that every one of S 1 to S 8 and instances S 9 are true in this interpretation that is why we see that say that I not is model for S. So that is an example of a first order proof system with a concrete gamma

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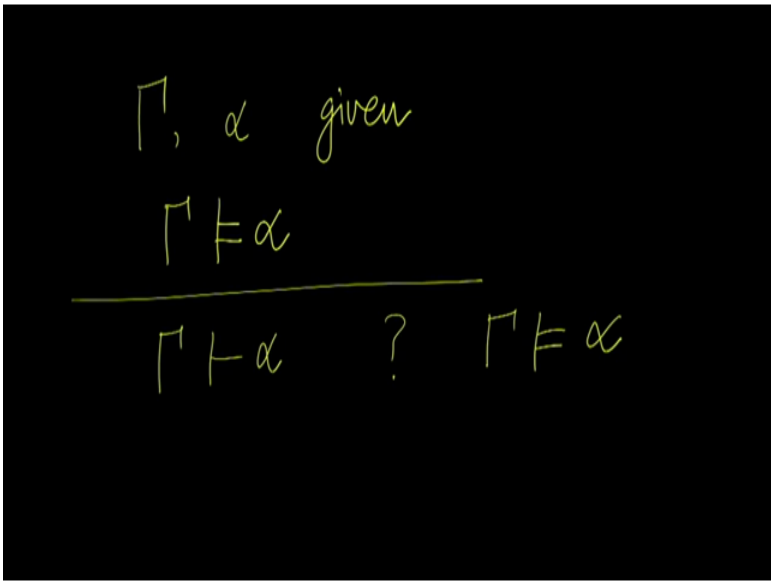

Now, let us go back to the original question we have a question of this form we are given gamma and we are given an alpha we want to answer this question really what we are interested in this alpha a logical consequence of gamma. We have designed a proof system to essentially answer this question and what we want is this is there any relationship with between the provability of alpha from gamma and the logical consequence of alpha being a logical consequence of gamma is there any relation between the two.

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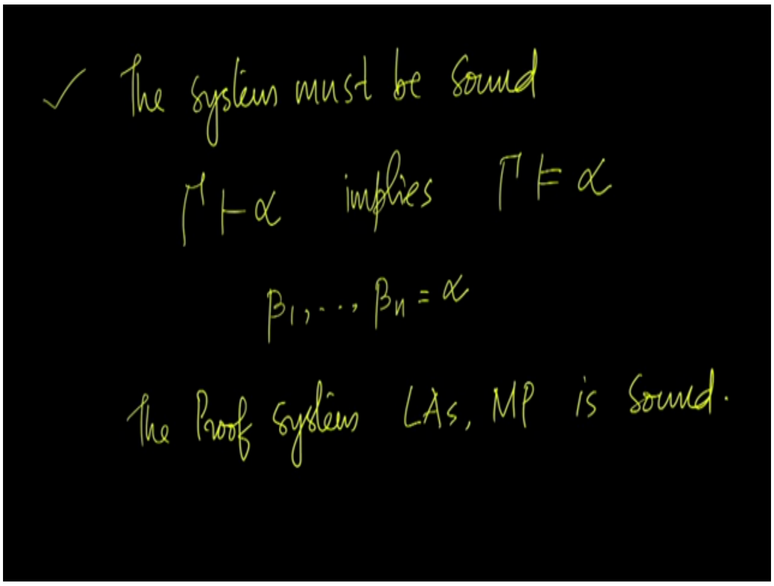

We would like this to notions to be identical. In other words we would like the system to be sound if the system is sound then anything that we prove is sound, in other words anything that we prove is logical consequences. So if the recces approve culminating an alpha with in this system where any of these statements you see either an axiom or follows from 2 of the previous statement by modus ponens then alpha is provable and then we would like to argue that alpha is a logical consequence of gamma.

So this is when we would say this system is sound it can indeed be shown that the proof system consisting of logical axioms and modus ponens is sound in the sense that you use any plug in gamma anything that is provable within the system would be a logical consequence of gamma. So the system that we have been discussing is sound.

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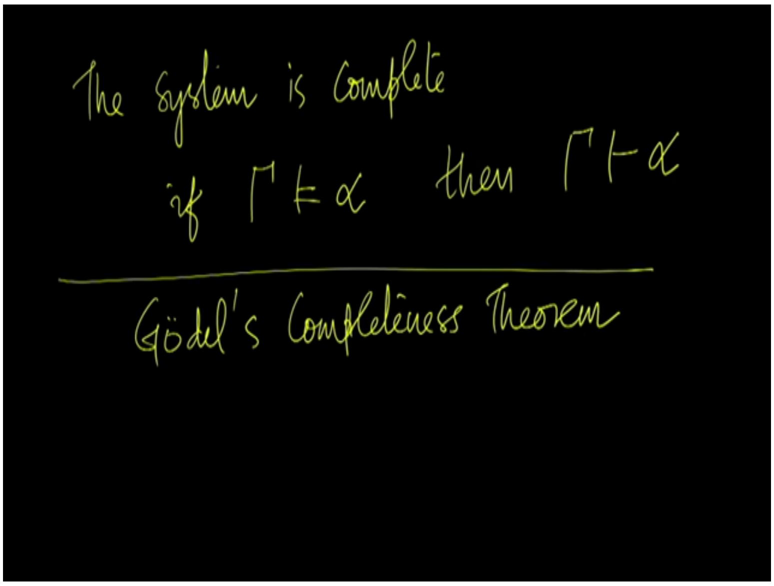

The other question is that the completeness, we would like to know that the system is complete, in other words we would like to know that if alpha is a logical consequence of gamma then alpha is provable.. Godel's famous completeness theorem proofs just this, it shows that this proof system is indeed complete that is when you use a proof system of the sought with axiom skimask A 1 through A 6 as a logical axioms and modus ponens as the rewriting rule then every logical consequence happens to be provable.

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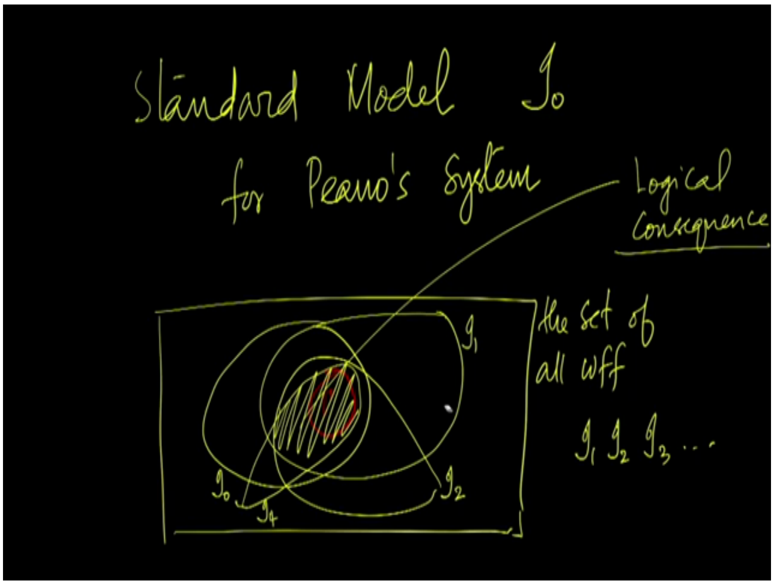

So the proof system is sound and complete but there are consider that standard model, I naught for Peano's system this is just one interpretation and in this one interpretation suppose this rectangle represents this set of all well-formed formulae, all syntactically correct formulae. So, let us say I naught makes some of these formulae true in particular it will also make all of gamma true consider another interpretation which will make some other set of formulae true of course this sets of formulae may overlap, suppose I 1 is also a model for gamma that is gamma is true under this interpretation I 1 as well.

Now imagine a third interpretation which will make yet another set of formulae true suppose this is also a model for gamma. So in this sense let me imagine all models for gamma there could be countably infinite number of models per gamma then the intersection of all of this will form the logical consequences of gamma that is because this statements the shaded portion within the given diagram.

When we have plugged in all imaginable interpretations we find that the intersection of all of them is exactly the logical consequences of gamma. So in all this interpretations gamma is true therefore all of them are models for gamma and in all these interpretations these shaded portion is also true.

Therefore we can say that any interpretation which makes the whole of gamma true will also make all these formulae true. So these are the logical consequences of gamma and these are indeed the statements which are provable within R system. So the shaded portion is what is provable within the system.

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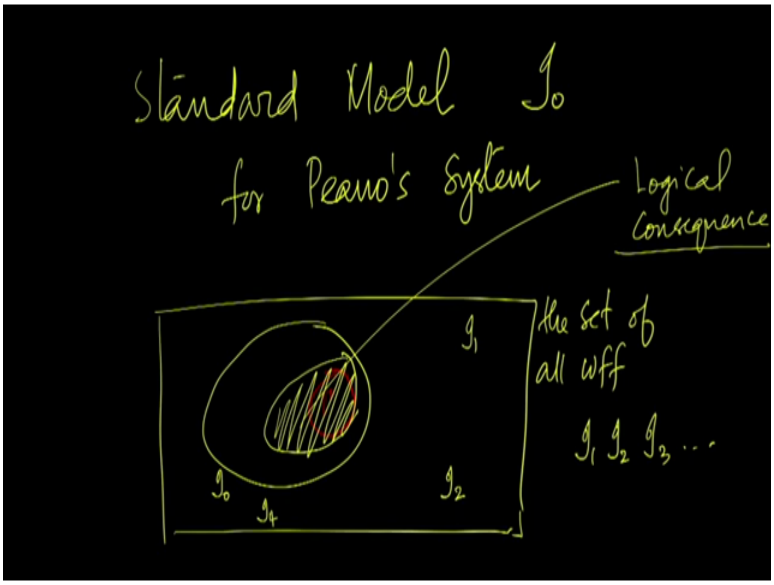

Now, let us consider I naught we are going back to I. not we find that the shaded portion is a subset of I naught now what is a circle I not it is supposedly the set of all statements that are true in the standard interpretation and the shaded portion of the set of all statements that are true in every interpretation that will make gamma true of course, I naught also makes gamma true but there could be more statements which are true in I naught or in other words

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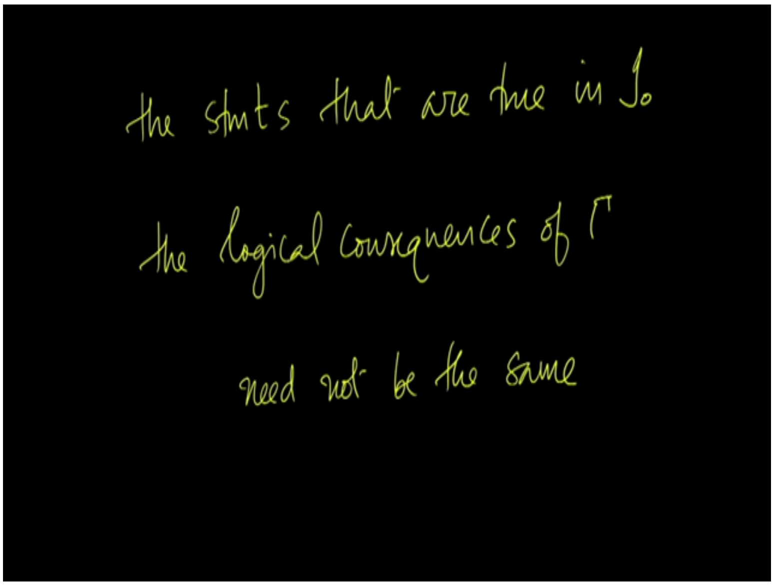

The statements that are true in I naught and the set of logical consequences of gamma need not be the same the need not form the same set or in other words there could be a statement that is true in I naught which is not a logical consequence of gamma clearly the second set is a subset of the first but what I am saying is the 2 sets need not be equal.

The second set could be a proper subset of first set in which case there would be I naught which is not a logical consequence of gamma.

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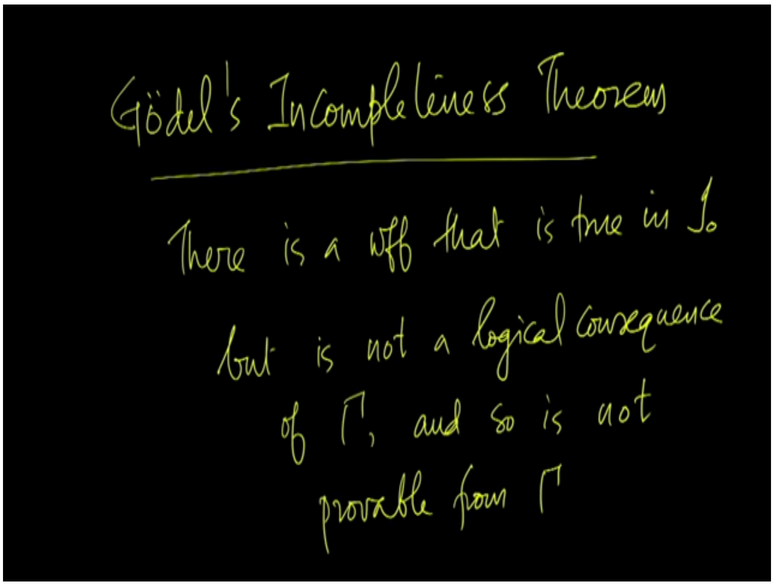

The famous Godel's incompleteness theorem establishes just this this statement shows that there is a well form formula that is true in I not but is not a logical consequence of gamma where gamma is the set of Peano's axioms and so is not provable from gamma That is because by the soundness and completeness of the Godel's system where we obtain completeness from Godel's completeness theorem what we know is that logical consequences of gamma are exactly the statements that are provable from gamma.

Therefore, if there is a statement that is true in I. which is not a logical consequence of gamma then this statement will not be provable from gamma but then what it is the significance I naught, I naught is the standard interpretation it interprets the domain of discourse as a set of natural numbers and the function symbols and the predicate symbols in the familiar way the we have the 0 symbol and the addition symbol and multiplication symbol and the successor symbol and when you read the axioms in this sense you realize that it is the set of axioms for the theory of natural numbers.

Therefore, there is a statement which true according to our intuitive understanding of numbers which is not provable from gamma. The demonstration of such a statement form Godel's incompleteness theorem.

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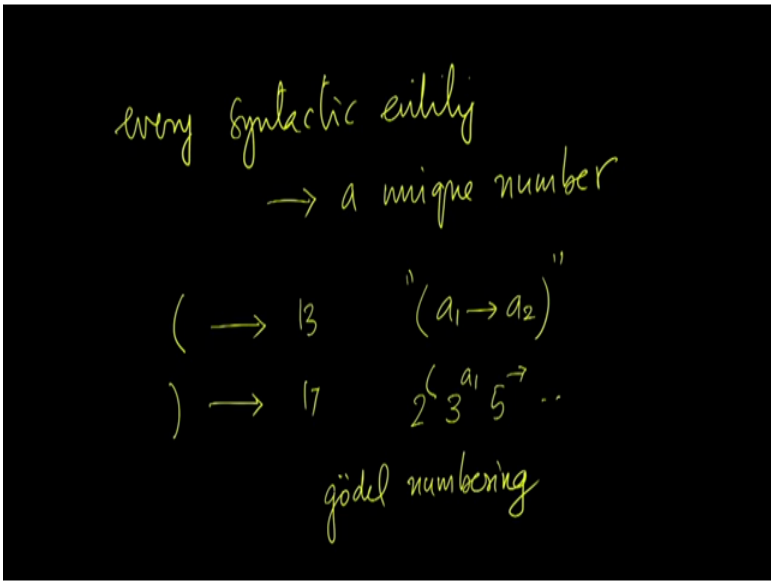

Yodels construction was roughly like this for syntactic activity we can define a unique number. This is a way of encoding the syntactic entity, for example, we can assign a symbol a number for the opening bracket let us say 13 for the closing bracket we can assign a number 17 and so on and then combining these we can assign a number, we can find a way of encoding every syntactic entity in to number.

For example, when I have a statement of this form the corresponding number for this could be the number corresponding to the opening bracket multiplied by the number corresponding to A1 multiplied by the number corresponding to implication and so on. This is one way of encoding and this encoding has the property that we can uniquely decode any such given number. So, Godel demonstrated that such an encoding is possible for the syntactic entities these are called Godel numbering.

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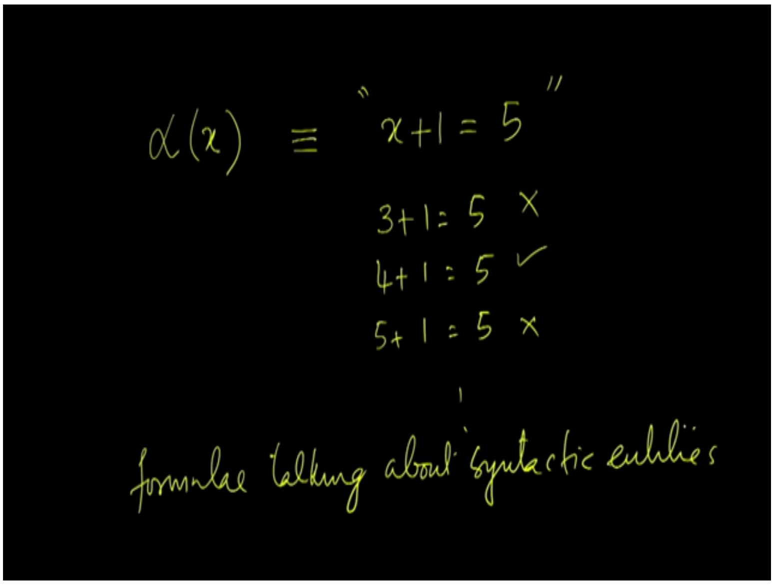

Once the syntactic entities are encoded are into numbers you can treat numbers as syntactic entities and then when you look at the statements within the system we find that we have well form formulae with free variables in particular if I look at well form formulae alpha with one free variable X it seems to be saying something about x statement could be something like this x plus 1 equal 5 so this a formula with one free variable, x by substituting for x I can get various formulae.

Some of them are true, some of them are false and so on. So, those are formulae with one free variables but then formulae with one free variables but then formulae with one free variables now can also be taught of as speaking about syntactic entities because now we have mapped syntactic entities and the numbers and the one to one manner therefore at least some of the numbers represents valid syntactic entities. So a formula with one free variable can be thought of as talking about a particular number but a number can be thought of as a syntactic entity as well so now we have formulae taking about syntactic entities.

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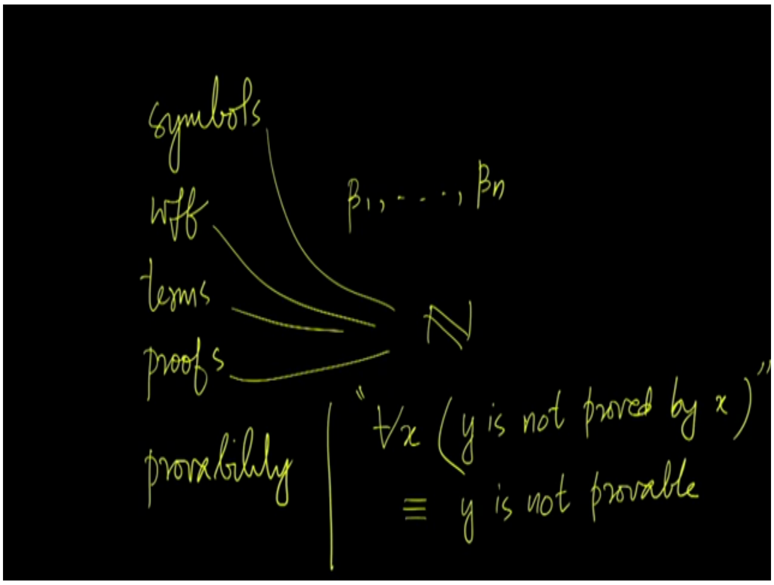

But what are syntactic entities the symbols of syntactic entities the function symbols the predicate symbols open closes brackets implication negation all these are syntactic entities and then well form formulae are syntactic entities terns are syntactic entities which form constituents of well form formulae and we can also consider proofs as syntactic entities a proof is nothing but a sequence of formulae so if you combine the Godel numbers of the well-formed formulae belonging to a proof in an appropriate manner.

We can also get device Godel number for a proof. So proofs are also syntactic entities now yodel numbering maps all of these into the set of natural numbers into 1 to 1 manner therefore, we can now have 1, statements that are talking about syntactic entities which includes symbols well form formulae, terms, proofs etc.

Therefore, we can have statements that are talking about true abilities. Roughly, when you have a statement which says that for every x, y is not proved by x, in other words for every x, x is not a proof of y, is essentially asserting that y is not provable. Now this statement itself has a Godel number and y is a free variable in that.

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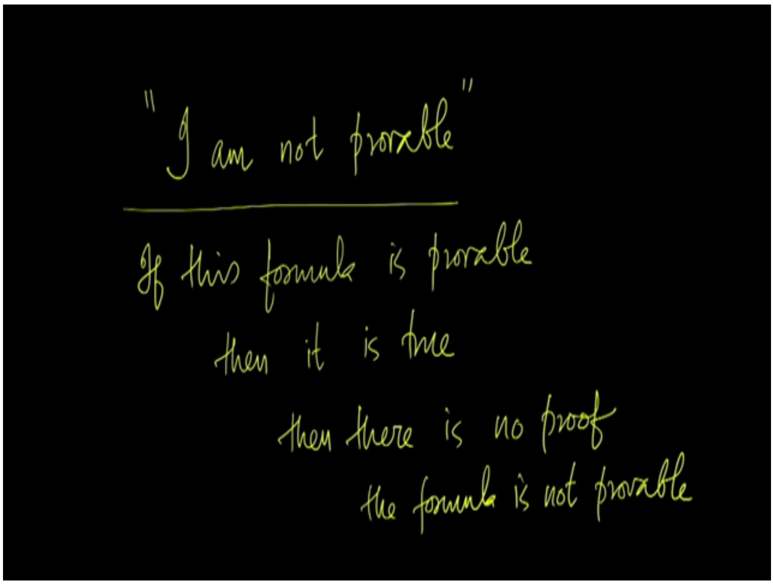

So by appropriate substitutions we could device an expression which essentially says that I am not provable, this is what Godel did in his incompleteness theorem, he devised a formula which essentially says I am not provable, the meaning of the formula is I am not provable when you consider the standard interpretation of natural numbers, that is from the perspective of our understanding of natural numbers, the formula can be thought of as meaning this the formula is self-referential formula it refers to itself .

It says that there is no y which is a proof of this statement. That is the statement the Godel number inverse of which is this formula itself, but then what would be the truth value of such a formula, will this be provable if this formula is provable, if this formula is provable then by the soundness we know that it is a logical consequence and therefore, it is true if this formula is provable then it is true.

But then what does the formula say, it asserts that there is no proof of this formula itself then there is no proof or in other words the formula is not provable, which is a contradiction if this formula is provable then it is false but it cannot be false because anything that is provable has to be true by the soundness of the system. Therefore, we have a contradiction therefore it is not possible for this formula to be false

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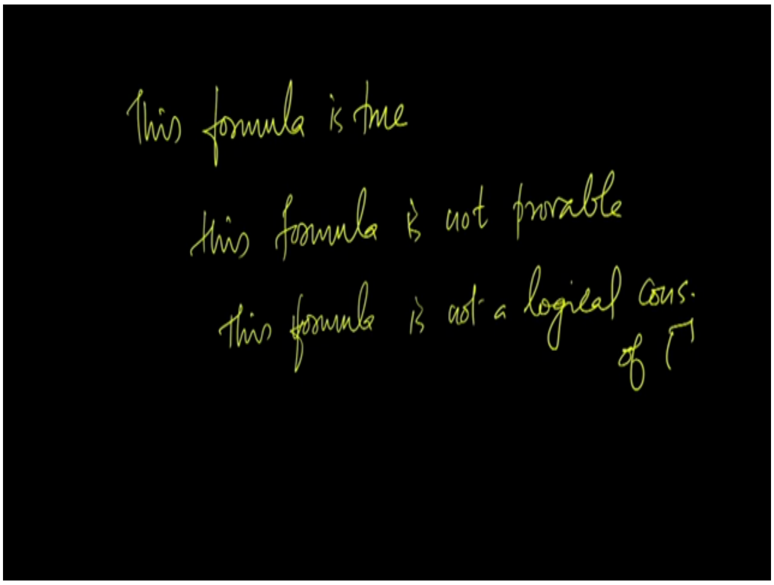

Therefore, in the standard interpretation this formula is true which means this formula is not provable in other words this formula is not a logical consequence of Peano's axioms in other words there is a true in the standard interpretation but is not a logical consequence Peano's axioms or in other words there is a statement which is true in the standard interpretation according to due to understanding of numbers this formula has to be true but it cannot be provable from the set of Peano's axioms.

In other words the proof system that we have laid out is not complete in that sense that is it is not capable of proving every statement which is true in the standard interpretation, it is capable of proving exactly the logical consequences of Peano's axioms.

So, Godel establish demonstrated that the recess of formula which is true according to our intuitive understanding of numbers and is not provable from Peano's axioms okay. So this is the end of discussion on mathematical logic and the end of this lecture. Hope to see you in the other modules. Thank you.