Discrete Mathematics
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Lecture 6
Mathematical Logic

Welcome to the NPTEL MOOC on mathematics, this is the sixth lecture on mathematical logic. In the previous lecture, we saw a formal discussion of propositional calculus. Today we shall have a formal discussion of system of logic which has more involved than propositional calculus.

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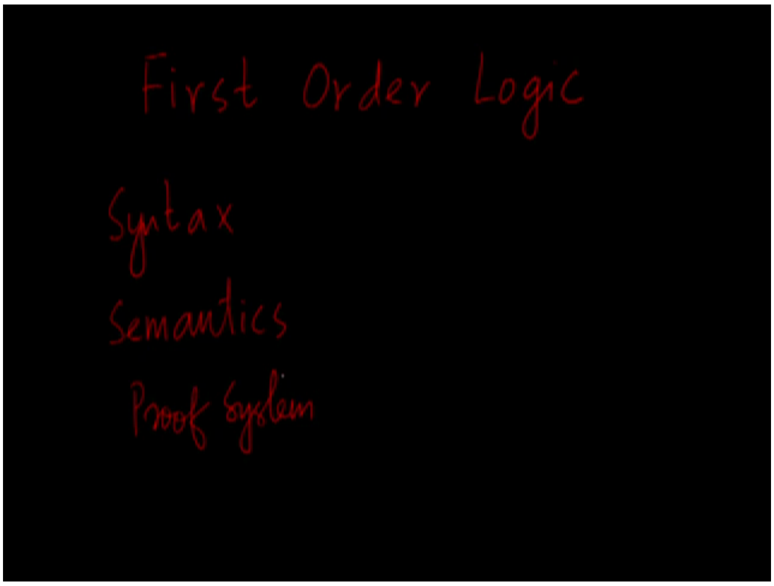

So, in first order logic exactly as in the case of propositional calculus we have a specification of syntax, semantics and a proof system.

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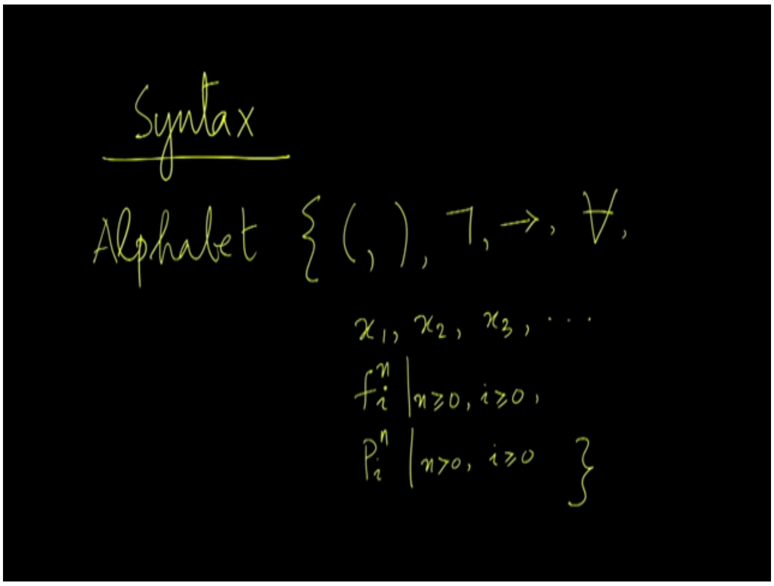

Let us begin with syntax. So, as in propositional calculus to specify syntax we require an alphabet first of all and using this alphabet we have to device formulae, so the alphabet of first order predicate calculus will consist of these symbols open and close brackets, the logical connectives, negation and implication exactly as in propositional calculus and we will have a quantifier the for all quantifier.

And then we will have a number of variables, the variables could even be infinite and then we have function symbols of the form where n is greater than or equal to 0 and i is greater than or equal to 0; F n i will denote the ith n-ary function symbol, so this is a function symbol which will have n arguments, so this is the ith such. So, we have such function symbols and we have predicate symbols P n i where n is greater than 0 and i is greater than or equal to 0.

So, this represents the ith n-ary predicate symbol. So, the alphabet of the language will consist of these symbols the open and closed bracket, the negation symbol, the implication symbol, the quantifier, the universal quantifier, a number of variables, a number of function symbols and predicate symbols.

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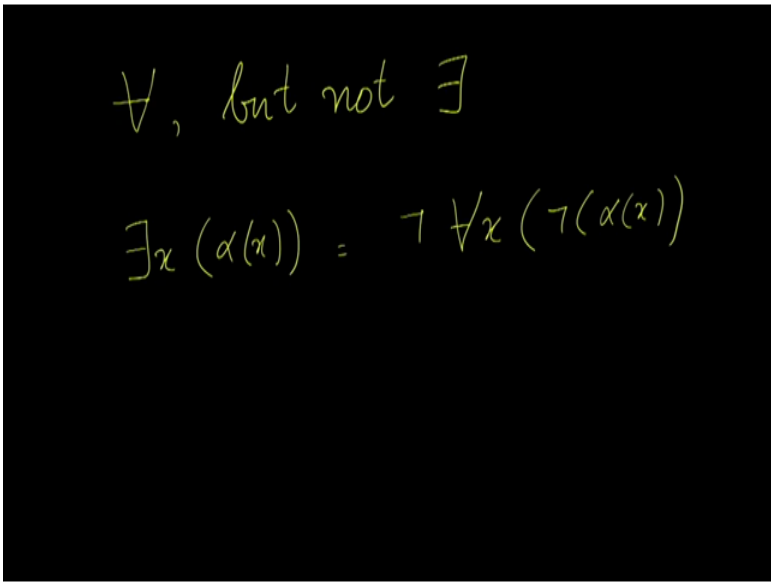

You notice that we have only one quantifier here, the universal quantifier but we have not included the existential qualifier that is because existential quantification can be expressed using universal quantification and negation by De Morgan's law. For example, a formula of this form can be written as the negation of alpha of x, the negation of for all x the negation of alpha of x by De Morgan's law.

Therefore, the existential quantifier can be expressed using the universal quantifier and negation and in our alphabet we have included both the universal quantification and negation. So, that is a sufficient set a special set.

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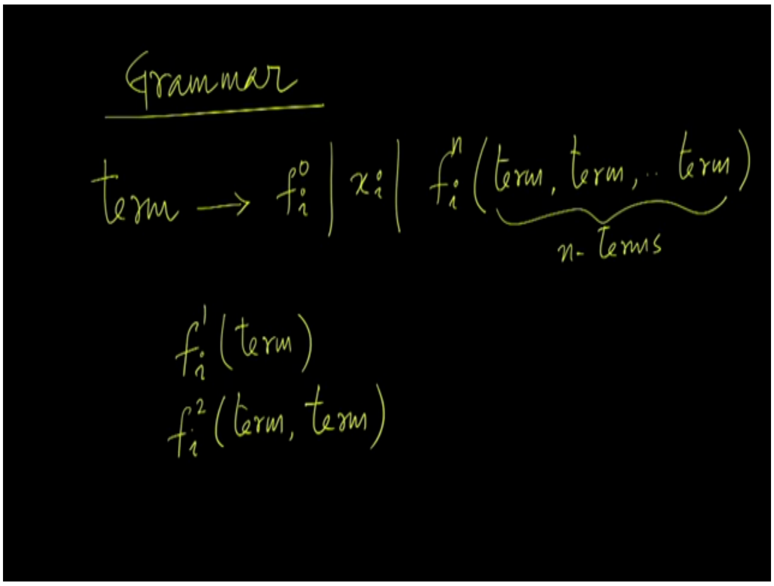

Now, let us define the grammar which governs the language. First, we define what is called a term, a term is an entity that is supposed to name individuals, so a term is either a function symbol of this form a 0 or a function symbol, we will also write a 0 argument function symbol as a I, we will use a i as a short form for F 0 i which is a function symbol that does not take an argument. So, a term could be one such or it could be a variable or it could be an n-ary function symbol applied to a number of terms.

So, a term can be constructed from other terms in this fashion, a term could be made up of an n-ary function symbol applied on n terms, so there would be n arguments here if we use an n-ary function symbol. In particular, if we use a 1 argument function symbol, we will have a term of this form, if you use a 2 argument function symbol we will use two terms as arguments. So, a term can be generated inductively in this form.

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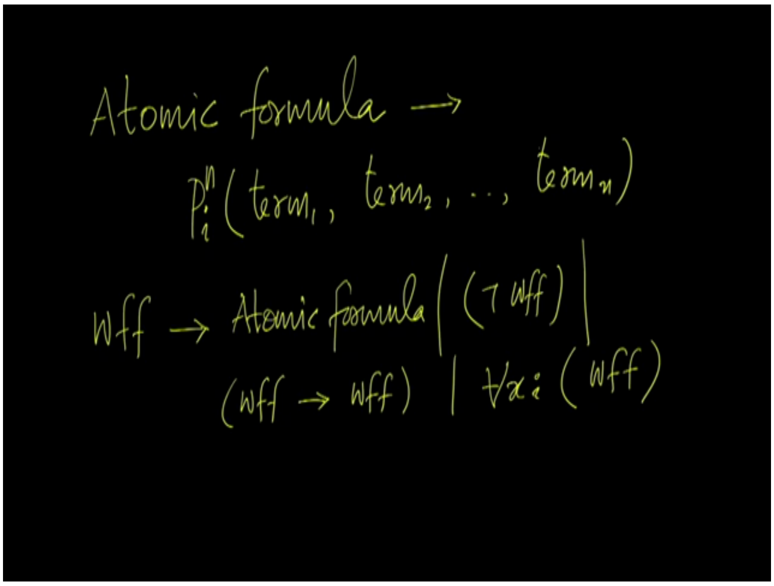

So, that is what a term is an atomic formula would be an n-ary predicate symbol applied on in terms. So, using the previous grammar rule we generate terms and using n such terms we have to use an n-ary predicate symbol to generate an atomic formula and then a well-formed formula would be either an atomic formula or a negation of a well-formed formula or a well-formed formula implying another, these two are similar to propositional calculus and then finally for any variable x i for all x i quantification applied on a well-formed formula will also be a well-formed formula.

So, a well-formed formula can be synthesized in this fashion. So, these are the rules governing the syntactic entities of first-order logic every syntactic entity can be generated using one of these rules from the alphabet.

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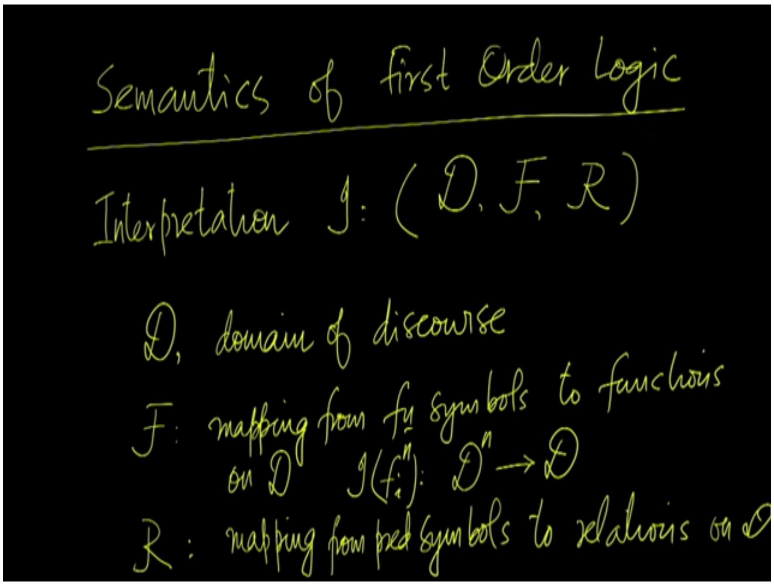

Now, we need to specify the semantics. To specify the semantics of first-order logic, we use what is called an interpretation. An interpretation I is a triplet, it specifies D a domain of discourse, this is the set of individuals about whom we talk using the system of logic and then we have a function F, F is a mapping from the set of function symbols to functions on D.

In particular an n-ary function symbol should be mapping D power n to D, it would take n individuals and map them to D that is the semantics that we assigned to F n i and n-ary function symbol, the meaning of this function symbol would be a function which maps a tuple of n entities from D to an entity from D and finally, we have the third component R. The third component R maps the predicate symbols to relations on D.

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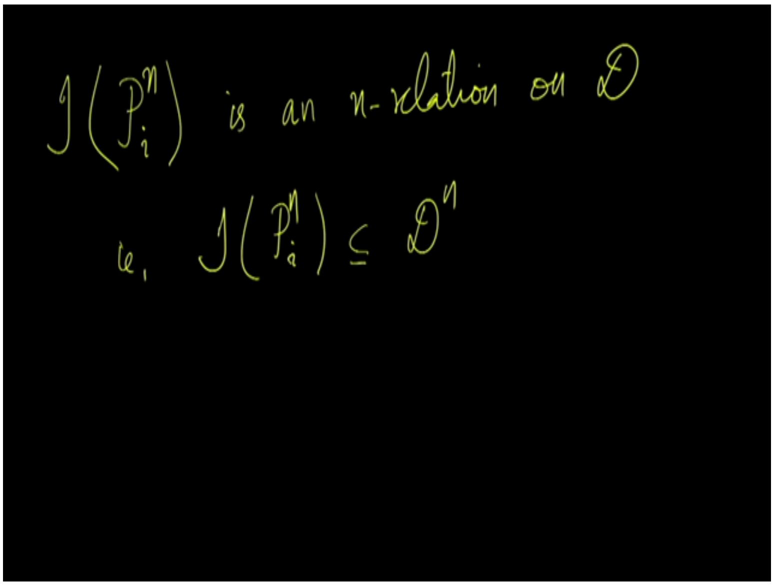

In particular, an n-ary predicate symbol will map to and n-ary relation on D that is the meaning of an n-ary predicate symbol is going to be a subset of D parent.

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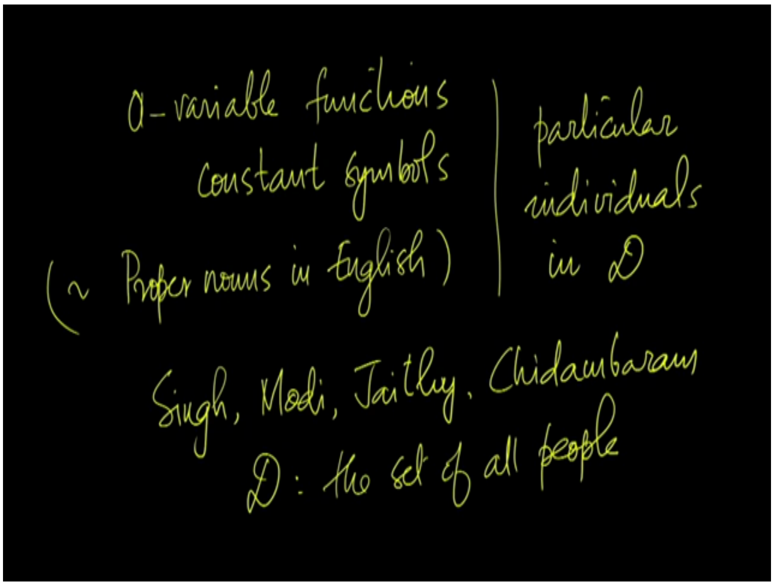

So, let us take an example, first of all we consider 1 variable functions or let us begin with 0 variable functions, these are also called constant symbols, these are akin to proper nouns in English, these are supposed to refer to particular individuals in the domain of discourse. Proper nouns like, Singh, Modi, Jaitley Chidambaram etcetera are proper nouns, they refer to individuals when the domain of discourses the set of all people.

So, when we have a first order system that is talking about the set of all people we would be referring to particular individuals using their proper nouns. So, these are all proper nouns using 0 variable functions which are constant symbols we refer to particular individuals belonging to the set D.

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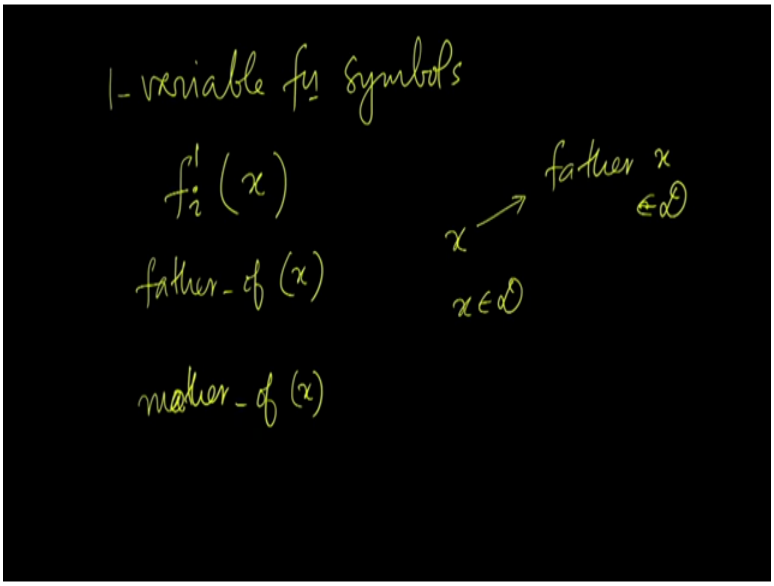

Whereas 1-variable function symbols of this form take an argument x and then map this argument to an individual belonging to D. For example, when we say father of x we take argument x and then use the father of function to map x to the father of individual x in the domain of discourse. So, x belongs to the domain of discourse and the father of x also belongs to the domain of discourse, another 1-variable function could be the mother of x. So, this is these variable functions map individuals to individuals. So, 1-variable function symbols are mapped to such functions.

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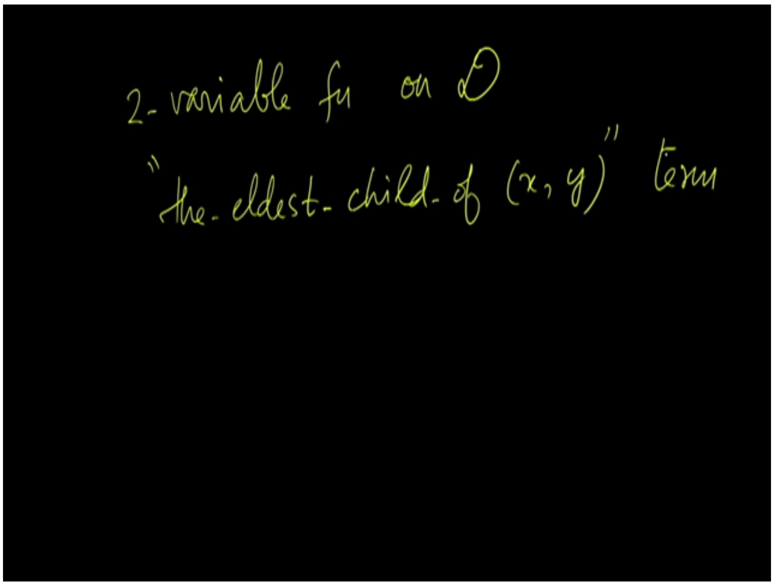

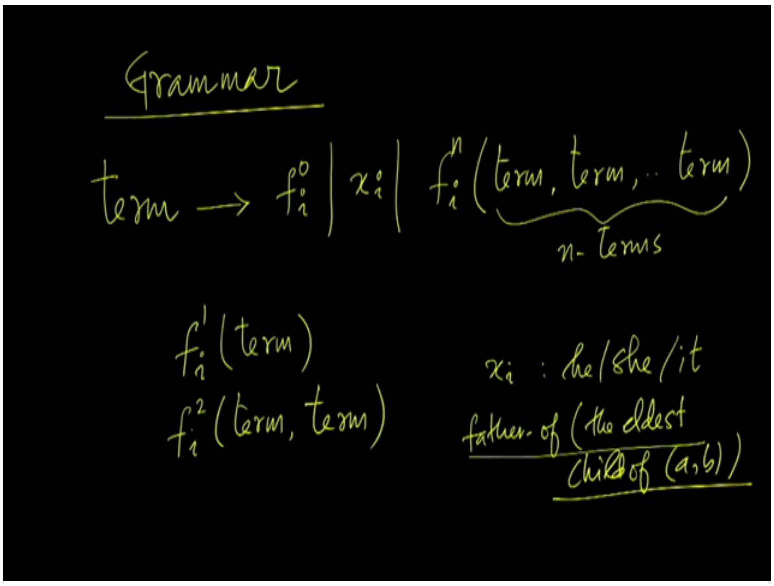

Then what could be a 2-variable function on D? 2-variable function symbols would be mapped to such functions. A 2-variable function on D where D is the set of all people could be the eldest child of x and y, you substitute appropriate individuals for x and y you get the eldest child of x and y as the meaning of this expression, so this is a term. So, if you look at the definition of a term that we had before, we find that a term as obtained in this manner.

A term could be a constant symbol that is it could be an individual specified using his or her proper noun or a term could be variable or a variable is rather like a pronoun in English he or she or it. So, an individual can also be referred using a pronoun. So, a term could be a proper noun or a pronoun or it could be a function symbol applied on a number of terms. For example, when you say father of the eldest child of a and b where a and b are proper nouns then you understand what is the meaning of this expression. So, you can construct terms in this fashion using function symbols and smaller terms.

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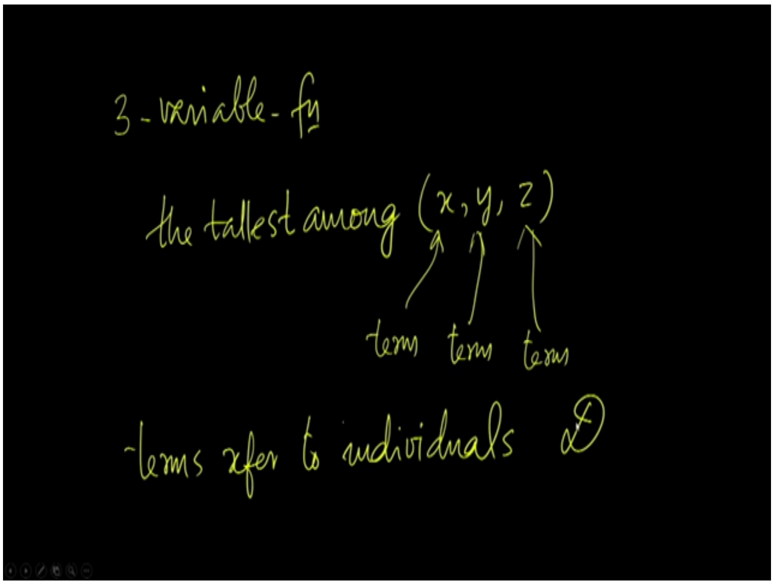

An example of a 3-variable function, again when D is the set of all people could be the tallest among x, y and z you could substitute appropriate terms for x, y and z to get terms out of this function. So, every term you can see is supposed to refer to individuals.

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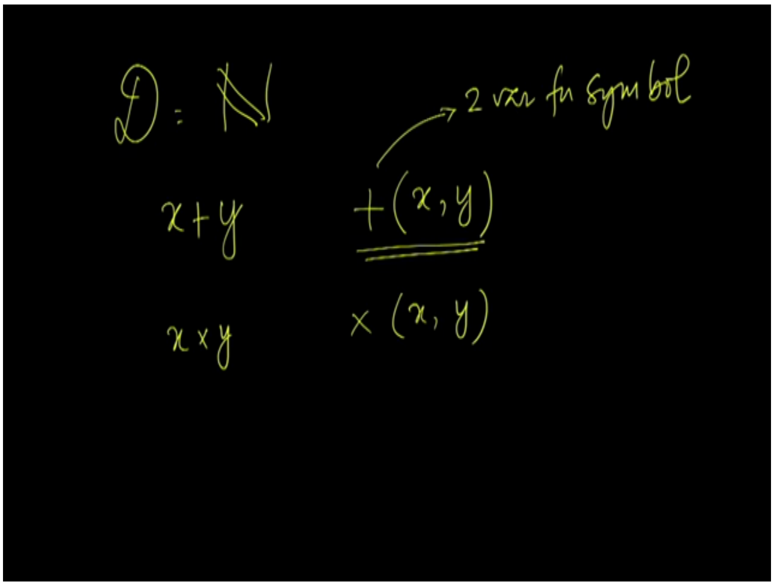

To take another example, if the domain of discourse where the set of natural numbers then x plus y is the function symbol plus applied on x and y only that the function here is applied in the in fixed format, you could use it in the usual prefix format like this plus applied on x and y this is this represents the sum of x and y. so, this is now a 2-variable function symbol and that is mapped to the addition function. Similarly, x into y written in prefix form would look like this, multiplication applied on x and y this is again a 2-variable function symbol. So that is how we construct terms out of smaller terms.

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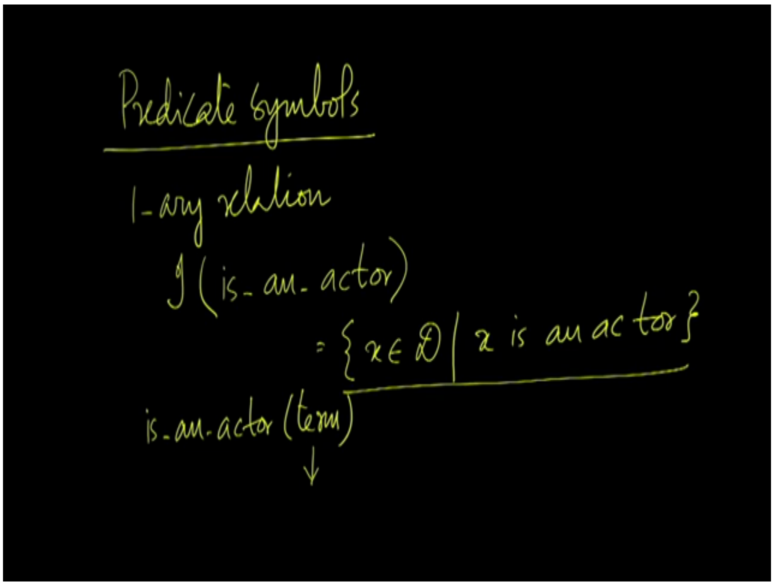

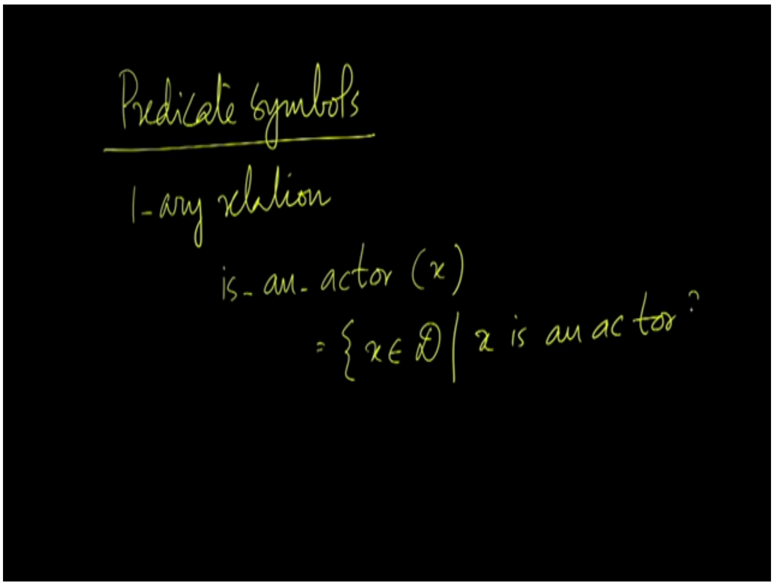

And then, predicate symbols, predicate symbols are mapped to relations, so what would be an example of a 1-ary relation? For example, is an actor is a 1-ary relation, so this is the set of all x belonging to D such that x is an actor that is the interpretation I maps the function symbol is an actor to this relation.

So, when we write is an actor being applied to a term to evaluate the truth value of this you have to first evaluate this term and then you have to check whether that individual the individual referred to by this term is indeed an actor that is it indeed belongs to this set. If that is the case then this predicate is true otherwise this predicate is false.

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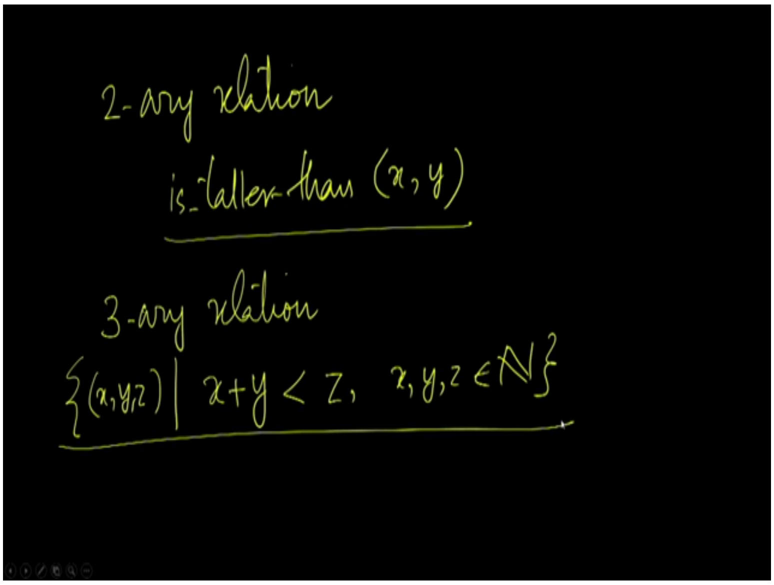

What would be an example of a 2-ary relation? On the set of individuals, you could say it is taller than you would want to say that x is taller than y, this is a 2-ary relation. A 3-ary relation on the set of natural numbers would be the set of all triplets, could be like this. So, we have n-ary relations that are mapped to predicate symbols with n variables.

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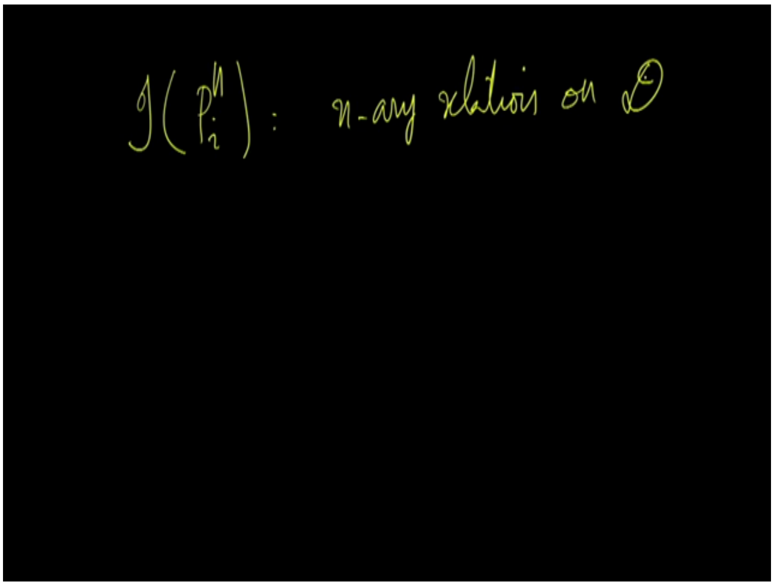

The meaning of a predicate symbol with n variables would be an n-ary relation only. So, that is how we define the semantics of the symbols.

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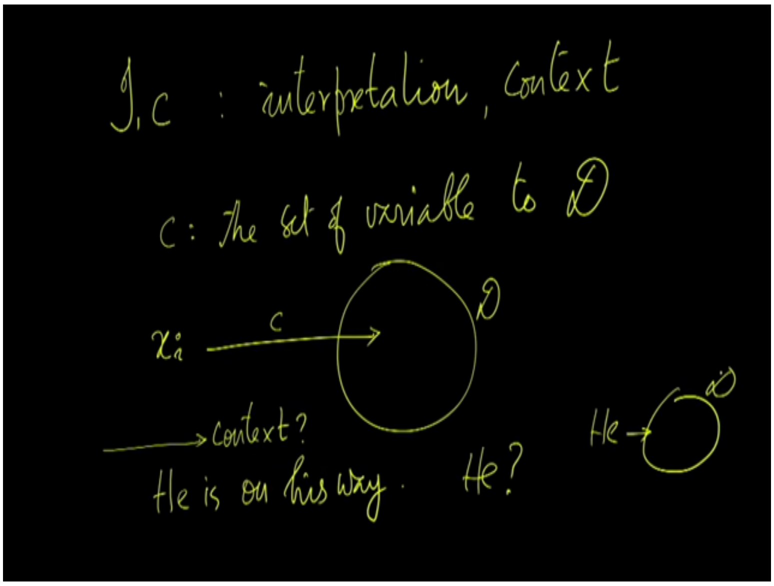

Now, we consider what is called a context? We will consider the context along with an interpretation. We call this an interpretation context pair, what the context does is to map this set of variables to the domain of discourse. So, for each variable x i the context specifies a member of the set D to take an analogy with English in a sentence when we say he is on his way, what does he refer to?

To understand who the person being referred to here is we will have to look at the previous sentence or we will have to understand the context. If you look at the context you will understand who this pronoun is referring to, to understand the meaning of a pronoun you have to look at the context. In the context, this pronoun will be mapped to a particular individual. So, he is mapped to some member of the domain of discourse which in this case is the set of all people.

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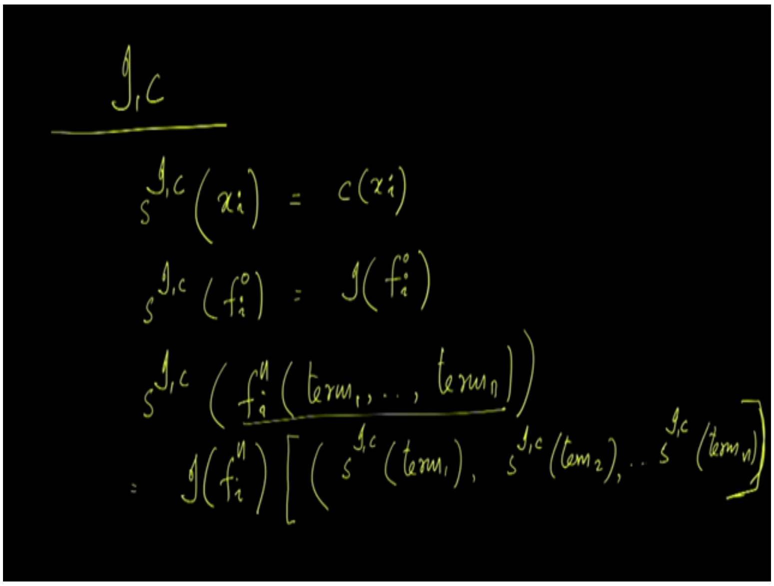

So, once we have given an interpretation and a context we can talk about the meaning of various formulae and terms. First, let us consider the meaning of terms, the meaning of variable x i in an interpretation and context is defined solely by the context that is as I mentioned the meaning of a pronoun will be given by the context in which the use of the pronoun occurs whereas the semantics of 0 variable function symbol would be specified by the interpretation itself.

The interpretation would say which individual each proper noun refers to? For example, we consider an n-ary function being applied to n terms, the semantics of such a term is defined inductively. First, we apply the interpretation on the n-ary function symbol which will give us an n-ary function this is applied on the n tuple of individuals that we obtain by applying this same function the same semantic function s I, c on these terms.

So, s I c term 1 will give us an individual, the individual who is referred to by term 1, s I c term 2 will give us another individual the individual referred to by term 2 and so on. We collect all these individuals from an n tuple of them and on this n tuple we apply the n variable function which is the mapping of F n i that would give us an individual, that individual is the person who is referred to by F n i on term 1 through term n. So, this is how we would synthesize the meaning of a term.

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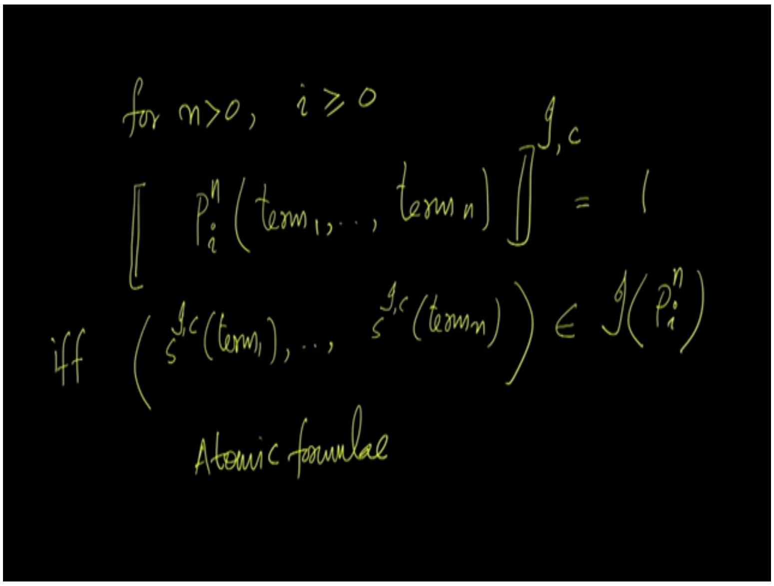

Now, coming to the semantics of formulae for n greater than 0 and i greater than or equal to 0 when we consider an atomic formula of the form P n i applied on term 1 through term n, we would say that this is 1 precisely when the n tuple obtained by applying the meaning function on these terms, this n tuple belongs to the n-ary relation which is mapped to the predicate symbol P n I, if this is the case we would say yes for P n i on term 1 through term n otherwise we will say no. So, this is how we define the meaning of all atomic formulae.

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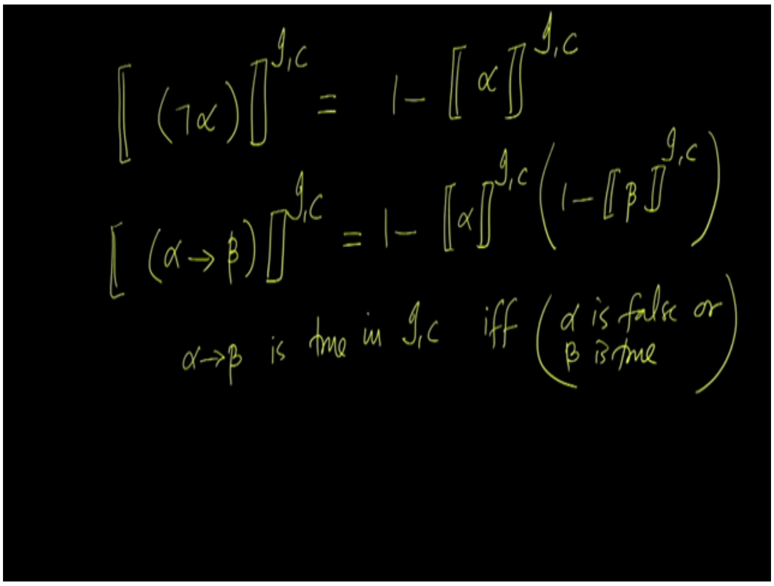

Now, let us consider a general formula. A general formula could be a negation of another formula this exactly as in the case of propositional calculus will be 1 minus then semantics of alpha in the context of high I n c. The semantics of alpha implies beta with reference to the interpretation context pair again would be as in propositional calculus 1 minus the meaning of alpha multiplied by 1 minus the meaning of beta, the truth value of beta. So, you can see that alpha implies beta is true in the interpretation the context pair if and only if alpha is false or beta is true exactly as it should be.

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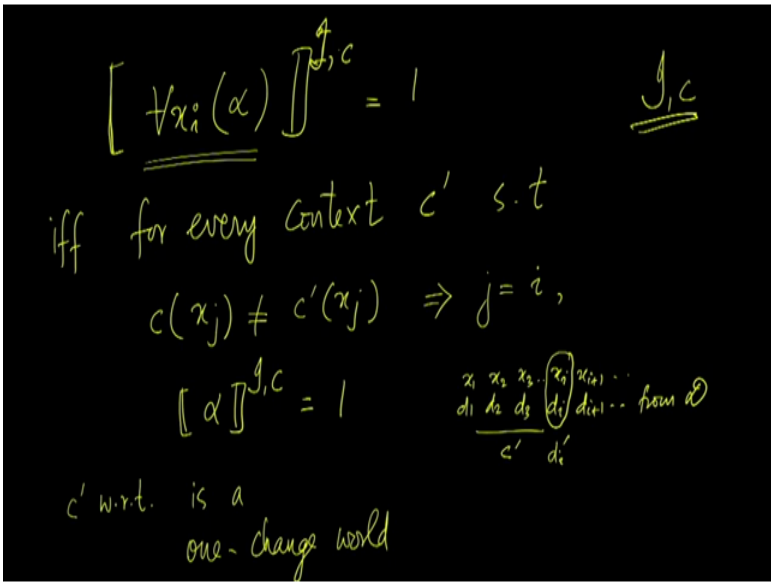

Now, coming to the last rule of formula synthesis, what would be the meaning of for (all i) for all x i alpha? We say that this is 1 if and only if for every context c prime such that on x j c and c prime disagree only if j equal to i. For this alpha i c has to be 1, so what does it say? We are now standing in context c, in context c variables are mapped in various forms, the variables x 1, x 2, x 3 etcetera x I, x i plus 1 etcetera are mapped to individuals d 1, d 2, d 3 etcetera from D.

So, that is what the context does, it maps an individual to each variable. Now, we are looking at a context c prime which is identical to the context c except for variable x I, what c prime should do is to map every variable except x i to the same individuals as c does but x i could be mapped to a different individual d i prime c i prime could map x i to a different individual d i prime.

So, c prime with respect to c we say is a one change world, the worldview of c prime is almost identical to that of c except that there is one change the pronoun x i refers to a different individual possibly that is it might refer to d i prime instead of d i which is where it is being referred to in c. so, c prime is a one change world, so what do we say here? What we say is that, for every one change world of c the statement alpha must be true which means when you stand in context c with respect to the interpretation I you should be able to say that irrespective of the meaning of variable x i alpha must be true that is precisely what we try to say here.

What we say is that alpha is true for every individual x i but in this context x i has a certain meaning but what we want to say is that even if the meaning of x i changes so as long as everything else remains the same alpha will still be true in whichever way x i changes alpha will still be true. So, in all one change worlds imaginable when you are standing in context c where only the mapping of x i changes alpha will still be true in all those contexts.

If that is the case then this will be 1 otherwise it will be 0 this is precisely our understanding of the universal quantification.

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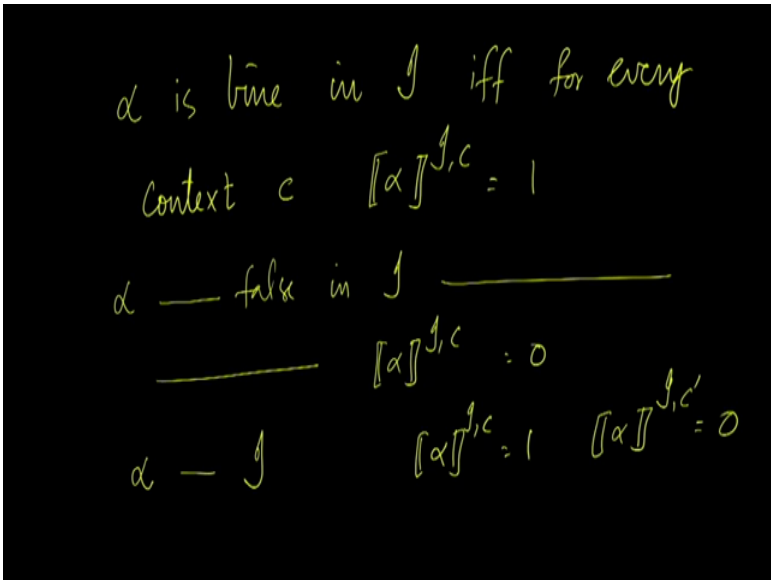

We say that alpha is true in an interpretation I if and only if for every context c alpha of I c is 1 that is irrespective of the context c alpha is true in that case we said that alpha is true and I. Analogously we can say that alpha is false in I if for every context c alpha of I c is 0 but of course you can see that for a formula alpha and an interpretation I alpha might neither be true in I nor be false in I, how is that possible?

It could be that alpha of I c is 1 for some context c but for the same in reputation if you take another context alpha might be 0 in which case alpha is neither true in this interpretation nor false in this interpretation.

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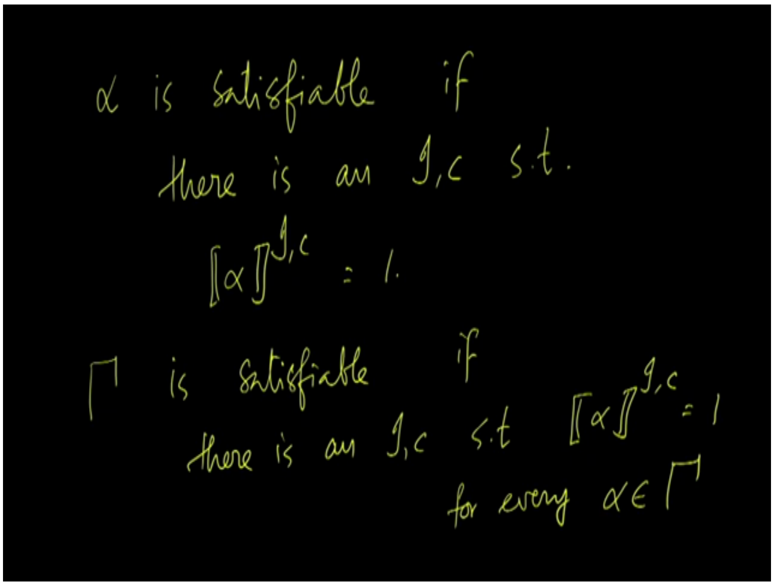

We say that alpha is satisfiable if there exist an interpretation context pair such that alpha is true in that interpretation context plant and generalizing on this notion we said that a set of formulae gamma is satisfiable if there is an interpretation context pair such that every formula alpha in gamma is satisfied by this interpretation context pair, this must be true for every alpha in gamma in that case we said that gamma is satisfiable.

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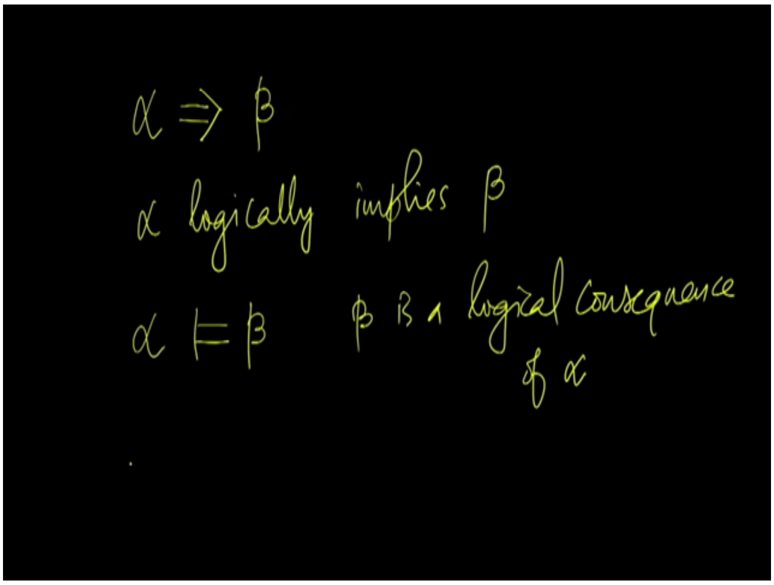

Analogous to what we did in propositional calculus we can now define the notion of a logical consequence. We said that alpha logically implies beta if every interpretation context pair which makes alpha true will also make beta true. In this case, we will also say that beta is a logical consequence of alpha.

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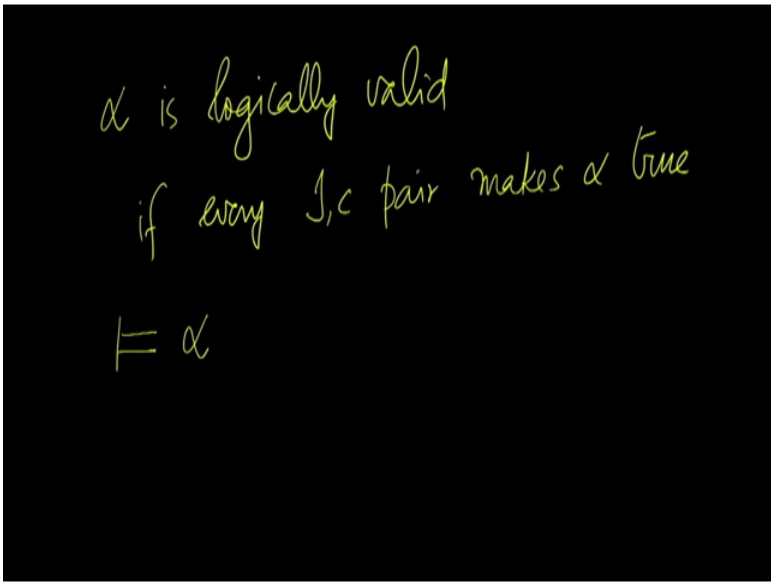

In particular, we said that alpha is logically valid if every interpretation context pair makes alpha true, alpha has to be true every interpretation context pair that is when alpha is logically valid this is analogous to the notion of a tautology in propositional calculus. We write like this to indicate that alpha is logically valid.

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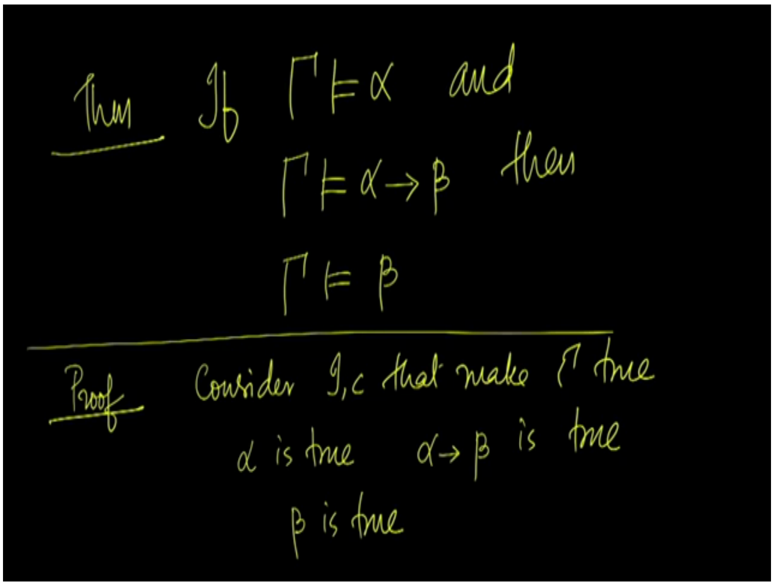

Again analogous to what we did in propositional calculus we can prove this theorem. If alpha is a logical consequence of gamma and alpha implies beta is a logical consequence of gamma then beta is a logical consequence of gamma. The proof is quite similar, if alpha is a logical consequence of gamma then there is an interpretation context pair, in any interpretation context pair which makes every formula of gamma true alpha is also true.

So, consider some interpretation context pair in which the whole of gamma is true, every formula in gamma is true in the interpretation context pair I c. So, in this interpretation context pair alpha is true because alpha is a logical consequence of gamma, alpha implies beta is true because that is also a logical consequence of gamma then necessarily beta has to be true because if beta were false then we would have that alpha is true and beta is false in which case alpha implies beta would have to be false therefore, beta is necessarily true.

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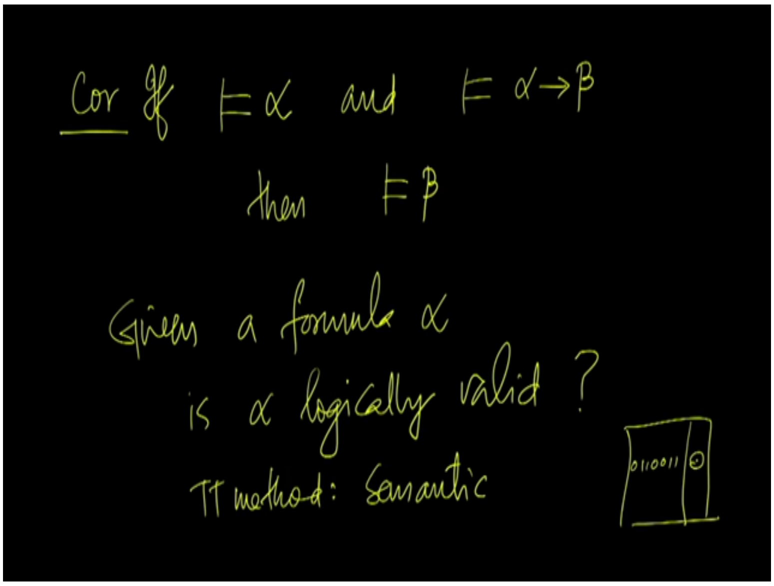

As a corollary to this, we can argue that if alpha is a tautology and for alpha is logically valid and alpha implies beta is also logically valid then beta is logically valid. So, now analogous to the case of propositional calculus we can post these questions, given a formula alpha is alpha logically valid.

In the case of propositional calculus, we could drop the truth table for alpha and check whether alpha evaluated to 1 in every single assignment but we cannot do that here for to show that alpha is logically valid we have to look at every possible interpretation context pair for the system but there could be an infinite number of such interpretation in context pairs. So, we do not have analogous semantic procedure here, the truth table method is a semantic method because it handles entirely the semantic entities that is the truth values.

In a truth table, what we do is to consider every possible assignment, the assignment is a set of truth values and then the function is evaluated for this particular assignment. So, we deal entirely with semantic entities. An analogous semantic method is not available for first order logic that is because the space that we are looking at is infinite.

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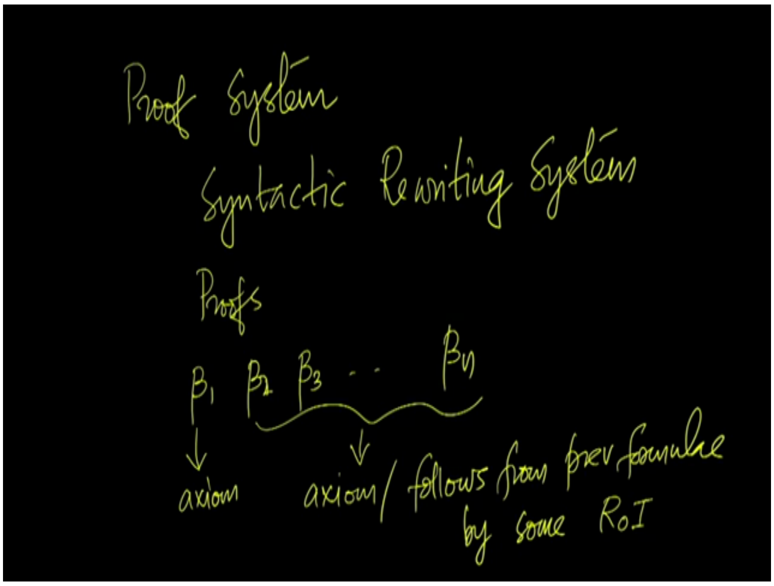

Therefore, we require a proof system. A proof system is a syntactic rewriting system and we have proofs in the system. The proofs are exactly analogous to what we saw in propositional calculus we have a sequence of formulae, beta 1 through beta n where beta 1 is an axiom every subsequent beta is either an axiom or follows from previous formulae by some rule of inference such a sequence of formula is called a proof.

What we want is this, for every statement which is logically valid, we should be able to start from a set of logical axioms and culminate in this formula through a proof. So, the proof is witness to the fact that the statement is logically valid. We would be very happy if every logically valid statement is provable in this fashion and everything that is provable is logically valid that is our system is both sound and complete.

The system is sound if everything that the system proves is logically valid and the system is complete if the system is capable of proving everything that is logically valid.

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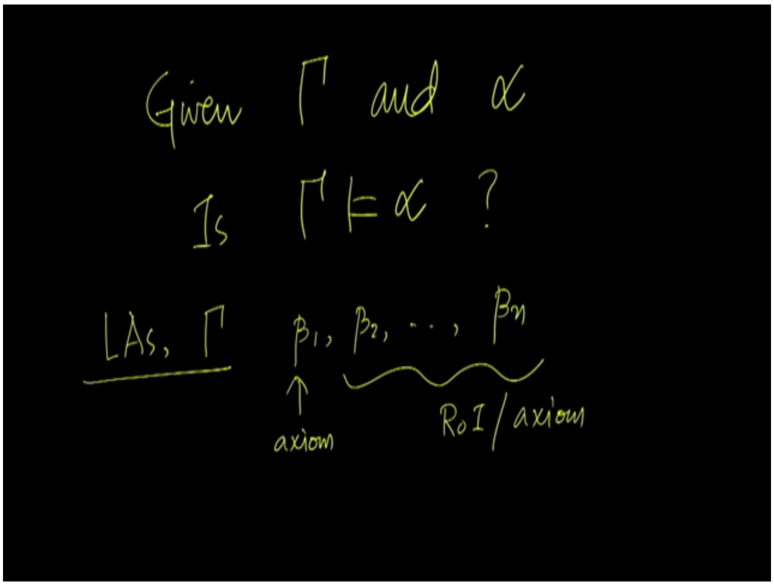

We would also ask questions of this form, given a set of formulae gamma and a formula alpha is it the case that alpha is a logical consequence of gamma, here also we would require a proof system. The proof system would help us in answering this question, what we do is this we take a set of logical axioms and the set gamma as a set of proper axioms and then from this we write a proof exactly as before only that now formulae in gamma could also be used as axioms.

So, beta 1 is an axiom necessarily an axiom and every subsequent formula is either an axiom or follows from two previous formulae or some previous formulae by some rule of inference. So, you obtain these formulae either by a role of inference or by invoking an axiom.

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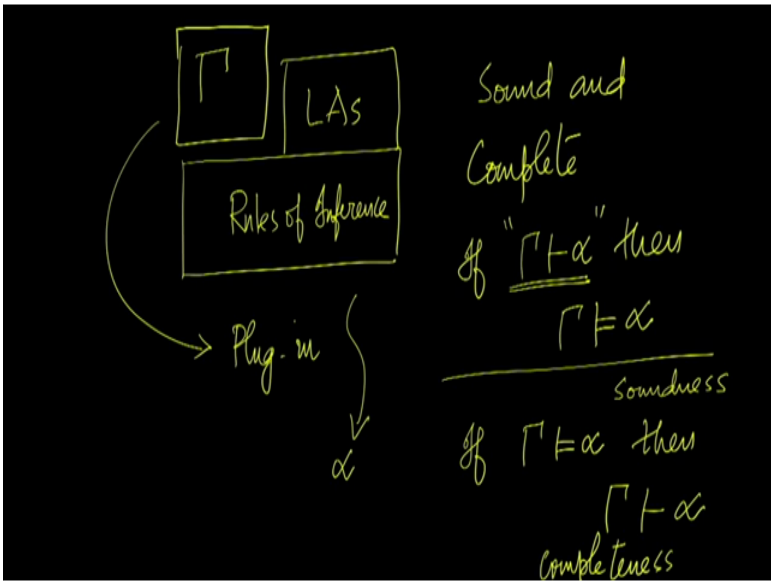

So, the proof system that we have in mind in fact has these components, it has a logical it has a set of logical axioms and it has the rules of inference and it has a set of proper axioms. The set of proper axioms is rather like a plug-in, you change the set of proper axioms you will have a different proof system and you will have different consequences what we want is that this proof system is both sound and complete.

We would be able to say that this proof system of sound and complete if alpha is proved in this proof system then alpha is a logical consequence of gamma when I write like this what I mean is that alpha is provable from gamma that is when I use gamma as the plug in here, gamma as the set of proper axioms here then I will be able to derive alpha from this proof system that is the assertion alpha is provable from gamma means, there exist a proof of alpha starting from the proper axioms set gamma.

So, this asserts that if alpha is provable from gamma then alpha is a logical consequence. So, this is an assertion of soundness, it says that whatever we prove is sound. The converse of this says that if alpha is a logical consequence of gamma then alpha is provable, this is the statement of completeness of the proof system this is what we would desire of the proof system we would want the proof system to be both sound and complete.

In which case we would be able to reach conclusions that are logical consequences without dealing with semantic entities since the semantic entities form an infinite space in any case we cannot have an algorithm which is analogous to the truth table method in the case of propositional calculus. Therefore, we do need a syntactic method and the pouf system will function as a syntactic method if this is the case.

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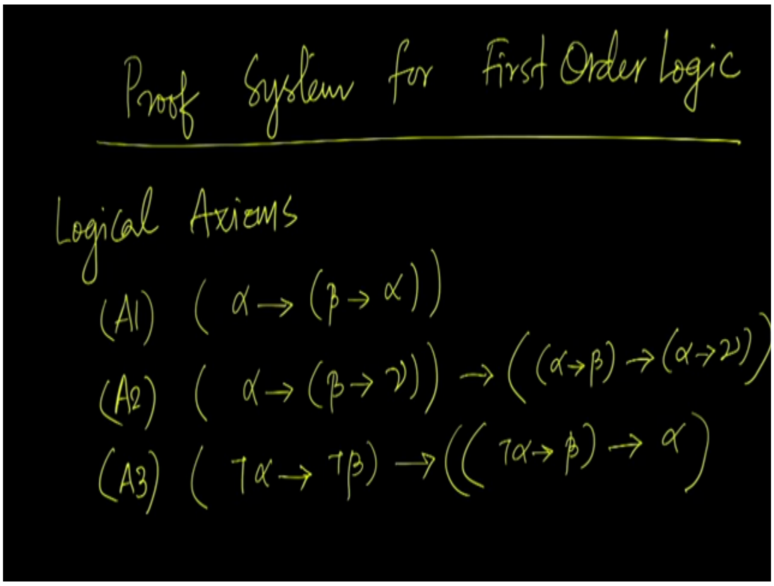

Now, let us see one such proof system, proof system for first order logic. First, let me see a set of logical axioms, so we have three logical axioms schemas exactly as in the case of propositional calculus. So, the first three logical axiom schemas are identical to that of propositional calculus that is why I said the first order logic system is an extension of the propositional calculus system.

So, the second axiom schema says that alpha implies beta implies gamma; implies alpha implies beta implies alpha implies gamma and the third axiom schema says that if not alpha implies not beta then not alpha implies beta implies alpha. In other words, if negation of alpha implies both beta and not beta which will be an inconsistency in which case alpha must be true. So, these three axioms are analogues to are exactly the same as in the case of propositional calculus proof system.

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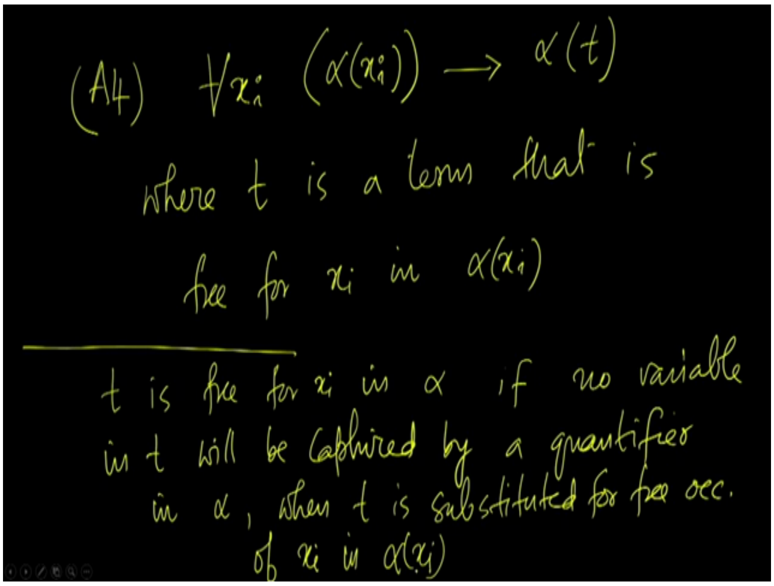

Now, we have some axioms which are different. The fourth axiom says that for all x i alpha x i that is alpha is true for every x i is what this assertion says, if this is the case then alpha is true for an individual referred to by term t where t is a term that is free for x i in alpha x i, what it means is that when term t is substituted for every free occurrence of x i within alpha of x i then none of those substitutions should have a variable that is caught by a quantification here.

So, let us take an example for this. So, first we said that t is free for x i in alpha if no variable in t we will be captured by a quantifier in alpha when t is substituted for free occurrences of x i in alpha of x i.

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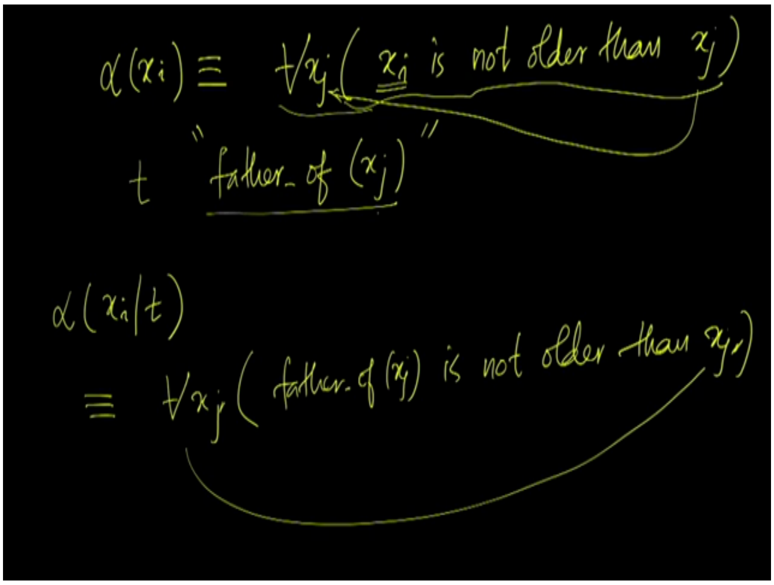

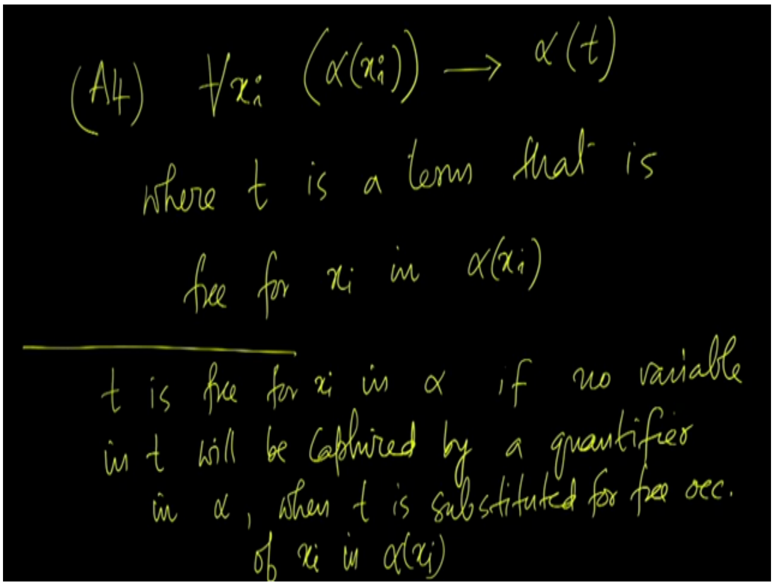

As an example consider this, suppose alpha of x i is this formula for all x j; x i is not older than x t, so this formula asserts that x i is not older than everybody, so in that sense x i is at least as young as anybody in the group that is what the statement asserts. Now, let us consider a term t which is father of x j when this term is substituted for every free occurrence of this formula, every free occurrence of x i in this formula so in this formula there is a bound occurrence of x j, this is the bound occurrence of x j it is bound to this quantifier but this occurrence of x i is free.

So, we are planning to substitute t for every free occurrence of x i in this formula, we could write it this way t being substituted for every free occurrence of x i would give us this formula for all x j father of x j is not older than x j which is a funny statement in the normal interpretation of this world but this was certainly not what was intended. So, if you were to substitute this term in a formula we should make sure that no quantifier existing in that formula will capture the free variable here.

So, this substitution is happening in some contexts in which x j has some meaning, x j is mapped to some individual but then that is defined by the context and therefore after the substitution also we should let the context define the meaning of this particular x j but when that substitution happens here we find that this free occurrence of x j is being caught by the already existing quantifier there.

So, to avoid this pitfall what we should do is to change the variable name? So, this bound variable x j can be changed to x j prime. So, once you do this then this association is not made, so what does it say now? It says that father of x j is not older than everybody. So, we might be considering a pool of others and what we are asserting is their father of x j is possibly the youngest in this group is at least as young as anybody in this group that meaning makes sense but then this requires a change of variable.

So, what the axiom that we have seen asserts that if alpha is true for everybody then alpha must be true for individuals too any particular individuals too. So, this allows for particularizations, when you have a statement which is universally quantified then you will be able to particularise the statement for a certain individual drawn from the domain of discourse.

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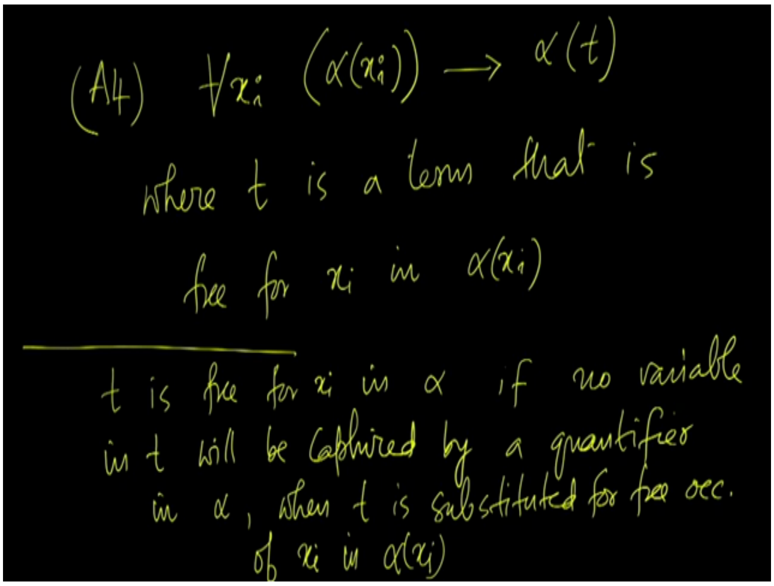

Now, we have two more axioms, axiom A 5 is a formula that we have seen before for all x i alpha implies beta implies for all x i alpha implies for all x i beta. In an earlier class we showed that this is a logically valid formula, so this is our axiom A 5. Then axiom A 6 says alpha implies for all x i alpha if x i is not free in alpha which allows for generalizations of formulae.

So, these axioms A 1 through A 6 are axiom schemas they use variables alpha, beta and gamma that stand for any well form formulae. So, if you substitute appropriate well form formulae alpha, beta and gamma then we would have instances of these axioms. So, we have a accountably infinite number of axioms obtainable in this manner.

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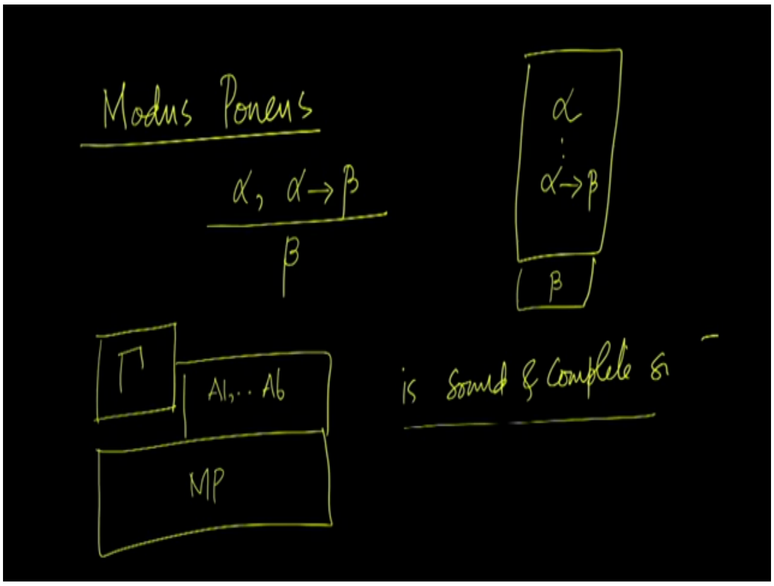

And we have one rule of inference exactly as in the case of the proof system for propositional calculus here also we have only one rule of inference namely alpha and alpha implies beta together allowing us to write beta in the proof. So, if you have alpha in the proof and alpha implies beta in the proof then you will be justified in writing beta as the next step in the proof, so this rewriting rule is called modus ponens.

So, we are considering a proof system in which the logical axioms are the axioms derivable using the templates A 1 through A 6, modus ponens is the only rewriting rule and then we have a set of proper axioms gamma from one proof system to another gamma could be different but the other components will all remain the same.

Now, we would like to assert that this is a sound incomplete system that is for any plugin gamma here what is provable from this system happens to be the set of logical consequences of gamma and every such logical consequences indeed provable in the system which would indeed be a nice property. So, this is what we would like to establish, okay that is it from this lecture, hope to see you in the next you.