Discrete Mathematics
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Lecture 4
Mathematical Logic

Welcome to the NPTEL MOOC on discrete mathematics, this is the fourth lecture on Mathematical Logic. In the last lecture, we started a discussion on 1st order logic we continue with this discussion; we saw what is a free variable and a bound variable in the last class.

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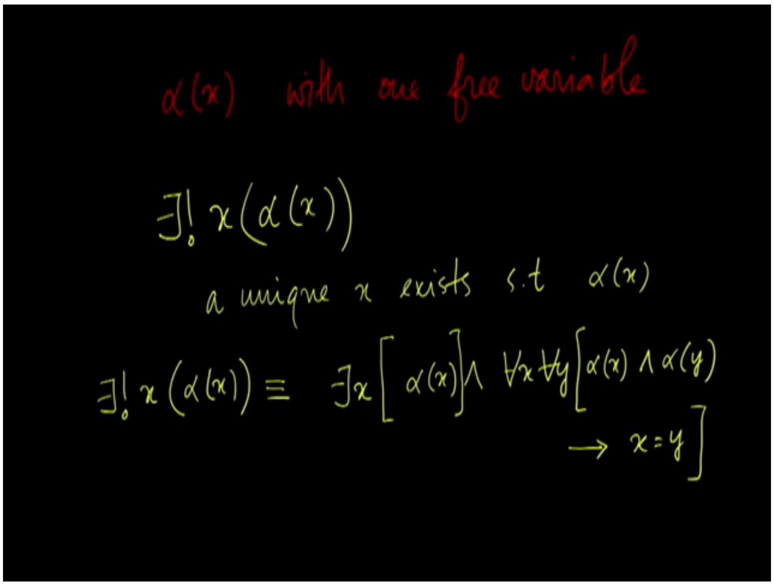

Let us consider a formula alpha of x with one free variable. So, x is the free variable here, in that case this notation says that there is a unique x, a unique x exists such that alpha of x. in other words, this formula is equivalent to saying that there exists an x such that alpha of x is true and for all x for all y alpha of x and alpha of y is true implies that x is equal to y, what does it say?

It says that, there is an x so that alpha of x is true and in addition to that for every x and y if alpha is true for x as well as for y then x must be equal to y, that is they are cannot exist two distinct x and y so that alpha holds for both of them. Though, so this is exactly analogues to saying that there is a unique x such that alpha of x.

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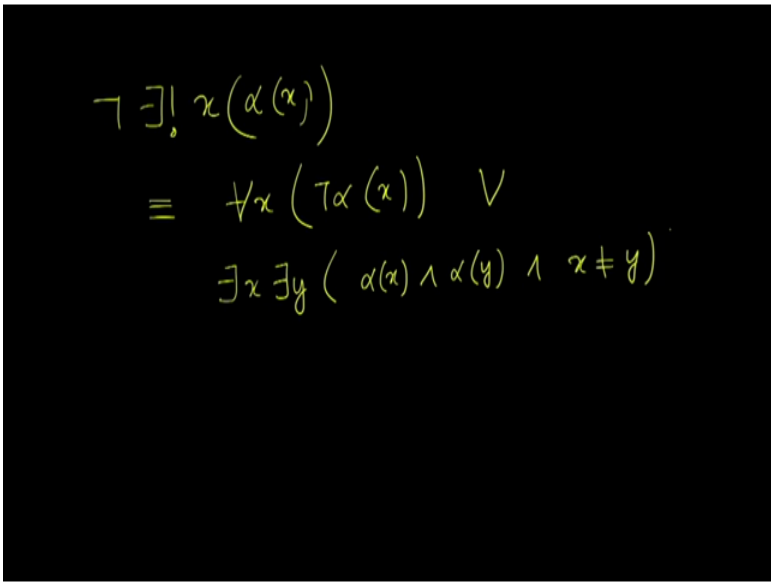

Let us consider the negation of this, we want to say that there does not exist a unique x so that alpha of x is true, this would be equivalent to if you apply De Morgan’s law repeatedly to the above formula it would be a disjunction in which the first term would be this for every x alpha of x is false, so this is one way of negating the assertion that there is a unique x which satisfies alpha of x.

Here what do we say? We say that, alpha is not satisfied by any x so this is one possibility. The other possibility is that, there exists x and y such that alpha of x is true for both x and y and x is not equal to y, so this is the other way of contradicting the statement that there is a unique x so that alpha of x is satisfied. So, alpha x is satisfied by some individuals in the domain but there are multiple individuals that satisfies alpha of x. So, the statement that there is a unique x so that alpha of x is true is not right.

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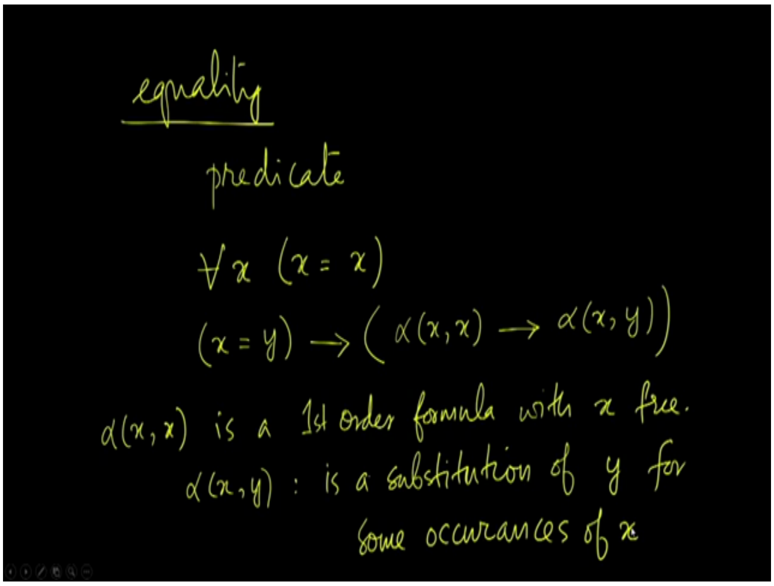

In many 1^{st} order systems, we have a special road for the equality predicate. So, equality is a predicate which satisfies some conditions, the conditions are this for all x we can assert that x is equal to x for every individual that individual is equal to itself. In addition to that, if x is equal to y then alpha of x, x implies alpha of x, y here alpha of x, x is a 1^{st} order formula with some occurrences of the free variable x. Alpha of x, y is a substitution of y for some occurrences of x.

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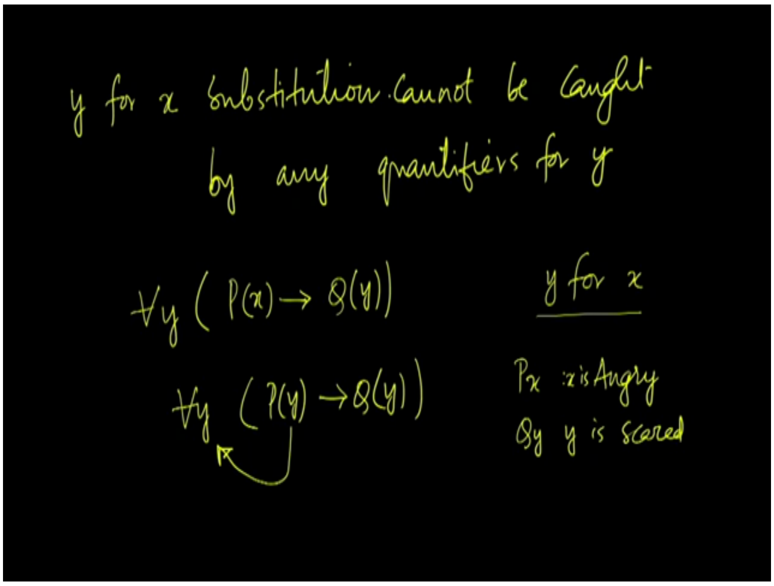

The only stipulation is that when you substitute here, when you substitute y for x this substitution should not be caught by any existing quantifiers for y because if such a caching happens then the intended meaning would change. For example, let us say we have a formula of this form, for all y; P of x implies Q of y and let us say we want to substitute y for x in this, x is a free variable here but if you substitute variable y for x here we would get the formula for all y; P of y implies Q of y but then this substitution is now caught by this quantifier which can change the intended meaning of the formula.

So, what does the 1st formula say? The 1st formula says that if P is true for x then for every y Q is true that is what it effectively says. In other words, if I take P of x as x is Angry and Q of y as y Scared then what is the 1st formula say? The 1st formula says that for every y if x is angry then y is scared, if x happens to be a despotic dictator then it could be true, if the dictator was angry then everybody is scared but then what does the second statement say?

With the quantifier for y catching the substituted occurrence of y, we have a statement like for every y, if y is angry then y scared which means? Any angry person is scared which is not true at all and that is not the indented substitution. Therefore, while substituting for one variable you have to be careful, the substitution cannot be caught by any existing quantifier.

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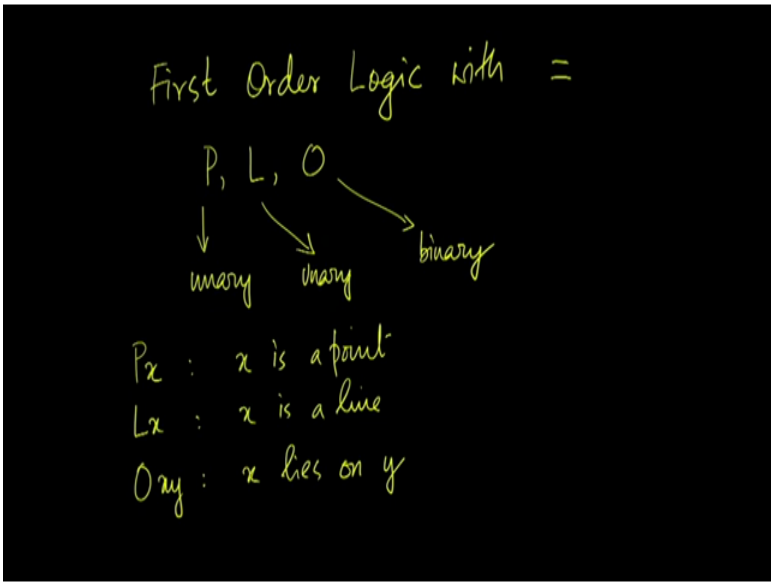

Let us take an example, which would explain the notion of logical consequences. So, we consider 1st order logic with equality, so equality predicate is available within the logic with the properties we mentioned. Let us say we have three predicate symbols P, L and P, P is a unary predicate, L is also a unary predicate and O is a binary predicate. We indent P of x to stand for x is a point, L of x stands for x is a line, O of x, y stands for x lies on y.

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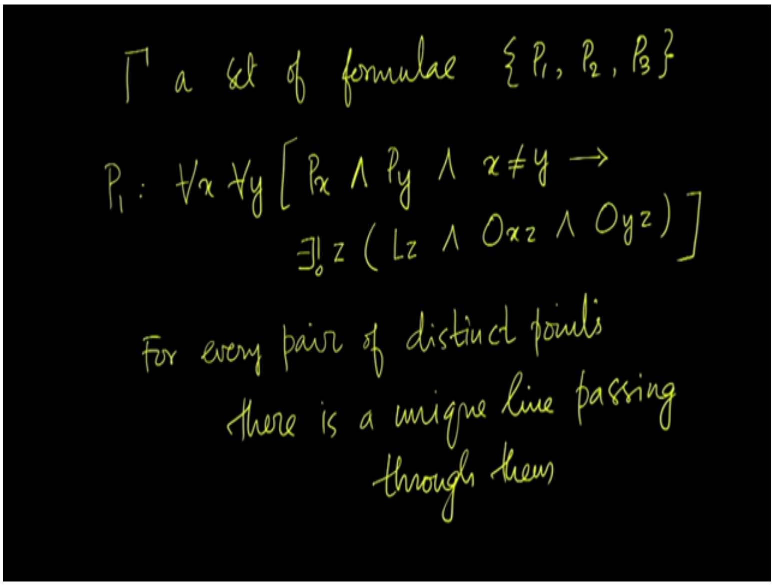

So, with these predicate symbols let us write a few formulae and see what the meaning is, by gamma we define a set of formulae, a set of three formulae P 1, P 2 and P 3. So, let us now see what these formulae are, P 1 is the formula which says that for all x for all y; x is a point y is a point and x not equal to y implies that there exists a unique z so that z is a line and x lies on z and y lies on z.

So, this is the formula that we denote as P 1, what does this formula say? For every pair of points x and y that are not the same as each other in other words for any pair of distinct points there exists a unique z that is a line so that x lies on z and y lies on z. In other words, for every pair of distinct points, there is a unique line passing through them which we all know is true in Euclidean geometry, for any pair of distinct points there is a unique line passing through them, so this is the formula that we call P 1.

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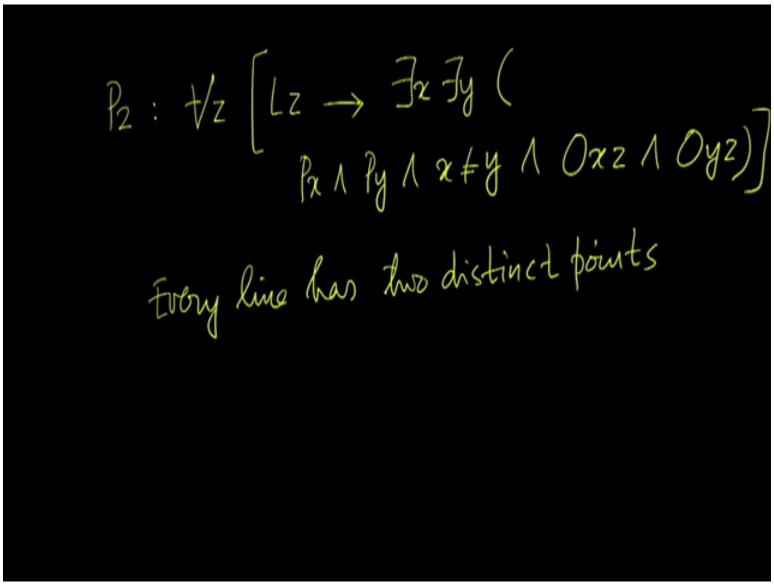

Now, let us see P 2, P 2 asserts that for every z, if z is a line then there exists x and y so that x is a point, y is a point, x is not equal to y and x lies on z and y lies on z. so, what does it say? For every line z there is a pair of points x and y so that x is not the same as y which means there exists a pair of distinct points so that x lies on z and y lies on z.

Or in other words, in every line there are at least two distinct points, every line has two distinct points this is again true in Euclidean geometry we have familiar with the fact that every line has an infinite number of points. So, here the assertion is only that every line should have two distinct points at least that is what assertion P 2 is.

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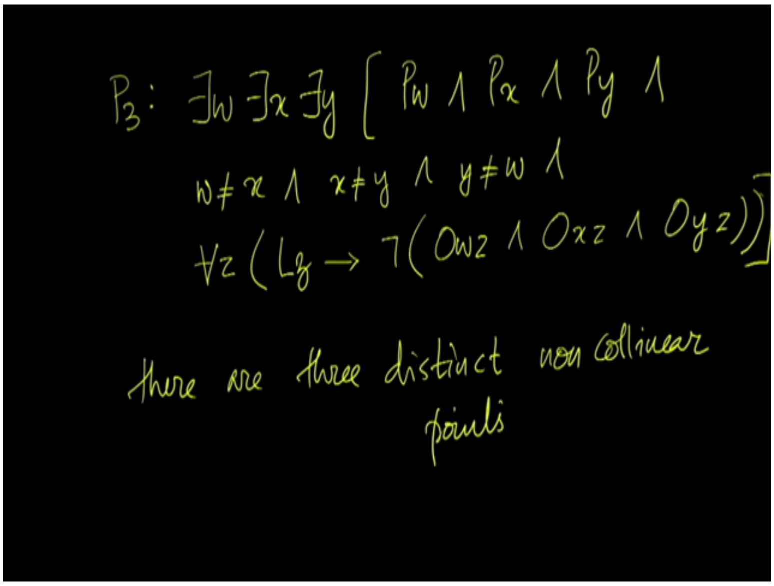

Now, coming to assertion P 3, what P 3 says? Is this, there exist is w, there exist x and there exist y so that w is a point, x is a point, y is a point so there are three points w, x and y. So, that w is not equal to x and x is not equal to y and y is not equal to w which means? The three points should be distinct. So, there exist three distinct points w, x and y so that for every z if z is a line then it is not the case that w lies on z or x lies on z or y lies on z.

Or in other words, there are three distinct non collinear points, in other words there exist three points w, x and y so that all three do not lie on the same line. So, check once again whether this is what the formula says, there exist 3 points w, x and y all three are points and all three are distinct so that for every z if z is a line then all three cannot be lying on z where is w lies on z, x lies on z and y lies on z all three cannot be true simultaneously. So, there is at least one triplet of points that do not lie on any line together.

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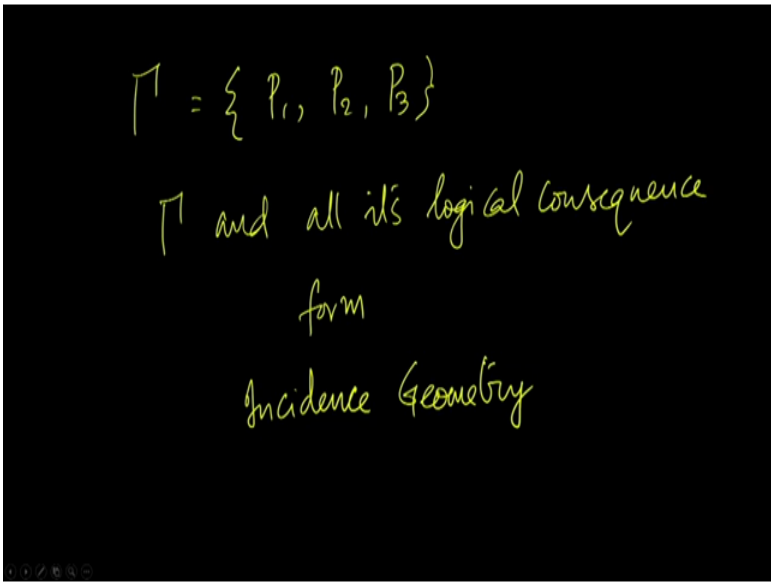

So, we have these three statements, the statements P 1, P 2 and P 3 together form the set gamma, we say that gamma and all it is logical consequences form Incidence Geometry.

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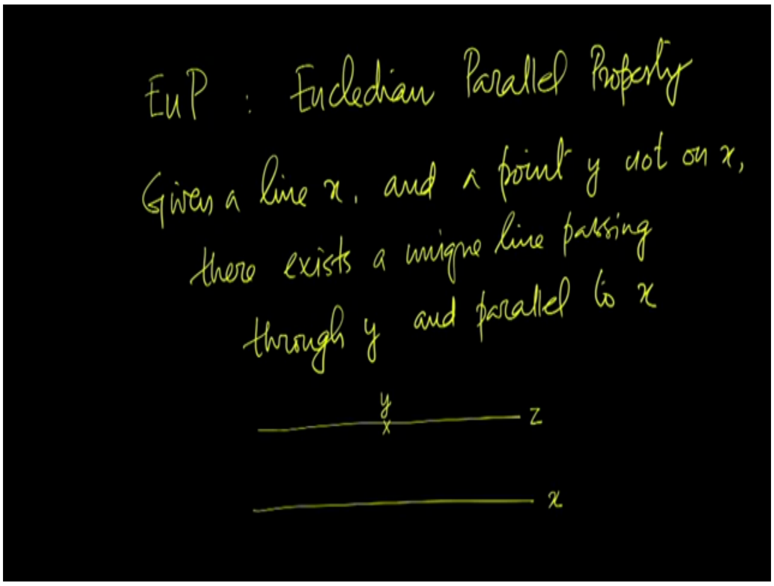

Now, the question we want to ask us this, we consider a fourth formula which we called Euclidean parallel property, what Euclidean parallel property? It says as this, given a line x and a point y not on x there exists a unique line passing through y and parallel to x, we know that this statement is true in Euclidean Geometry given a line and a point which is not on the line through that point we can draw a parallel to the original line.

So, this is our line x and this is the (line) this is the point y through y we can draw a line z which is parallel to x, z and x do not intersect. So, you could (paraphrise this) paraphrase the statement like this, given a line x and the point y not on x there exists a unique line z passing through y so that x and z do not intersect.

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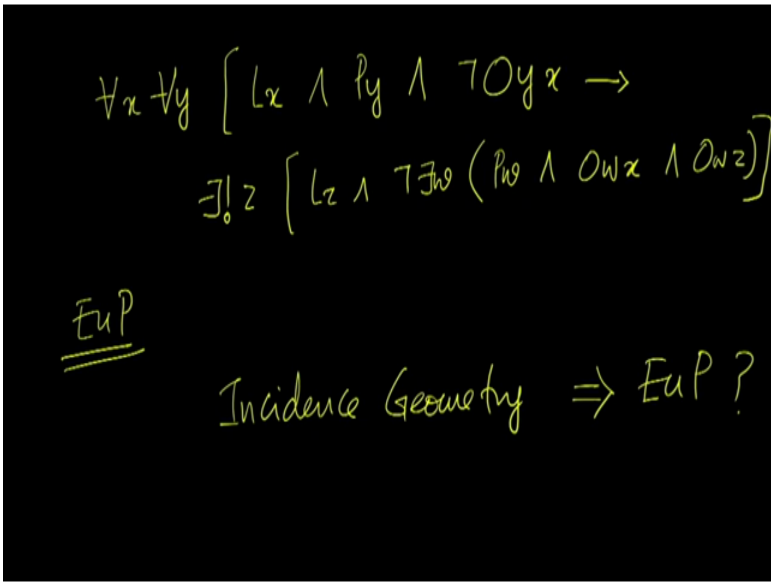

So, let us write the first order representation of the statement, we say for every x and every y so that x is a line and y is a point and y is not on x there exist a unique z so that z is a line and it is not the case that there exist w so that w is a point which lies on both x and z when two lines x and z intersect there is a point of intersection, if two lines intersect then there exists some point w that lies on both the lines.

So, that is what we negate here for every x and y so that x is a line and y is a point and y does not lie on x there exist a unique z which is a line so that there does not exist a point w that lies both on x and z. so, this is the Euclidean parallel property then there is this question, is Euclidean paddle property necessarily true in incidence geometry? Is Euclidean parallel property a logical consequence of Incidence Geometry as it happens? It is not.

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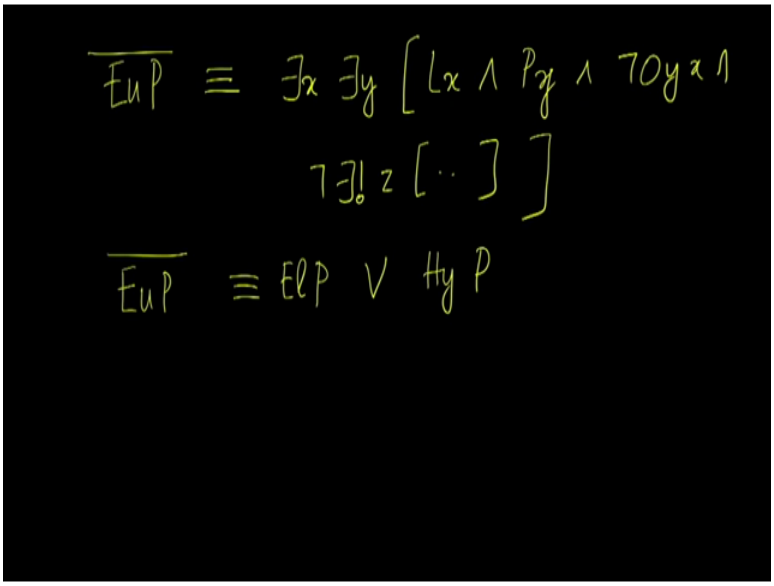

How do we do show this? Let us consider the negation of Euclidean parallel property, so from the formula that we have just written we find that the negation of Euclidean parallel property can be written in this fashion by using De Morgan’s law this will be the general structure of the formula so you can fill the formula. You find that this is actually the or of two statements, one is the elliptic parallel property and the hyperbolic parallel property. So, what are these statements, the elliptic parallel property and the hyperbolic parallel property?

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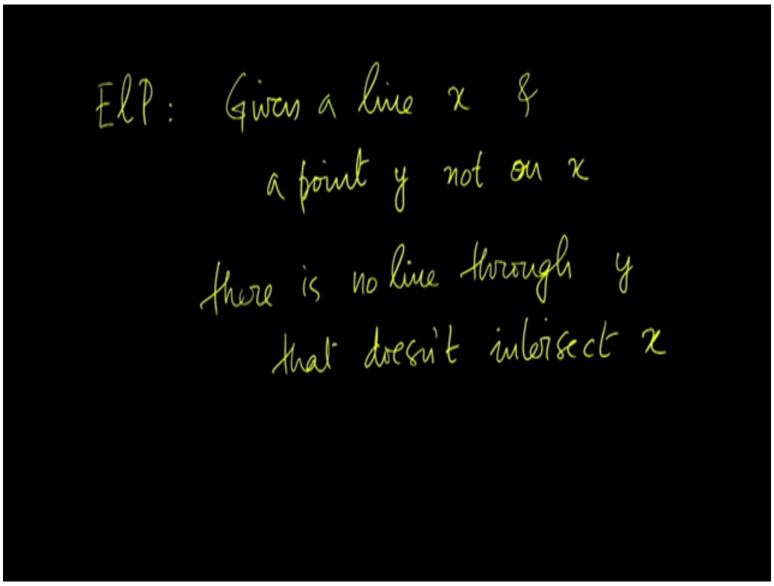

The elliptic parallel property says that, given a line x and a point p and a point y not on x there is no line through y that does not intersect x. In other words, given a line x and a point y not on x, there is no parallel line through y for x this is what is called the elliptic parallel property.

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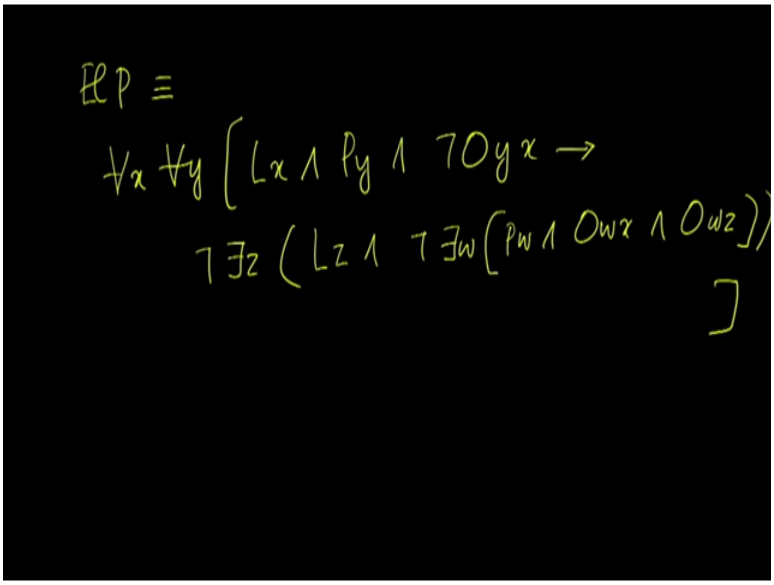

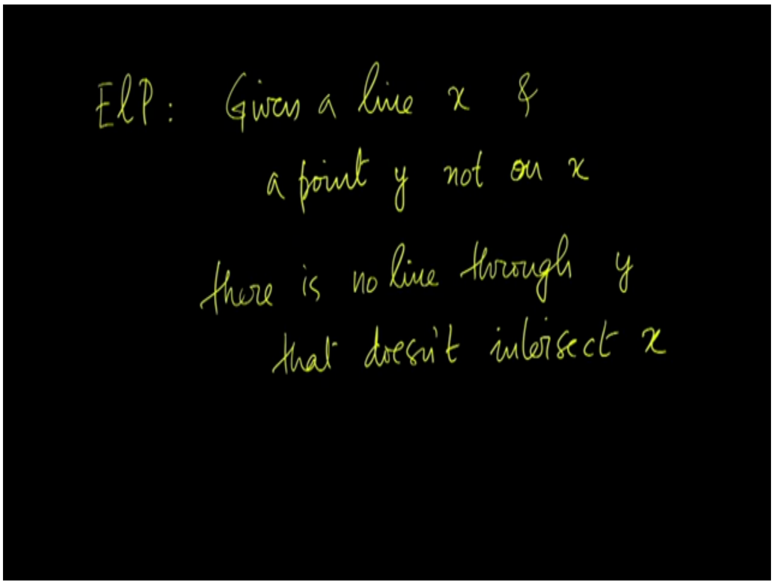

Elliptic parallel property could be written in this manner, there exist x and there exist y so that x is a line, y is a point and y is not on x and it is not the case that there exist is a z where z is a line and there is no w. Check once again, if the statement is right, what does it say? For every x and y where x is a line and y is a point and y does not lie on x it is the case that there does not exist a line z so that there is no w on which both (x and w) x and z lie that is x and z do not intersect.

So, there is no point w which is on both x and z that is when x and z would intersect. So, the meaning would be exactly what we have seen given a line x and a point y not on x there is no line through y that does not intersect x.

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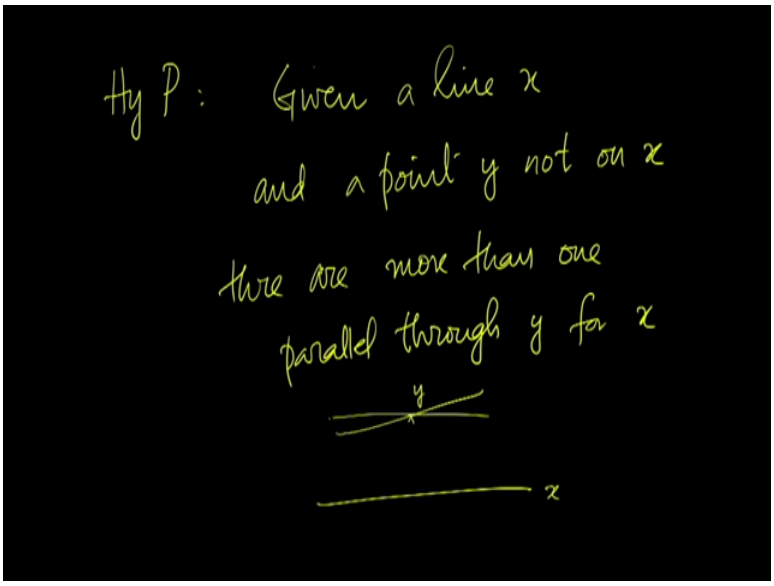

The hyperbolic parallel property would contradict the Euclidean parallel property in another way, it says that given a line x and a point y not on x there are more than one parallel through y for x, there is given a line x and a point not on y there are multiple parallels to x through y such a statement is what is called the hyperbolic parallel property. So, along the lines of the Euclidean parallel property and the elliptic parallel property you can write the formula which corresponds to the hyperbolic parallel property which I leave to you as an exercise.

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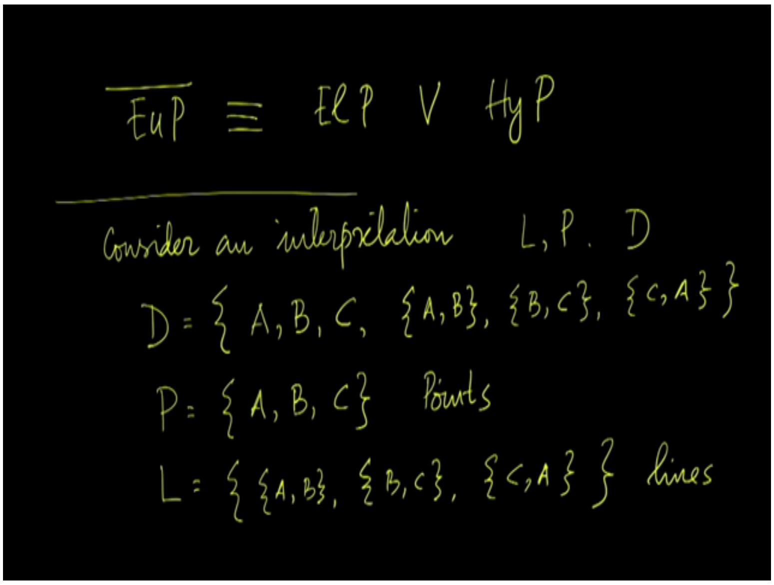

So, you can see that the Euclidean parallel property when contradicted gives us the OR of elliptic parallel property and the hyperbolic paddle property whereas Euclidean parallel property can be contradicted either by holding elliptic parallel property or by holding hyperbolic parallel property. Now, coming back to our question, is Euclidean parallel property a logical consequence of incidence geometry?

We will show that it is not, the proof goes like this. Let us consider an interpretation, an interpretation for incidence geometry to interpret incidence geometry we have to define what lines are, what points are so for doing this we need a domain of discourse D. Let the domain of discourse D consist of entities of this form three entities A, B and C and sets of pairs of these entities, the entities A, B, C and sets A, B; B, C and C, A all belong to the domain of discourse.

Then the predicate p is associated to the set A, B and C are the points, so we say the set of points is A, B and C and we say the set of lines is A, B; B, C and C, A this is a departure from Euclidean geometry. So, the definition of points and lines are different here, a line in particular is just a two member set.

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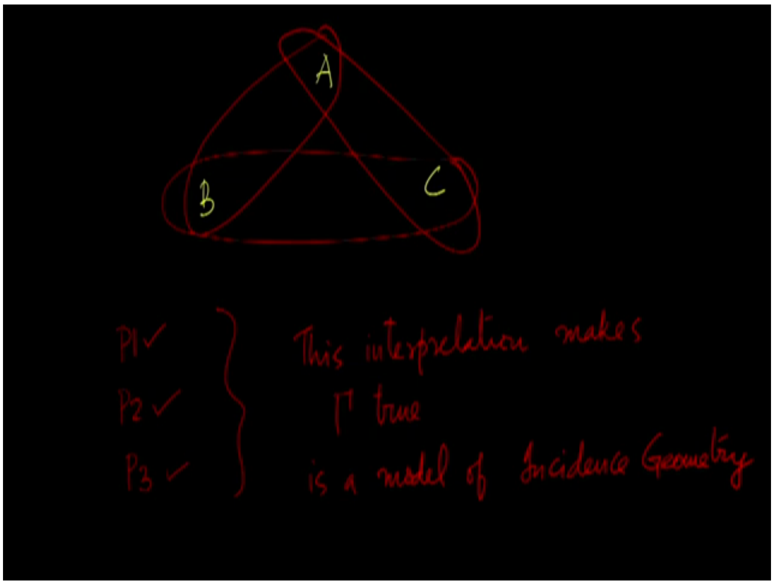

So, we have three points A, B and C and we have lines of this form, the set which contains A and B is a line this line has exactly two points A and B, the set which contains B and C forms another line this set has exactly two points B and C and then the set which contains A and C also forms a line this that also contains exactly two points. So, these three are r lines, so if we define A, B, C as the points and the sets A, B; B, C and C, A as the lines then let us see if the three formulae P 1, P 2 and P 3 are satisfied.

So, what does P 1 say? P 1 says that, there is a line a unique line passing through any distinct pair of points, any pair of distinct points. So, let us take A and B, A and B are two distinct points A is not equal to B and there is a unique line A, B that passes through them, so in this case there is a unique line L which contains B. So, the statement P 1 is true for A and B and the statement is true for B, C as well you take the two points B and C there is a unique set B, C which contains them among the lines, so there is a unique line that contains both B and C.

Similarly, there is a unique line which contains both A and C. So, the first proposition is true P 1 is correct, what about P 2? P 2 asserts that, in any line there are at least two points which is again the case here there are only three lines here A, B; B, C and C, A are the three lines in each of these lines there are exactly two points so P 2 is also satisfied, what about P 3? P 3 asserts that there are three non collinear points, so though must exist three points, all three with all three not lying on the same line which is the case here.

Indeed there are three points A, B and C and there is no line which contains all three that is trivially so because every line contains exactly two points here so P 3 is also true. Therefore, this interpretation makes the whole of gamma true, every formula in gamma namely P 1, P 2 and P 3 are true in this interpretation therefore we say that this interpretation is a model of Incidence Geometry.

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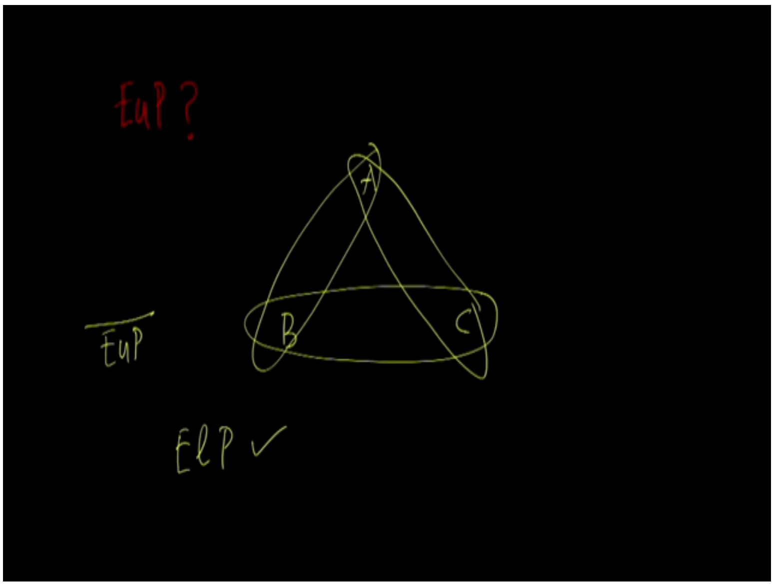

But now, what about the Euclidean parallel property? We find that the Euclidean parallel property is not true in this interpretation that is because when we consider line B, C and appoint A not on B, C for Euclidean parallel property to be true there should be a unique line through A which does not intersect B, C here there are two lines through A one is A, B which intersects B, C; A, B intersection B, C is B.

Similarly, the other line through A which is A, C also intersects B, C so Euclidean parallel property is violated here. In particular here we find that the elliptic parallel property is true, for any line and a point which is not on the line there is no line parallel to the first line through the point in this case you take line B, C and point A which is not on B, C there is no line through A which is parallel to B, C a line is parallel to B, C if that intersects with B, C a line in this case is a set of two points.

So, if you interpret a line in this sense then elliptic parallel property holds and therefore in this model of incidence geometry elliptic parallel property is true, Euclidean parallel property is false but then is elliptic parallel property a logical consequence of incidence geometry? That also is not true that can be shown with another interpretation.

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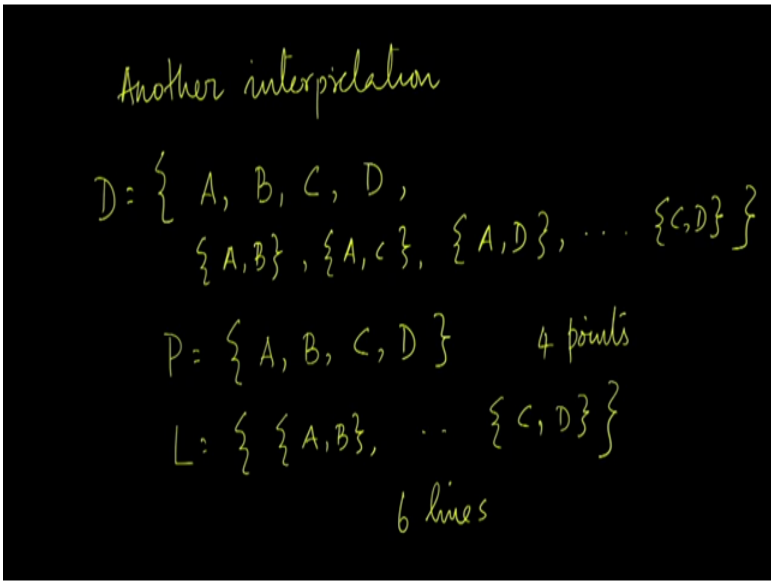

A second interpretation uses a different domain of discourse, here we have four points A, B, C, D and we have lines of this form A, B; A, C; A, D; B, C; B, D; C, D that is we consider every two member subset of the set A, B, C, D. So, the domain of discourse is made up of all these as in the previous interpretation we define P the set of points as A, B, C, D and L the set of lines as all to member subsets A, B; A, C; A, D; B, C; B, D and C, D. So, we have four twos, two namely six lines, there are four points and six lines.

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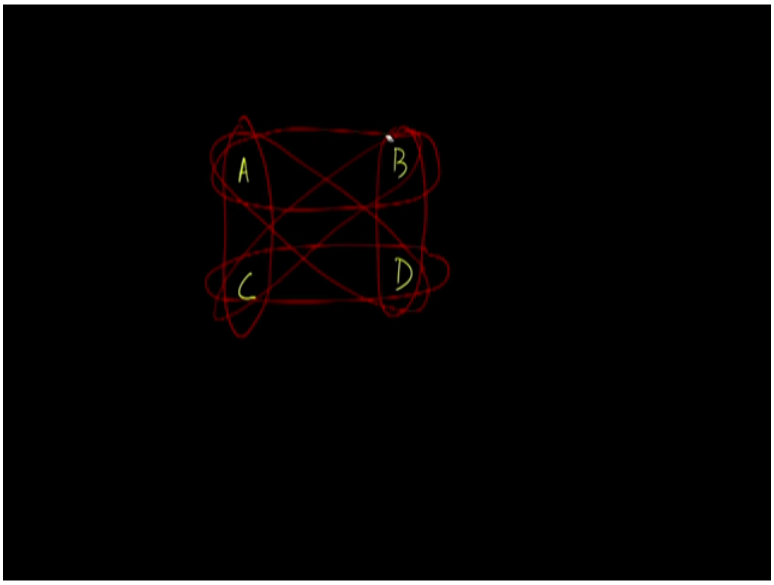

Once again, we can verify that propositions P, P 2, P 3 are true here, so here we have four points A, B, C and D and the lines are two member subsets A, B is a line, B, D is a line, C, D is a line, A, C is a line so are A, D and B, C.

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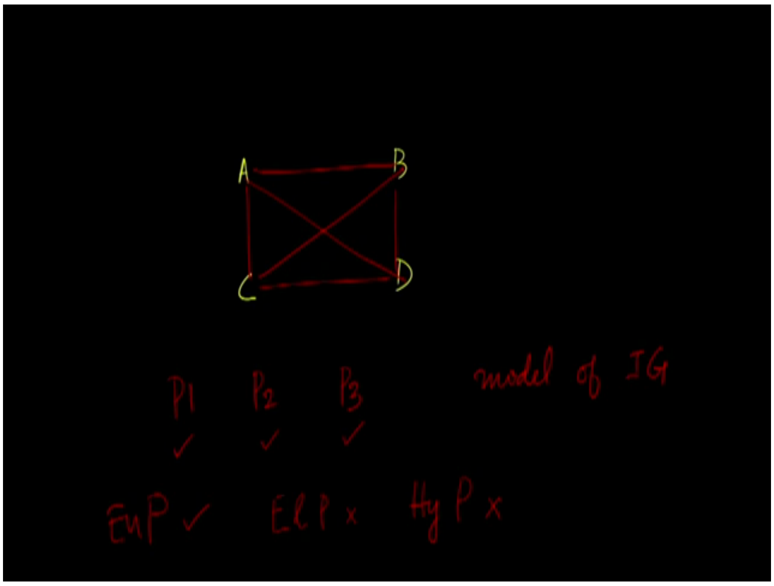

Or to simplify the diagram, we could just draw lines this corresponds to the set A, B and this corresponds to the set C, D this is B, D this is A, C; B, C and A, D. So, in this interpretation there are four points and six lines and we find that the statements P, P 2 and P 3 are all satisfied given two distinct points B and C there is a unique line which passes through the two of them namely B, C and P 2 asserts that in every line there are at least two points which is indeed the case.

A line is made up of two distinct points and there are three non collinear points and it reflect that can be chosen from here for example A, B, C do not lie on the same line that Is because the line has only two points. So, P, P 2, P 3 are all satisfied therefore this interpretation is a model of Incidence Geometry, what about the Euclidean parallel property? We find that the Euclidean parallel property is true here, take any line B, C and a point A which is not on B, C then there is a line A, D which passes through A and this line does not intersect B, C does not intersect.

In the sense that, these two lines A, C and B, D do not have a common element, mind you even though I have drawn the sets using lines they are in fact two member sets the line A, D contains exactly two points A and D and the line B, C contains exactly two points B and C, these two sets do not intersect. So, we can say that A, D is parallel to B, C because they do not have an intersecting point.

So, the line B, C and A which is not on B, C shows us that there is a line A, D which is through A and does not intersect B, C and you can verify that this is the case for every line and a point which is not on the line for example, if you take C, D as a line and A as the point, A is not on C, D and there is A, B which passes through A and it is parallel to C, D therefore Euclidean parallel properties satisfied here.

Since Euclidean parallel property is satisfied, elliptic parallel property and hyperbolic parallel property will be false in this interpretation. Therefore, we find that elliptic parallel property is not a logical consequence of incidence geometry either where is there is an interpretation in which all the axioms of incidence geometry namely P, P 2, P 3 are true but in this interpretation elliptic parallel property is not true. The first interpretation that we saw was a model of incidence geometry but in that interpretation the Euclidean parallel property was not true.

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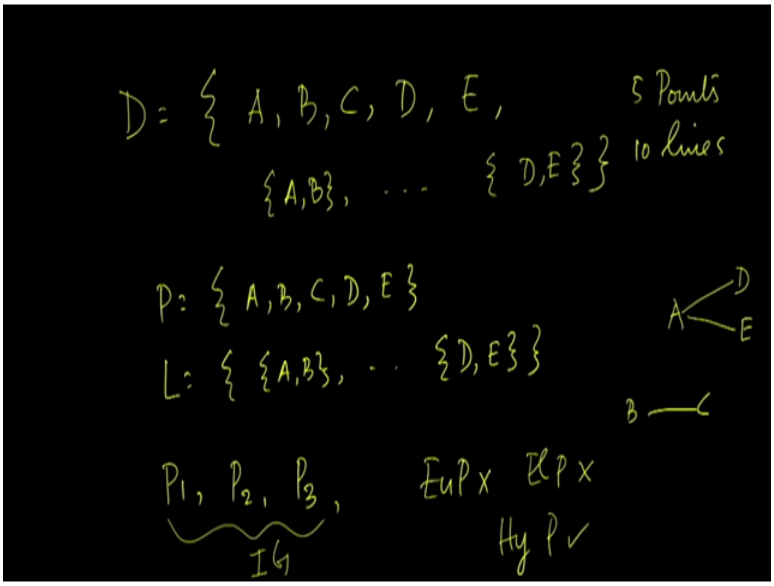

Now, coming to the hyperbolic parallel property. Let us consider, a different the third interpretation in which we have five points A, B, C, D, E and all two member subsets namely A, B; B, C; A, B; A, C; A, D; A, E; B, C; B, D; B, E and C, D; C, E and D, E there are 10 of them, 5 choose 2 there are 10 lines, so we have 5 points and 10 lines. So, P is A, B, C, D, E the set A, B, C, D, E and L is the set of all pairs of points, unordered pairs of points.

Again you can verify that P, P 2, P 3 are satisfied exactly as before but we find that Euclidean parallel property is not satisfied that Is because if you take a line B, C and a point A which is not on B, C there are two lines A, D and A, E passing through A both of which are parallel to B, C, parallel to B, C in the sense that those two lines do not intersect with B, C mind you a line is just a set of two points.

So, set A, D does not intersect with B, C and set A, E also does not intersect with B, C therefore we can say that there are two distinct sets containing A that do not intersect with set B, C therefore Euclidean parallel properties not satisfied elliptic parallel properties also not satisfied but we find that hyperbolic parallel property is satisfied. So, this is again a model of incidence geometry since P, P 2, P 3 are all satisfied but in this model of incidence geometry Euclidean Parallel property and elliptic parallel property are not true but hyperbolic parallel property is true.

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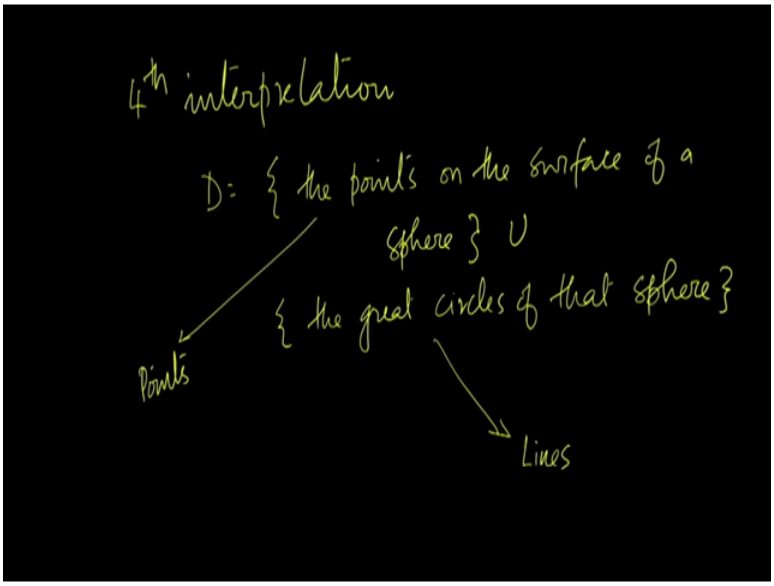

And finally we will look at one more interpretation, the fourth interpretation in which P, P 2, P 3 all need not be true. So, here D is the set of all points on the surface of a sphere along with the great circles of that sphere. Here the first set forms the points, the set of points in this interpretation is the set of all points on the surface of a sphere and the set of all great circles on that sphere will form the lines.

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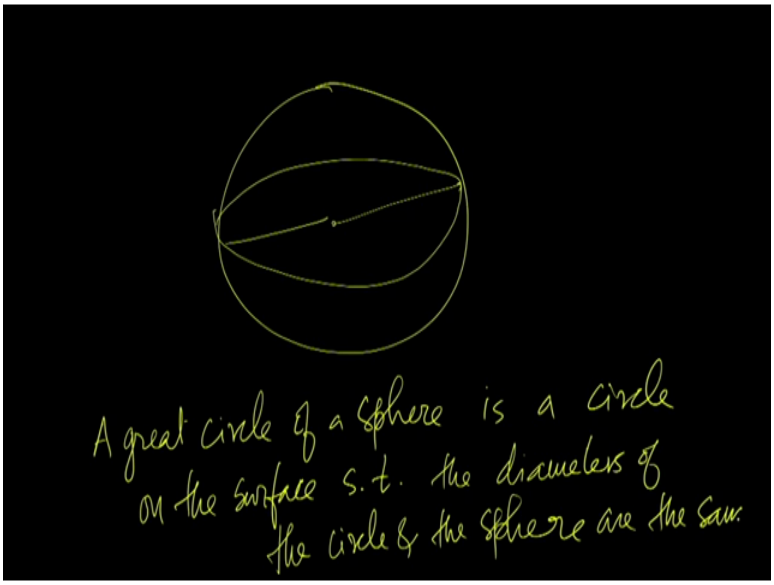

Now, what is the great circle of sphere? When you have a sphere and we consider a circle drawn on the surface of the sphere, so that the diameter of the circle is the same as the diameter of the sphere then the circle is a great circle, a great circle of a sphere is a circle on the surface of the sphere so that the diameters of the circle and the sphere are the same.

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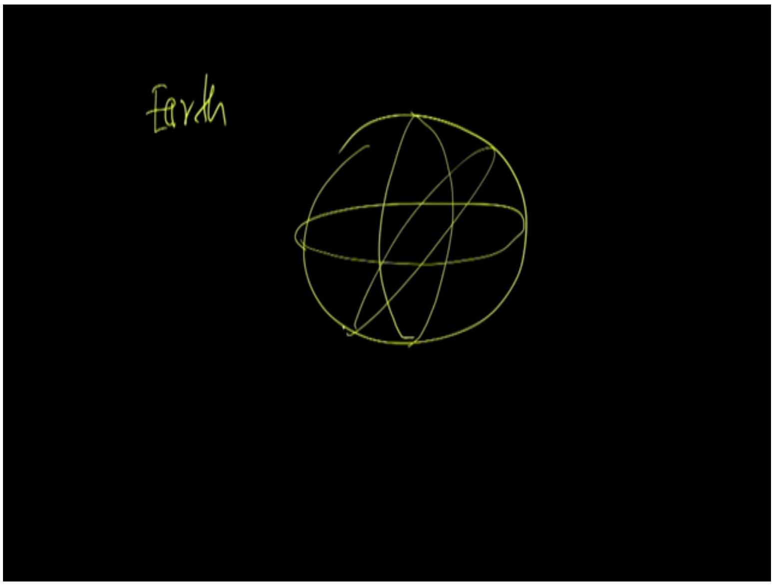

So, if you consider the earth as a sphere any meridian along with it is opposite meridian will form a great circle and the equator is also a great circle of course these are not the only great circles by any means you can draw any number of great circles. So, in this interpretation the points on the surface will form the set of all points and the great circles will form the set of all lines

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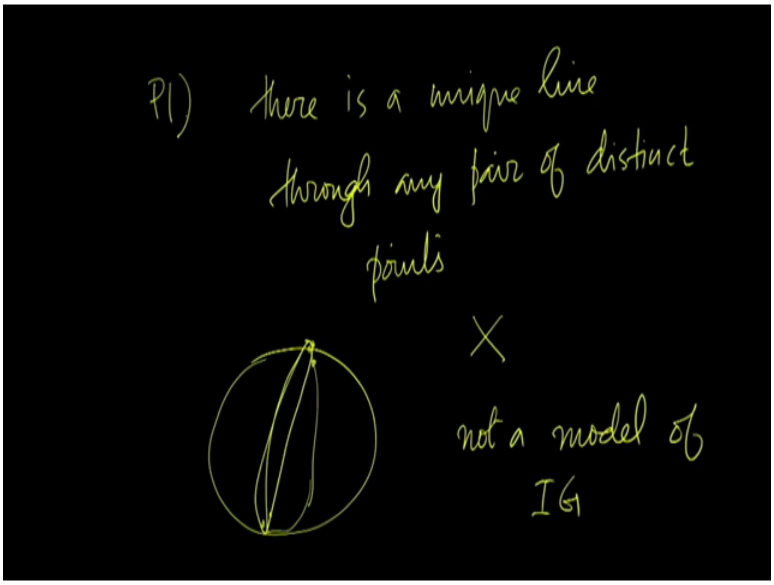

Now, let us see if this is the model of incidence geometry, we find that P 1 is not satisfied here so what does P 1 say? There is a unique line passing through any pair of distinct points which we find that is not true in this case, if the points that we take happen to be the polar opposites then we find that there are any number of great circles passing through them. In particular, any meridian along with it is opposite one will form a great circle here, that is if you take the two points as the two poles, the north pole and the south pole on the sphere which is the earth then any meridian along with the diametrically opposite to meridian will form a great circle.

So, there is an infinite number of great circles passing through this pair of distinct points that is through the south pole and the north pole we have an infinite number of meridians passing. The pair of points that we take need not be the poles, you can take any pair of diametrically opposite points and there would be an infinite number of great circles passing through them. Therefore, P 1 is violated here therefore this interpretation is not a model of incidence geometry.

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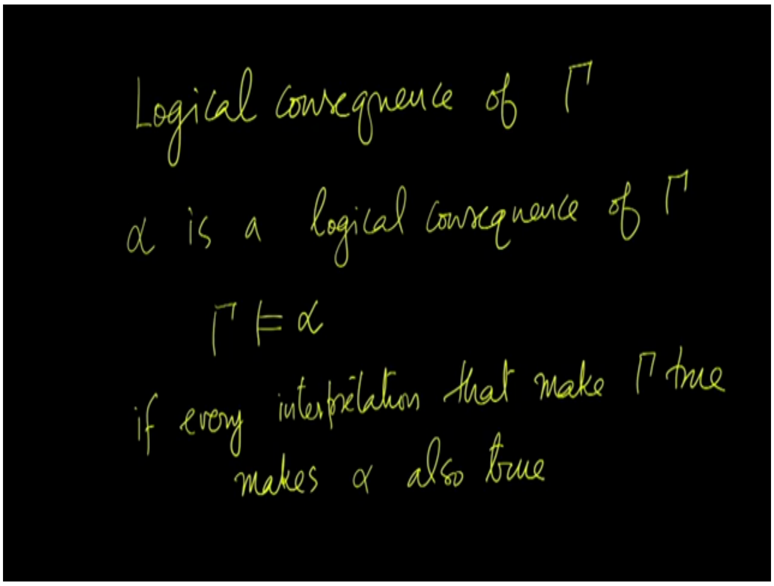

Then let us formally ask this question, what is a logical consequence? We have a set of formulae gamma and we want to define the logical consequences of gamma, we say that alpha is a logical consequence of gamma denoted like this, alpha is the logical consequence of gamma if every interpretation that makes every formula in gamma true makes alpha also true.

So, for incidence geometry we have seen three interpretations in which P, P 2, P 3 are all true for Euclidean parallel property to be a logical consequence of incidence geometry it would have had to be true in all these three interpretations but we find that it is true in only one of the interpretations therefore Euclidean parallel property is not a logical consequence of incidence geometry.

Similarly, elliptic parallel property is also not a logical consequence of P, P 2 and P 3. The last interpretation that we saw did not satisfy P 1 therefore that is not a model of incidence geometry. So, when we investigate Euclidean parallel property we need not consider this last interpretation because this is not a model of incidence geometry.

What we have to verify is that in every interpretation which makes every formula in the set gamma true namely P, P 2 and P 3 must also make alpha true if that is the case we say that alpha is a logical consequence of gamma. So, it is in this sense that we said that gamma along with all its logical consequences forms incidence geometry, ok that is it from this lecture, hope to see you in the next thank you.