Discrete Mathematics
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Lecture 33
Rock placement problem

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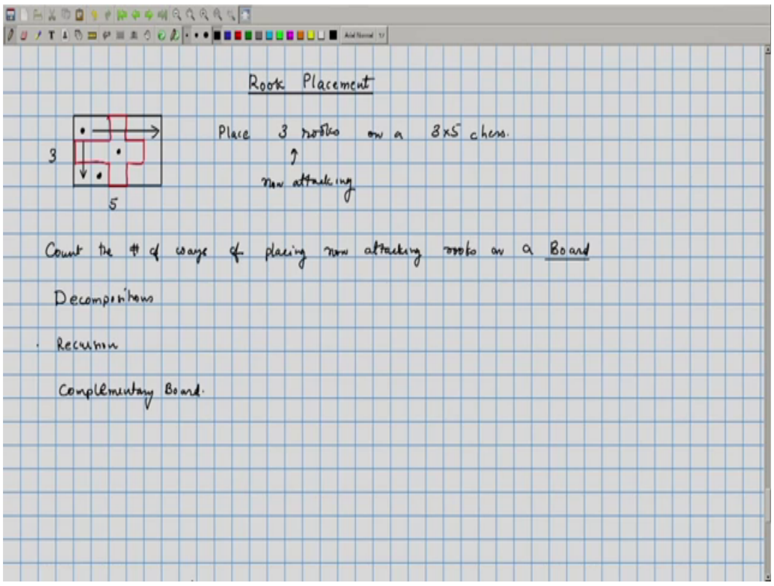

This lecture we will see about placing rooks on n cross n chessboard. So, let us look at this problem with more carefully. And, this will use many other techniques that we learnt so far. We will use the principal of inclusion exclusion, we will all so use the principal of method of generating function and so on. So, let us start with an example so this is a 3 into 5 chessboard. So, place 3 rooks on a 3 into 5 chessboard this is the problem. That we will began our discussion with.

Rook is a piece in chess and the property of that piece is if you place rook in a particular position. Every position in that row and that column is under attack by that rook. So, you cannot place any other pieces on along that row or column. Those will be positions which are attack by this particular rook. So, we want to place rooks on chess board such that this 3 positions are non attacking. So, one way of placing it would be in this particular manner. So, you can see that if, you look at any of the rooks there are not under attack from any other rook.

So, what is the total number of ways of doing it? And, note that we can not place more than 3 rooks the maximum number of rooks that we can place is going to be 3. Because, there are only 3 rows and if you place more than 3 rooks. Some row is going to contain more than 1 rook and those are going to be attacking. So, our objective is to count the number of ways of placing non attacking rooks on a board. Now, what is a board the simplest board is an n cross m board which is n rows and m columns.

But, the board could be much more complicated than that for example the red region that we are just marking out is also a board. Now, in we could look at the problem of trying to place the rooks just in the positions indicated by that particular by that mean that particular board which consist of the square indicated. Now, how do we count this we will learn 3 methods. One is based on say decompositions and another is based on recursion and third method will be based on complimenting.

So, given a particular board that is subset of squares of n cross and board if you need place non attacking rooks. The number of ways of doing that we need to count that we will use three generalized principals. One is the principals of decomposition then another is a recursive way of counting and the third would be to look at the complimentary board. So, this three methods is what we will look at in this particular lecture.

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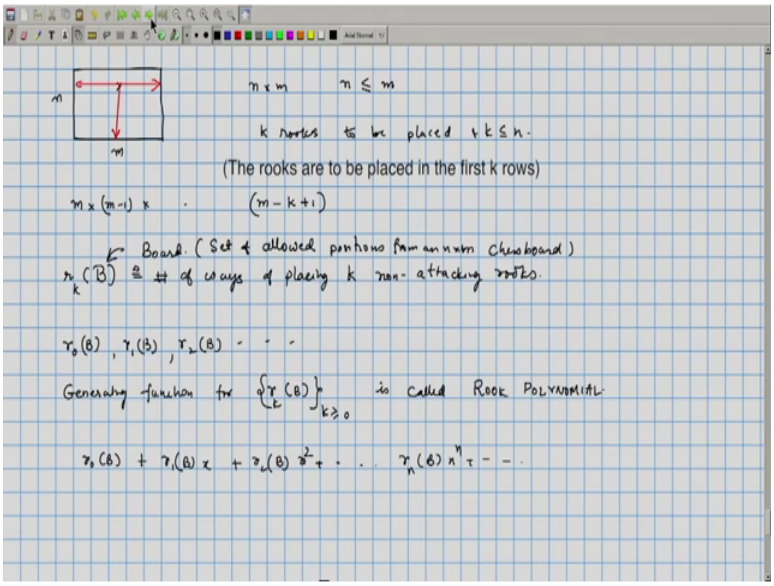

So, let us first look at the simplest case where the board is n cross m board. So, if you have a n cross m board we will assume that n is less than or equal to m. And, we need to place some number of rooks k rooks to be placed and k is less than or equal to n. The number of ways to do this what we need to find. When it is regular rectangular board with all positions being allowed position the first rook can be placed in the first row in m ways. And, the second rook so say place it here any of the other rows you can choose any column other than the one indicated by the first rook.

So, the red squares are gone the other columns you can use. So, the total number of ways of placing the second rook is m minus 1 and all the way up to m minus k plus 1. If you have to place k rooks and if k was n this would have been the following fact to real up to n. So, that is the number of ways placing k rooks on a very regular rectangular board. We will introduce certain notations so small rk B, so B is basically a board. And a board by a board what we mean is a set of allowed positions from n cross m cross chessboard.

So, given a particular collection of allowed positions the number of ways of placing k rooks on that board is denoted by rk. So, this defined as the number of ways of placing k non attacking rooks. Whenever, rook is placed you cannot placed anything along the same column, you cannot place anything else along the same row as well. And, if we look at the sequence r0 b r1 b r2 b and so on. This gives the number of rooks I mean the number of ways in which you can place k rooks in a board b.

So, if you consider this sequence it is going to be finite sequence because, the number of allowed positions is surely an upper bound on the length of the sequence. All the other terms after that basically becomes 0. And, the generating function for rk B, k greater than or equal to 0 is called the rook polynomial. So, look at the number of ways of placing k rooks on aboard B if you vary B if you vary k we get a sequence. The generating function of that sequence namely r0 b plus r1 B times x plus r2 B times x square and so on, r n b times x way is to n. So, this sequence is sequence is called as the rook polynomial.

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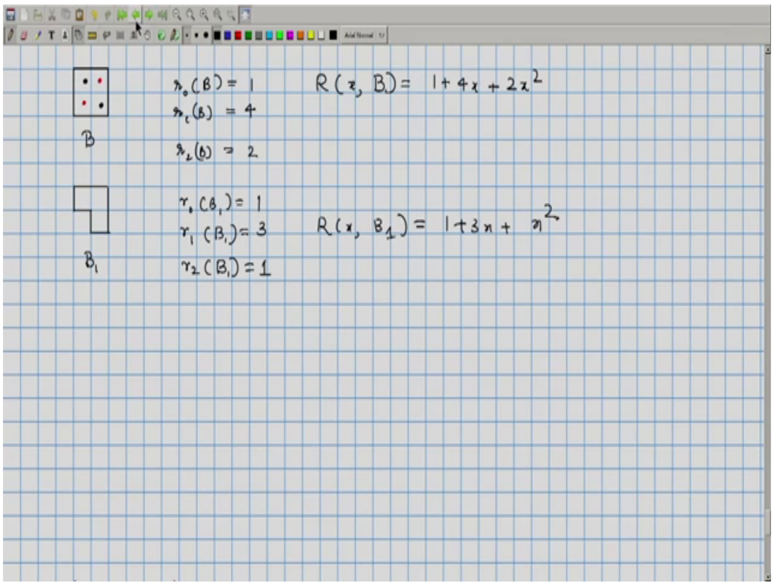

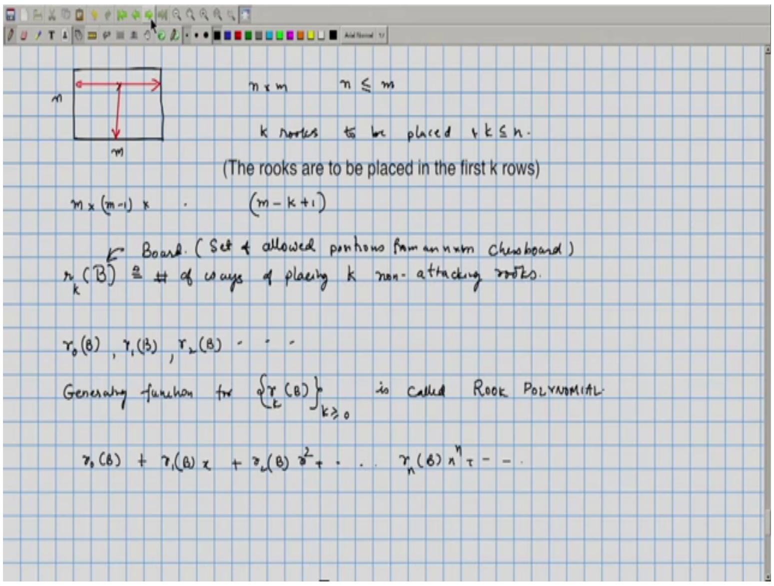

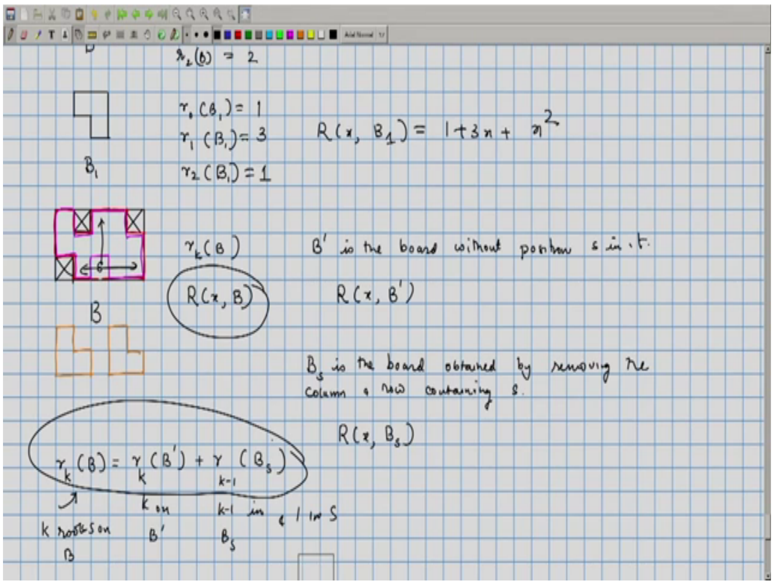

For example, let us do a couple of example, if we look at this particular board there are 4 positions and let us say all the positions are allowed positions. The rook polynomial for that r0 b is going to be 1 because, there is only one way which you do not place anything. And r1 b this is going to be equal to 4 fact for any board r1 b is the number of placing 1 rook will be equal to the size of number of allowed positions. And r2 b is going to be equal to you place a rook at the left side top corner and then you only place the replace a second rook is on the other diagonal.

So, this is one way and the other way of placing it is indicated by the red coloured placements there are only two ways of placing it. And therefore we will get r2 b will be equal to 2. And for the rook polynomial of this board. Because, this board is b the rook polynomial will denote by r and the variable is x. So, rx b will be equal to 1 plus 4x plus 2x square. If you take a slightly different board. Namely a board with three allowed positions now r0 let us now called this is b1 r0 b1 is going to be equal to 1.

And, r1 b1 is going to be 3 because there are three positions you can keep a single rook any of this positions and r2 b1 the only way placing 2 rooks by placing it on the available diagonal. There is only way of doing it so rx b1 is going equal to 1 plus 3x plus x square. The first thing that we will look at is how to compute the rook polynomial.

So, we could of course compute the values of rk b for all possible values of k and from that compute the rook polynomial. But, the purpose of rook polynomial is basically do the reverse thing. That is can it help us in figuring out the number of configuration possible. A number of non attacking rook configuration possible. So, we will look at ways of generate the rook polynomial. And, once we have generate the rook polynomial we will use that to compute the coefficient.

So, let us say we have a board so we will just think of this is an n cross m board. In which certain positions are marked as not allowed. So, these are the non allowed positions and all the other positions are the allowed positions. And, we need to compute r if you call this board as b. We are interested in calculation rk B in order to compute rk B we will compute r x B namely the rook polynomial of this particular board. And, once we have computed the rook polynomial from that we can derive rk B.

The first the thing that we will do is based on recursion. So, we will identify one particular position let us say this is a position s in the board. And, all the placements of rooks on this table would involve either a rook on as or it will not have a rook. There it will be two possibilities there is a rook position as or there is no rook. So, we will generate each board let us draw this board in red. So, a given board a given position so now we are interested in calculating the number of ways of placing k rooks on this particular board that is given.

The red outline basically indicates the allowed position the marked positions the cross positions are the forbidden positions. Now, let us marked out one particular square namely s and look in terms of whether s is a part of an arrangement that is whether a rook is place it or not. So, if you look at all the placements there are placements which involves a rook at position s and there are the others where there is no rook at s. So, we will construct two sub boards one is the one where mean the first one will consider is the board where we will ignore the position s.

So, that we will called as the board B prime. So, B prime is the board without position s in it. And, that will have its own rook polynomial. So, its rook polynomial will be rx b prime also we could have rook at position 1. In which case mean suppose we place a rook at s then, what we will have is all this positions will now be gone let us draw it separately. So, if you do the remaining positions would be everything the row and column would essentially began.

So, this would be the remaining board because, this positions would be ruled out if we have a rook on s. So, that will be our board Bs so Bs is the board obtained by removing the column and row containing s. And, that will also have a rook polynomial let us call that as Rx Bs our objective was to compute Rx B. But, let us just look at in terms of the number of ways of placing k rooks on this board b. Now, rk b is going to be rk b prime plus rk minus 1 bs. Let us see why this a recursive formula is true.

The number of ways of placing k rooks on the board b is equal to the number of ways of placing k on b prime that is after removing position s. You do not put anything on position s and it place k rooks on the remaining positions or you put 1 rook on position s. And then put k minus 1 in the remaining board so k minus 1 in Bs and 1 in s. So, this recursive formula we have justify why it is true. Now, form this recursive formula we can compute the rook polynomial we can give an expression for the rook polynomial of B.

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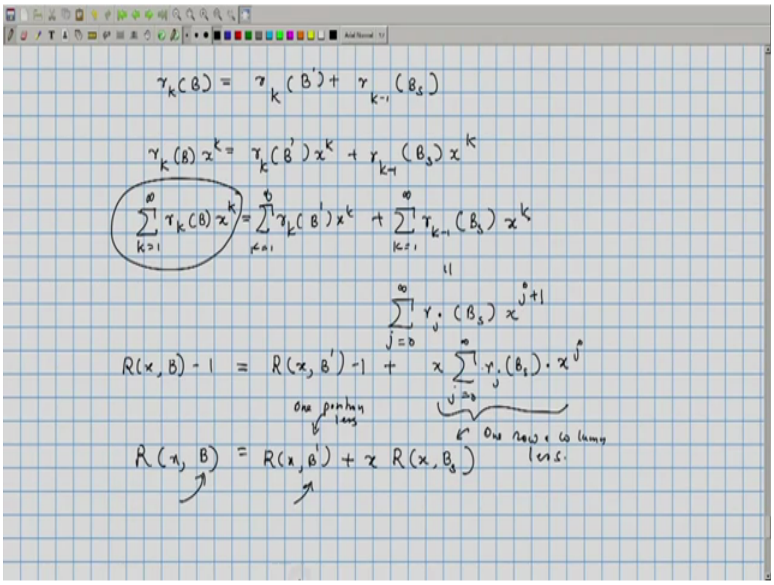

So, what we knows rk B is equal to rk B prime plus rk minus 1 Bs. We can multiply this expression by x raise to k will get rk b x raise to k is equal to rk B prime x raise to k plus rk minus 1 bs x raise to k. Now, you can sum this over all possible values of k going from let us say because we do not want to take 0. Because, r minus 1 does not make sense so summation k equals 1 to infinity rk Bx k is equal to rk B prime xk again sum from k equals 1 to infinity plus summation k equals 1 to infinity rk minus 1 bs x raise to k and this term is equal to summation j is equal to 0 to infinity rj bs x raise to j plus 1.

And, the left hand side term is almost the rook polynomial of b but just the first term is missing we can write it as rx b minus r0. R0 is always 1 and the right hand side first term is going to be the rook polynomial of b prime. So, rx b prime minus 1 plus x times summation j equal 0 to infinity rj of bs into x raise to j and this is rook polynomial of bs. So, this is equal to rx b prime so we can write this as rx b is equal to rx b prime. The minus 1 minus 1 get cancelled plus x times rx bs. What we have managed is in order to calculate the rook polynomial of b. We can calculate we can write in terms the rook polynomial of board with 1 less position.

And another one which has significant which is one row and one column less. So, one position is less and here one row and column gone. So, we write the rook polynomial of b in terms of its constituent terms. And, we can do this recursively and compute the rook polynomial of the entire board. This could be very cumbersome if there lot positions so, we will look at other methods. So, it can be useful for certain kind of boards we will see other methods as well.

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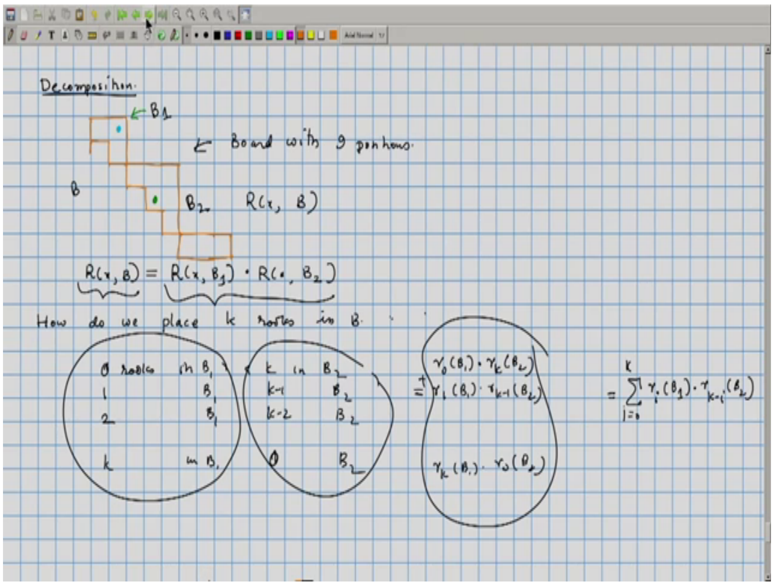

The next method is we will see is called as decomposition. So, let us look at the special kind of a board. So, let us say our board contained this positions so this is 6 positions. So, this is the board with 9 positions. And let us call this is a b and we need to find out the number of ways of places a rook on this board. We now that it is sufficient to find out the rook polynomial of this board. So, what is Rx b now note that this board is some sense be clear it naturally breaks into two parts. So, what does it mean to break into two parts or decompose?

So, the formal definitional it means the board can be split in to two positions. Such that rooks in one position do not attack rooks in the other board. For example if I place a rook here and if place a rook anywhere else in the other part. They can never be attacking so if you can break down the board in to constituent parts. Such that any position in one of the part do not attack any other squares on the other part. Then we have called that is a decomposition.

By decomposition is helpful to us is because of the following fact if the board decomposes into parts the rook polynomial of the board is basically the product of the generating function or is the product of the rook polynomial of the constituents. So, let us call this board as B1 and this is B2 we will see that RxB is going to be equal to Rx B1 into Rx b2. The proof is simple if you look at the definition of rook polynomial RxB is equal to R0 plus R1 B times plus x plus R2 B times x square and so on.

And, here the number of ways of placing a rook in B1 k rooks in B1 that is given by the rook polynomial. So, that is going to be let us say I will write as a r1 to indicate they are looking at board 1 so r0 plus r1 1. So, also ask this question how do we place k rooks in B? So, we could think of it as 0 rooks in B1 and k in B2 or 1 in B1 and k minus 1 in B2, 2 in B1 and k minus 2 in B2. All the way we have to k in B1 and 0 in B2 and all of this arrangements are going to be different arrangements. Because, they have different number of rooks in b1 and b2 and notice that we could do this part and this part independently.

Because, any place went of rooks in B1 is not going to attack any positions in B2 and this is exhaustive because, if you have places rooks in B there has to be some number of rooks in B1. And, some number of rooks in B2 and their sum should be k. And, therefore the total number of ways is going to be equal to r0 b1 into rk B2 plus r1 B1 into rk minus 1 B2 all the way up to rk B1 into r0 B2. And, this basically is just the condition of the two sequences. If you look at the sequence corresponding to B1. So, r0 r1 up to rk and the sequence corresponding to B2.

And, if you conval them in convolution of the two you will get the kth term of b. So, this is summation I going from 0 to k r I B1 times rk minus I B2. And therefore many of the terms have this particular property then we know that it is just the product of the I mean if you look at the rook polynomial or the generating function. That is just going to be the product of the constituent generating functions. So, this is a very useful thing if the boards are can be naturally decomposed in to parts.

And it applies even if you have let us say more part if you had a board where there were elements the three elements here. Again if this is B3 the rook polynomial will be just the product of the rook polynomial of the constituents.

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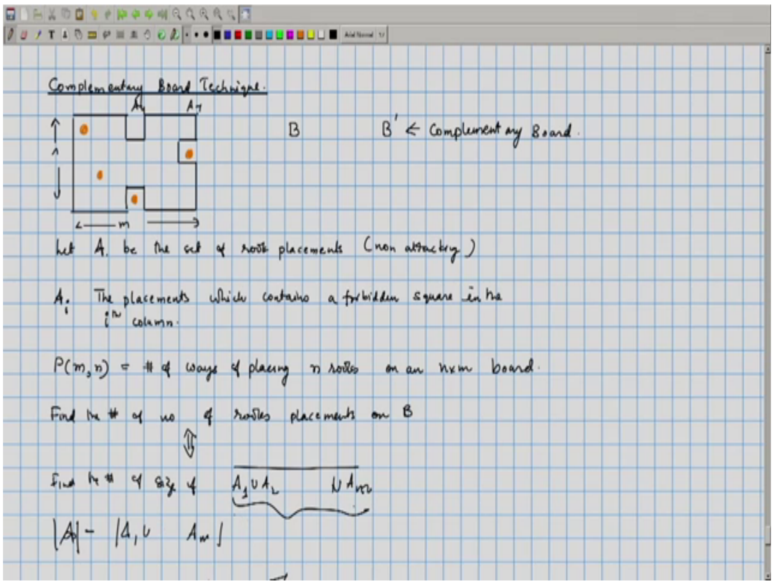

The third technique that we would see is something called as taking the complimentary board. So, here we will use the principle of inclusion exclusion. So, in general the number of allowed positions if they become too large. Then the decompositions are also going to become difficult and recursive formulation is also going to be difficult. Because, there are too many positions to handle. But, if the board is having a lot of allowable positions. And if the compliment has the significantly less than we can use the complimentary board technique.

For, example you had a board with all this positions but very few missing positions. If you have a board of this format you can see that the complementary board is a very small one as a just three allowed positions. So, we will denote a board by b and for this discursion b prime denotes the complimentary board. The complimentary board basically contains all the positions which are absent in the current board. We can look at rook place and then complimentary board as well it is just another board.

Let us look at this how this techniques is used. So, we will introduce some notations. So, let A i so let A be the set of in all rook placements. So, A is the collection of all rook placements for example if this was an n cross m board. We do not look at the forbidden positions. I mean we will allow rooks and all place all positions the only requirement is there should be at non attacking. So, A is what we denote that set by. The set of all possible rook placements without considering what is allowed and what is not allowed.

So, we need to think of a bounding box for the board rectangular bounding box. And in that the set of all possible placements is what we will call as A. And A i will denote the placements which contains forbidden square in the ith column. So, look at any placements supposed we had placed rooks in the following manner one here one here and one here. This will be rook placement but it is not an allowed I mean it is not a placement which we want to include in our counting.

Because there are two rooks in forbidden positions of course this is not even valid placement. So, consider the orange positions that is the placement of rooks. And, this is not to be included in our count because, there are two rooks at forbidden positions. So, if we consider A4 and A7 so this would be placement which will belong to the set a 4 and the set a 7. Because, in the fourth column there is a forbidden square on which there is a rook. And the seventh column also there is a forbidden square there is a rook.

The total number of rook placements was easy to count we can denote that by P m n. So, this is the number of ways of placing n rooks on m columns. Or placing n rooks on an n cross m board. We can assume that m is going to be larger than n. If m is less than we will be can not really place n rook that count will automatically be 0. So, A is just going to be P m n. Now, we need to basically find the number of rook placements on b. And, that is basically the same as find the number of find the size of A1 union A2 union A m the whole compliment.

So, A1 union A m if you take this set the union of that basically consist of all placements in which a forbidden square is used. If you take the compliment of that none of the rooks will be on a forbidden square. And this sets size is what we need to consider. And that is where we will use inclusion exclusion principal. We know the set of the size of a. So, A minus A1 union A m and if you the compute the of size of A1 union A m. And, subtract it from the size of a you will get the number of ways placing rooks on A on to the board b. So, how do we compute these things?

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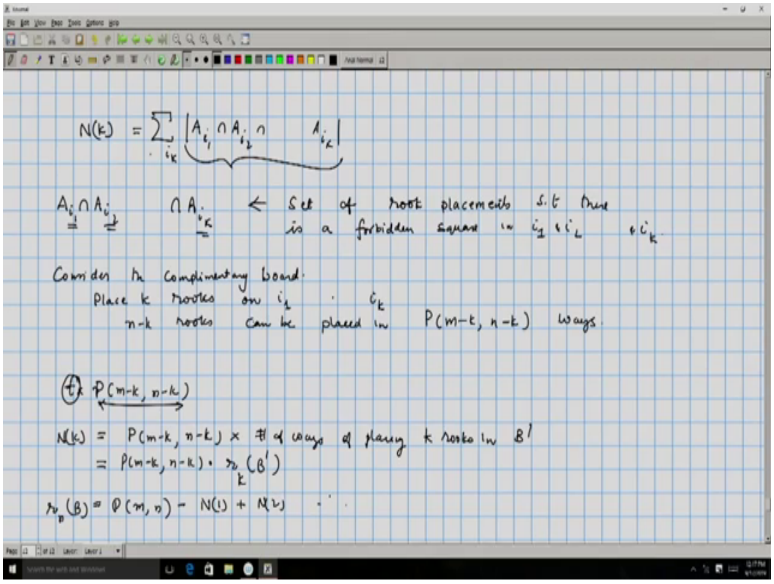

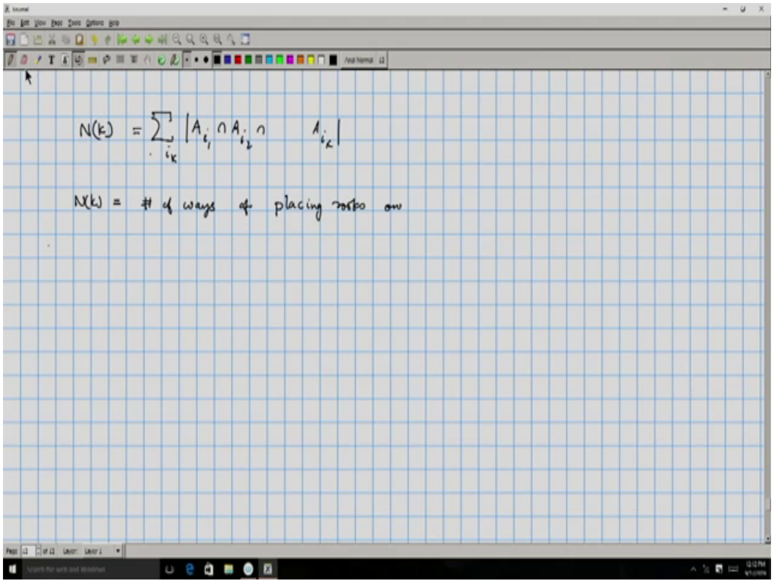

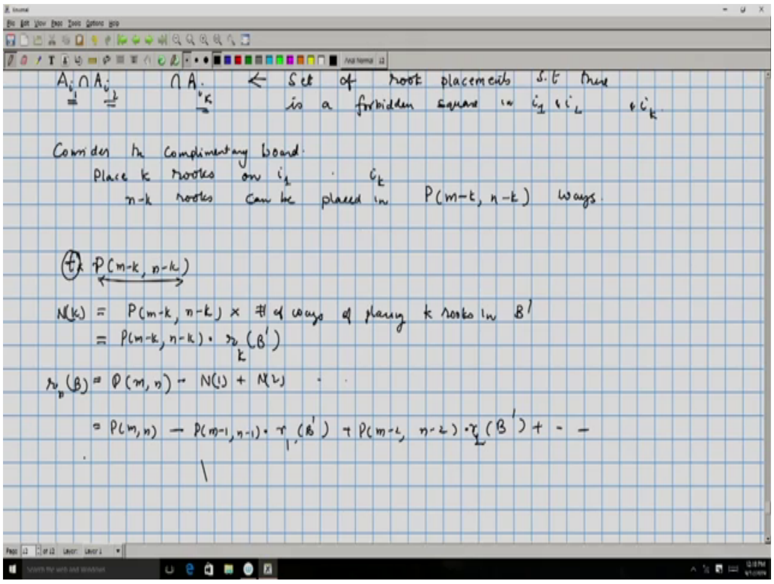

So, we will basically will be using the inclusion principle so while we are using inclusion exclusion principle. The terms that we need to estimate are let say A 1 or let say i1 intersection A i 2 intersection A i k. Look at the size of the set is what we need to estimate. And this summed up over all subsets. Summed up over i 1 i k. Let us denote this by Nk, Nk is what we need to find out. So, if you can find out the value of Nk. Then the count that we want is just the size of a minus this I mean this plus or minus nk by means of inclusion exclusion.

So, Nk denotes the number of ways of placing rooks. So, look at the inner term A i 1 intersection Ai2 intersection A i k. So, this is a set of rook placements such that there is a forbidden square in i1 and i2 and i k. So, if we look at the complimentary board these positions are valid positions for those are positions those are allowed positions. So, we are placing k rooks on i 1 to i k. And the remaining n minus k rooks can be placed in n minus k p n minus k time. So, A 1 intersection A k basically denotes the sets of rook placements such that the forbidden squares in i 1, i 2, i k etc are being used.

Now, if you consider complimentary board these positions are valid positions or allowed positions in the complimentary board. For this particular choice i 1 i 2 i k there were let us say x there were t ways of doing that the total number of ways of placing rooks would be mean placing n rooks would be t times P m minus k times n minus k. Because, if k rooks have already been placed we have taken away k rows and k columns. The remaining m minus k columns and n minus rows could be can used.

So, t into Pm minus k n minus k ways are there to place rooks and that this t is just the number of ways of placing rooks on the complimentary board. This is for 1 particular i 1 i 2 i k if you submit up over all possible i 1 i 2 i k that is going to give you the total number of ways of placing k rooks on the complimentary board. So, we can write Nk as equal to P m minus k n minus k this is going to be the common factor multiply it by the total number of ways of placing k rooks in B prime.

So, this is going to be equal to Pm minus k n minus k times r k B prime. Therefore, r n b we can just write it as P m n this is the total number of ways. Now, we can apply inclusion exclusion minus N 1 plus N 2 and so on. And N 1 and N 2 we already know this is going to be P m n minus P m minus 1 minus and minus 1 into r k B prime plus P m minus 2 n minus 2 times r k instead of r k this will be r 1 and r 2 b prime plus and so on. So, if we can compute the rook polynomial of the complimentary board or the number of ways of placing rooks on the complimentary board from that count we can compute r and N B by using the principal of inclusion exclusion. We will stop here.