Discrete mathematics

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So in earlier classes we have learned about product of generating functions.

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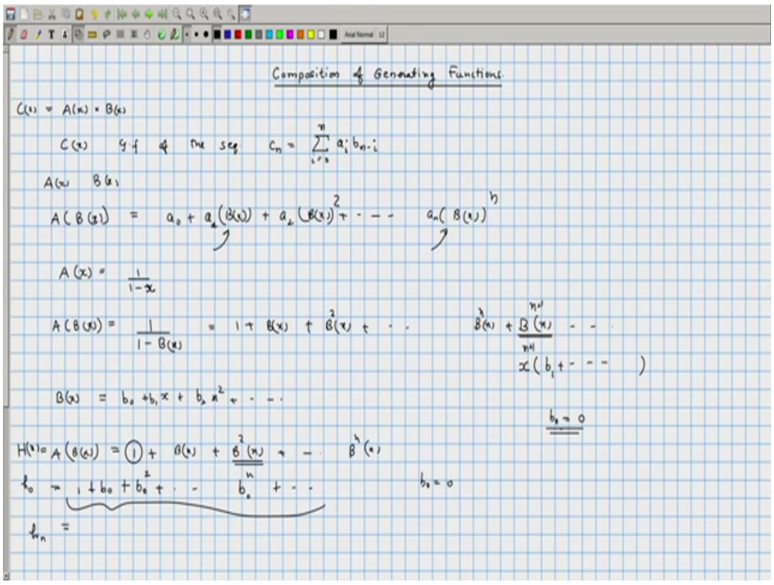

In particular, if we look at generating function Ax and the generating function Bx and multiplied them the sequences and if you call this as Cx. Cx basically corresponds to the generating function of the sequence C small n given by summation i going from 0 to n ai bn minus i. Or in other words you take the convolution of 2 sequences and what you get is, if you look at the sequence obtained by the convolution of 2 sequences.

Its generating function is going to be equal to the product of the generating functions of the underlying sequences. We want to understand more operations under generating function in particular we want to look at 2 generating functions let us say Ax and Bx and make sense of let us say A composed with B. So if you look at ABx this is A 0 plus A1 instead of X you have now Bx plus A2 Bx square plus so on.

When does this even make sense? Because now each term if you look at the nth term An, Bx raise to n this itself is an in finite series. Bx raise to n is going to be product of 10 copies of Bn and you are adding infinitely many of them, does it even make sense? When can we make sense of this kind of objects and what combinatorial objects do they represent? This is what we want to understand today.

So let us just take some simple examples. So let us say if we look at, let us say Ax is equal to the simplest of generating functions 1 by 1 minus x. And then A of Bx would be equal to 1 by 1 minus Bx and this we can think of as, let us say 1 plus Bx plus B square x, Bx the whole square and so on. What object does this denote? That is the first thing that we would like to understand and let us look at this Bx a little more carefully.

Suppose we call it as b0 plus b1x plus b2x square and so on and if you look at the constant term of ABx. So look at ABx, its constant term would be what? Well ABx is equal to 1 plus Ex plus B square x and so on. So the constant term from this is going to be one. The constant term from Bx is going to be b0. The constant term from B square x is going to be the product of the 2 terms that is b0 whole square and so on. And if you look at Bnx there is going to be b0 raise to n. And this goes on.

So if this b0 was some nonzero term then there are going to be infinitely many terms in this expression. So this is going to be the, so if you denote this by let us say H. The h0 term would be this. So in order to compute any particular coefficient you might have to sum up an infinite series and that may not really be convergent sequence. In our previous applications when we were looking at adding generating functions, multiplying generating functions and so on. Each term of the resultant generating function could be computed.

The nth term of the resulting generating function could be computed by just doing constantly many operations. Here that's not the case because we might have to do infinite number of operations and that may or may not be convergent… It may not, if you have infinitely many terms to add up then that might not converse to any suitable value. But here if we insist that b0 is equal to 0 then things fall in place properly.

And not just for this particular term, if you look at the nth term, let us say hn is going to be, let us say, so that is going to be coming from all these individual terms. But what we can say is if you look at Bnx. Now in Bnx and every term that comes after Bnx, you can see that since the constant term was 0, every term was going to be multiplied by an x raise to n.

So if you take let us say n plus 1 and that is going to be multiplied by x raise to n plus 1. Because Bx can be written as x into B1 plus something and when you raise it to power n plus 1. You have an n plus 1 term and therefore all the contributing terms in the formal power series will have x appearing as a power which is greater than n plus 1 and therefore they won't contribute to the term hn.

So in each of this expansion there are only finitely many terms 1Bx, so the last term you might have to consider is Bnx. So there are only these many generating functions. Finitely many generating functions to reconsider and aired up and therefore things work out very well when b0 is equal to 0. So while studying these compositions of generating functions we will assume that b0 is 0 for all of them.

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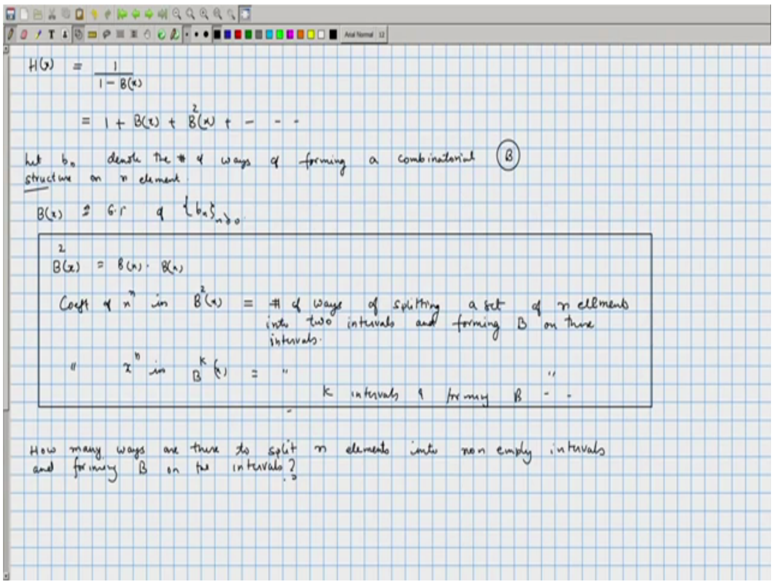

So now let us understand what is this 1 by 1 minus say Bx looking like? So what does the generating function if you call this as let us say Hx. What quantity does this denote? Clearly Hx is equal to 1 plus Bx plus B square x plus so on. So we will describe combinatorial quantity and show that, that combinatorial quantity or object will have generating function Hx.

So let bn denote the number of ways of forming combinatorial structure on n elements. So you take a set of n elements and the total number of ways of forming particular combinatorial structure could be let us say graphs and connected graphs on n vertices, could be number of trees on n vertices. Whatever combinatorial structure that you can imagine on n elements.

Let us say means some particular combinatorial structure on n elements. The number of ways of doing that on some set of size n we denoted by Bn and Bx is the generating function of that particular sequence given by Bn. Now our product rule says that if we look at B square x that is going to be just Bx times Bx. And the coefficient of x raise to n in B square x is nothing but number of ways of splitting a set of n elements into 2 intervals and forming a structure.

So let us call this combinatorial structure as, we will give it a name, we will call it as B. So forming, so there is lot of overloading of notations. So the generating function is Bx the combinatorial object we will call it by the name B and the number of combinatorial structures that you can form on n element that we call it by small bn. So B square X is the number of ways of splitting a set of n elements into 2 intervals and forming B on these intervals. One of the intervals could be empty in which case the other interval would be the full set.

So this is what we know by the product rule. So generalization of this would say that B raise to kx and the coefficient of x raise to n in B raise to Kx reason number of ways of splitting n elements into k intervals and forming B on these intervals. So that will be the coefficient of x raise to n.

Now the particular combinatorial structure that we want to count is, how many ways are there to split n elements into nonempty intervals and forming B on the intervals? So if you take n elements set the number of ways of doing this we will denote it by hn. And if we denote the sequence by hn it's generating function we can denote by capital HX and we will show that this is equal to 1 by 1 minus Bx. So let us see a more concrete example wherein we spell out what is a particular combinatorial structure B.

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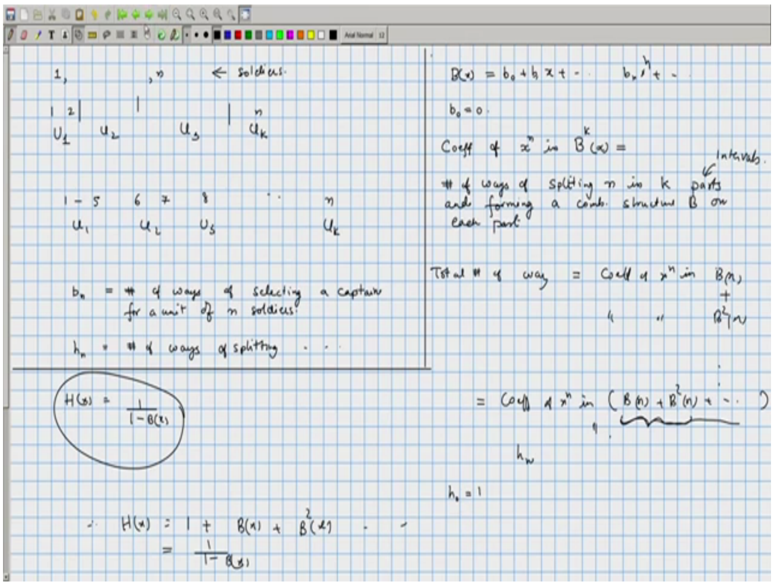

So here is a concrete combinatorial problem, we have let us say soldiers number 1 to n and they are placed on a straight line 1 to n and this generally wants to split this into some number of units. This is the first unit, this is the second unit, third unit and its k units. So k could be anything going from 1 to n. But each of them are each unit should be an interval. You cannot have let us say all the even numbered people going into 1 unit and odd numbers going to another unit that is not allowed.

Because the units themselves have to be intervals. So 1 to 5 could be 1 unit and 6 to 7 could be one unit. 8 could be in the single unit, it could have k units and k is a, you can choose k to be whatever you want. And once we are split this into units, you need to pick a unit captain. So the question is how many ways are there of splitting bunch of soldiers, splitting n soldiers into some number of units and choosing captain for each unit.

So bn would denote number of ways of selecting a captain for a unit of n soldiers. And note that since we are living the number of units unspecified it could be any number they will have to insist that each of the units are nonempty. Because if the units were allowed then we can have infinitely many such splits because there is no bound on the number of empty units.

So each unit will now have to be nonempty I suppose to the earlier case where we are splitting into 2 units and we are splitting into 2 units we cannot have let us say infinitely many empty units because the number of units is bound. So bn is this and hn is a number of ways of doing this of splitting. Now we need to argue that Hx is equal to 1 by 1 minus Bx. Why is this so?

So we will argue it for the general case. What we will see is, we will argue this for the general b and then we will solve our particular problem using this particular method and count the exact value of Hn using the generating function methodologies. So let us look at Bx is equal to b0 plus b1x plus bnx raise to n plus so on. Here we can assume that b0 is equal to 0 the number of ways of selecting a captain for unit with 0 soldiers, we will just assume it to be 0 because it does not make sense to, there is no national meaning for b0, so we can assign it to be 0 arbitrarily.

So what we have argued so far is, the coefficient of x raise to n in B raise to kx this will be equal to number of ways of splitting n into k units or k parts and forming combinatorial structure B on each part and parts here or intervals. So total number of ways of splitting, so where the part could be anything will be equal to coefficient of x raise to n in plus coefficient of x raise to n and B square x plus so on.

So that is equal to coefficient of x raise to n in Bx plus B square X plus so on. This is equal to hN. So coefficient of x raise to n and this expression is equal to hN for N greater than or equal to 1 by our assumptions we will take h0 is equal to 1 and therefore Hx is equal to 1 plus Bx plus B square x plus so on and this is equal to 1 by 1 minus Bc.

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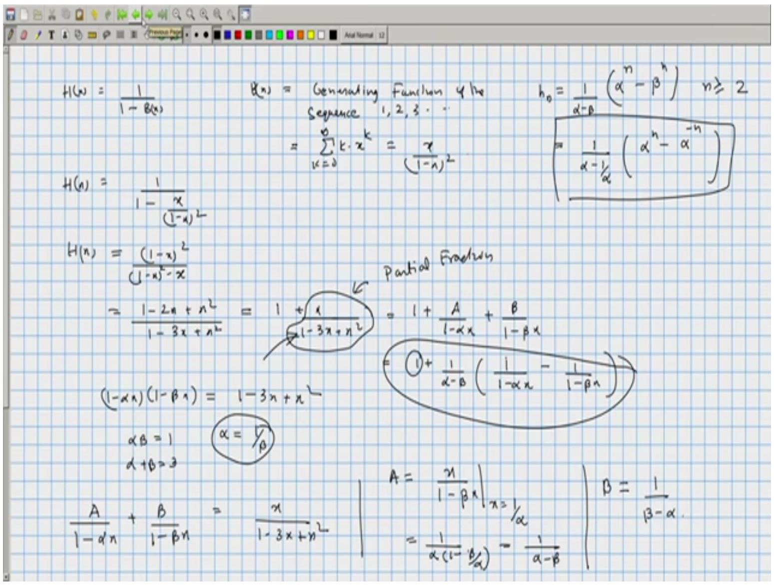

So in our particular case where we were looking at splitting you group of soldiers into n units we can say that Hx will be equal to 1 by 1 minus Bx and Bx is nothing but generating function of the sequence 1, 2, 3 so on because this is the number of ways of selecting a captain for a group of it k elements. So that can be run in k ways and this is nothing but summation k, x raise to k. k going from 0 to infinity and that is going to be equal to x by 1 minus x the whole square.

So therefore Hx will be equal to 1 by 1 minus x by 1 minus x the whole square multiplying with 1 minus x the whole square on both numerator and denominator. What we get is, 1 minus x the whole square by 1 minus x the whole square minus x is Hx that is equal to 1 minus 2x plus x square divided by 1 minus 3x plus x square that is equal to 1 plus x by 1 minus 3x plus x square.

Now this term we can do partial fractions on this, compute the partial fractions. So this is equal to 1 plus A by 1 minus Alpha x plus B by 1 minus beta x where 1 minus Alpha x into 1 minus beta x should be equal to 1 minus 3x plus x square. So comparing coefficients that would mean Alpha beta is equal to 1 and Alpha plus beta equals 3. So Alpha is equal to 1 by beta that is something that we know, and if we plug that in here, what will be the value of A?

So A by 1 minus Alpha x plus B by 1 minus beta x should be equal to x by 1 minus 3x plus x square. If we multiply both sides by 1 minus Alpha x and put x is equal to 1 by Alpha we get the value of A. So A is equal to x by 1 minus beta x evaluated at x is equal to 1 by Alpha. So that is going to be equal to 1 by Alpha into 1 minus beta by Alpha that is equal to 1 by Alpha minus beta.

Similarly, if you do, B will be equal to 1 by beta minus Alpha. So the entire expression Hx is equal to 1 plus 1 by Alpha minus beta into 1 by 1 minus Alpha x minus 1 by 1 minus beta x. So from this we can simply write the nth term, so Hn will be equal to, this is a constant term for n greater than or equal to 2 we can write it as 1 by Alpha minus beta into alpha rays to n minus beta raise to n.

So that will be the final formula where I mean if you substitute Alpha is equal to 1 by beta what you will get is, 1 by Alpha minus 1 by Alpha into alpha raise to n minus Alpha raise to minus n. So this will be the final expression where Alpha you can think of as the larger root of 1 minus 3x plus x square is equal to 0. The larger root of this particular…

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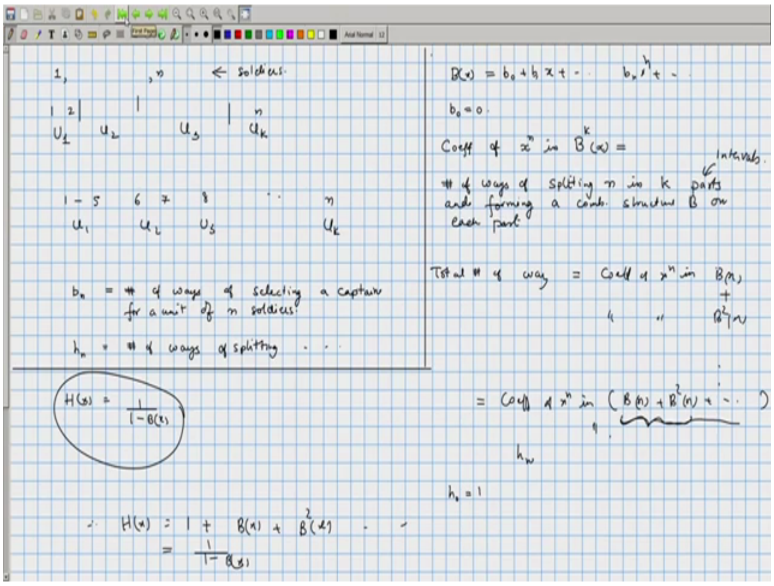

Now we will look at a slightly more general problem. Since here what we did is, we look at a collection of n soldiers who were arranged linearly and then we split it into intervals and on the intervals we were picking the captain. Now suppose we could do additional operation on the intervals themselves. For example, for each interval we could say that we want to either take that unit for my duty or not give night duty to them.

So how many ways are there to do that? So clearly you can split it and then the total number of ways of splitting you will get some number and for each split you can do additional operations on them like choosing them for special duty.

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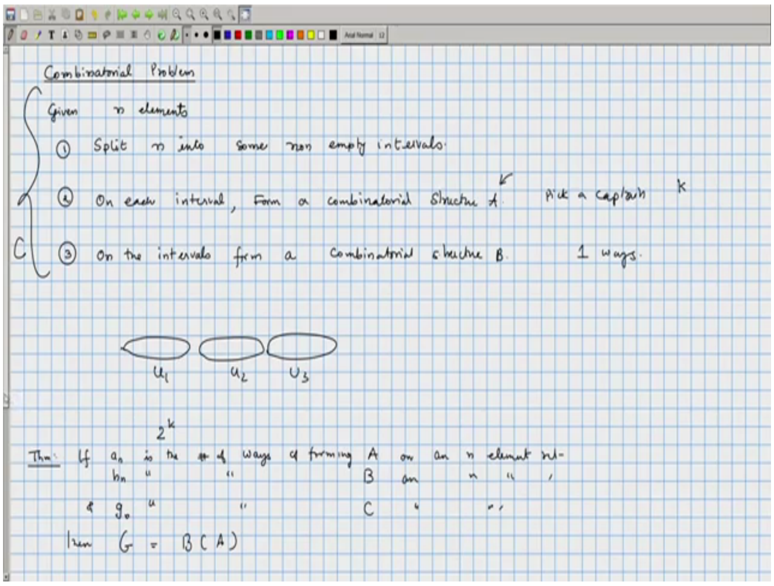

So let us look at that particular problem. So we are given, this is the combinatorial problem that we are interested in. There are n elements. The first is to split n into let us say some non-empty intervals and then on each interval we may form particular combinatorial structure. Form a combinatorial structure which we will call it as a and suppose the split of n into non-empty intervals thereof k intervals.

We will say on the intervals from a combinatorial structure B. So you can think of n as the soldiers and A the combinatorial structures that we were interested in is. Pick a captain. So there were k ways to do, mean each of the interval, if the interval was of size k, there are k ways of doing this. And on the interval we have formed this combinatorial structure namely we had only the trivial structure initially that means there is just one way of doing things.

Now suppose we had more ways of performing combinatorial structures on the intervals themselves. So if we had split the soldiers into 3 blocks. We had just one way of finding we did not have any extra (()) (26:37) on this. We were happy with the split but now we are saying that look each unit, let us call it as U1, U2 and U3, some of them could be chosen for special duty.

Let us say U1 alone was chosen or let us say U2 alone was chosen, the ways of doing it. There are 2 raise to k ways of choosing the second combinatorial structure B. So this is a general question that we want to show and we will see that if we take the compositions of the generating function for combinatorial structure A and B in the correct order then we will get a generating function for this particular combinatorial problem.

So if Ax this is our theorem. If An is a number of ways of forming A on an n elements that. bn is the number of ways of forming B on an n set. And let us say gn is a number of ways of forming this complex combinatorial structure involving both A and B, so let us call that as C then gn will be equal to B composed with, so the generating function for gn, g will be equal to B composed with A.

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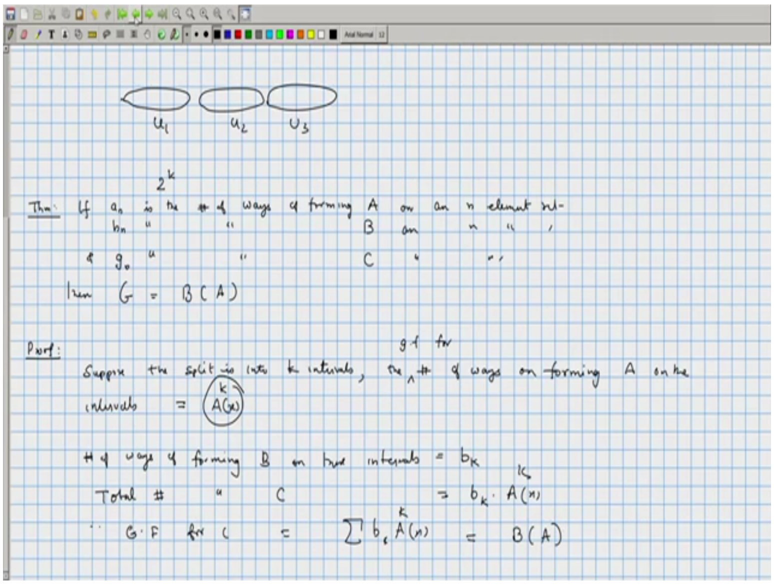

How do we see that, so suppose we split it into, so what we have is we could split it into any number of intervals. Suppose we had splitted into k intervals. So suppose this split is into k intervals. The number of ways of forming A on these intervals is, that will be equal to means the generating function for that. So this will be equal to generating function for the number of ways of forming A on the intervals will be equal to Ax raise to k.

Number of ways of forming B on these intervals is equal to bk, so if you had just k intervals there are bk ways of forming combinatorial structure B and since there are these many ways of splitting the total number… So the generating function for the total number of ways of forming C will therefore mean if you splitted into k intervals that will be equal to bk times Ax raise to k.

And therefore the generating function the complete generating function will be equal to, so GF for c will be equal to summation bk Ax raise to k. And this is nothing but B composed with A, so that the proof. And let's see how we can solve our problem of selecting.

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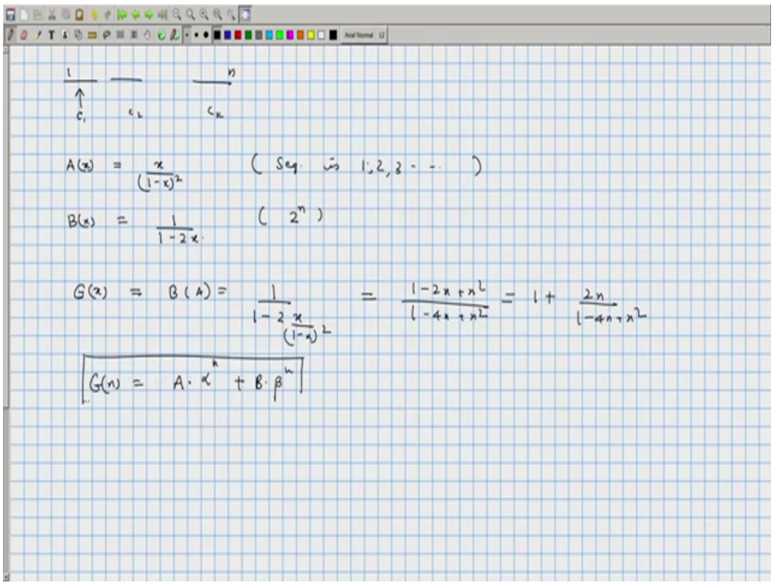

So now the question that we have the combinatorial problem, the concrete problem is we have n elements 1 to n. We need to split them into some number of intervals and each interval is given a captain, it will say C1, C2, Ck are the captain's and each interval additionally told whether they will be doing night patrolling or not. So the total number of ways of doing this, so Ax is the number of ways of selecting the captain in a group of n people, so this would be equal to x by 1 minus x whole square.

So the sequence is 1, 2, 3 and so on. And Bx is the number of ways of assigning duties, special duties, so that will be group can either be given night patrolling duty or not be given. So there are 2 raise to n ways, so the generating function corresponding to that would be 1 by 1 minus 2x. And the total number of ways of forming the complex editorial structure which takes into consideration both x would be x, if you denote it by Gx there is going to be decomposed with A that is equal to 1 by 1 minus 2 instead of x we need to put x by 1 minus x whole square.

So that is going to be equal to multiply 1 minus x the whole square numerator and denominator will get 1 minus 2x plus x square divided by 1 minus 4x plus x square that is equal to 1 plus 2x by 1 minus 4x plus x square. Again we can use the partial fractions method and show that Gx is going to be equal to some of 2 exponentials. If you solve the partial fractions correctly we will get Gx is equal to A times Alpha raise to n plus B times beta raise to n, So that will be the end of this lecture.