Discrete Mathematics
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Lecture 30:
Product of Generating Function

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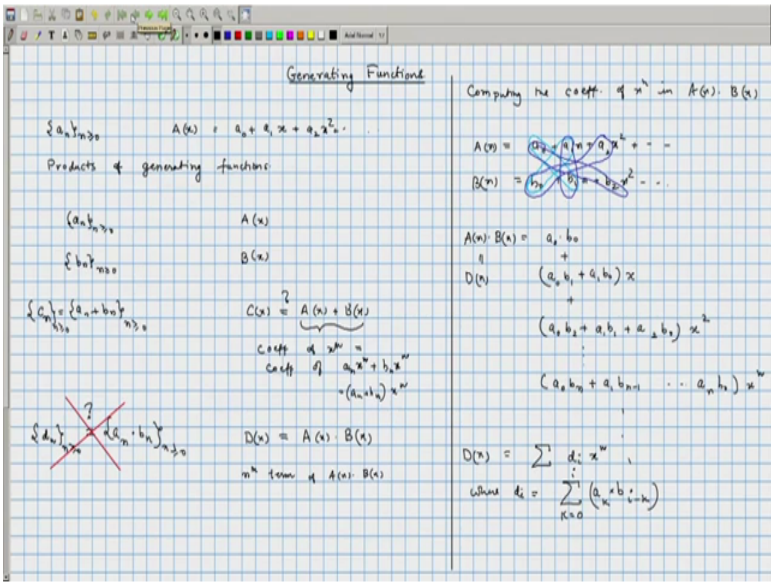

So, earlier we learned about generating functions, so if you have a sequence, let see we denoted by an and greater than or equal to 0, we can capture the information that is there in that particular sequence of real numbers by its generating functions which we denote by A x. So, Ax is nothing but, the formal power series A0 plus A1 x plus a2 x square and so on.

So, we saw an example where we looked at a sequence coming out of some combinatorial property, computed its generating functions and used the generating function to obtain a closed from solutions for, the nth term of the sequence. So, today, we will look at combining generating functions, in particular we will look at products of generating functions, and understand their significant from a combinatorial stand point.

So, let us look at the following sequence, look at 2 sequences one is a n and its generating functions let see we denoted by a x and there is another sequence b n and its generating functions we denote by, Bx. Now, if we define a new sequence Cn which is equal to a n plus bn so, the nth term of the sequence is a n plus bn and the sequence is denoted as C n, so for this C n what will its generating functions be. So, will Cx be equal to Ax plus Bx and it will be because if you look at Ax plus Bx the n their term.

The coefficient of the n th term, of x raise to n is equal to so, a n x raise to n and Bn x raise n are both present, then Ax plus Bx and their sum which is An plus Bn is going to be the company efficient of x raise to n. So that so to add to generating functions is simple, so if we add 2 sequences pointwise addition of the sequences will result in a new sequence whose generating function is simply, the sum of the generating functions of the previous sequences.

Now, if you take the product of generating functions, what happens, so instead of looking at Ax plus Bx, let us say we have Dx is equal to Ax into Bx, so this is going to be Dx is clearly a generating function of some sequence, so if we denote Dn by the nth terms coefficient, will this be equal to a n into Bn. So, this unfortunately not the case, but whatever is Dn we will understand that in more detail today.

And see some examples of using this understanding to solve some combinatorial problems. So, the math is very simple, we need to determine the nth term, of Ax times Bx or rather the coefficient x raise to n in Ax times Bx. So, if we look at, so we need to compute the coefficient of x raise to n and Ax times Bx. So, if Ax by definition it was a0 plus a1x plus a2 x square, and so on and Bx is equal to b0 plus b1x plus b2x square and so on.

When you take the product Ax times Bx, which is what we called as Dx this is going to be the coefficient of the constant term is just a0 times b0. The coefficient of x raise to 1 can come from a0 b1 plus b0 a1, both these products are going to result in x and their sum is going to be the coefficient of x raise to 1.

So, we will write this as a0 b1 plus a1 b0 times x and the next term, the second term or the coefficient of x square will essentially be a0 b2 a1 b1 and a2 b0 so there is nice pattern in the sequences. So, the next term will be a0 b2 plus a1 b1 plus a2 b0 the whole times x square. And x cube term would essentially be a0 bn plus a1 bn minus 1 up to a n b0 multiplied by x raise to n and so on.

So, Dx can be written as summation of di x raise to n where di is equal to summation K going from 0 to I ak bi minus k. Okay, so and this is, so the sequence of obtained in this particular manner so called as a convolution of a and b. So, when you multiply the generating functions, the sequence that you will get from the product of the, 2 generating functions will be a sequence which is the convolution of the 2 underlying sequences. So now, let see some particular applications of this.

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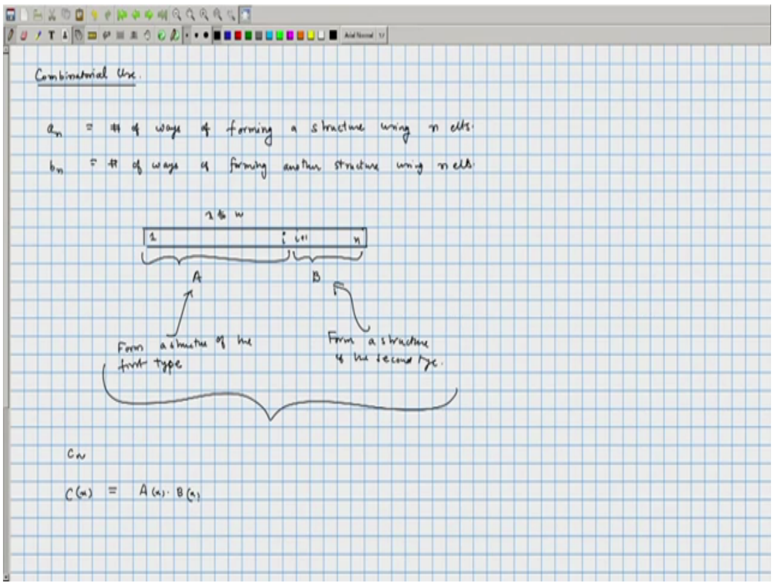

We will see combinatorial use, so let a say for us a n we will denote the number of ways of forming some particular object, some particular combinatorial objects or structures. Using say n elements and the number of ways of forming a certain structure using these n elements. So, this n could be a letters means, 10 letter or n letters of some particular alphabet and you want to look at words which can be formed without containing a let a say certain pattern.

And bn is also, so this is for the number of ways of forming different structures, could be even the same structure so, is called as another structure, you will see n elements. Now, let us look at the following problem, we have given elements 1 to n, we are allow to form a subset A containing elements 1 to let us say i and B containing elements i plus 1 to n and on this set A form a structure of the first kind or we will call it as type A and in this you will form a structure of the second kind.

And how many ways are there of forming this, and what we are interested in us, this whole new object, there are how many ways to form this kind of objects that is what we want to count. And that can be, we denote that by, it is a CN, so CN has a number of ways of splitting a set of n elements into 2 parts so that the first part is 1 to let say k, this k is your choice and then the second part is from k plus 1 to n and you need to form a structure using the first k elements and the another structure using the next elements.

The total number of doing this is what we denote by CN and what we know is the generating function of CN which we denote by Cx is going to be equal to Ax times Bx so, this is the combinatorial use of whatever we had learned about generating function earlier. Let see here more concrete applications.

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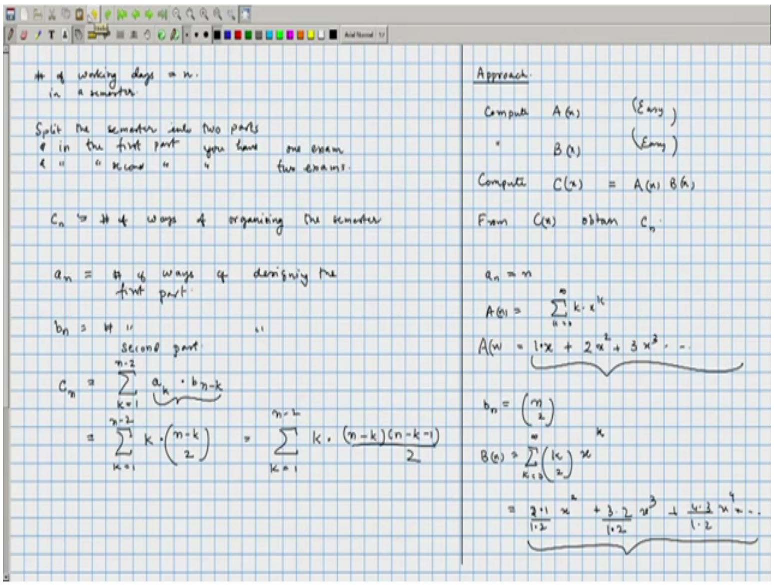

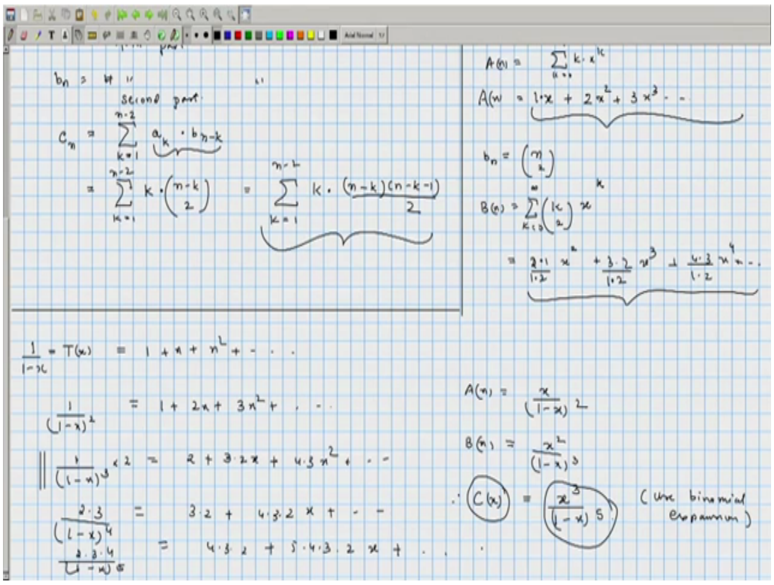

So, let a say that you studying in university and there are n working days, and what is required you have to split the semester into 2 halves and in the first half you have 1 exam, and in the second half you have 2 exams so, semester consist of n days and out of these n days you have to select 3 days for conducting examinations, and out of these 3 exams 1 exam should be in the first half so there is designated first half and there is a designated second half.

The first half will have precisely 1 exam and the second half will have precisely 2 exams. You are free although we call it as half we can just split it into 2 parts so, the first we should probably effort towards the first part the first part will have 1 exam and second part will have 2 exams. There are how many ways of designing the semester with these exams. That is what we need to compute let Cn denote the number of ways of organizing the semester in this particular way.

Now if you denote a1 or ai so a n denotes a number of ways of designing the first part and bn denotes the number of ways of designing the second part. And Cn we can simply by looking at the problem say the Cn will going to be first we have to split the semester into 2 parts and for each part you can design the examination days and whatever way please and then you will get the total numbers.

So that is going to be summation over k going from the first part k is the number of elements or the number of days in the first part so k go some 1 to n minus 2 because the first part contains n minus 1 days the second part can contain only 1 way so it is not going to give a meaningful split with 2 days or 2 examination days and therefor k go some 1 to n minus 2 and ak times bn minus k.

So that is going to be summation K is equal to 1 to n minus 2 ak is the number of ways of picking 1 day out of k days so that is going to be k times take this as n minus k, n minus k choose 2 is the total number of ways.

So this, is what we need to compute k times n minus k, choose 2. You can expanded out and work out so this is going to be summation k equals 1 to n minus 2 k into n minus k, into n minus k, minus 1 by 2 so you can expand out the terms. N is fixed so you can take an outside so there will going to be 6 terms on the expansion, you can expand out the terms and you will have a k rest 1 and k square and k cube appearing, you can sum it up, and you will get some close form solutions.

In this part what we will see is, how to do the same calculations without using the formulas for summing up polynomials of degree 2 and 3, etc. you will do it with the help of generating function. Because generating functions for a n and bn are fairly easy to compute. So, what we know is, the following and so this our approach, compute Ax compute Bx this will be easy to compute we will see, and then Cx since Cn is equal to Ak times bn minus k. what we know about the part of generating functions, we can say that Cx will be equal to Ax times Bx.

And then, from Cx obtain Cn because Cn is the convolution of a n and bn. We can just compute the generating function Cn by just taking the product of Ax and Bx. So, let us see what is Ax now, a n is equal to k, therefore, Ax is equal to k into x raise to, so this is going to be summation K x raise to k, k going from 0 to infinity. Or this is going to be equal to 0 term does not contribute, x plus 1 times x plus 2 times x square plus 3 times x cube and so on. And bn is equal to n choose 2.

That is the number of ways of picking 2 days choose 2, that is number of ways picking 2 days for exam, and therefore, Bx is equal to k choose, so greater than an equals k, an is equal to n or ak equals k, the x equals k chose 2 x raise to k, summation K equals 0 to infinity. So k equals 1 k equals the 0 terms are going to be absent, so this is going to be equal to 2 into 1 by 2 into x square plus 3 into 2 by 2 into x cube plus 4 into 3 by 1 into 2 into x raise to 4 and so on.

So all these expressions we need to somehow obtain a nice close form solution for this. And that is what we will do, so all these are very nice expressions whose close forms can be easily obtained from the following generating function. Let say, Tx is this particular sequence 1 plus x plus x square so on. And Tx clearly is 1 by 1 minus x now, if you differentiate both side what we will get is, 1 minus x raise to 2 is equal to 1 plus 2x plus 3x square and so on.

That is pretty much same as Ax just 1x is missing, so therefore, from those we can conclude that Ax is equal to x by 1 minus x the whole square. And if we differentiate this once more, what we will get is, 1 by 1 minus x raise to 3 into 2 is going to be equal to 2 plus 3 into 2 x plus 4 into 3 x square and so on. So this sis almost this is same as Bx, but there are few missing terms, so if you just apply the missing terms when all the ones has to do is multiply by x square.

So multiply both side the x square by 2 we will get Bx is equal to x square by 1 minus x the whole cube, and therefore, Cx is just going to be product of Ax and Bx that is going to be x cube by 1 minus x raise to 5. So now, what we know us, this summation that we are looking at which gives us the nth term of the number of ways on splitting the semester is a sequence whose generating function has a simple form. x cube by 1 minus x raise to 5.

Now, how do we find the n th term of 1 minus x raise to 5, you can use the generalized binomial expansion, so we just need to expand 1 by 1 minus x raise to 5, and then shift everything by 3 because there is an x cube. We need to find the x raise to nth term inside this, if you find x raise to nth term inside this, that is going to be equal to the x raise to n plus 3^{rd} term in 1 minus the expansion of 1 minus x raise to minus 5.

So, let us look at how we can write this as a sequence, so again we will look at this particular equation to differentiate it once more what we will get is, 2 into 3 by 1 minus x raise to 4 is equal to 3 into 2 plus 4 into 3 into 2 into x plus 1. To differentiate yet another time we will 2 into 3into 4, by 1 minus x raise to 5, that is the left hand side that will be equal to 4 into 3 into 2 plus 5 ,4 ,3, 2 into x plus so on. And what we need is, this multiplied by x cube, so we can write this in the following manner.

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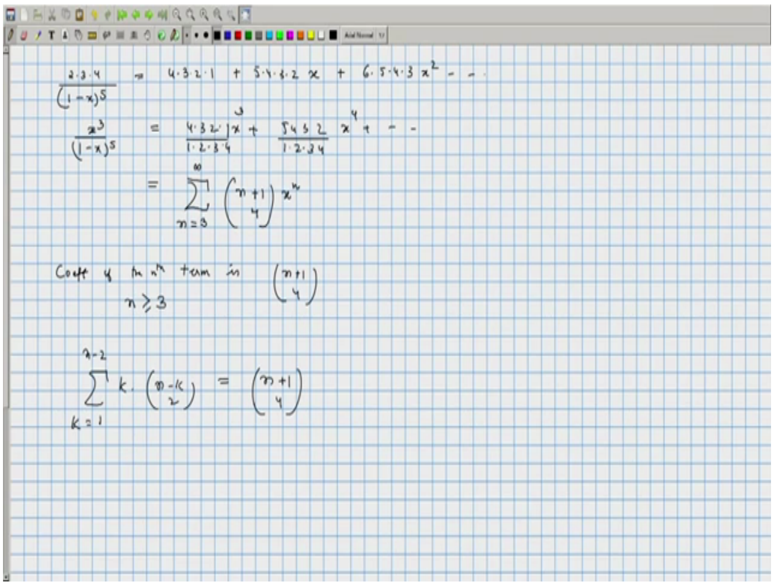

2 into 3 into 4 by 1 minus x raise to 5, is equal to 4, 3, 2, 1 plus 5, 4, 3, 2 into x plus 6, 5, 4, 3 into x square and so on, so if we just multiply by x cube, on both sides and divide by 2,3,4 what we will get is, 4, 3, 2, 1 divided by 1, 2, 3, 4 plus 5, 4, 3, 2 by 1, 2, 3, 4 into there be an x cube, into x raise to 4 and so on.

So, basically this is going to be equal to summation over n going from 3 to infinity. X raise to n, n plus 1 choose 4. So that would mean that the coefficient of the nth term is n plus 1 choose 4 so this will apply only when n is greater than or equal to 3. The other terms are going to be 0, rightly, so because we want to split with the first part having 1 holiday and the second part having 2 holidays you need at least 3 terms.

So that is basically the answer, of that combinatorial identity summation K into n minus k, choose 2 k going from say 1 to n minus 2. This will be equal to n plus 1 choose 4. Let look at one more problem.

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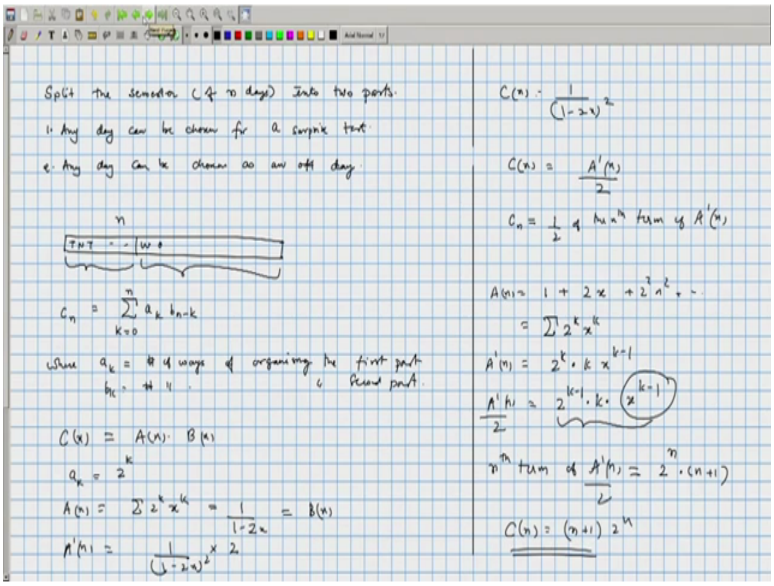

We will not do it in these details; we will just quickly rush through the problem. We again need to split the semester in the 2 parts, so semester has n days and now, any day of the first part can be chosen for a surprise test, so the possibilities is that we have us the, there could be surprise test on all days, there could be surprise test, there could be no surprise test every day, I mean you do not have any surprise test so that is a possibility and the, in the second part there could because e a surprise holiday any day can be chosen as an off day.

Another strange way to have a semester, but that are problem we have, so we have n days and these n days must be split into 2 parts, and the first part, so you could have let a say test or no test, any choice is okay, and in the second part, you could have a working day or an off day and that could be chosen in any way, the way it pleases. So, we want to know how many ways are there to organize such a semester.

So, again Cn is the number of ways, Cn is going to be summation K going from 0 to n, a n so ak bn minus k where ak is equal to number of ways of organizing the first part and bk is the number of ways of organizing the second part and what we know is, see generating function of Cn that is going to be equal to Ax times Bx. so Ax is generating function of the sequence a n and a n is going to be 2 raise to n because if you had n days or k days, every day you could either have the test or no test.

So there are 2 possibilities, total there are 2 part or k possibilities, therefore, Ax is equal to summation 2 raise to k x raise to k, k going from 0 it is going to be 1 by 1 minus 2x and Bx is also the exactly same thing instead of surprise test, we have an off day or working day, so this is go also going to be equal to Bx. And therefore, we can write Cx is equal to 1 by 1 minus 2x the whole square.

Now, from this how do we extract out the nth term, we would again use binomial theorem or the binomial expansion and from that we can infer, but here there is an easy way, so if we denote so Cx is equal to A prime x by 2. So, if you take A prime, so A prime is nothing but 1 by 1 minus 2x the whole square into 2, so Cx is also equal to A prime x by 2. So, Cn is equal to half of the nth term, of A prime x and the Ax is equal to 1 plus 2x plus 2 square x square and so on.

2 raise to k x raise to k summation, so A prime x is equal to 2 raise to k into k into x raise to k minus 1, so A prime x by 2 is equal to 2 raise to k minus 1 into k into x raise to k minus 1. And therefore, the nth term, of this of A prime x by 2 is nothing but coefficient of x raise to n that is going to be equal to 2 raise to n into n plus 1. So, we can conclude that Cn is equal to n plus 1 into n. So, we have seen 2 examples where the use of products of generating functions to compute combinatorial quantities.

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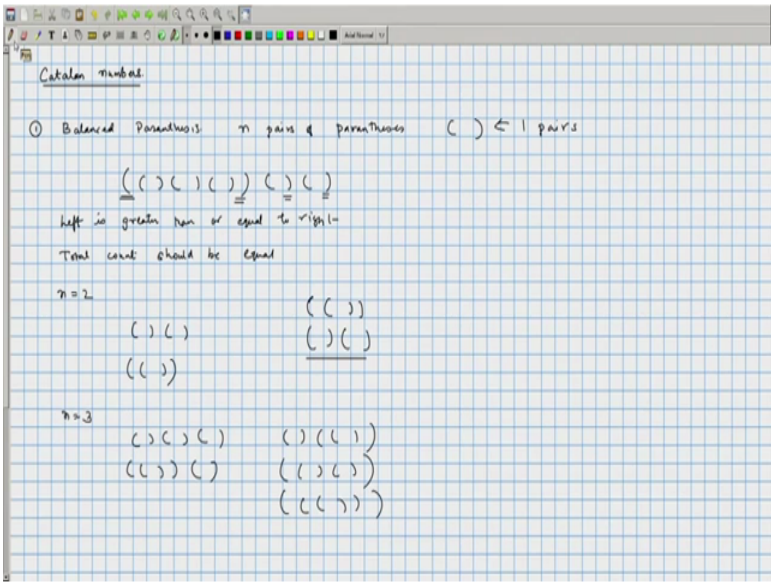

So, the third example is a more classical example, this involves, what is the known as the Catalan numbers. So, Catalan numbers arise in a wide variety of context, here we will see 2 examples, both of them are essentially the same combinatorial object mass spreading different thing, first thing is balanced parenthesis, so we have let a say n pairs of parenthesis, so left and right parenthesis, so there are so this is 1 pair and we have n such pairs. And we want to rearrange them in any manner but, the result should be a balanced parenthesis.

That means that should come out of parenthesizing some expressions in particular all it means as every left parenthesis i mean a right parenthesis should come only after the left parenthesis an the totally number of left parenthesis should always be greater than or equal to the number of right parenthesis okay, so left is greater or equal to right, if we look at the number of left parenthesis that is appeared in any prefix your parenthesis the number of lefts are going to be greater than or equal to the number of right and total count should be equal.

So this is an example of a balanced parenthesis. So here, I mean at the start there is 1 and the number of left is always greater, but except at the very end where they become equal, and if we take this here, the count becomes the left minus right becomes 0 at this point, 0 again here and finally when the full expressions is right at that point also it is 0, so we want to find the number of ways of arranging this parenthesis so that it is balanced.

So, let us take the example where the number of pairs n is equal to 2, so this is one way and another way would have been and these are the only possible ways because it should begin with the left, so the next one can either be a right or a left. If the next one is a left, so this is possibility if it is a right, the only way we hold expression can be made balanced by having. So, these are the only 2 possibilities.

And n equals 3 you have this as one possibility, this is yet another possibility, so when n is equal to 3, there are these 5 possibilities and we need to compute the value on the number of balanced parenthesis for the general n.

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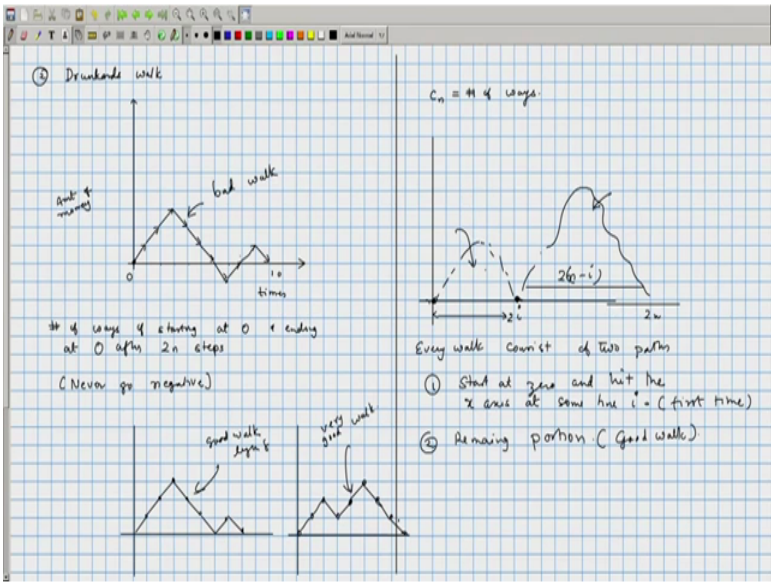

This is very similar problem, which is also known as the Gambler Ruing or the Drunkards walk, so this is the x axis denotes time and y axis denotes the amount of money or the position from your, from there you started, just call it as amount of money and you start with 0 units of money and at each time the total amount of money that you can have rather go up by unit or come down.

So can go up and again up and then going up, and then you could come down, so here it is a sequence of steps first that you started, at time 0, with the 0 amount of money and after 10 units of time you are still at 0, but in between you had gone negative. We want to find the number of ways of starting at 0, when a time 0, 0 amount of money and ending at 0, after it say 2n step it has to be always even number of steps because starting at some place and ending at the same amount of money.

The total number of steps that you would have taken should be even. So after 2 end steps how many ways are there, by which you can reach the same starting position and the additional constraint is we should never go negative. So both these problem the think of this Drunkard walk, it is essentially at Drunkard starting at some particular position, each step he takes he either goes one step close at it is one dimensional walk he is going towards whom or coming back it is never allowed to be at distance greater than what he was from his home at any point of time.

So under that restriction what is the I mean he is never allowed to go in towards to have the access, so how many ways are there to do this. So gain we will use the notion of generating functions but, it is slightly more trickier than what we have done in the earlier case. So, Cn let a say denotes the number of ways, so look at all possible steps that you can take, so this we will call as a bad walk, whereas if you had the following walk, this is a good walk.

Of length 8 so we want to find the total number of good walks we will also define what is known is a very good walk let me just give an example of a very good walk. So this is an example of a very good i will formally define what is the very good walk, we will do that shortly, so let us look at any of these walks we can split it into 2 parts the first part will consist of a very good walk, and the second walk will be a good walk.

So we essentially be taking the convolution of these 2 things to get your answer so let look at some particular walk, 2 things can happen, it starts at 0, and then it hits the x axis at some point of time and there again goes up, or it hits the x axis only at the very end. So, let look at the first time when it hits the x axis, so look at the first time when a walk comes back to x axis, now if we look at the remaining portion of the walk, that is essentially just a good walk of the same length.

So we can think of every walk as split into 2 parts, the first part is start at 0, and hit the x axis at some time i, so this i is it seating the x axis for first time, an then remaining portion now the remaining portion has to be a it has to be exactly same type of the walk it just has to be a good walk. What about the first portion, the first portion, has also have good walk but, it has a special kind of good walk.

In the sense this portion this, means so if you take this as let say if you do not by i the time when at first hit the x axis now, any walk of length 2 n minus i. You take any good walk of length 2 n minus i, put it here in this region between i and 2n and take these 2 portions you will essentially get a good walk. Whereas if you take this region 1 to i, and put, replace it with a good walk.

You will not get something of the kind we talking about because what we want here is at the walk should not have touch the x axis at any point of time. So the number of ways of constructing walks by combining these 2 walks, it is not going to be just the product of 2 simple good walks, it is a product of a simple good walk and something which is very good in the sense, it is never hit the x axis okay.

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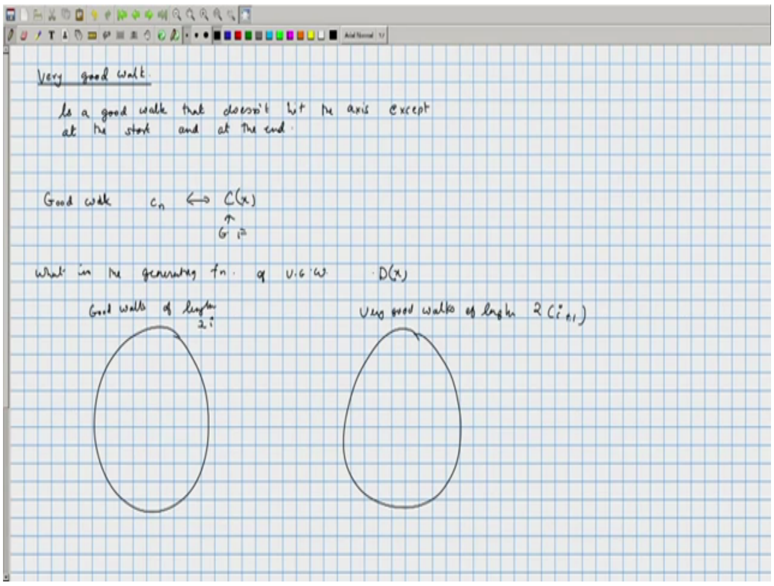

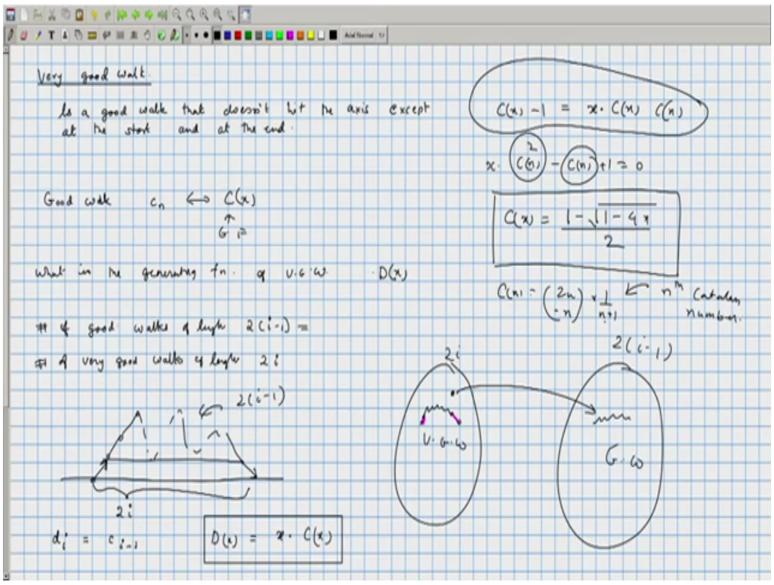

We will formally define what is a very good walk? So, very good walk is a good walk that does not hit the axis, except at the start and at the end. So, good walk its generating functions is the number of good walks of length Cn, its generating function is going to be let say Cx. What is the generating function of very good walk? That is what we need to determine.

And if we say that the generating function of very good walk, suppose we call it as Dx then since the first for n equals 0 the split does not work we will have to write Cx minus 1 is going to be equal to Cx times Dx. Now, from this equation we can solve for Cx but, assuming we know what is Dx. So, let first compute the generating function of Dx, so let us look at the good walks of length Dx of length 2i and these are the very good walks of length, let a say 2 times i plus 1.

What we will do is, now to compute this we will show that the number of good walks of length 2 times i minus 1 is equal to number of very good walks of length 2 I, why is this so. So, let us take a very good walk of length 2 I, so that we will start any walk, good or very good, has to start with an upper with the positive arrow and the last one of that should be a downward arrow. First one should be upward and the last one should be downward and this entire length is going to be let a say 2 i and the very good walk has the additional properties as it never touches the x axis anywhere in between.

So let us take a very good walk and split it off the first and the last moves, what you will get is some walk of length 2 times i minus 1. So, if we look at the set of all very good walks of length 2 i and all good walks of length 2 times i minus 1. We can have a one to one correspondence, we take a particular walk and strip of its first and last moves, so if you strip this off you will get something on the other side.

If you take 2 distinct elements of the very good walk and strip it off its first and last elements what you get will be distinct elements of the good walks. Further you take any good walk and add these first and last moves, you will get a very good walk. So, every element of GW can be generated from VGW and every element of VGW gives rise to a unique GW.

And therefore, these sets are in one to one correspondence so, that will also mean that the generating functions can be quickly computed. So now if you denote d i by the i th element when at the, if you denote d i by d i the number of very good walks, of length 2 i and d i is going to be equal to Ci minus 1. And therefore, the generating function Dx just going to be equal to x times Cx.

So that is the key property so from this we can write Cx minus 1, minus 1 because split works only for walks of length greater than or equal to 1, so this is going to be equal to x times Cx times Cx. So, we can rewrite this as x times Cx square minus Cx plus 1 equals 0, think of Cx as a variable and if you solve the quadratic equation involving Cx we will get Cx is equal to 1 minus under root 1 minus 4x by 2.

So when you solve the quadratic equation there 2 possible routes but, only one route will make sense for this particular generating function and you can check then it is the route with the negative sign. Now, we have the generating function for Cx from this we can obtain the close form solution. So, 1 minus 4x raise to half you can use binomial expansion that will have a one as the leading term and that will subtract from this one and therefore, the constant term vanishes, now when the constant term vanishes every other term will be an even term and the you can show that Cx will be other form.

2n choose n into 1 by n plus 1. So this is the n th Catalan number, this number has a name, the Catalan number we will see that many other recurrences of which gives rise to generating functions of this kind gives Catalan number as the answer. We will stop here.