Discrete Mathematics
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Lecture 3
Mathematical Logic

Welcome to the MOOC on discrete mathematics, this is the third lecture on mathematical logic. In the previous lecture, we talked about propositional logic. In propositional logic, we have propositions and truth values to propositions. We saw, how propositions can be combined to form larger composite propositions and how the truth values of the component propositions will combine to form the truth values of the larger propositions.

But not every logical statement can be captured using the apparatus of propositional calculus there are some arguments for which propositional calculus are not adequate.

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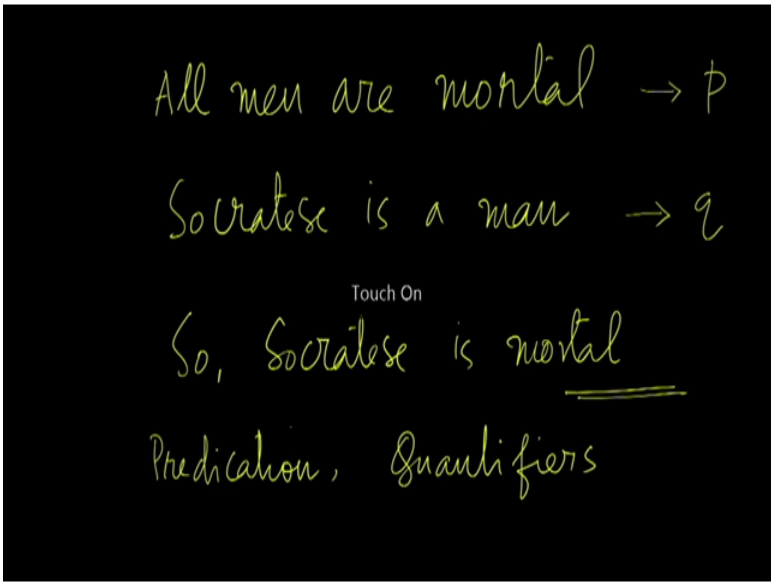

For example, consider the statement of this form, all men are mortal, Socrates is a man, so Socrates is mortal. In this statement, we form the conclusion from the first two propositions. So, if we call these propositions, let us say this is proposition p and this is proposition q whether proposition p and q are true or false will not help us in forming the third conclusion. So, even if we assume that the first two propositions are true, there is no way we can conclude that the third proposition is true using the apparatus of propositional calculus that is because the statements include predication and quantifiers.

So, let us see what we mean by this.

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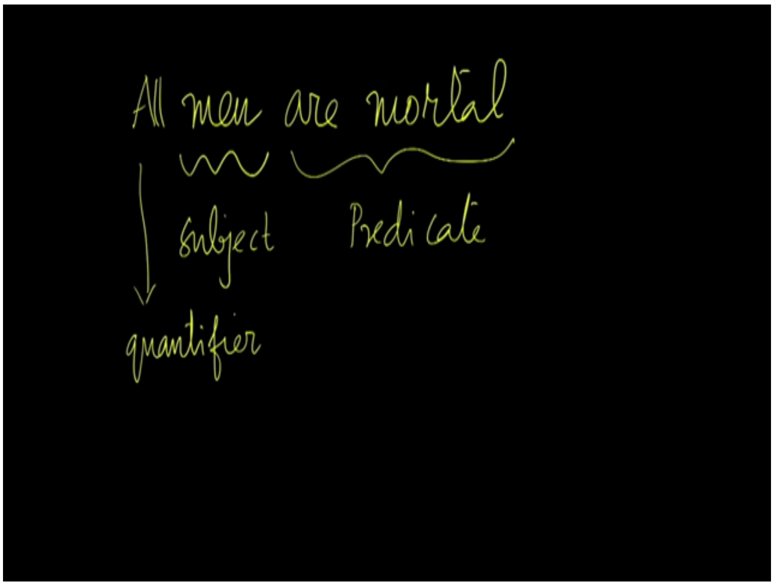

In the statement, all men are mortal, men form the subject of the sentence and are mortal form a predicate or a sub quantifier.

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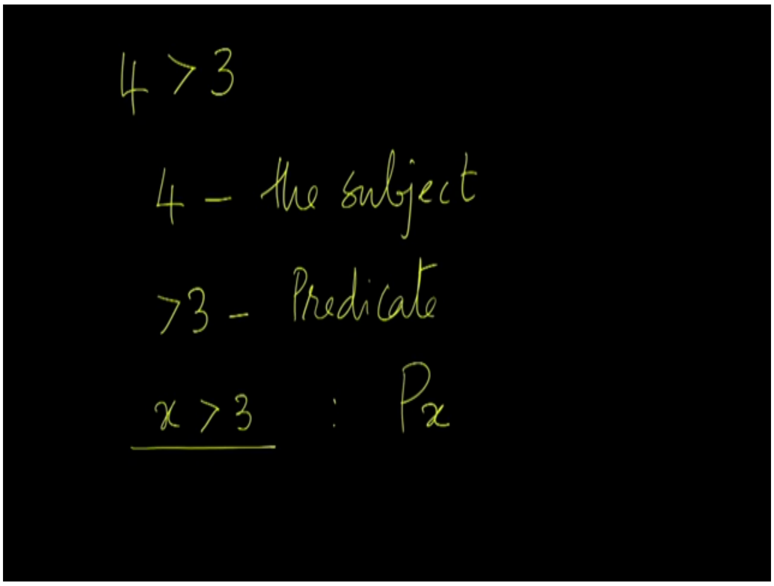

In general, when we have a sentence of the form, 4 greater than 3, we can say that 4 is the subject of the statement and greater than 3 is the predicate of the statement, from this statement we can abstract the subject away and write in this form, I use a variable for the subject and we say that x is greater than three. Let us say, we denote this symbolically in this manner, suppose P of x denotes x greater than 3, we might want to abstract away the other constant 3 as well.

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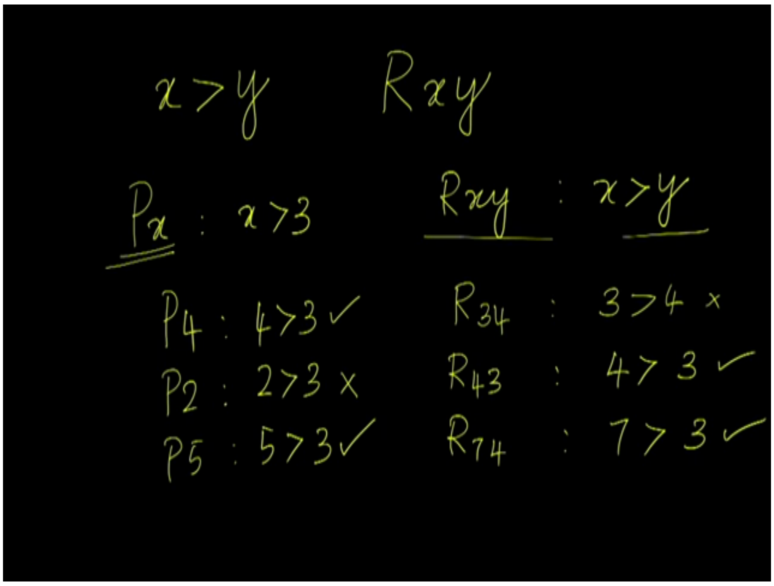

So, if you abstract that away too then we will have two variables then we will have a sentence of the form x greater than y where both x and y are unknown, we could denote this as R of x y. So, now we have two predicates P of x which says that x greater than 3 and R of x y which greater than, which says that x greater than y. If you substitute constants for the variables in these predicates, in these formulae by substituting 4 for x, we have P 4 which is 4 greater than 3, this is correct.

On the other hand if you substitute 2 for x, we have 2 greater than 3 which is false. If you substitute 5 for x we have 5 greater than 3 which is true. So, depending on the value that you substitute for x here, P of x may be true or false. Similarly, in the case of R of x y you can substitute various values for x and y, you can substitute 3 and 4 which then would say 3 greater than 4 which is false, if you substitute 4 and 3 you will have 4 greater than 3 which is true, if you have 7 and 4 you will have 7 greater than 3 which is true and so on.

So, now we have a way of abstracting individuals away and replacing them with variables.

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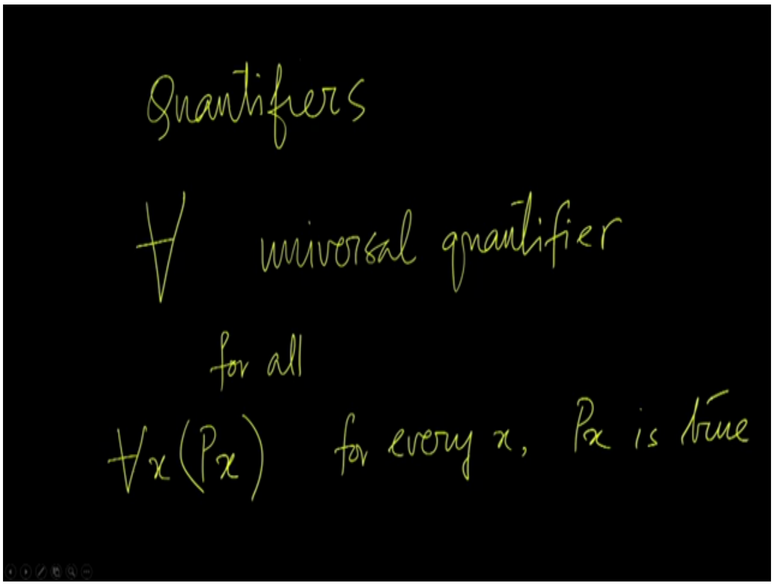

Now, let me introduce what are called quantifiers. The first quantifier is the universal quantifier. A universal quantifier stands for the expression for all, for example, when we say for all x P x what we mean is that? For every x, P x is true.

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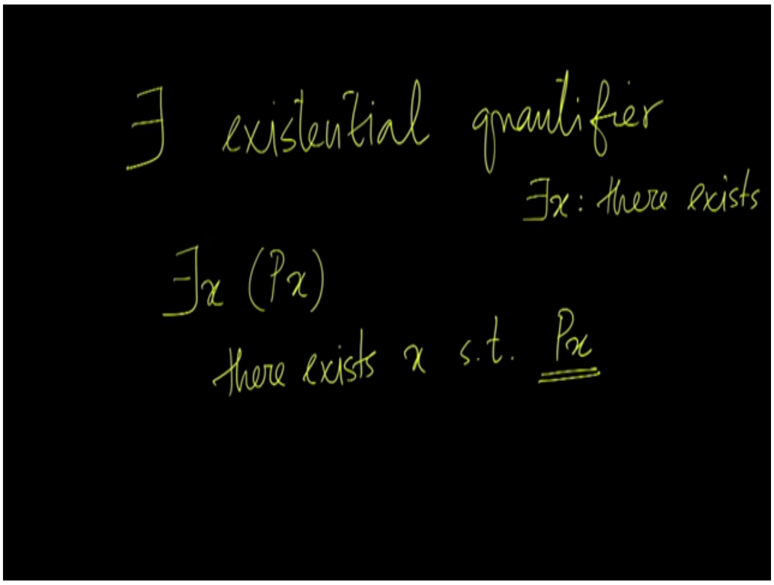

The other quantifier is the existential quantifier. Using an existential quantifier, when we say that when we write a formula of this sort here, this is supposed to stand for there exists, what we essentially say is that, there exists an x such that P of x is true. So, P x is a predicate with an argument supplied x is the argument here, so that will take on a truth value as we have seen before.

So, this statement is supposed to say that the there exists an x such that P of x is true.

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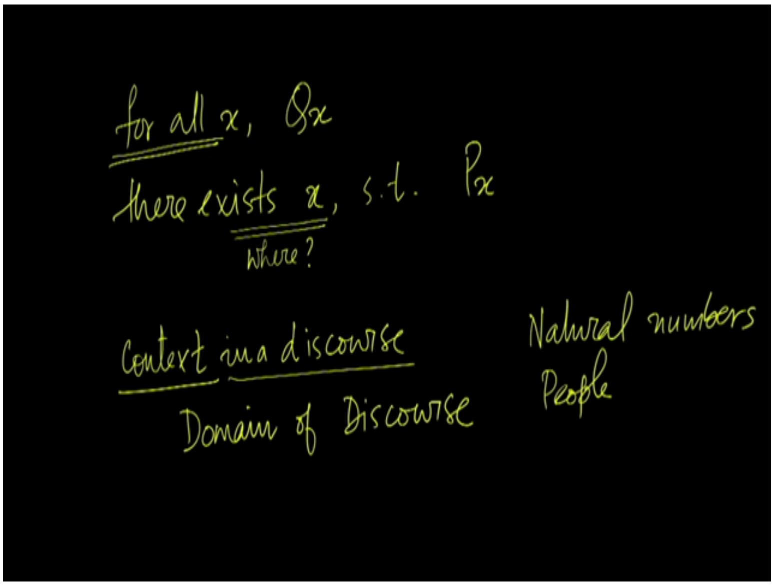

Now, in these statements, we say for all x or there exists an x such that some predicate is satisfied or here for all x, so that some predicate Q x is satisfied but then what do we mean by for all x? What kind of x do we talk about here? And here there exists an x where this question is not clear when we say for all x or there exists an x. Now, these quantified statements happen in a context, in a discourse, these happen in a context in a discourse.

So, from the context of the discourse it should be clear, what is the domain of the discourse? The domain of the discourse is the set of elements about which we are conversing at the moment for example, we could be talking about natural numbers or we could be talking about people. Depending on the domain of discourse, the statement that we say might make sense.

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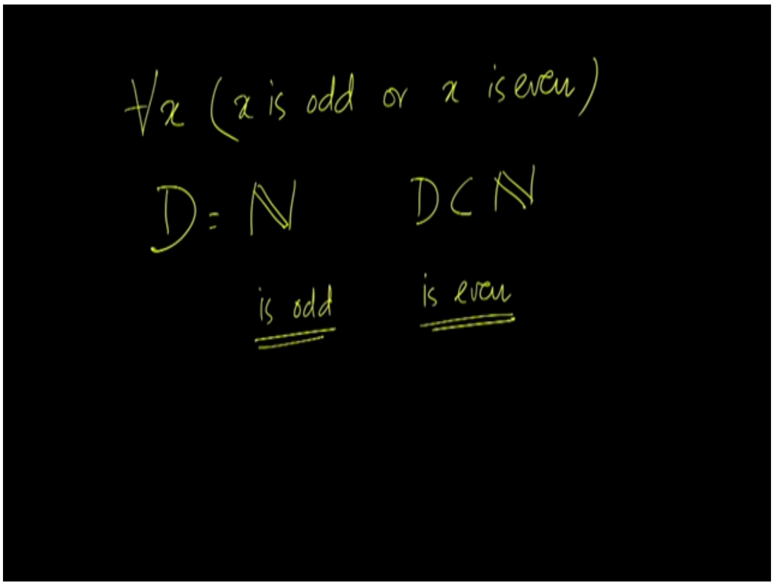

For example, when we say for all x, x is odd or x is even makes sense if the domain of discourse D is the set of all natural numbers, every natural number is either odd or even you can classify natural numbers as odd or even or D could be a proper subset of N, so in these context the statement x is odd or x is even makes sense because is odd or is even predicates do apply to natural numbers.

But if you are talking about people, these statements need not make sense, to make sense of these statements we will have to interpret the predicates is odd and is even in a manner which is appropriate to the members of the domain of discourse. So, if domain of discourse is the set of all people then these will have to be interpreted appropriately in terms of the people for this statement to make sense.

So, for a first orders statement to make sense, we will have to first fix the domain of discourse. So, we assume that in the context in which our conversation is happening the domain of discourse is fixed and in that context we quantify.

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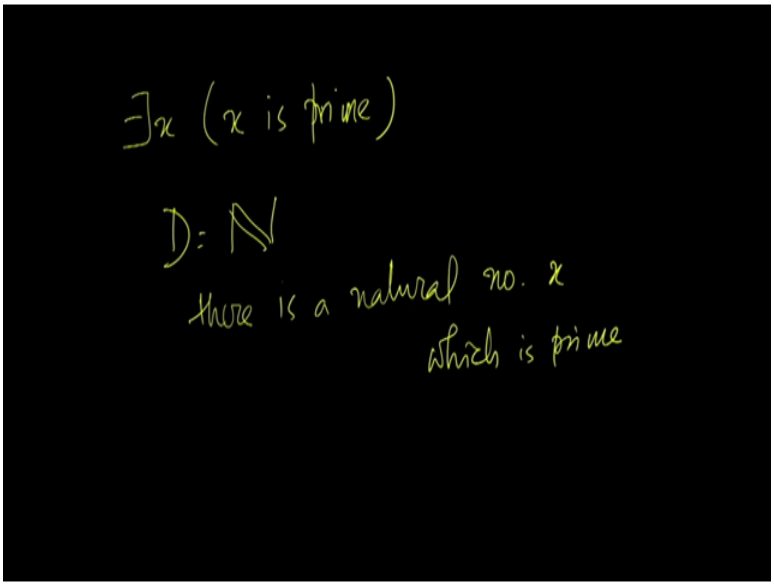

Similarly, when we say there exists an x such that x is prime, if the domain of discourse is the set of natural numbers then what do we assert? We assert that there is a natural number x which is prime, so once the domain of discourse is fixed and the predicate is prime as understood then the sentence makes sense then you can assert whether the statement is true or false.

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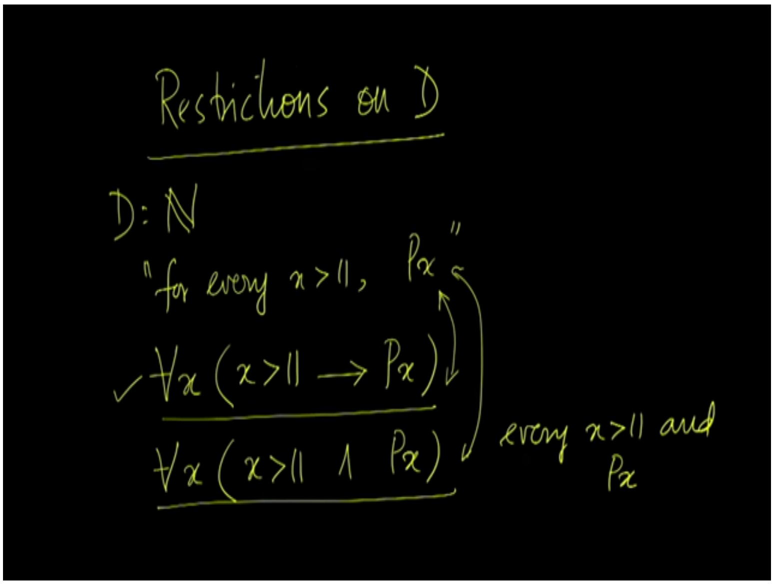

Sometimes, we want to make restrictions on the domain of discourse. Suppose, D is the domain of discourse and let us say we want to make restrictions. So, let us assume the D is the set of all natural numbers and let us say we have a statement of this form, for every x greater than 11, P of x that if we want to assert that the predicate P of x is true for every x which is greater than 11.

How would we write this in our logic using the quantifiers? You have to write this way, there exists an x, so that when x is greater than 11, if x is greater than 11 then P of x is true this is the correct representation of this statement, this cannot be paraphrased as this is an often made mistake people often write this way. The second statement, says that (there exists) the for every x, x is greater than 11 and P of x this would be true if and only if every x is greater than 11 and for every x, P x is true that is not what we intend to say, what we intend to say is that for every x which is greater than 11 P of x is true, these two are not the same at all but these two you can see are the same.

So, this is the correct paraphrasing of the sentence, the quantified sentence for every x greater than 11 P of x.

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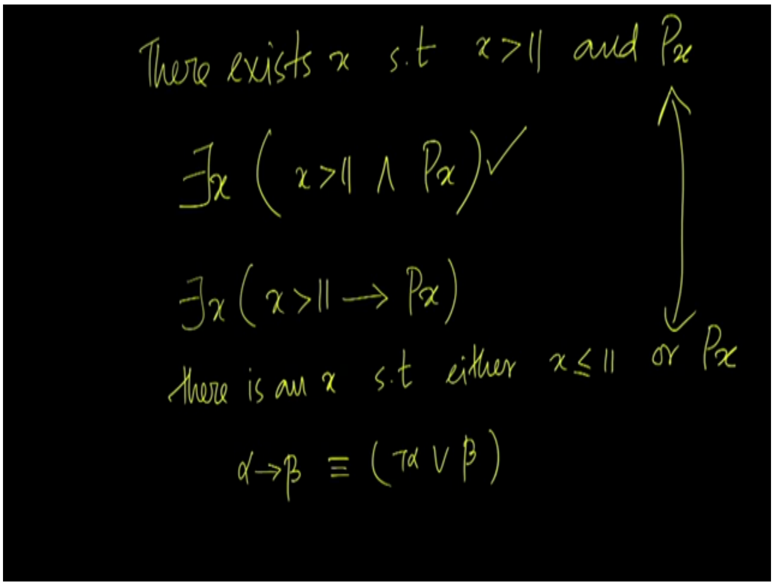

And then you can make an assertion of the sort there exists an x such that x greater than 11 and P of x then we would write this as there is an x greater than 11 such that P of x is what we want to say and we would write it in this way, you should remember that here x greater than 11 implies P x will not do, that is because this says that there is an x such that either x is less than or equal to 11 or P of x, what we assert is that x greater than 11 implies P of x.

So, from what we saw in the last lecture, we know that alpha implies beta is logically equivalent to negation of alpha or beta therefore x greater than 11 implies P x would paraphrase as x less than or equal to 11 or P x but these two statements are not the same at all. Therefore, this is the correct representation of the top statement.

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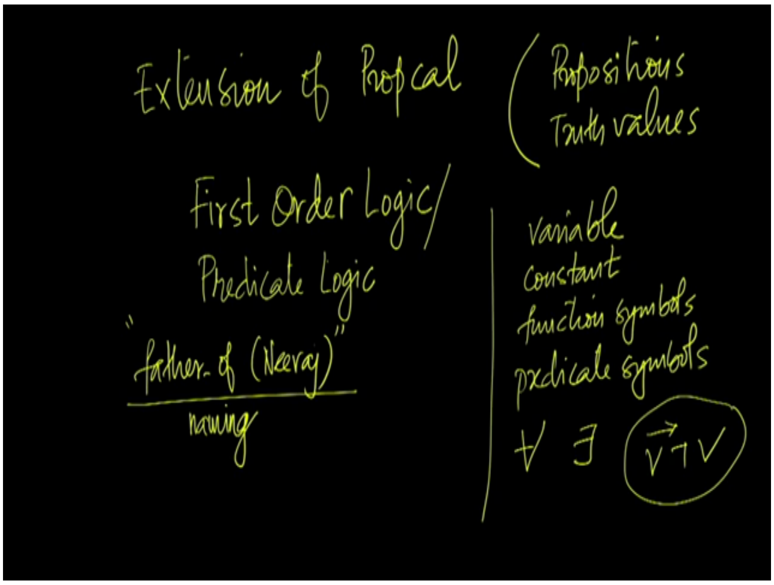

So, we can write statements of this sort using quantifiers, so the extension of propositional calculus. In propositional calculus, we have propositions these are the syntax entities and composite propositions which are made from atomic propositions and the truth values which are the semantic entities. So, these are what we deal with in propositional calculus but when we come to this logic which we called first order logic, recall propositional calculus is also called 0th order logic as supposed to that here we have first order logic which is also called predicate logic.

So, when we extend a propositional calculus to first order logic or predicate logic we have variables, variables are akin to pronouns in English. Constants, constants are similar to proper nouns in English. Function symbols, function symbols are used to create entities which can be used as names of objects, for example, father of Neeraj, this is a naming mechanism father of is a function which is applied to the constant Neeraj to create the phrase father of Neeraj, so this phrase is a naming phrase.

So, we can have function symbols that serve this purpose and we have predicate symbols and we have quantifiers for all in the there exists and we have all the apparatus of propositional calculus for example, the logical connectives, AND, OR, NOT, etcetera implication and what not. So, the apparatus of propositional calculus are still with us along with these additions.

So, this richer logic is called the first order logic or predicate logic.

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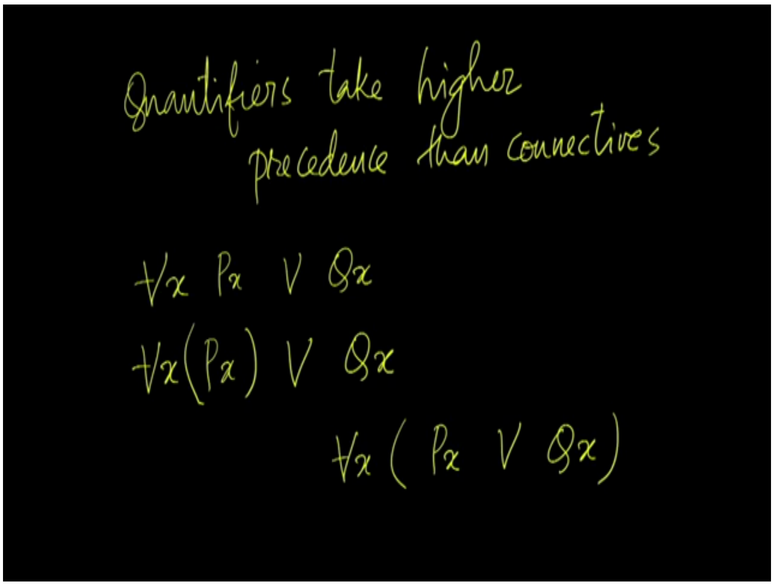

Quantifiers take higher precedence than connectives. In the last lecture we discussed, precedence of connectives, we saw that negation has the highest precedence and double implication has the least presidents but quantifiers take a precedence which is higher than that of all the connectives including negation. Therefore, a statement of the form for all x P x or Q x should be interpreted as for all x P x or Q x that is for all x applies only on P x not on Q x.

This is as supposed to for all x P x or Q x that is the scope of a quantifier is the immediately adjacent predicate unless otherwise specified using parentheses.

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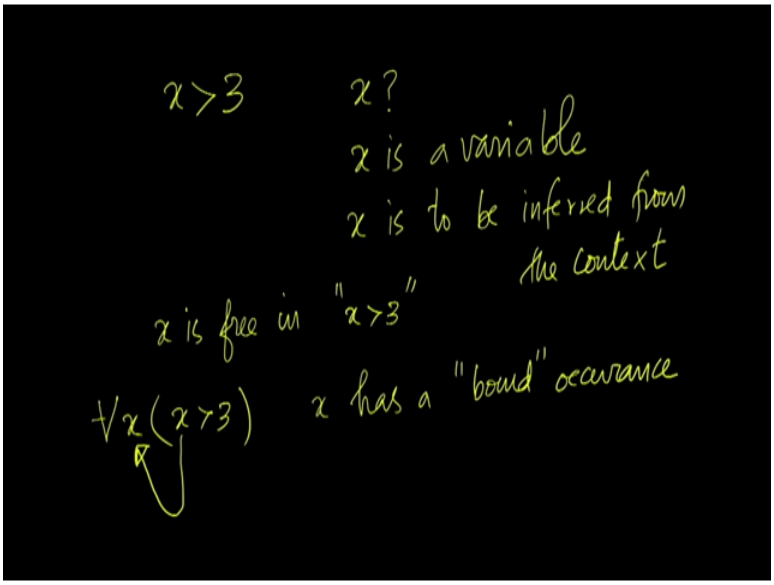

Consider the formula x greater than 3, here we say that x is greater than 3 but what is x? x is a variable. So, when you look at this formula, we do not know what x is, so x is to be inferred from the context, so in that sense x is like a pronoun in English. We say that x is free in the statement x greater than 3, the occurrence of x is free in x greater than 3. As supposed to this when we say for all x x greater than 3, of course the statement would be false if we are talking about natural numbers but never mind we are not talking about the truth value of the statement.

Look at the statement, the form of this statement, in the statement we say for all x x greater than 3, so in this statement x has a bound occurrence, this occurrence of x is bound to this quantifier. So, variables can have a free and bound occurrences, it is also possible to mix free and bound occurrences within the same statement.

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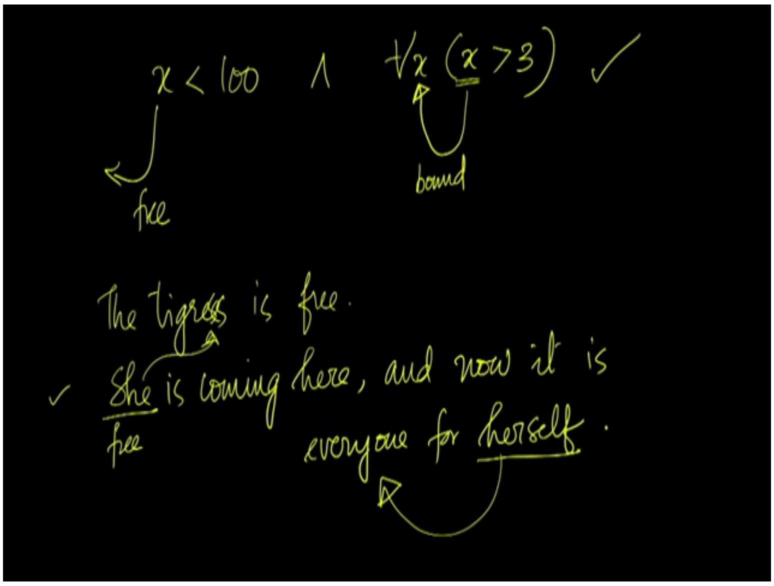

For example, when I have a statement of this form x less than 100 and for all x x greater than 3 this occurrence of x is bound to the x in the quantifier, so this is a bound occurrence of x. Whereas, this is a free occurrence, so this x is talking about some individual which is known only from the context, so it is rather like a pronoun whereas this second x is bound to the x in the quantifier, so that does not depend on the context.

This is similar to a statement of the form, the tigress is free that is one sentence that provides a context and let us say, in the second (state) the sentence we have, she is coming here and now it is everyone for herself. So, consider the second sentence, in this sentence the pronoun she occurs in two places that is similar to x in the statement x less than hundred and for all x x greater than 3.

The first she is a free occurrence of the pronoun, the meaning of this she has to be inferred from the context now what is the context? In the context, the previous sentence says that the tigress is free, therefore this she refers to this tigress but the she in herself is a bound occurrence, it is bound to everyone, so we have a group of women facing the tiger, so this quantification is over this group of women facing the tiger, so everyone refers to the individuals within this group, so the she in herself is bound to this occurrence of x.

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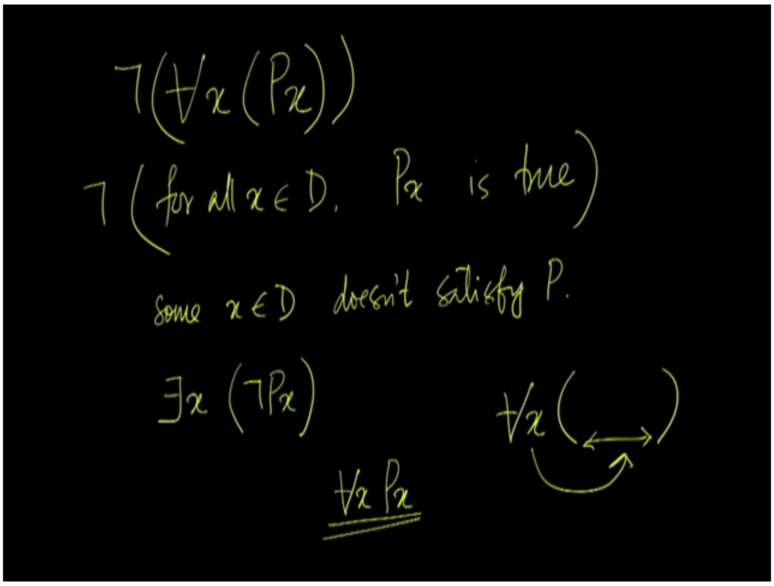

Consider the statement, for all x P of x, so as we said before this says that for all x in the domain of discourse D, P of x is true. Suppose we want to negate this, then we want to say that this is not the case, suppose we want to negate this, we want to say that it is not the case that for every x P of x is true then clearly somebody violates P of x that is if you go to every individual belonging to the set D we would (sign) find that P of x is not satisfied by everybody.

So, there is somebody who does not satisfy P of x. In other words, some x belonging to D does not satisfy the predicate P or in other words, there exists an x within D, so that P of x is not satisfied. So, we find that these two statements are equivalent there is a negation of for all x P x is the same as there exists an x NOT of P x, of course parenthesizing correctly, we will use this convention of parentheses, you will find in literature that there are different ways of parenthesizing quantified statements, we will always use this notation.

A quantifier will be immediately followed by a parenthesized statement, the scope of the quantifier will be defined by the parentheses, if such a parentheses parenthesization is not done then for all x will associate to the nearest relegate, it has the highest precedence as we said before.

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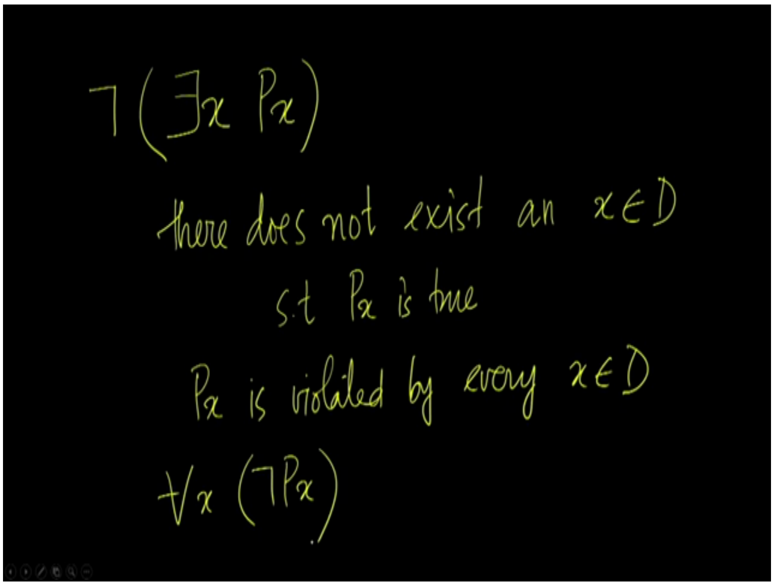

Similarly, let us try to negate this statement there exists an x so that P of x, let us say we want to negate this. So, what this asserts is that there does not exist an x in D, so that P of x is true or in other words if you go to the individual members of D, we will find that P of x is violated by every x in D or in other words for all x P of x is violated.

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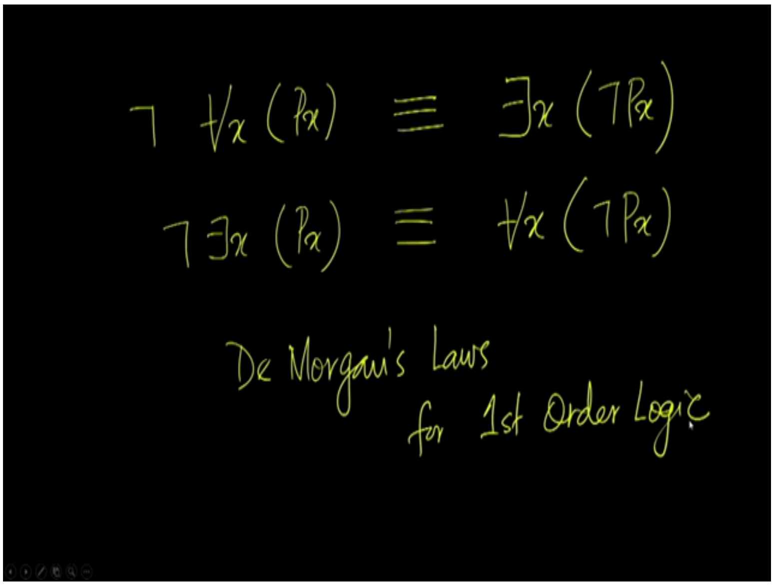

So, these two equivalences, you can avoid these parentheses and simplify the expression it says that, there exists x so that P of x is violated if it is not the case that P of x is true for everybody then there must be some x for which P of x is violated. Analogously, if there does not exist an x, so that P of x is true then for every x P of x must be false, these two are called De Morgan's laws for the first order logic.

Let us try a few examples, paraphrasing sentences in English into sentences in first order logic.

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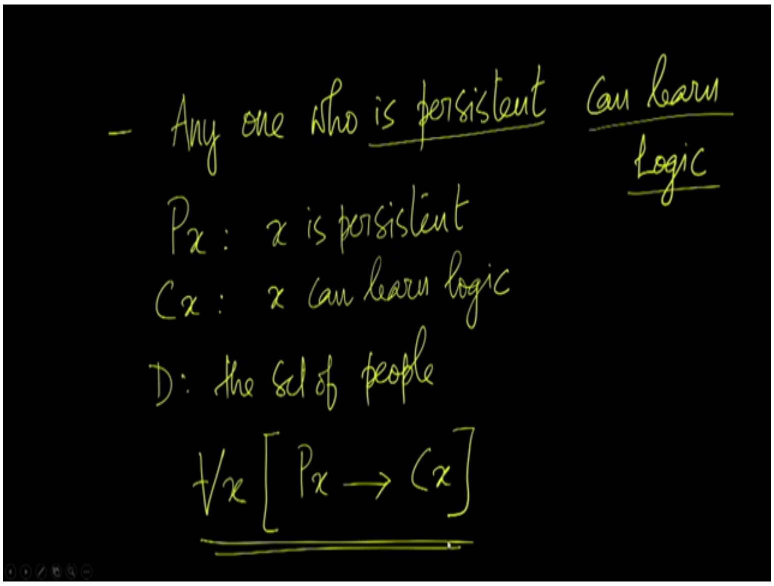

These examples are from the textbook of Mendelssohn, anyone who is persistent can learn logic. We want to translate this sentence in English into a first order formula. So, let us consider the predicates here, is persistent is one predicate, can learn logic is another predicate, so we can have P of x stand for x is persistent, we can have C of x stand for x can learn logic then what we essentially assert is that any person who is persistent is capable of learning logic.

In other words, for every x when x is a person that is our domain of discourse is a set of people, for every x where x belongs to D that is understood, the domain of discourse is understood, for every x if x is persistent then x can learn logic, this would be the first order representation of the sentence.

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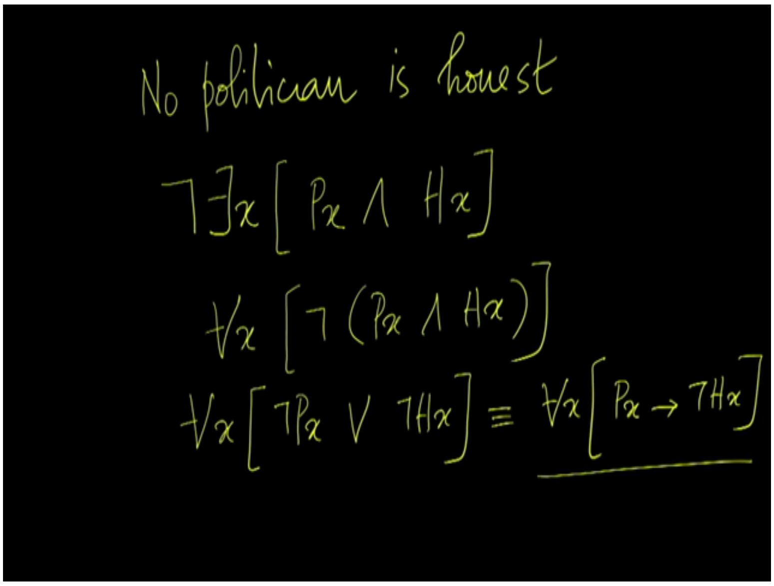

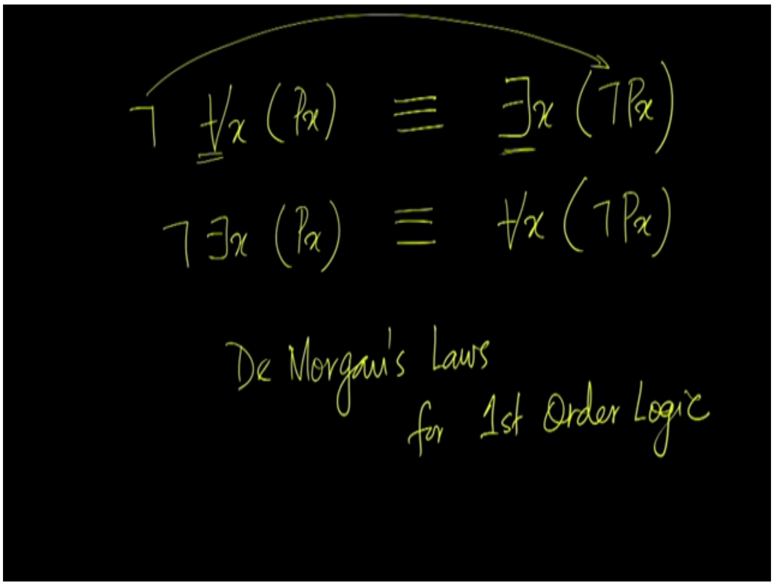

Consider another statement, no politician is honest, a debatable statement but there we have it. Let us consider the compliment of this statement, the compliment of this statement would say that some politician is honest, some politicians are honest or in other words there exists of politician who is honest. So, let us say there exists an x in D, the domain of discourse is the set of people here again.

So, there exists an x, so that x is a politician so in this case P stands for the predicate is politician, so P x means x is a politician and honest x. so, we have the statement there is some x who is both a politician and honest that would be a negation of this statement. Now, what we want here is to negate this, no politician is honest. So, here we have a negation of quantified statement then we can apply De Morgan's laws to take the negation inside.

So, from the De Morgan's laws, we know that when negation is taken inside a quantified formula it changes the quantifier for example, when a negation travels over a universal quantifier into the parentheses then the universal quantifier changes into the existential quantifier, this universal quantifier changes into an existential quantifier when the negation travels inside the brackets.

Similarly, when the negation travels over an existential quantifier, inside the parentheses it converts the existential quantifier into a universal quantifier. So, let us use that here and take the negation inside then this becomes, for all x and we have the negation of P x and H x, but the negation of P x and H x can be found using the De Morgan's laws of first order logic which would be here we have the negation of a conjunction, the negation of a conjunction is the disjunction of the negations.

So, we have negation of P of x or negation of H of x which is logically equivalent to saying this, so what does it say? For every x if x is a politician then x is not honest, x is dishonest. So, that is tantamount to asserting that every politician is dishonest which is logically equivalent to saying that no politician is honest.

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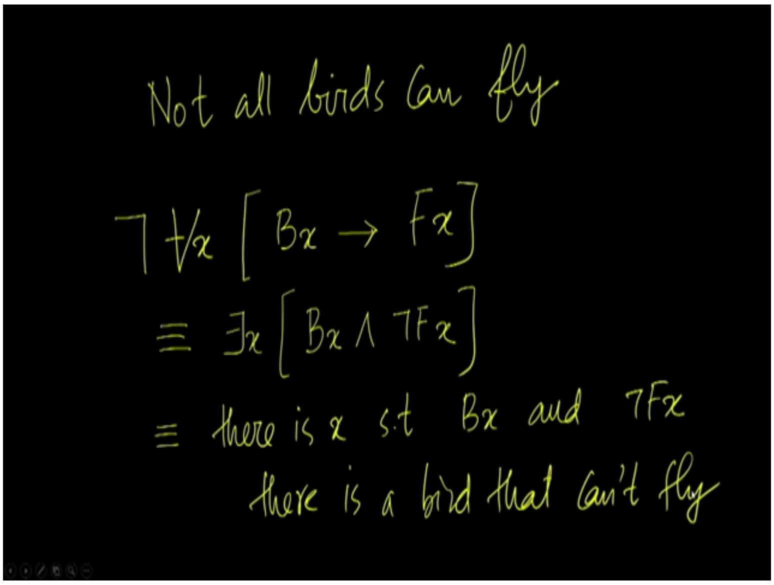

Similarly, consider the statement not all birds can fly. Suppose, we want to say that every bird can fly, then we would say for every x if x is a bird then x can fly, here B of x stands for x is a bird and F of x stands for x can fly. So, the statement asserts that every bird can fly, suppose we want to negate this then we would have the required assertion so that says that not all birds can fly.

Once again, if you take the negation inside the brackets the quantifier flips, we have there exists, then we have the negation of the implication B of x implies F of x but the negation of an implication is the conjunction of the antecedent and the negation of the consequent which means, we have B of x and F of x, what does this say? There exists an x, there is x such that bird of x and not of F of x.

In other words, there is a bird that cannot fly, you see that this is logically equivalent to our original statement, not all birds can fly.

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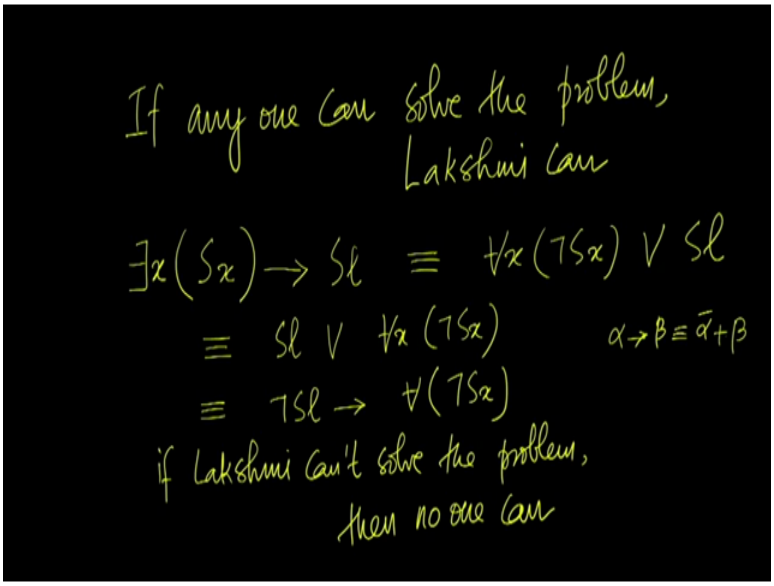

Another interesting example, if anyone can solve the problem, Lakshmi can. Let us say, S of x denotes the predicate x can solve the problem, so if anyone can solve the problem translates into this quantified statements there exists an x, so that x can solve the problem, this asserts that someone can solve the problem. Now, we have an implication if anyone can solve the problem in other words, if there is someone who can solve the problem then Lakshmi can solve the problem.

Let small l denote the individual Lakshmi, so the statement now asserts that if there is some x that can solve the problem then Lakshmi can solve the problem. So, this is a translation of the given statement, let us take the logical equivalents of this, the logical equivalents of an implication would be the negation of the antecedent and the consequent. So, the negation of the antecedent here would be for all x, not of S of x and then or S l which by commutativity of or can be written like this, which is logically equivalent to saying this, that is because alpha implies beta is logically equivalent to alpha bar or beta, we are invoking that in the reverse here.

So, what does this say? If Lakshmi cannot solve the problem then no one can, which is exactly the first assertion. The first assertion and the last are logically equivalent.

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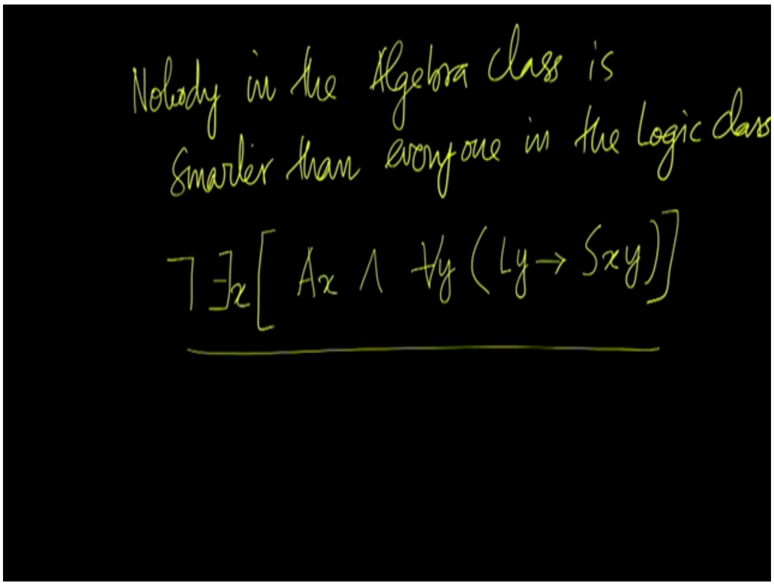

One more example, nobody in the algebra class is smarter than everyone in the logic class. So, to paraphrase this we would write this way, first let us assume that there is somebody in the algebra class who is smarter than everyone in the logic class. So, we would say there exists an x, so that x is in the algebra class and for all y, if y is in the logic class then x is smarter than y.

So, what it asserts is that, there is some x, who is in the algebra class and is smarter than every y in the logic class, this is what we want to negate. So, if you put a negation symbol here, we are asserting that nobody in the algebra class is smarter than everybody in the logic class. So, this is the first order translation of the above sentence given in English. So, now that gives you an idea as to how English sentences can be translated into first other sentences.

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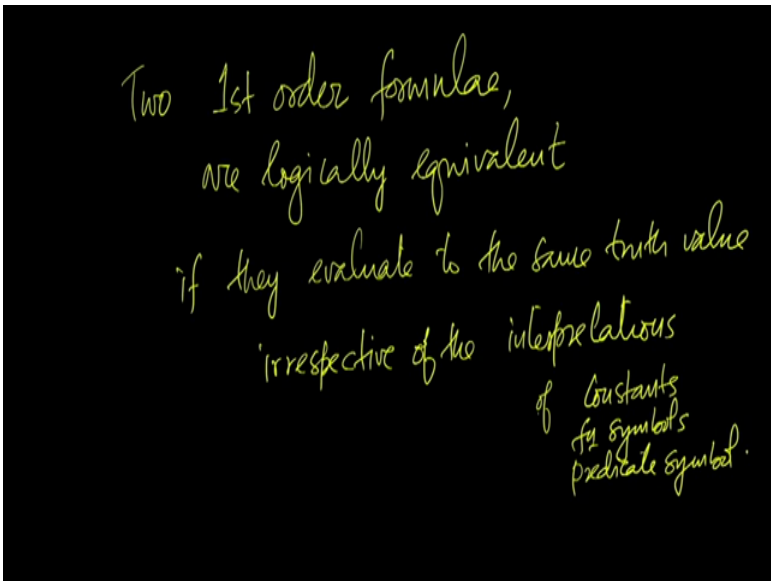

We say that, two first order formulae, I have not formally defined a formula yet which we will do that later, at least now you know you have an idea about what a first order formula is. Considered two first order formulae, two first order formulae are logically equivalent if they evaluate to the same truth value irrespective of the interpretations, interpretations of constants, function symbols, predicate symbols, etcetera.

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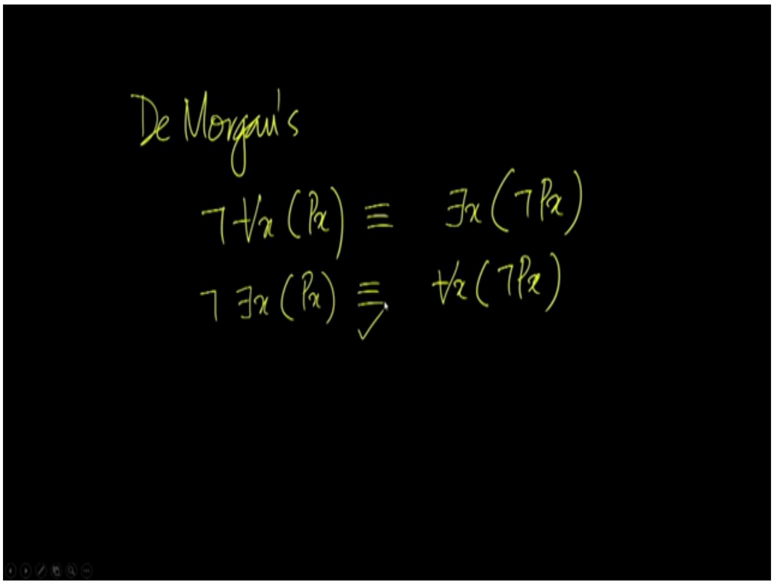

For example, by De Morgan's laws as we saw just now, negation of for all x P x is logically equivalent to there exists x negation of P x. Similarly, negation of there exists x P x is logically equivalent to for all x negation of P x. So, these are logical equivalences.

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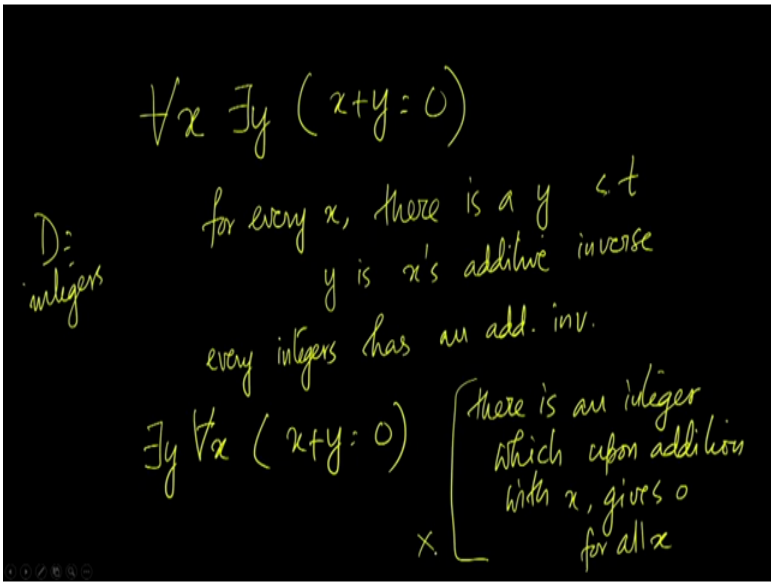

We can have quantifiers nested within one another but then when universal quantifiers and existential quantifiers are nested within one another, the order in which we nest them is significant. So, if our domain of discourse is the set of natural numbers then what does this statement say? It says that, for every x there is once you fix the x, there is a y such that y is x is additive inverse, when x and y are added together we get 0 or y is the negative of x.

In other words, we say every natural number has an additive inverse or every integer, we would of course be making the statement correctly only if we are talking about integers, that is the domain of discourse will have to be the set of integers. Compare those two, this statement if there exists a y, so that for all x, x plus y equal to 0, what does this say? It says that that there is a number, there is an integer which upon addition with x gives 0 for all x but this is patently false.

So, we see that the two statements mean entirely different things. So, in a sequence of universal quantifiers and existential quantifiers, if you change the order the meaning of the statement would change.

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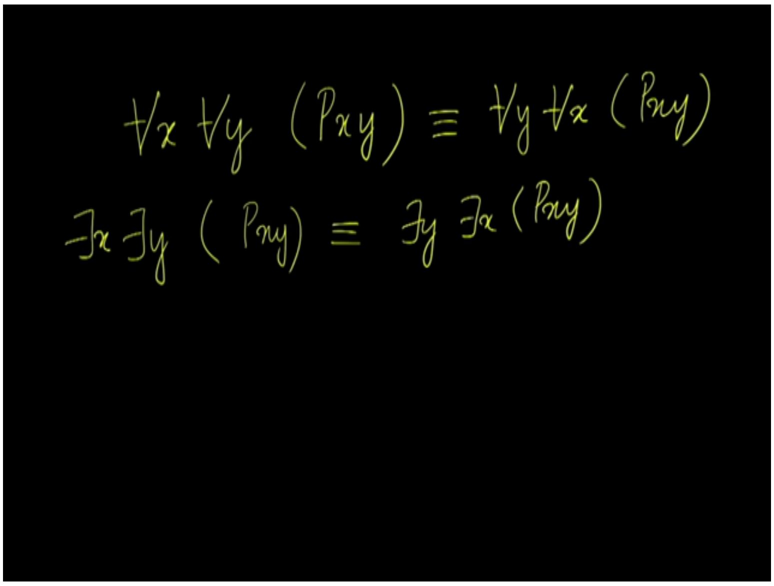

But that is not the case with a sequence of universal quantifiers when we have an assertion of this form for all x y P x y what we want to assert is that for every ordered pair drawn from the domain of discourse for every ordered pair x y, P is true for x and y, this would be exactly the same even if we change the order of x and y, as you can verify. Therefore, in a sequence of universal quantifiers we can change the order of the quantifications. Analogously there exists x, there exists y P x y is logically equivalent to there exists y, there exists an x P x y.

In a sequence of existential quantifiers too, we can permute the order of the quantifications.

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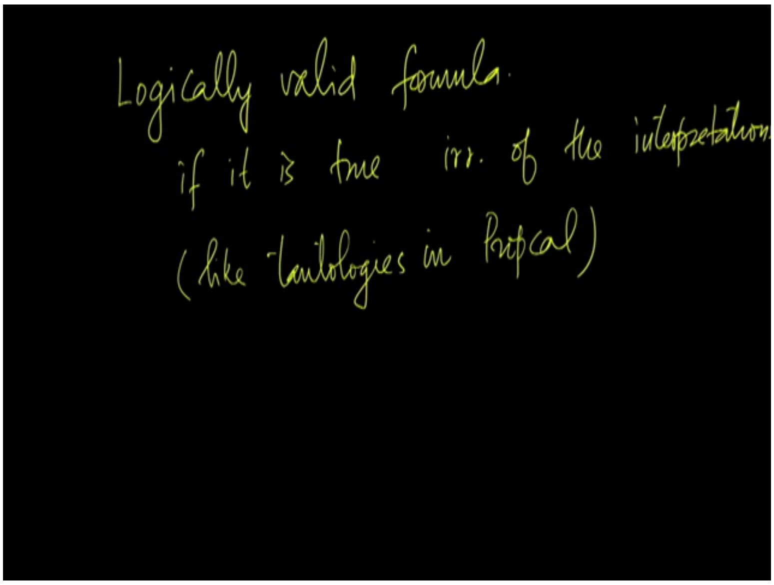

We say that a formula is logically valid if it is true irrespective of the interpretations of the function symbols, predicate symbols, constants, variables, etcetera. So, a (logical) logically valid formula is akin to tautologies. Tautologies is in the context of propositional calculus, that is a formula which always evaluates to true. A logically valid formula in first order logic is similar, it always evaluates to true irrespective of the interpretation that you place on the various symbols of the language.

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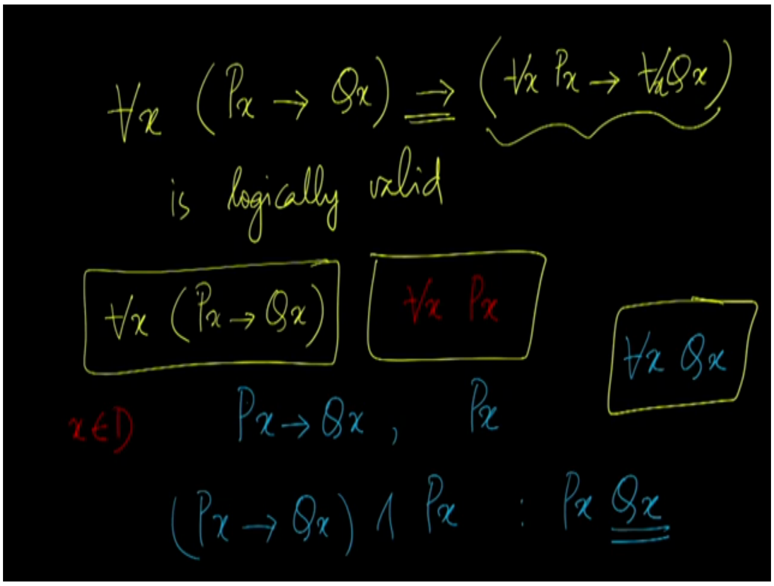

For example, consider this statement for all x P x implies Q x implies for all x P x implies for all x Q x, I want to claim that this is logically valid that is irrespective of the interpretation that is placed on P and Q this statement will always be true, how do we argue this? To argue this, let us look at the structure of the sentence, this is an implication. So, this is the implication at the topmost level.

So, this implication has an antecedent and a consequent, we want to assert that this implication is always true. In an implication, if the antecedent is false the statement anyway evaluates to true, so we do not have to worry about the situation where the antecedent is false. So, let us consider only the case where the antecedent is true. So, let us assume that for all x P x implies Q x is true then for the implication to be true the consequent will have to be true that is when the antecedent is true the consequent will have to be true for the implication to be true.

Now, we want to show that the consequent is true. Now, let us look at the consequent, the consequent itself is an implication and we want to claim that it is true. So, for an implication to be true the antecedent has to be true the antecedent has to be false or the antecedent and the consequent both have to be true. So, here again let us assume that the antecedent of this implication is true.

So, we make these two assumptions for all x P x implies Q x is true and for all x P x is true then consider any x belonging to D, for this x we have that P x implies Q x is true and P x is true, you can readily verify that P x implies Q x and P x together ensures that P x and Q x are both true or in particular Q x is true. Therefore, this is true for every single x, we have taken an arbitrary x and D therefore we can assert that for all x Q x there is an x here.

So, we have shown that for all x P x assuming these two. Therefore, the formula has to be logically valid that is in that implication the antecedent and also the antecedent of the consequent are both true and we show that the consequent within that global consequent is also true therefore the formula is true always, that is irrespective of the interpretation that you place on P and Q the formula will be true.

So, this is an example of a logically valid formula, but if you take the converse of the formula that will not be true.

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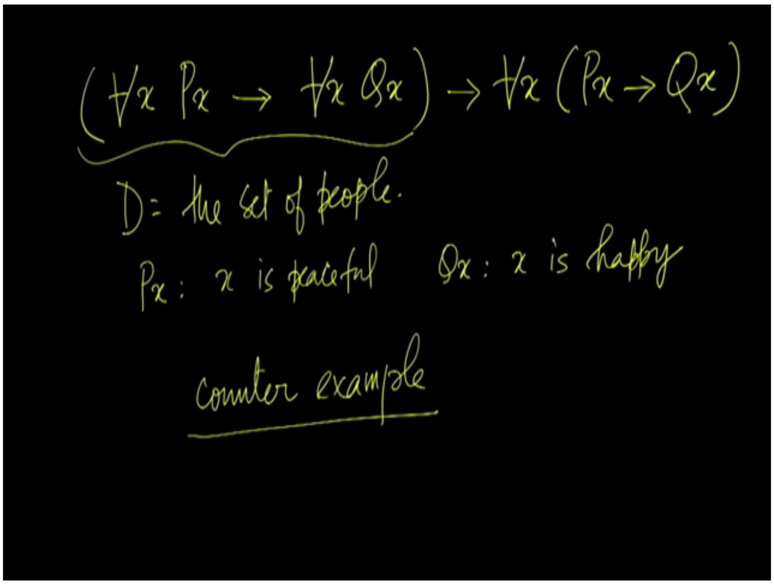

For example for all x P x implies for all x Q x implies for all x P x implies Q x need not be true, that will depend on the interpretation for P and Q. Let us consider an interpretation which will make this formula false, let us say the domain of discourse is the set of people, let us say P of x stands for x is peaceful and Q of x stands for x is happy, so what does this statement assert?

It asserts that if all are peaceful, all are the implies that all are happy then for every individual x if x is peaceful then x is happy that need not be the case because even if the antecedent is true, that is if all are peaceful then peace will prevail within humanity and that is sufficient for all to be happy, still the consequent does not follow, what does the consequent say? It says that for every single individual, if that individual is peaceful then he is happy, that may not be the case because this individual might be surrounded by quarrelsome people, so even if he holds the peace the his neighbours may not therefore he may not be happy.

Therefore, this is a counter example to establish that this statement is not logically valid. To prove that first order formula is logically valid you have to argue in terms of all interpretations, you have to show that this formula has to be necessarily true in every single interpretation. On the other hand, to prove that formula is not logically valid all that you have to do is to come up with a counter example, come up with one particular interpretation in which this formula will not be true.

So, in this case you have to come up with a counter example in which P and Q are universal properties but if P is a universal property then Q is also an universal property, so you have to assume that about the properties P and Q but then it should still be the case, it should still be the, be such that if P is held only by one person then that person need not satisfy Q. If you can find such an interpretation then you have a counter example and that is what we have just done.

So, that is it from this lecture, hope to see you in the next, thank you.