Discrete Mathematics
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Lecture 28 - Stirling Numbers, Bell Numbers

In today's lecture we will look at the problems involved in balls and bins. The basic question that we are trying to study is, how to distribute n balls into k bins? And counting the number of ways of doing this. Now stated in this format it is a bit under specified. So, we could say things like the balls are distinguishable. The bins are distinguishable or the bins need to be nonempty and you impose lot of conditions on how the distributions should be. And we will look at many variants of that and that will give us different kind of problems.

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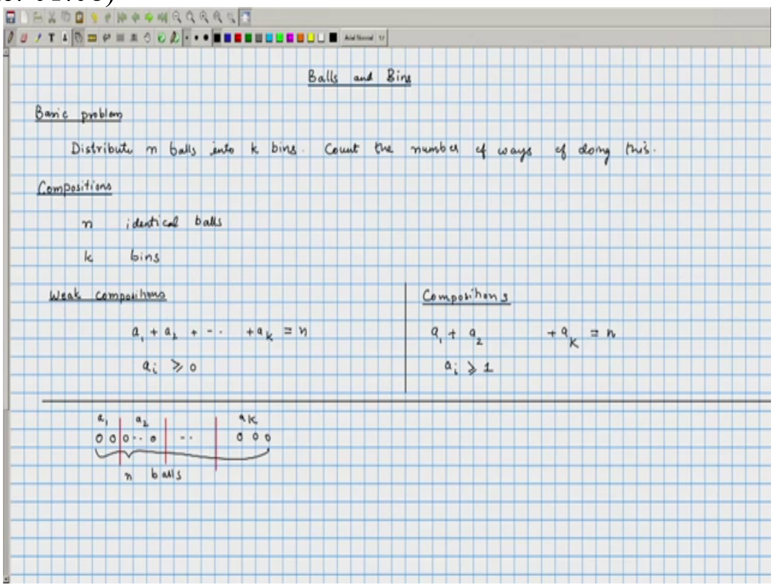

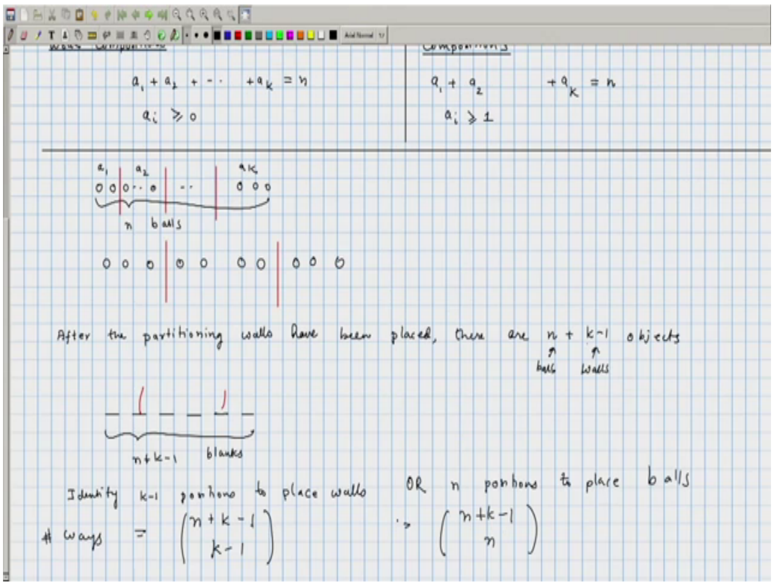

So, the first type of problem that we will discuss is counting the number of compositions. So here what we will assume is that we have n balls which are indistinguishable or n identical balls. And there are k bins. So these bins, you can think of them as numbered from 1 to n or 1 to k. And they are distinguishable. You can think of distributing identical toffees to children being distinguishable. You can say that is a similar problem, ok.

So, how do we do this? So let us, there are two variants of these as well wherein the bins could be empty and nonempty. So the first problem that we will address is what is known as weak compositions. So weak composition is basically a split of n identical balls into k distinguishable bins and the bins could have zero balls in that. So, basically we need to find out numbers a 1 to a k such that a 1 plus a 2 plus all the way up to a k is equal to n, they add up to n. And each a i is greater than or equal to zero. So the weak comes from the fact that some of these a i's could actually be zero, ok.

And if we are talking about compositions, what we mean is we still want k numbers k positive integers such that they add up to n and further each a i is greater than 0 or we can say is greater than or equal to 1. I mean every bin should be nonempty. There should not be any empty bin. And we need to look at the number of ways of counting this, ok.

So, let us first look at the problem of weak compositions. So, we look at this problem in following way: We have all these balls, let us say n of them. And in order to split them into k bins we basically draw partitioning walls in between them. Ok. So, if we have to split it into k bins k minus 1 partitioning walls are required, ok. And once the partitioning walls are put in place, whatever is between the start and the first partitioning block can be thought of as a 1. And between the first and second can be thought of as a 2. The last can be seen as a k. Between the k minus 1 towards the end can be seen as a k.

And you can see that this is a one to one correspondence. If you draw these balls in this particular way, any arrangement any distribution of and balls into k bins can be viewed in this particular format, ok. Now so as an example we say that there are 10 balls. And if we wanted to split it into three blocks, all that we have to do is two partitioning walls, ok. And this gives one particular split, ok. So, basically we want to count the number of ways of placing partitioning walls when you have an arrangement of balls along a straight line.

We could also think about the same thing in a slightly different manner. So note that after this partitioning walls have been placed, there are precisely n plus k minus 1 objects in the arrangement. So this n corresponds to balls plus k minus 1 corresponding to the partitioning walls. So there are n plus k minus 1 objects placed along a line. Okay. So we will exploit that observation to count the number of ways of distributing balls into bins. So let us say we have n plus k minus 1 blanks. And out of these blanks some blanks would be selected as positions where you can put balls and the others will be where you can place the partitioning walls, ok.

So, if we were just placing the partitioning walls we just need to identify k minus 1 positions to place walls or equivalently n positions to place balls, so they are identical in some sense. I mean they count or the number of ways of doing this are exactly the same. So the number of ways of choosing k minus 1 positions out of n minus k plus 1 position that count is the number of ways for doing this is equal to n plus k minus 1 positions are there, out of which you have to choose k minus 1 positions.

So, if we choose one set of positions then you have the remaining positions as positions of balls. So the total number of ways of doing this would be n plus k minus 1. Choose k and this is as you can see it will be also same as n plus k minus 1. Choose n, these quantities are equal. So the number of weak compositions of n into k parts is n plus k minus 1. Choose n minus 1. So we can write this as a theorem.

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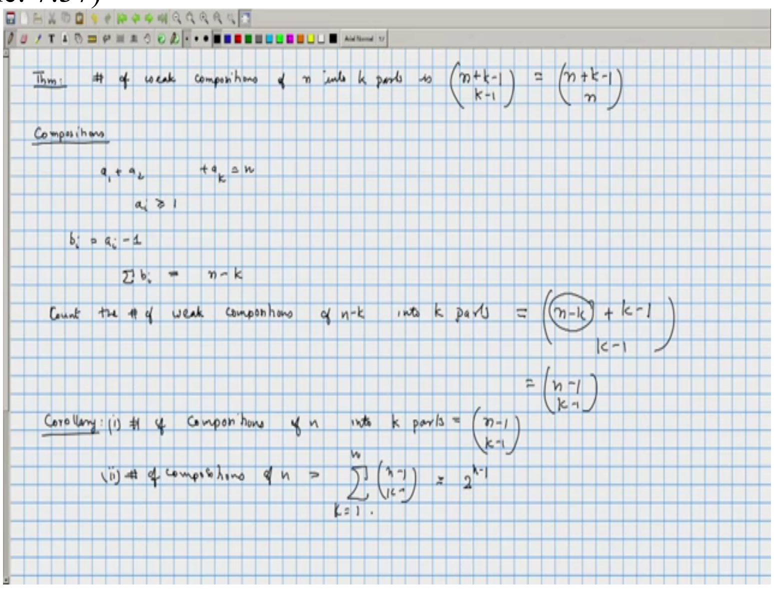

Number of weak compositions of n into k parts is n plus k minus 1, choose k minus 1. This is also equal to n plus k minus 1, choose n. So now we are in a position to find or count the number of compositions. So when we were counting compositions, what we were interested in is a 1 plus a 2 plus a k should be equal to n. And each a i should be greater than or equal to 1. We define b i to be equal to a i minus 1. Then summation b i should be equal to n minus k.

Ok. So, we can look at a different problem. Instead of trying to count the number of ways of obtaining a 1 to a k such that they add up to n and each a i greater than or equal to 1, we could look at count the number of weak compositions of n minus k into k parts, ok.

So, look at a weak composition of n minus k into k parts and for each of them, for each of the part of the composition if you add 1 what you will get as a composition. Ok. And if you have a composition of n, you can you can convert that into a weak composition of n minus k, so their numbers are equal. So and this counting that is counting the weak composition is something that we have already done and this is equal to n minus k that is the n minus k plays the role of n now, plus k minus 1, choose k minus 1 and that is equal to n minus 1, choose k minus 1. So this is the number of compositions of n. So we can write it as a corollary. Number of compositions of n into k parts is equal to n minus 1, choose k minus 1.

This would also mean that the total number of compositions if you do not say how many parts you have to split them into, you just split it into arbitrary number of parts. Now when we are splitting it an arbitrary number of parts we cannot really count the weak compositions because if each part will allow it to be empty you can have say one empty part, two empty but there are infinitely many ways of doing that. So the total number of compositions makes sense whereas the total number of weak compositions does not really yield anything meaningful.

So, if you look at the total number of compositions the number of compositions of n that is, is equal to summation n minus 1, choose k minus 1 where k varies the number of parts at least 1, 1 to n and that is going to be equal to 2 raised to n minus 1 that is just the binomial identity, ok. So, now that we have done compositions we will move onto slightly different problem, ok.

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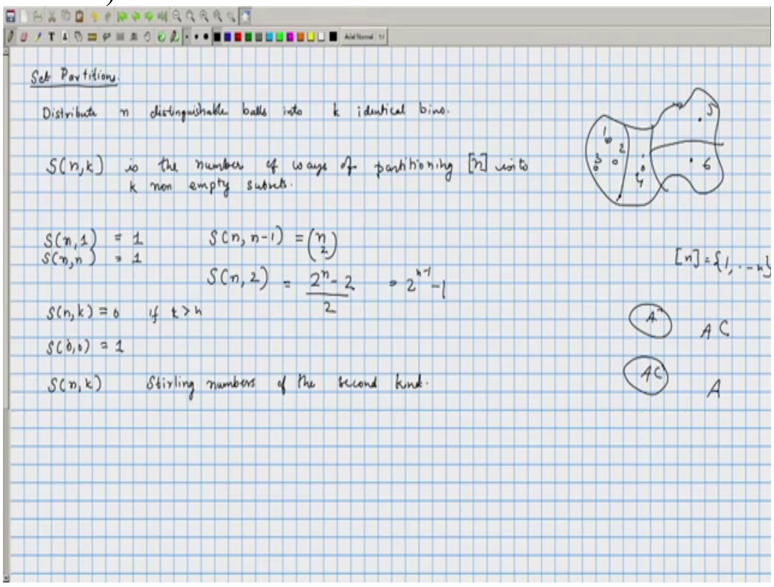

So, this is known as set partitions. So here, so in the earlier case what we had was the balls were indistinguishable whereas the bins were distinguishable, means you are distributing identical objects to k people. Here the balls are distinguishable in the sense as red balls, green ball. They are different colours, ok. But we are putting them in let us say cartons which are indistinguishable from each other, ok. So, there is no first box, second box and third box, all the boxes look identical. Ok. After you have, so suppose you have two boxes. Ok. And let us say we had put three items here and four items here.

Now it is crucial as to which three items you had put. Suppose you had 7 balls, we could choose 3 out of 7 and put that into here. Each of those choices would give a distinct arrangement. But whether these three that you have chosen goes in the first box or the second box is not, it does not really matter because they look identical. So you can think of them as identical boxes and they are shuffled around after the balls have been distributed.

So how many ways are there of doing this? So again we will have n balls and k bins. So, the problem we are looking this is: distribute n distinguishable balls, so n distinguishable balls into k identical bins, ok. Or partition n objects into k bins. So you can think of it as let us say there is a set of objects.

We can think of them as set of objects because each object is distinguishable from the other. Okay. And then we need to split it into some number of say partitions. So this will be a split into 1, 2, 3, 4 parts, six objects are being split into four parts. So we will define the count or the number of ways of doing this as S n, k. So by S n, k we mean what we mean is the number of ways of partitioning n into k non, so we will write square bracket n to denote the set of numbers from 1 to n. So we can either think of them as distinguishable balls or when we write square bracket n what we mean as a set of numbers from 1 to n. So, that set is being partitioned into k nonempty subsets.

So, let us see some example. If we look at S n, 1, that is a number of ways of partitioning n into one subset. So that is precisely one way of doing it. If we look at S n, n that means the number you have n objects and split them into n nonempty bins and there is only one way of doing it. Whereas if you look at S n, n minus 1, this is little more interesting. So here you have n objects and you need to split it into n minus 1 boxes, nonempty boxes. So we can think of it as all except one box would contain single element and there will be precisely one box which contains two elements. Okay, so which two elements goes into that box containing two elements, that can be decided in and chose two ways. So that is the total number of ways as well. So this is equal to n choose 2.

Whereas if you look at S n, 2 this is going to be something different, the total number of ways of splitting n objects into two parts, ok.

So, you can think of one part as a set as a subset and the other as a complement. So there are two raised to n ways of choosing a subset. But this subset had to be nonempty. So we have to remove 2 because if you take the full subset or the empty subset, then they would result in one of the parts being empty. So that has to be avoided. And does not matter whether you take the subset or the complement, so the first part is that is if the set that you have chosen is A and the complement is A complement or of the set that you have chosen A complement and the complement is A, both are counted as the same. So, there is by 2. So this will be 2 raised to n minus 1 minus 1, Ok.

What is a general formula for S n, k? We could say that S n, k is going to be zero if k is greater than n. So the only interesting case is where k is less than n and by convention we could choose S 0, 0 as 1. This will just make our formulas look better later on. So, this means if you have zero objects and you want to split it into zero parts, by convention we are saying that there is only one way of doing it. So, we need to find a formula for S n, k. So, S n, k is also known as the Stirling number of the second kind. We need to find a formula for S n, k.

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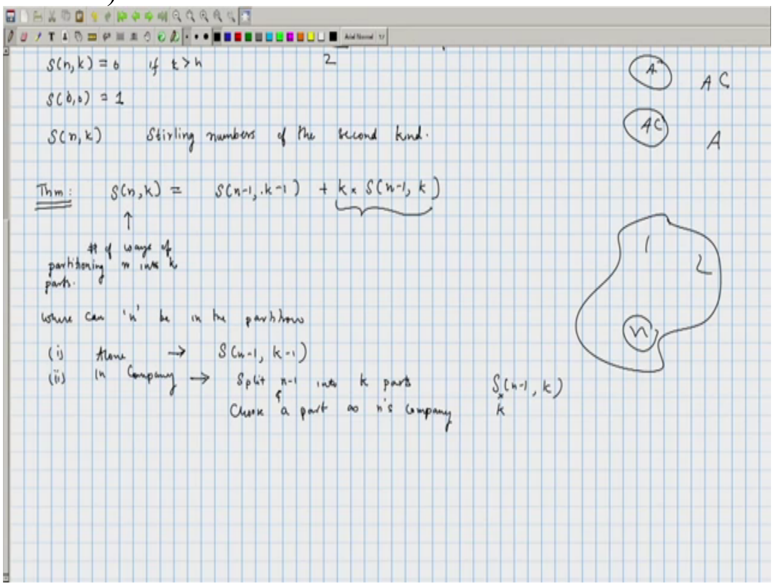

So the following theorem gives a recursive formula to obtain S n, k. So, S n, k is equal to S n minus 1, k minus 1 plus k into S n minus 1, k, Ok. We need to prove that this theorem is correct. So like many other problems in Combinatorics when you have, so there is a combinatorial identity. When you have a combinatorial identity, what you can do is try to find sets whose cardinality is same, ok?

So, if you can find a set and find the number of and find two different ways of counting those sets and show that these two distinct two different ways essentially counts the same object. Ok. So, let us try to do that. So if we look at S n, k, S n, k is basically the number of ways of partitioning n into k parts. Now when you are splitting n into k parts clearly the number n has to go into one of the parts, ok. So, based on that we are going to count. So this is a set of numbers 1 to n.

Now if you look at the number n, this might either be in a part of its own. Or it might be with some other elements. These are the only two choices. So, there can n be in the partition. First alone, second in company, ok. If so if you look at the total number of ways of partitioning says that n is alone and the number of ways of partitioning such that n is in company and you add them up, we will get the total account, ok. If it were alone then it means the remaining and minus 1 objects have to be split into k minus 1 parts. So this alone corresponds to S n minus 1, k minus 1. So, in company should basically be k times S n minus 1, k.

Let us see why that is the case. So, we know that n is in company. So let us forget about n and the remaining elements and if you just take n out of the partition, what you can see is that the remaining n minus 1 elements are being sent into k distinct parts. They are sent into k parts and into any of these parts if you put k if you put n, you will get a different partition. Ok. So, in company the total number of ways of doing it is split n minus 1 into k parts and choose a part as n's company. So, splitting can be done in S and minus 1, k ways and the choosing of n's company can be done in k different ways because there are k parts.

So the product is the total number of ways of doing it and therefore the total number of ways of partitioning and into k parts is the sum of these two ways and that, so that proves the theorem. We will see some interesting consequences of this identity.

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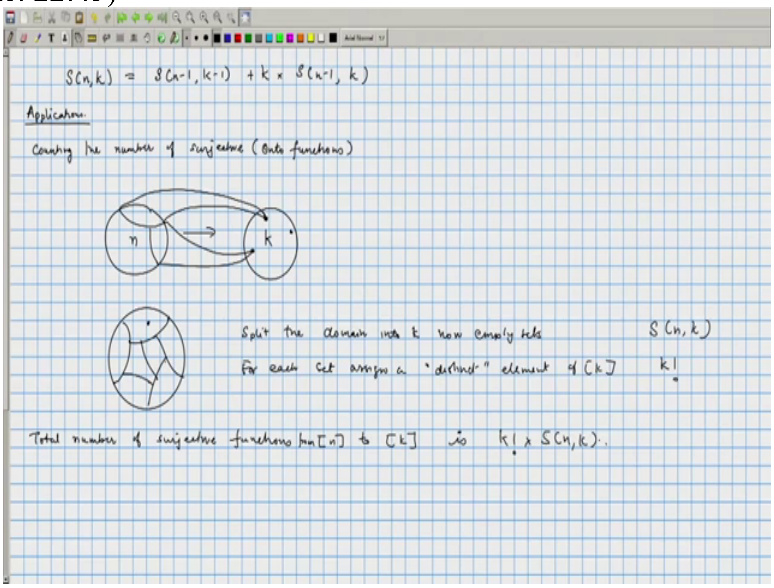

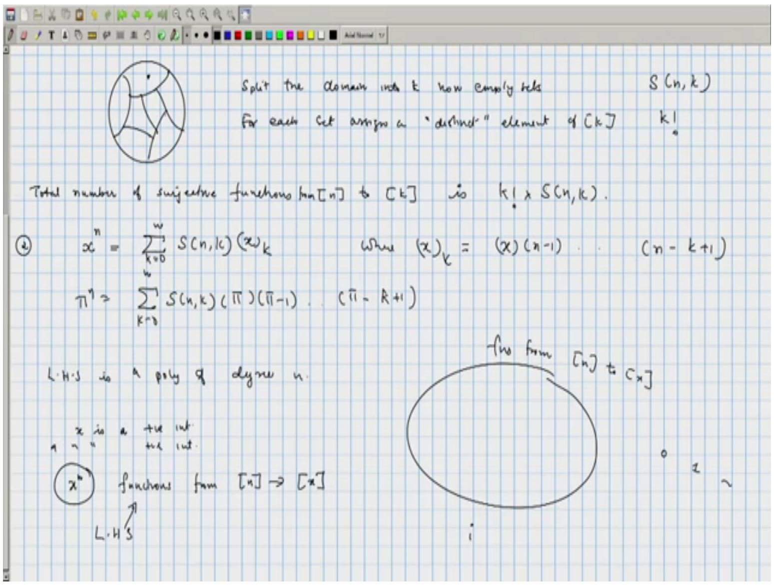

So, S n, k is equal to S n minus 1, choose k minus 1 plus k into S n minus 1, k. So, let us look at the number of surjective functions. Ok. So, we will use this as applications, surjective on two functions. So, we are interested in functions from an n element set to a k element set, ok such that it is an onto function. Every element of the image every element of the codomain is part of the image. So how many ways are there of doing it? So if you look at any such function we can think of it in the following way. So look at any element. That has a pre-image. So let us say all these map to this particular element and maybe these map to some other element and so on.

So, if you look at the pre-images they basically split the domain into different parts. So, there you obtain a partition of n, ok. So, if you want to count the total number of surjective function you can basically count the partitions. So you take one particular partition it can be converted into a function, of course you have to decide this particular part is assigned to which particular number. If you look at any particular surjective function it basically induces a partition on the domain. Ok.

So, the way of the one way of constructing surjective functions would be first choose the partition of the domain and then for each partition assign a particular number from the range. So split or partition the domain into k on empty parts. For each set assign a distinct element of k. You cannot assign the same element because that would not make the function surjective, ok.

So, the number of ways of splitting the domain is basically S n, k. And if you have to assign distinct numbers from 1 to k to these parts, that is going to be k factorial ways. So the total number of surjective functions from n to k is k factorial times S n, k. And as a corollary of this we can say. We can prove the following polynomial identity. So x raised to n is equal to summation S n, k. x k where k is the following factorial. So that is x into x minus 1 into x minus k plus 1. Ok.

Now note that this is a polynomial identity. This is not just for integers or just natural numbers, ok. So, this would also say that pi raised to n is equal to summation is over, k equals 0 to n. This would also say that pi raised to n is summation k equals 0 to n S n, k into Pi into Pi minus 1 into pi minus k plus 1, ok. So, this is a fairly complicated expression. But this is true for, what this means is that this Polynomial identity it is true for all real numbers. Ok. So, from basic so in the sense it is an interesting formula. What we want to prove is something, it is from the Combinatorics of finite objects, we will show something, is true for say much larger class of objects. Ok.

The idea is very simple. If you want to show that two polynomials are equal the only thing that you have to show is they are equal at some large number of points, ok. So, here you have a polynomial of degree n, if so the LHS is a polynomial of degree n. And if you can show that the RHS which is also a polynomial of degree n if you can show that they agree at n distinct positions then they must be same for all different positions, ok. So, how do we show that they are the same at n different positions? Ok. We will show that these equations hold at all positive integer value. Once it is true at all positive integer value it must be true for all real numbers.

So, let us say x we will, we can now assume that x is a positive integer and n is a positive integer, so then we can use our combinatorial insights into proving this. x raised to n is just nothing but functions from n to x, choose numbers from 1 to n and for each of them choose an image, that would be a particular function. And that in that way you can find all functions. So the total number of ways of doing that is x into x into x n times, so that is x raised to n. So this is the LHS, number of functions from n to x. Now we want to find these we want to count this set in a different way. Ok. So, if you look at the set of all functions, we could look at functions whose image is of size 0, 1, 2 and so on. 0 will of course be 0, there are no functions of size 0. I mean there no functions whose image is a set of size 0, ok.

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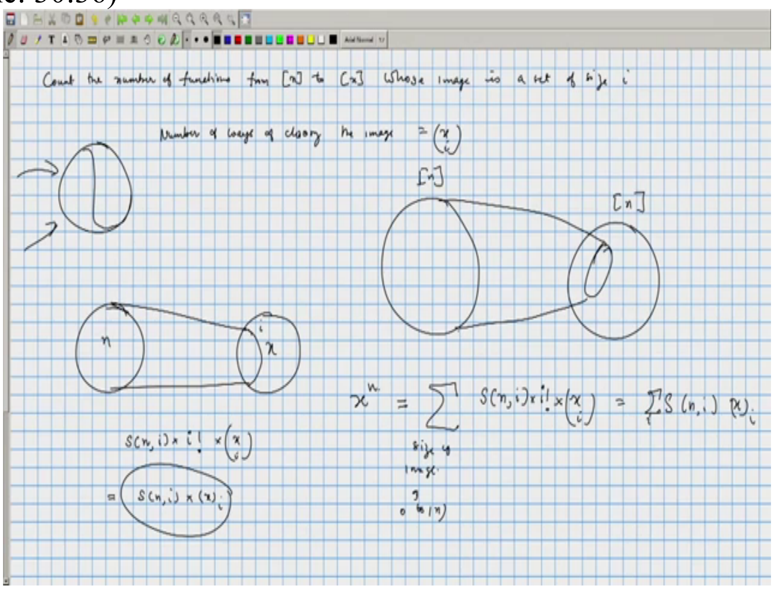

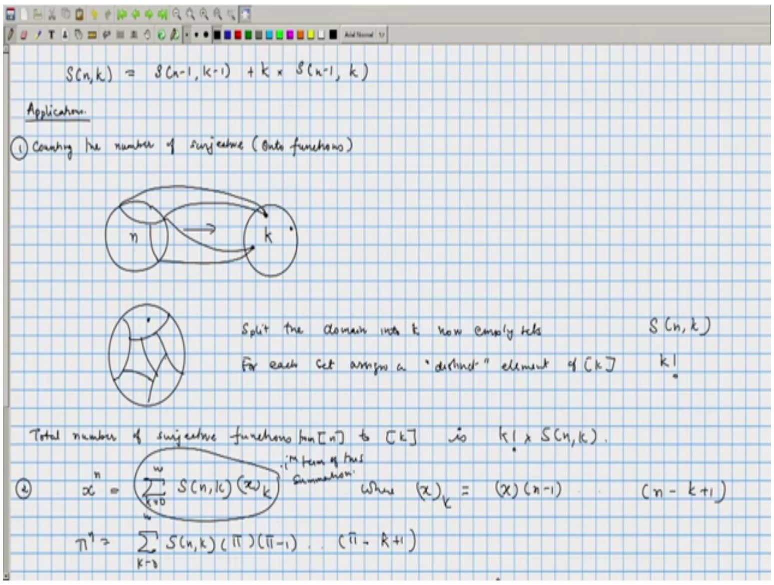

So, if you look at functions from x from n to x whose image is of size is i, how do we count that? So, the first step would be count the number of functions from n to x where image is a set of size i. So basically that quantity would be the ith term in the RHS. So, this we will show as the ith term in the RHS. So we want to count the number of functions from n to x whose image is a set of size i, ok. So, x is this particular set. And we were looking at functions which map into this particular set.

Now if the image is of size i, then so that is a subset of size i. That can be chosen in x choose i ways. So number of ways of choosing the image is equal to x choose i. Once that has been chosen we had the set n and we wanted to map it to x. We first restricted that the subset that it maps into a set of size i, that could have been chosen next choose i ways, ok. And then we need to find a surjective function from this particular set n to the subset that we have chosen.

So the total number of ways of doing that would be S n, i into i factorial. Ok. Because that is the total number of ways of, that is the total number of surjective functions, ok. And into the image could have been chosen in x choose i ways. Ok. So, this is equal to S n, i multiplied by falling factorial x. So when so this is the ith term in the summation. So total number of ways, so what we did is instead of looking at all possible functions from let us say n to x, we summed over the size of the images. So if the image is of size 1, then how many functions are there? The image is of size 2 then how many functions are there? So sum over sizes of image.

So, the size of the image can vary from 1 to n, ok. Or we may assume that x is great x is less than n, so the size of the image varies from 0 to size of x, ok. Once we have fixed the size of the image, we could choose the image in x choose i ways. The image set could be chosen in x choose i ways and then we just need to look at the subjective functions from n to that particular set and the surjective function is equal to S n choose i into i factorial into x i. And that is equal to S n, i falling factorial i and this summed up for all values of, i is basically equal to the total number of function and that is equal to x raised to i. So, that concludes the proof of this particular identity.

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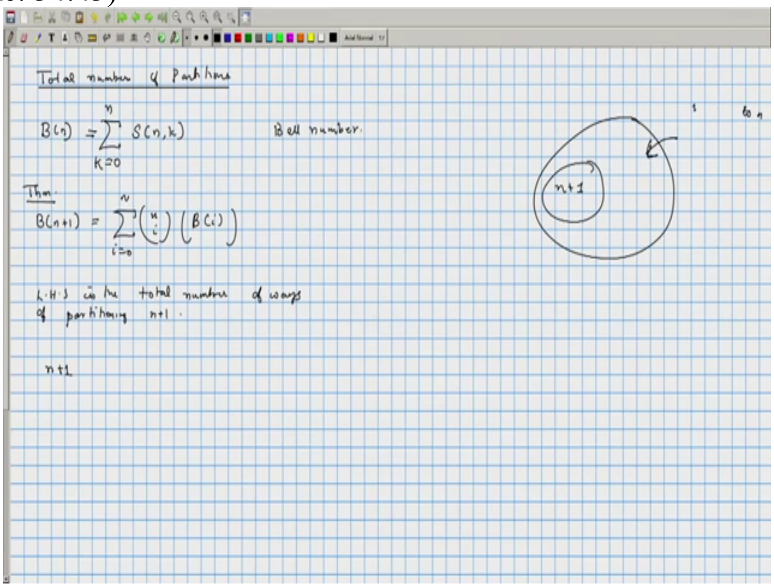

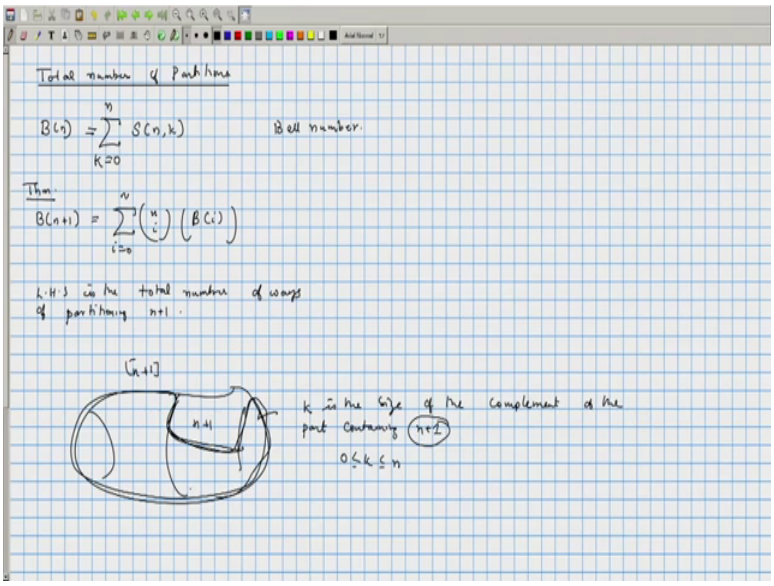

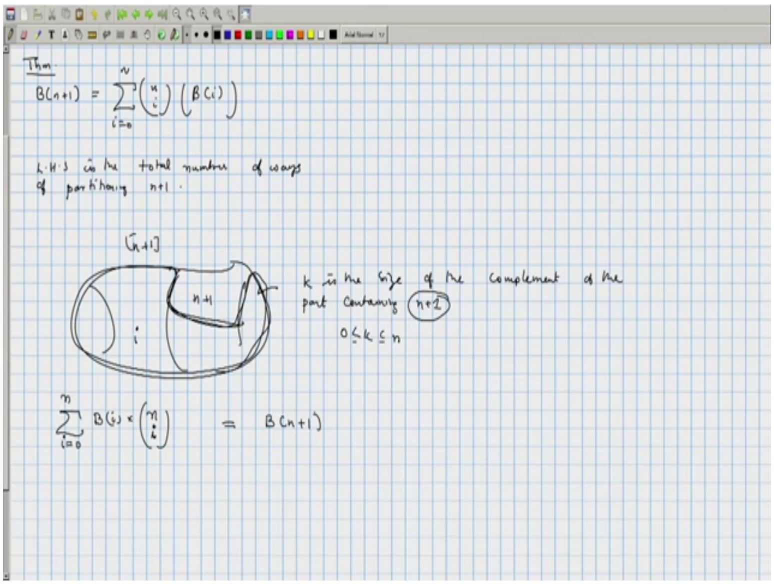

The next object that we will see is the total number of partitions. So, when we talked about S n, k, S n, k was nothing but the number of ways of splitting n into k parts. If we sum this up over all parts, all values of k, so k going from say 0 to and, that is the total number of ways of partitioning and this we will denote it by a special number called as B n or the Bell number.

So, the next thing that we have on agenda is to show that the Bell numbers satisfy the following identity: so B n plus 1 is nothing but summation over I going from 0 to n, n choose i, B i. Why is this? So, LHS is the total number of ways of partitioning. So, we need to look at the number of ways of partitioning n plus 1 and show that the RHS also counts exactly that, ok. Again, we can look at the element n plus 1 and there are many possibilities, the element n plus1 could be in some block with many other elements, ok.

So, let us let us say that so if we were partitioning n plus 1 in some particular block and the others are in some other block. So the possibilities are that the complement is a set of size 1 to n, ok. So, what we are looking at is the block in which n plus 1 is present and its complement block. So in order to prove this theorem, we look at the element n plus 1. In any partition n plus 1 should be in one of the parts and the remaining elements if we consider they can be of size 1 to n, ok. So, this is what we are saying. We have split n plus 1 into some number of parts. And n plus1 is in one of those parts.

If you look at all the other elements together that is going to be a set of size k. So k is the size of the complement of the part containing n plus 1. And this k can vary from says 0 because everything could be n plus 1 could be in a block which contains everything else or it could go all the way up to size n, it cannot be n plus 1 because one element is taken off. Namely n plus

1, ok. So, and these other elements now has to be split into some number of parts. They have to be partitioned, that is all. So, there are how many ways of doing this?

So if the complement is of size i, so let us say if the complement is of size i, the total number of ways of splitting that is going to be B i in whatever was the complement if it is if its size was i then the number of ways of splitting it is B i. But these i elements could have been selected into n choose i ways, ok, n plus 1 surely is not there. The remaining n elements out of them i had to be selected. So B i into n choose i is the total number of ways of splitting the n plus1 elements into different parts provided n plus 1 is in a different block, ok. And if you sum this over all possible values of i going from 0 to n, we will get the total number of ways of splitting. So that basically proves this combinatorial identity. Let us stop here.