Discrete Mathematics

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Lecture 23

Lattices

Welcome to the NPTL MOOC on Discrete Mathematics. This is the seventh lecture on set theory. Here we continue the discussion on partial order in relations that we started in the sixth lecture.

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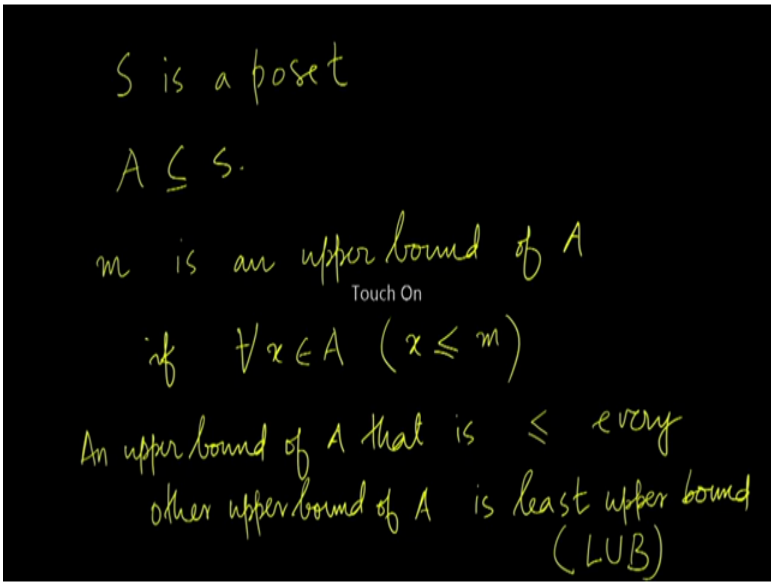

Let us say S is a poset. Let us say A is a subset of s. We say that m is an upper bond of A. If for every x belonging to A, it is the case that x is less than or equal to m or x precedes m. An upper bound of A that is less than or equal to every other upper bound of A is a least upper bound or LUB for short.

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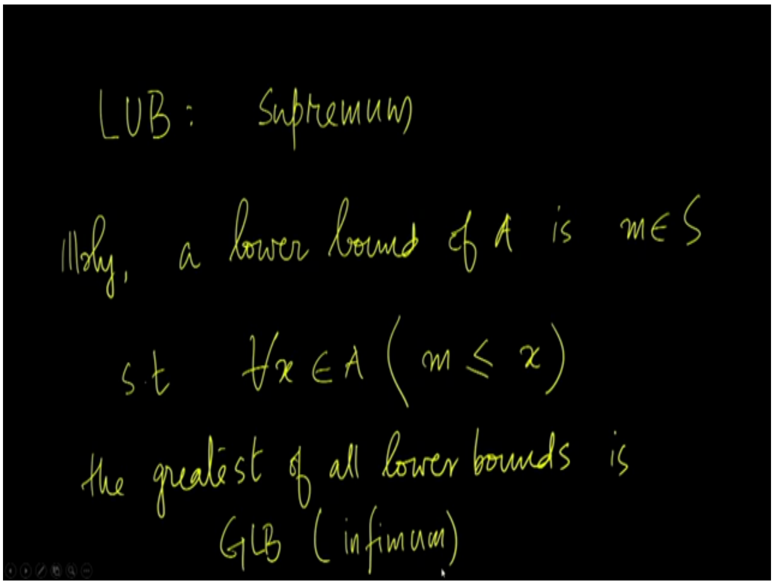

Where least upper bound is also called supremum. Similarly, a low bound of A is some m belonging to S such that for every x belonging to A it is the case that x is less than or equal to, x is greater than equal to m. It is the case that m is less than or equal to x. The greatest of all lower bounds is the greatest lower bound is also called the infimum.

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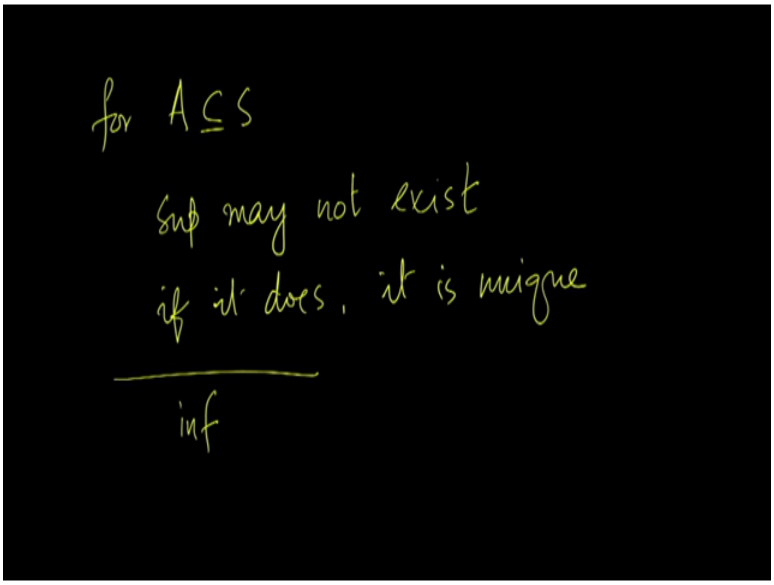

In particular for some A, for an arbitrary A, supremum may not exist, but if it does exist, it is unique. We can say the same thing about an infimum, the infimum may not exist, but if it does exist, it is unique.

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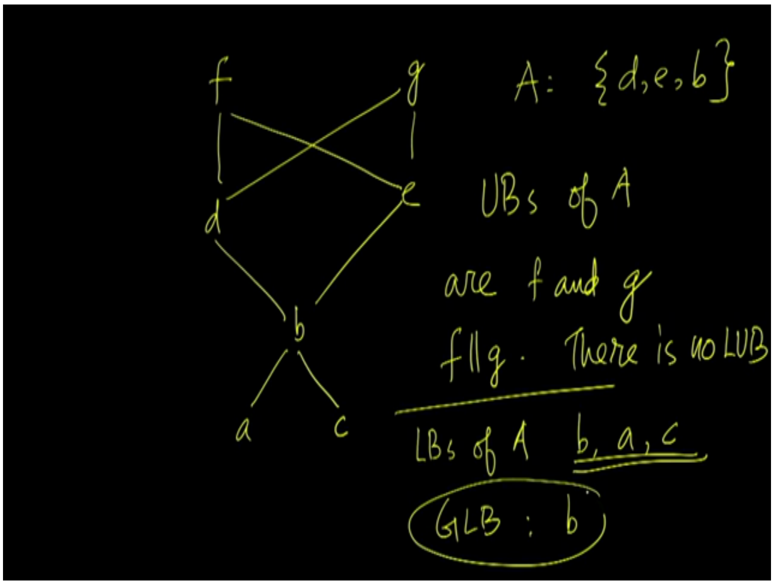

For example, consider this partial order denoted by a hasse diagram. Let A equals d e b, the set d e b, then the upper bounds of A are f and g. A has to upper bounds, but since, f and g are incomparable there is no LUB. The set A has no supremum. On the other hand, the lower bounds of A are b, a, and c. All these three are less than or equal to every member of A. So, these are all lower bound of the set A. The greatest lower bound is the greatest of them, that is unique which happens to be b. So, the set has a greatest lower bound but it is without a least upper bound.

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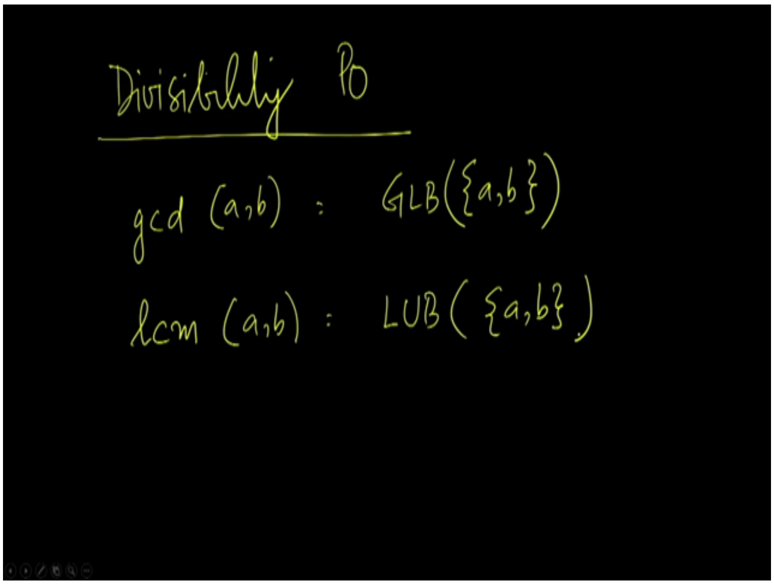

Consider the divisibility partial order here g c d of a and b is the greatest lower bound of the two member set a b. l c m of a and b is the least upper bound of a and b.

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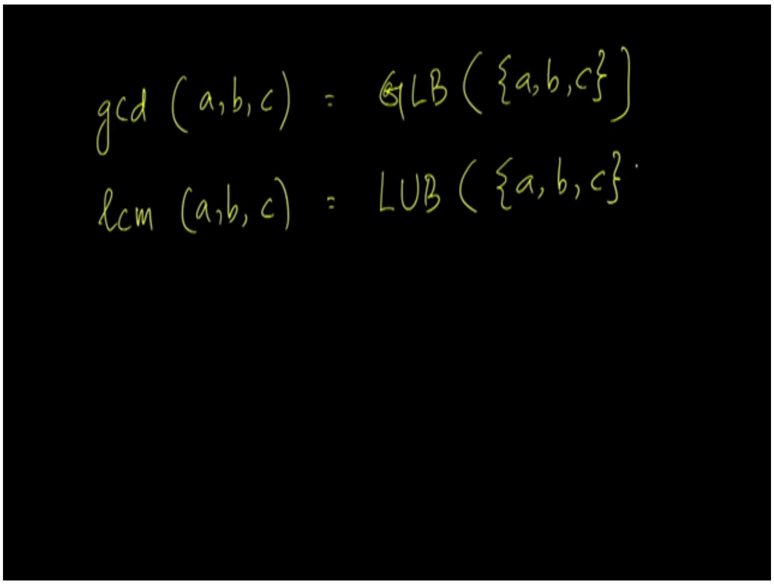

We can, of course, extend those to more elements, for example, the g c d of a, b, c would be the greatest lower bound of the triplet, the three member set a, b, c. Similarly, l c m of a, b, c, the least common multiple of a, b, c would be the least upper bound of this set.

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Now, let us study what are called lattices.

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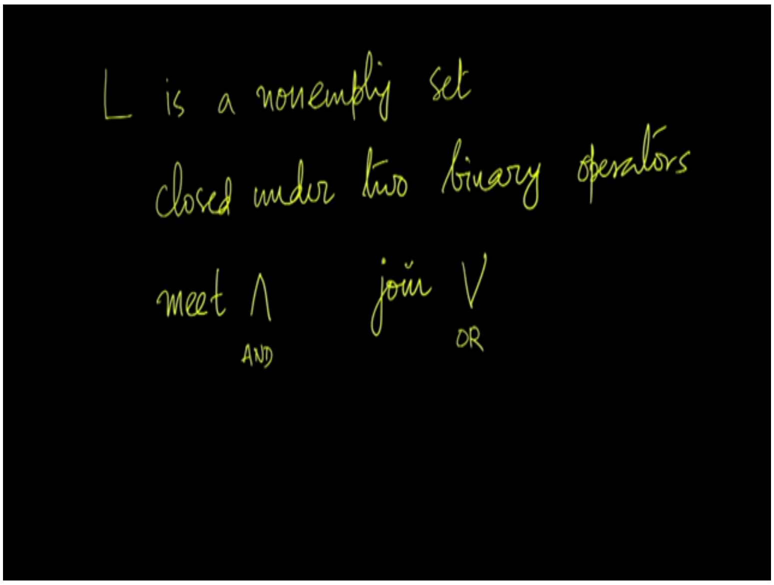

Suppose, L is a non-empty set closed under two binary operators, operators of functions. So these two operators are named meet and join. It is customary to represent meet use in this symbol, and join using this symbol.

So, they are similar to the AND and OR symbols in Boolean Algebra. There is not without reason. Therefore, in our subsequent discussion, I will use the word meet, AND as synonyms and join and OR as synonyms. But meet is with more general connotations than and similarly join. So, we consider a nonempty set L with is closed under two binary operators.

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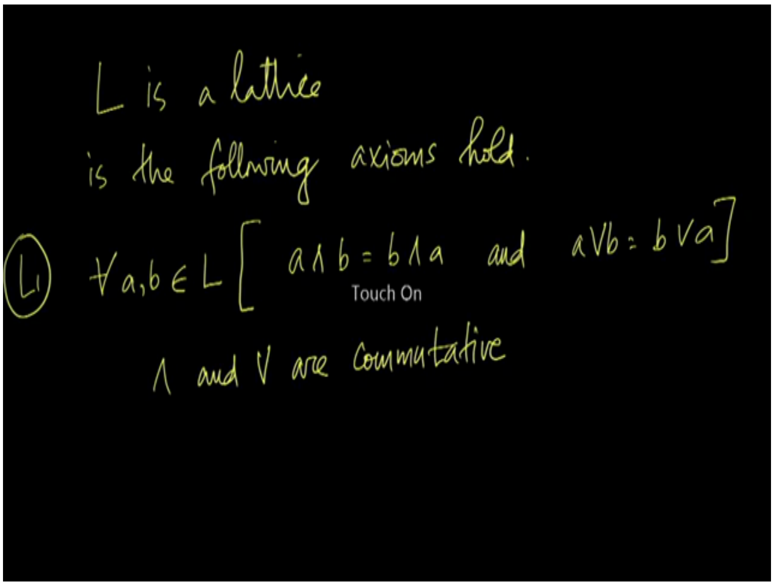

We say that L is a lattice if the following axioms hold. The first axioms says that for every a and b belonging to L, a meet b is equal to b meet a and a join equals b join a, which means meet and join are commutative operators. This is the first axiom, let me call it L 1, the first axiom of lattice is.

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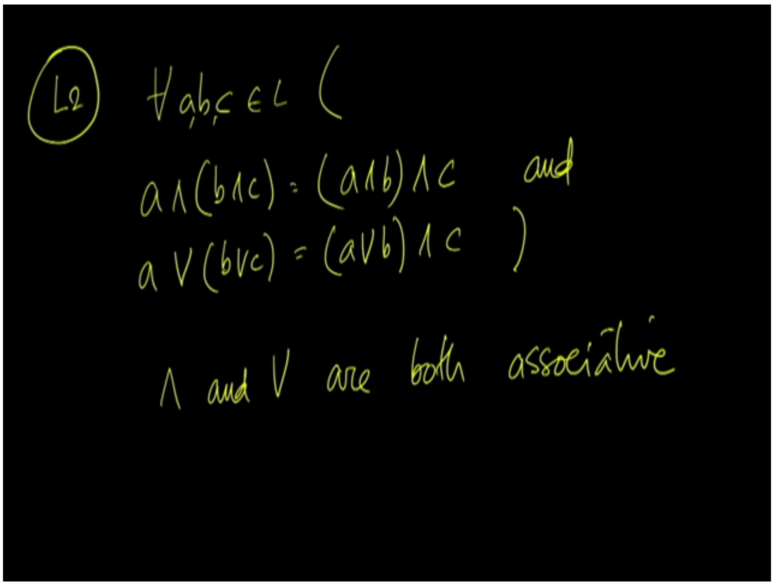

The second axiom says that for every a, b, c belonging to L it is the case that a meet b meet c is equal to a meet b meet c, and a join b join c is equal to a join b join c which means meet and join are both associated. This is the second axiom.

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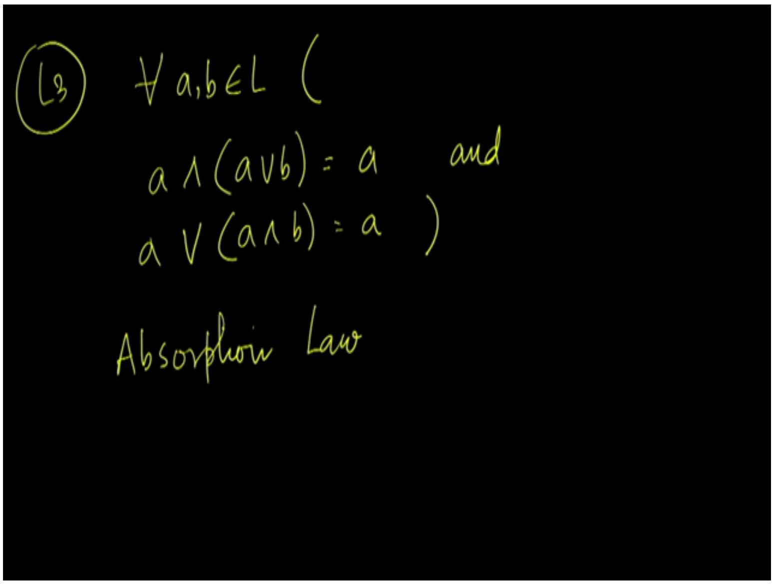

The third axiom says that for every a and b belonging to L, a meet a join b is equal to a and a join a meet b, this a again. This is the absorption law. So, let us assume that our nonempty set L which is closed under the two operators meet and join satisfy these three axioms, commutativity, associativity and absorption.

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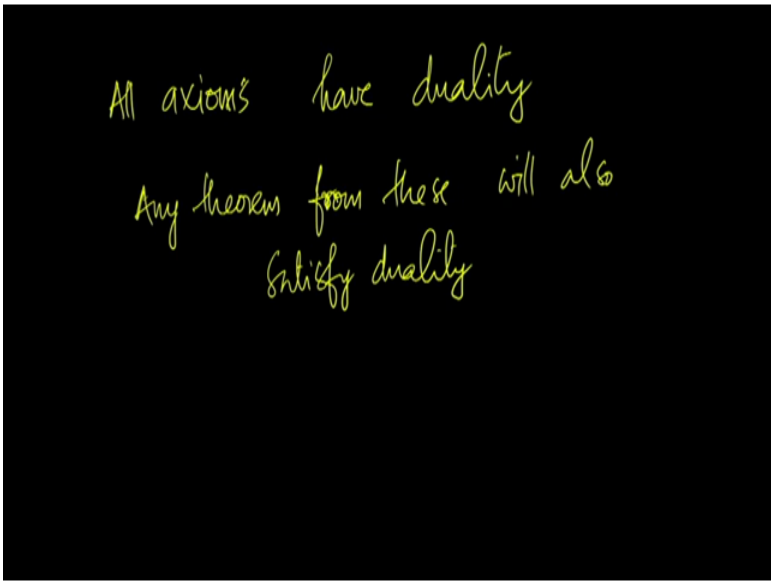

Now, you can say that all axioms have duality. That is in the commutativity axiom when you substitute meet with join and join with meet, we again get another commutativity axiom. Similarly, for associativity, and absorption as well. So, all axioms satisfy duality, therefore, any theorem that is proved from these axioms will also satisfy duality. That is you take a proof in this system and in this proof if you substitute meet with join and join with meet throughout then you find that you conclude the dual of the original conclusion.

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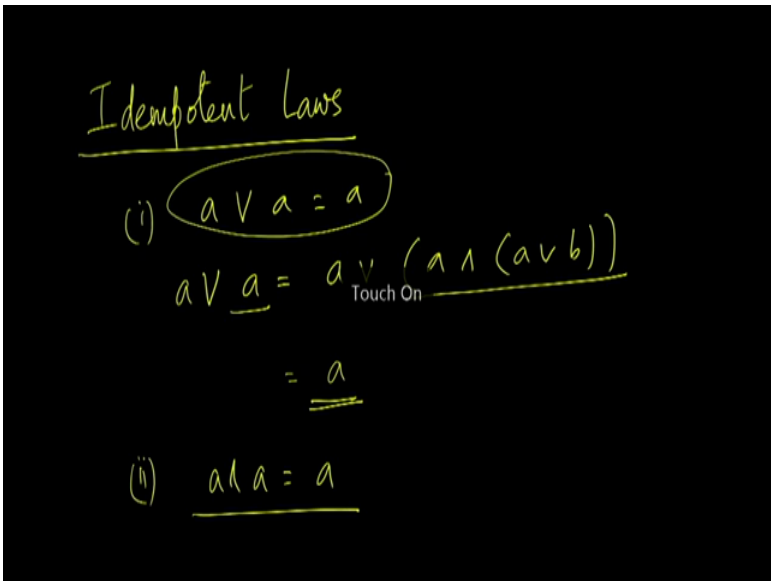

Let us derive the idempotent laws. The first idempotent law says that a or a equal to a. You can prove this way a or a is, a or a and a or b, that is because the underlined portions are equivalent by the absorption law. But here we have a or a and something which by the dual of the previous absorption law, the absorption law which we used in step one is a indeed, therefore, we have proved the idempotent law for join. The dual of this proof will give us the idempotent law for meet, a meet a is a.

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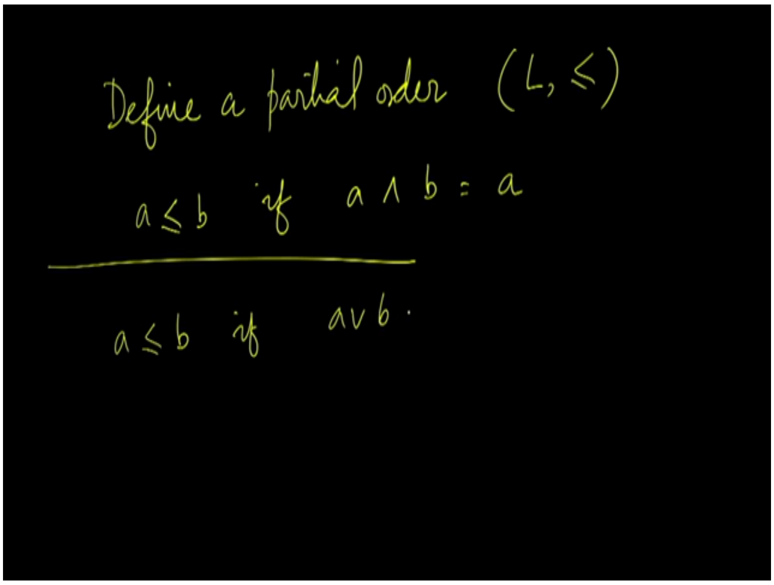

Now, let us define a partial order on set L, let this relationship be called less than or equal to or the precedes relation. In this we say that a is less than or equal to b if a meet b equal to a, this is how we define the partial order. Then we can see immediately that a is less than or equal to b if a or b is b.

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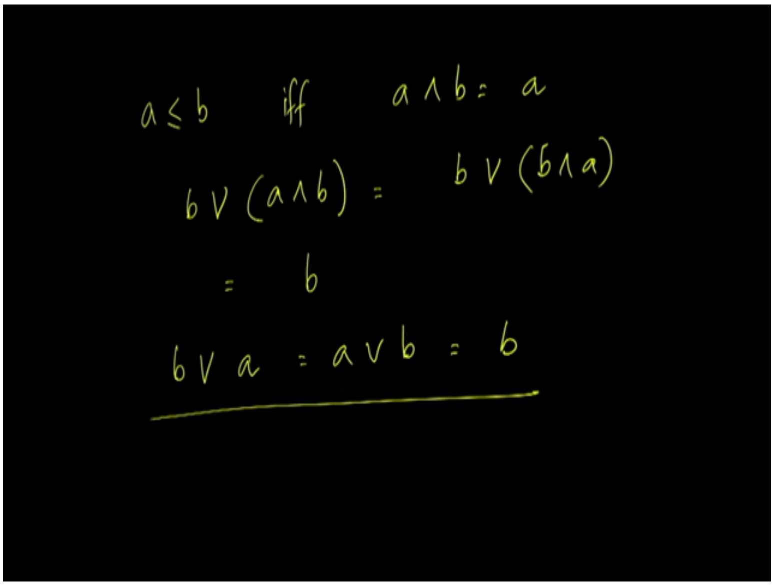

Why is this so? We know that a less than or equal to b if and only if a meet b is a by definition, but if a meet b is a then b join a meet b is b join b meet a by commutativity, which is then b by absorption. Which means b join a is a join b, which is b. Therefore, we conclude that a join b is b.

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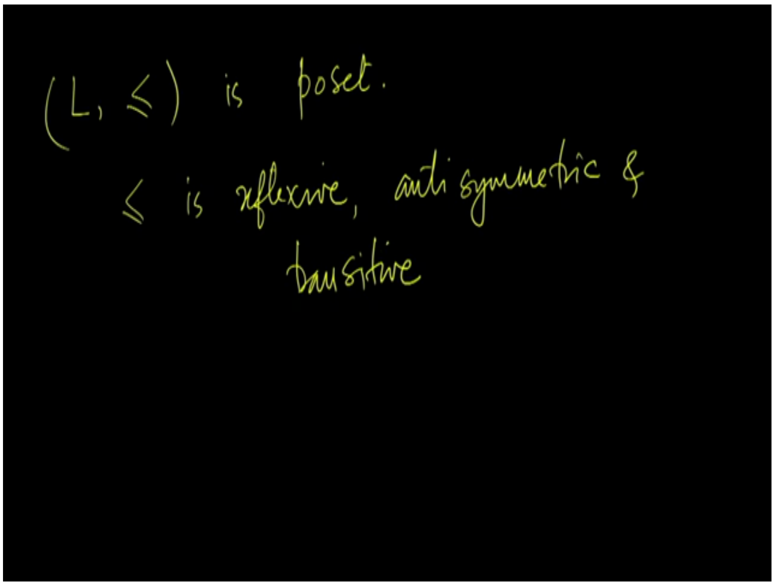

Then, we see that this less than or equal to relationship along with L forms a poset that is because the less than or equal to relationship is reflexive, antisymmetric. and transitive.

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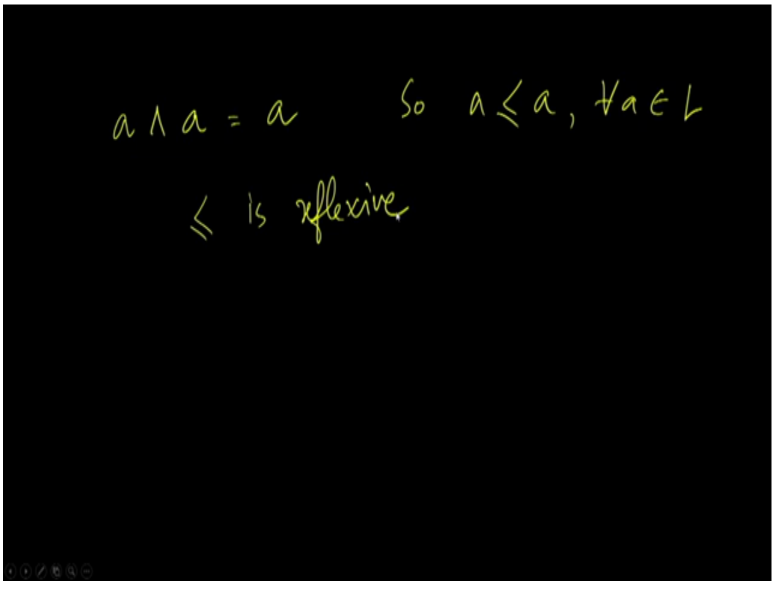

Why is this so? a meet a is a by the idempotent law we have just shown. So a is less than or equal to a by the definition of the less than or equal to relationship, this is the case for every a in L. Therefore, the less than or equal to relationship is reflexive.

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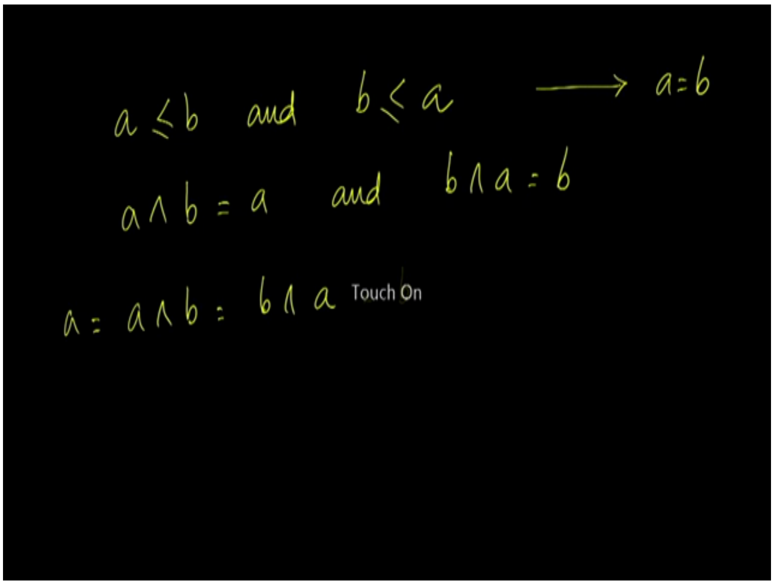

Suppose a less than or equal to b and b less than or equal to a, then by the definition of the less than or equal to relationship, we know the a meet b is a and b meet a is b, but then a meet b is b meet a by the commutativity the meet operation which is b so we have a equal to b, therefore, we have the antisymmetric property. We have concluded that a equal to b, if a is less than or equal to b and b is less than or equal to a, then a equal to b. So the less than or equal to relation that we are talking about is antisymmetric.

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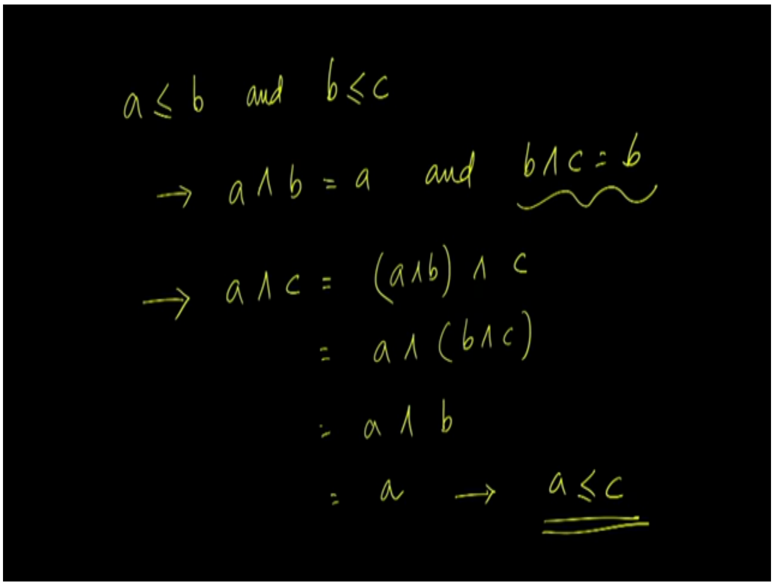

Now, let us say a less than or equal to b and b less than or equal to b, then by the definition of the less than or equal to relationship we have a meet b equal to a and b meet c equal to b. Therefore, a meet c would be, since, a is a meet b, those would be a meet b meet c, but by associativity those would be a meet b meet c, but then from above we know that b meet c is b, so this is a meet b, but a meet b is a, so a meet c is a which implies that a is less than or equal to c, thereby establishing transitivity. So, the less than or equal to relationship is reflexive antisymmetric and transitive.

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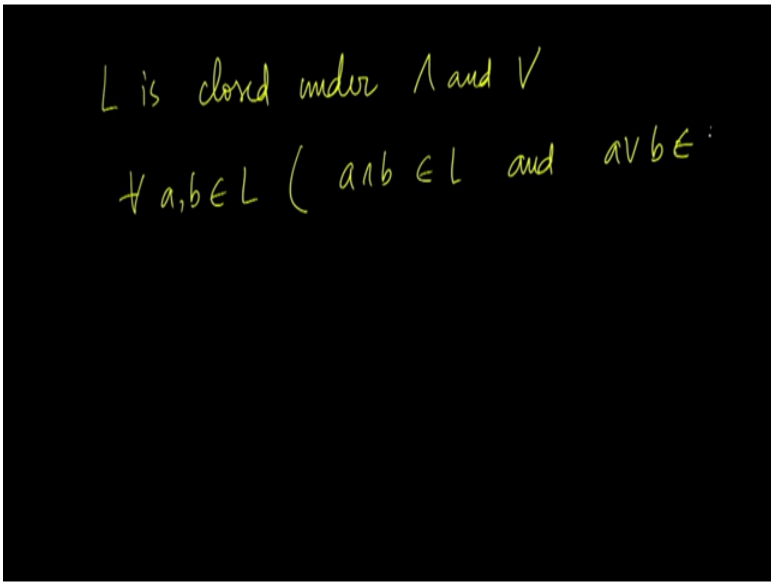

Now we started out by saying that meet and join operations are defined for every L. We started out by saying that L is closed under the operators meet and join, in other words for every a and b belonging to L, a meet b belongs to L and a join b belongs to L.

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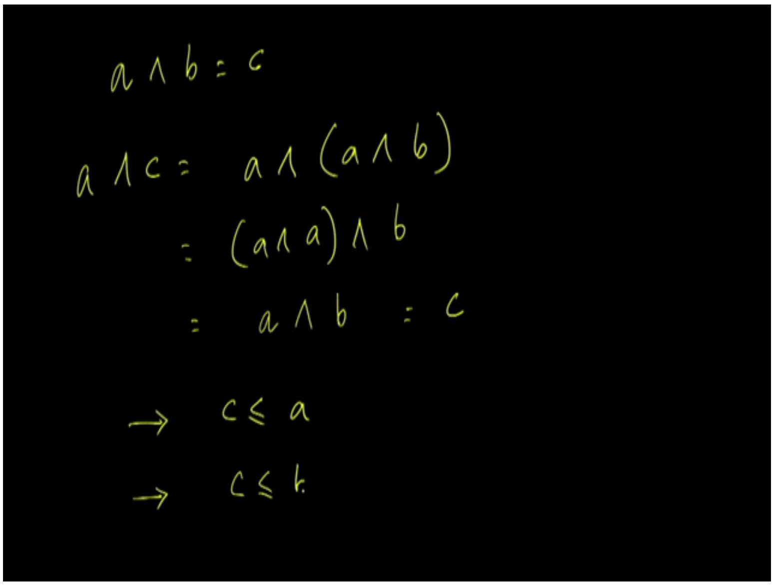

For an arbitrary a and b, let us say a meet b equal to c, then a meet c is a meet a meet b, which is a meet a, by associativity this is a meet a meet b. But a meet a is a by the idempotent law, so this is a meet b, which is c. So, we find that a meet c is c or in other words, c is less than or equal to a, similarly, we also have that c is less than or equal to b.

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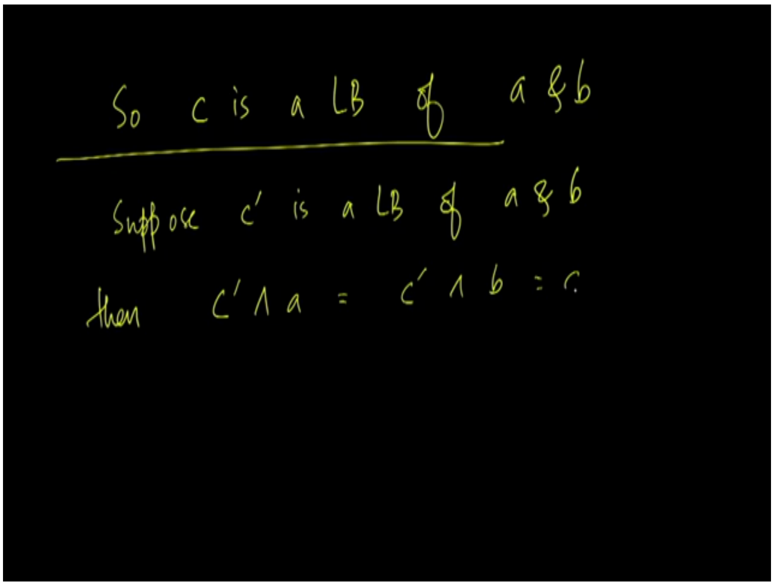

So, c is a lower bound of a and b but, of course, we do not know that c is the greatest lower bound, what we know is that c is a lower bound. Suppose c prime is another lower bound, c prime is a lower bound of a and b, then clearly c prime meet a equals c prime meet b equals c prime.

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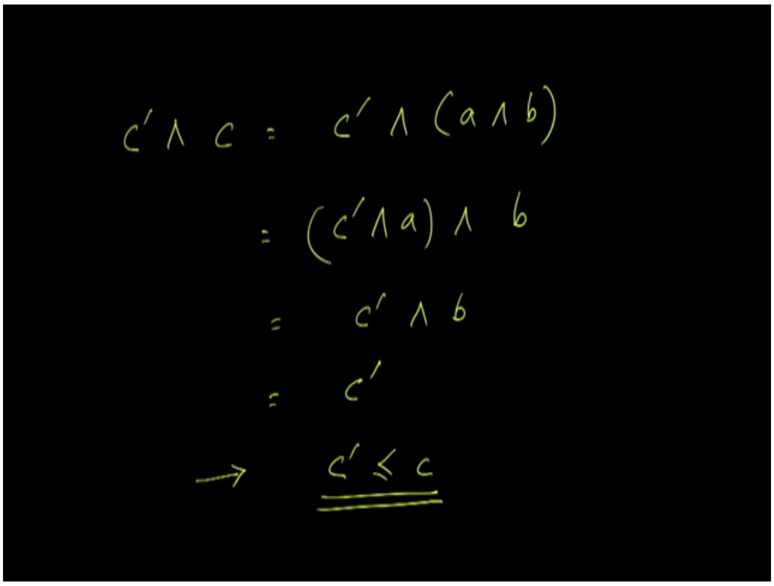

Then what would be c prime meet c this would be c prime meet a meet b, remember c is a meet b, but this would be c prime meet a meet, but c prime meet a is c prime so this is c prime meet b, which is the same as c prime again or in other words, c prime is less than or equal to c. So, if there is another lower bound to a and b then that lower bound is less than or equal to c.

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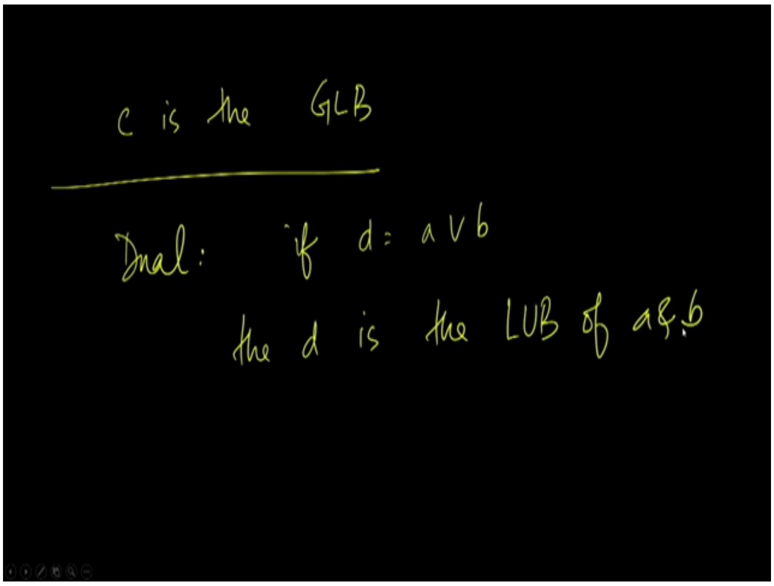

Or in other words c is the greatest lower bound. By the dual of this argument we know that if d is a join b then d is the least upper bound of a and b.

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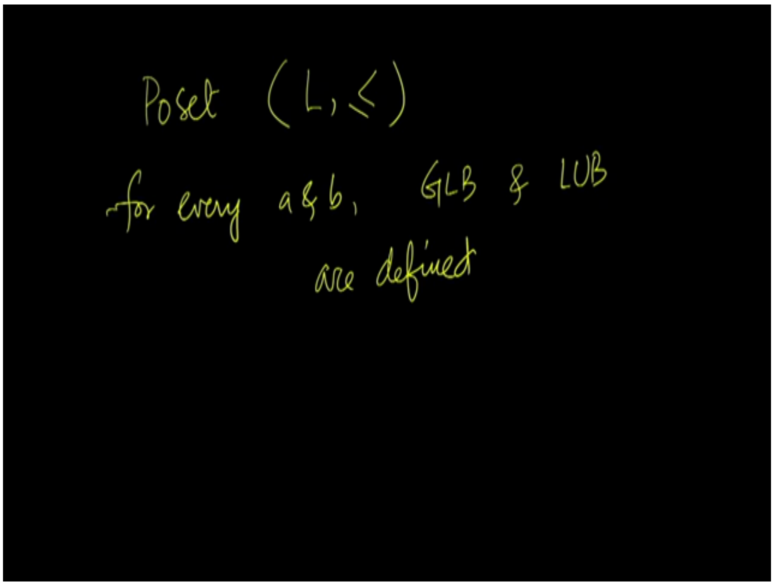

So, what we have concluded is that in the partial order in relation less than or equal to defined on L, we have this property for every a and b, the greatest lower bound and the least upper bound are defined. They happen to be the meet and join of a and b respectively.

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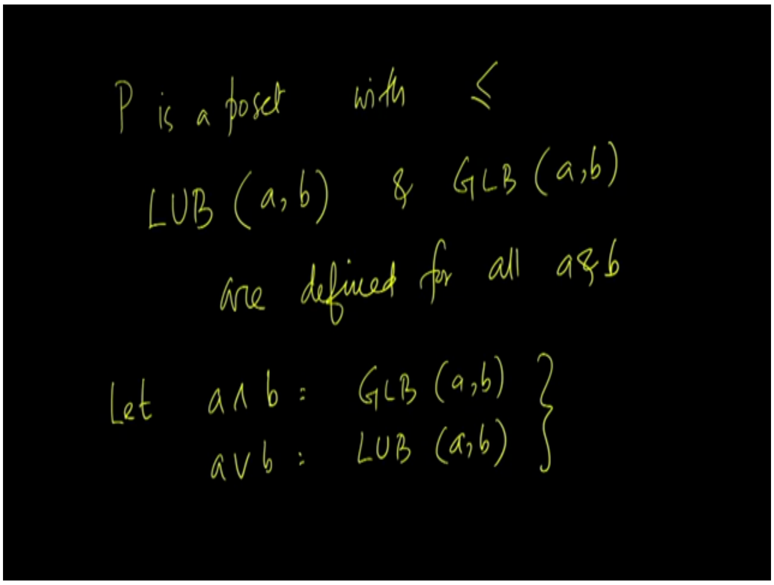

On the other hand, suppose p is a poset with less than or equal to relationship, let us say LUB of a and b and GLB of a and b are defined for all a and b. Let a meet b equal to GLB of a and b and a join b equal to LUB of a and b. So, if you do this, then for every a and b GLB and LUB are defined or meet and join are defined, so meet and join are closed operators for p.

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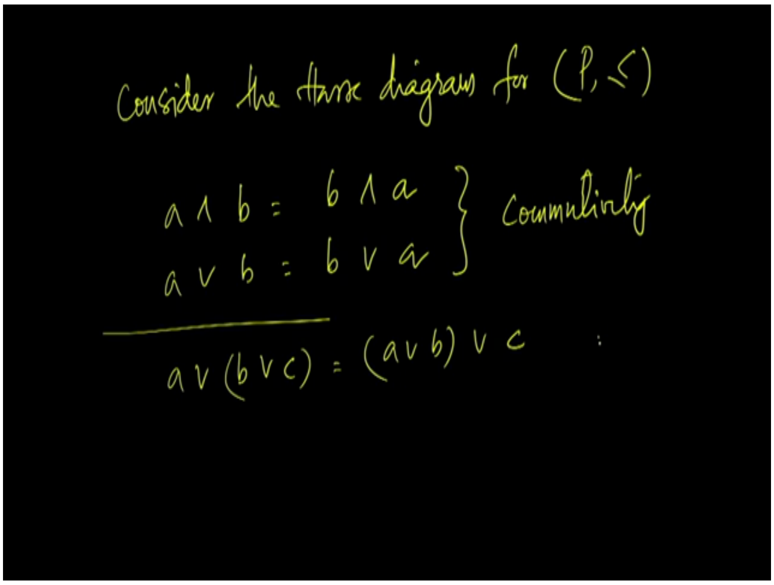

Then consider the Hasse diagram for this poset, from this we find that a meet b is b meet a and a join b is b join a, therefore, commutativity hold for both meet and join. Similarly, form the diagram we can also conclude that associativity holds and the dual of it too, so associativity also holds.

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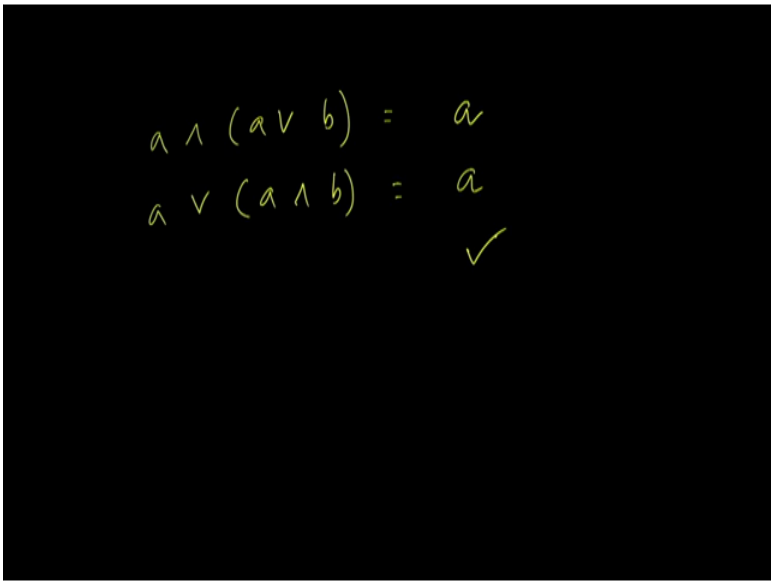

And absorption also would hold, so arguing this from the Hasse diagram this straight forward I leave it as an exercise.

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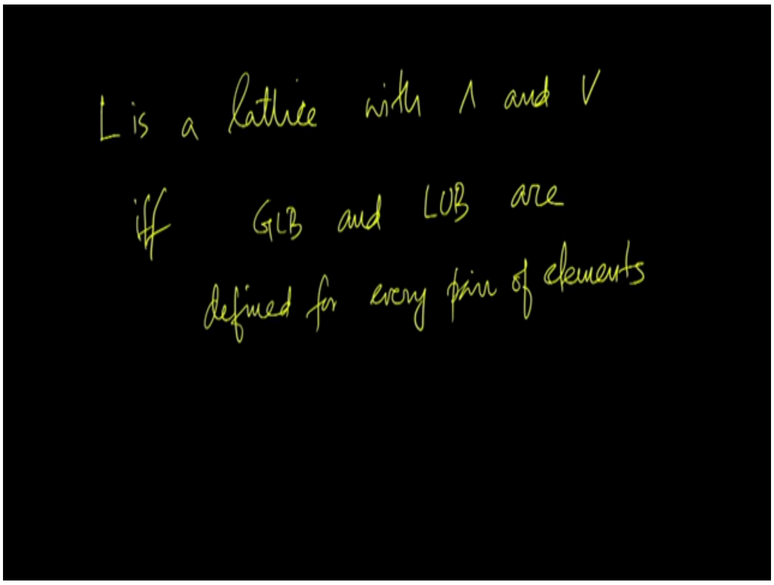

So with this we find is that if L is a lattice with meet and join, then greatest lower bound and least upper bound are defined for every pair of elements. This is, of course, and if and only if relation, L is a lattice if and only if the greatest lower bound and least upper bound are defined for every pair of elements.

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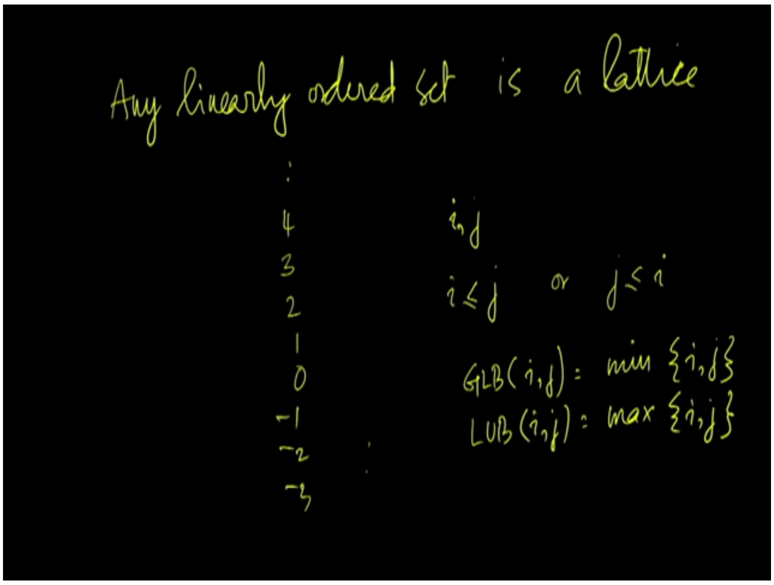

A lattice is a poset in which every pair has a greatest lower bound and least upper bound. Some examples, any linearly ordered set is a lattice, in particular consider the set of integers. Take any two integers i and j, either I less than or equal to j or j less than or equal to i or both. So, the greatest lower bound of i and j would be the smaller of the two and the least upper bound would be the larger of the two, so GLB and LUB are defined for every pair of elements, therefore, this is indeed a lattice.

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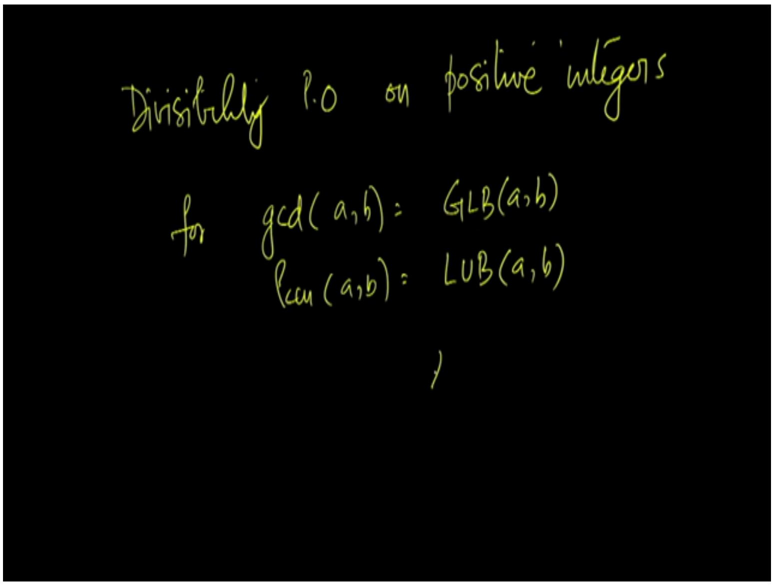

The divisibility partial order on positive integers: For any a and b, gcd of a and b happens to be the greatest lower bound, and lcm of a and b as we saw is the least upper bound, therefore, GLB and LUB are defined for every pair of integers the positive integers. Therefore, on this partial order is a lattice.

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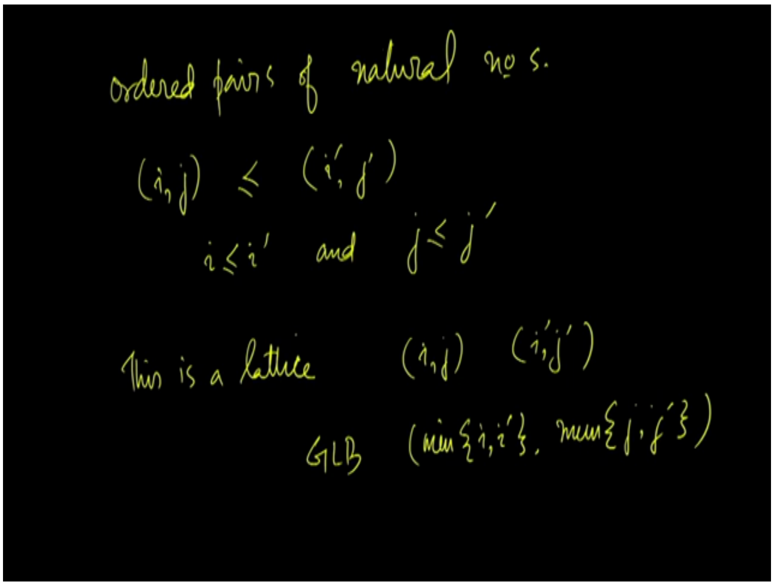

Consider ordered pairs of natural numbers, with that ordered pair i j is less than ordered pair I prime j prime less than or equal to, if I is less than or equal to i prime and j is less than or equal to j prime. This is also a lattice because given two ordered pair i j and i prime j prime, minimum of i i prime, minimum of j j prime is a greatest lower bound. Similarly, maximum of i i prime and maximum of j j prime will form a least upper bound. So, every pair of ordered pairs has a greatest lower bound and a least upper bound so this is a lattice.

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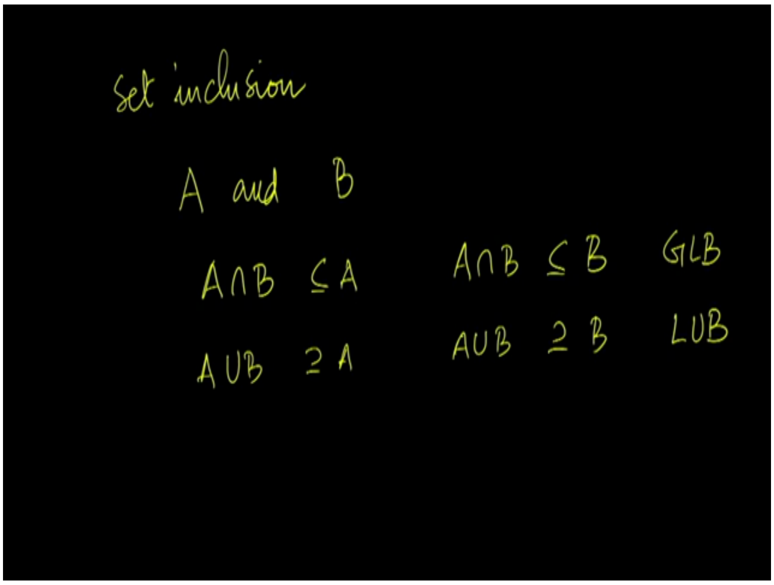

Set inclusion: Let us say we have a family of sets given two sets a and b, a intersection b is a subset of a and a intersection b is a subset of b as well. Therefore, a intersection b is a lower bound of both a and b and it also happens to be the greatest lower bound. A union b is a super set of a and a union b is a super set of b as well and this happens to be the least upper bound. So, every pair of sets has a greatest lower bound and least upper bound so this is a lattice again.

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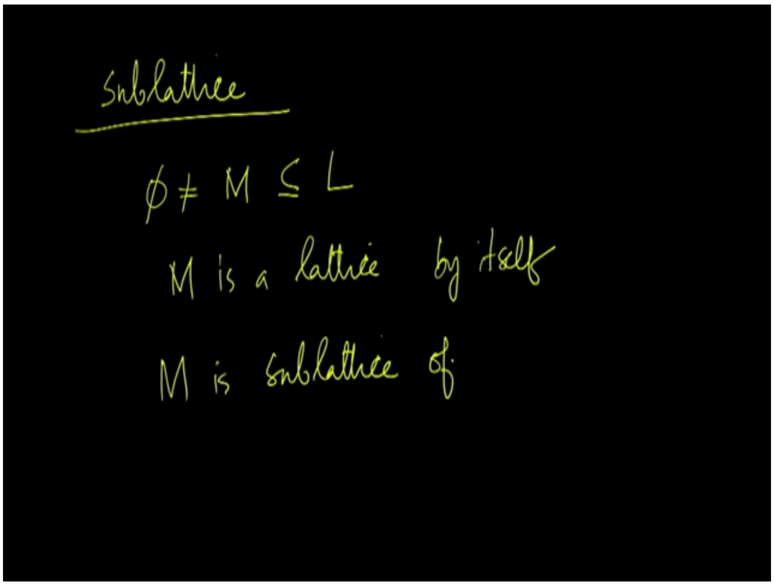

Now, let us define what is called a sub lattice. Let us consider nonempty set m which is a sub set of L where L forms a lattice, suppose m is a lattice by itself. Then we say m is as sub lattice of L.

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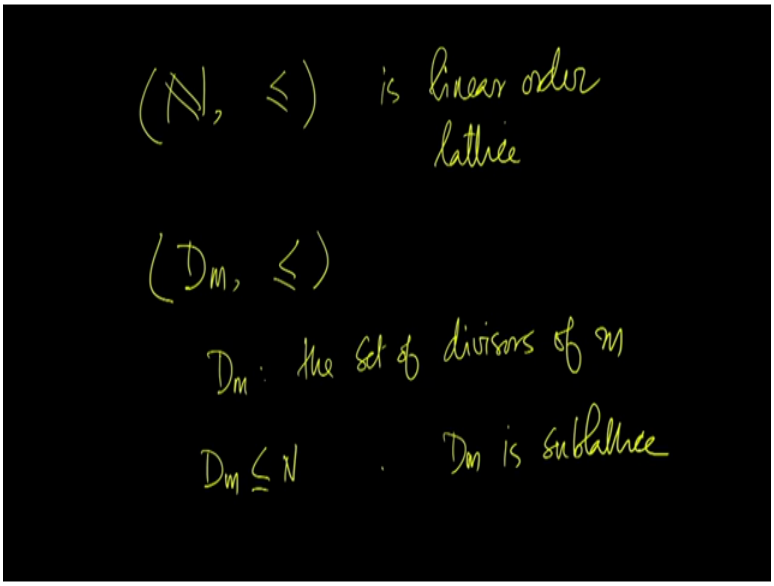

For an example, we saw the set of natural numbers and the less than or equal to relation is a linear order and the for a lattice. Let us consider Dm and the less than or equal to relation, where D m is the set of divisors of m, so clearly D m is a subset of n. Therefore, D m is a sub lattice.

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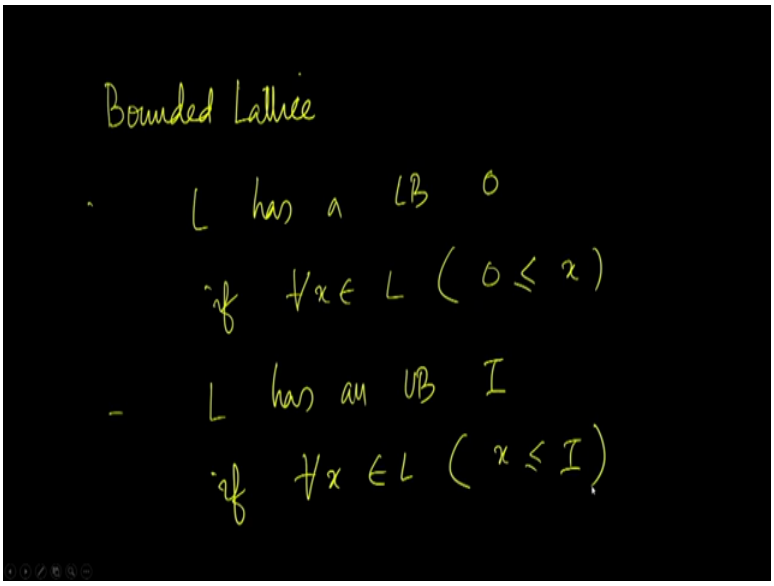

We say that a lattice is Bounded Lattice if L has a lower bound 0. We say that L has a lower bound 0 if for every x belonging to L it is the case that 0 less than or equal to x and we also say that L has an upper bond I, if for every x belonging to L it is the case that x is less than or equal to I.

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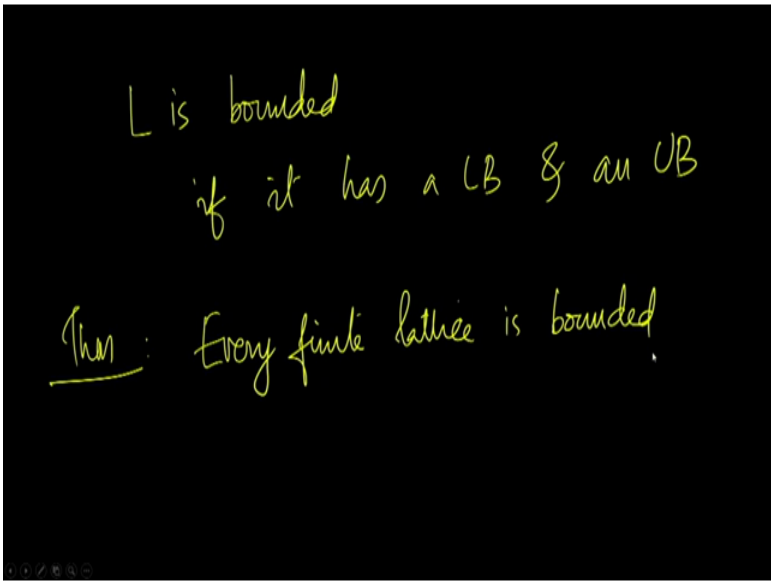

L is bounded if it is both lower bound and an upper bound. Every finite lattice is a bound. I will not prove it here, I am leaving the proof as an exercise to you.

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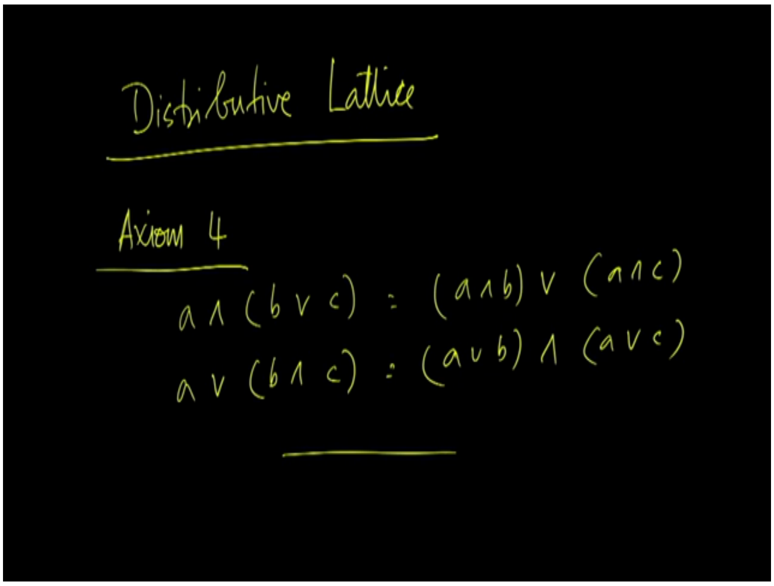

Now to our final topic in this module - Distributive lattices. Distributivity is the fourth axiom, if the fourth axiom which is called the distributive axiom satisfied, then the lattice is called a distributive lattice. That is this axiom should be satisfied in addition to the first three axioms. So, those says that meet distributes over join and join distributes over meet so the duality still holds, meet distributes over join and join distributes over meet.

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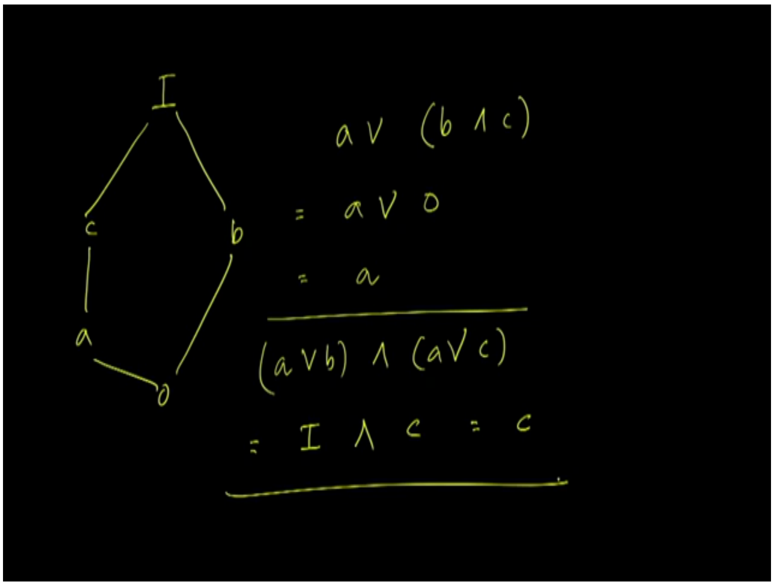

Look at this lattice, this is not non distributive that is because in this a join b meet c is a join b meets c, in this case happens to be 0 the greatest lower bound of b and c that is the downward part from b and c meet and 0, therefore, b meet c is 0 and a join 0 is a. Let us consider the right hand side of the distributivity axiom which happens to be a join b meet a join c, what is a join b. What is a join b? That is I and a join c happens to be, a meet c a join c happens to be c, which is c. So, we find that the left hand side and the right hand side are identical. Therefore, the distributivity law does not hold in this case.

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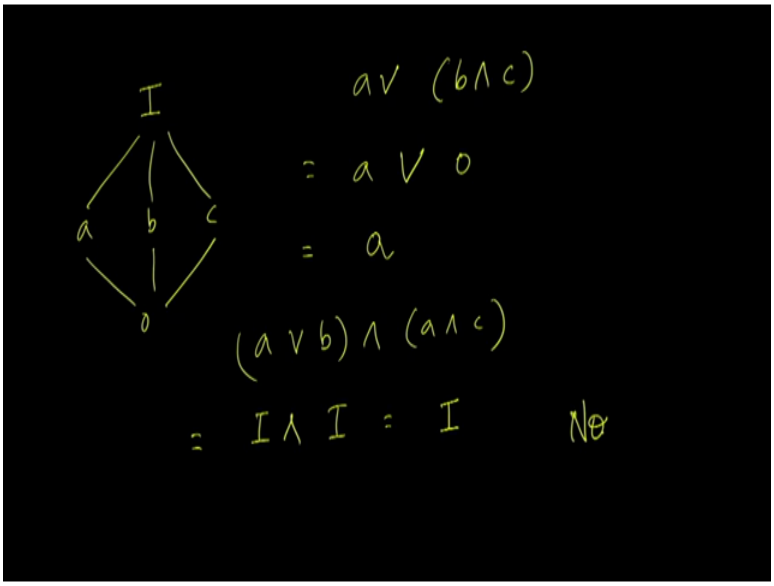

Similarly, this lattice is also not distributive that is because a join b meet c in this case is a join the meet f b and c is 0. That is on the downside b and c would be meeting at 0 so b meet c is 0, a join 0 is a. The right hand side of the distributive law is a join b meet a join c but a join b is i and a join c is again i. Because the upward path from a and c join at i this is i. So, once again the left hand side and the right hand side are not the same so there this is non distributive again.

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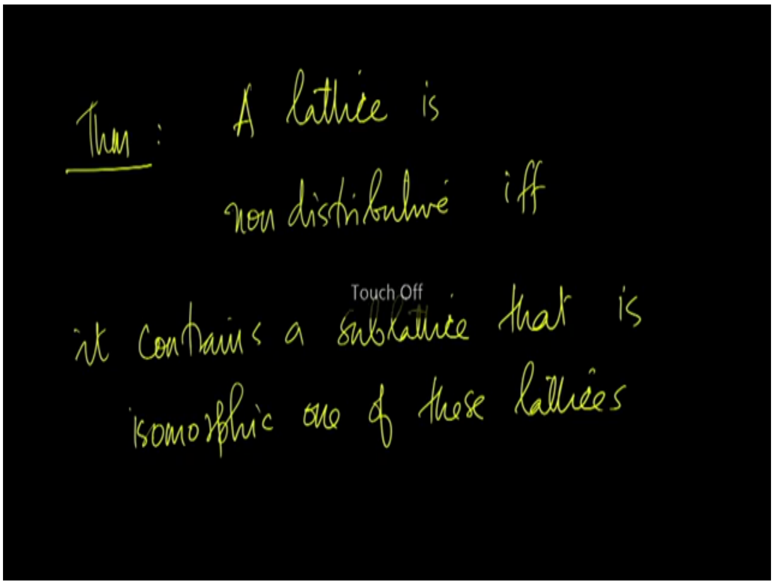

There is an interesting theorem which says that well lattice is not distributive if and only if it contains a sub lattice that is isomorphic to one of these lattices. That is one of these two lattices will have to be isomorphic to some sub lattice of a non-distributive lattice and there is a two way implication. The proof of this theorem is outside this scope of this discussion here. So, we come to the end of the discussion on set theory here. Hope to see you in the other module, thank you.