Discrete Mathematics
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Lecture 6
Set Theory

Welcome to the NPTEL MOOC on Discrete Mathematics, this is the sixth lecture on Set theory.

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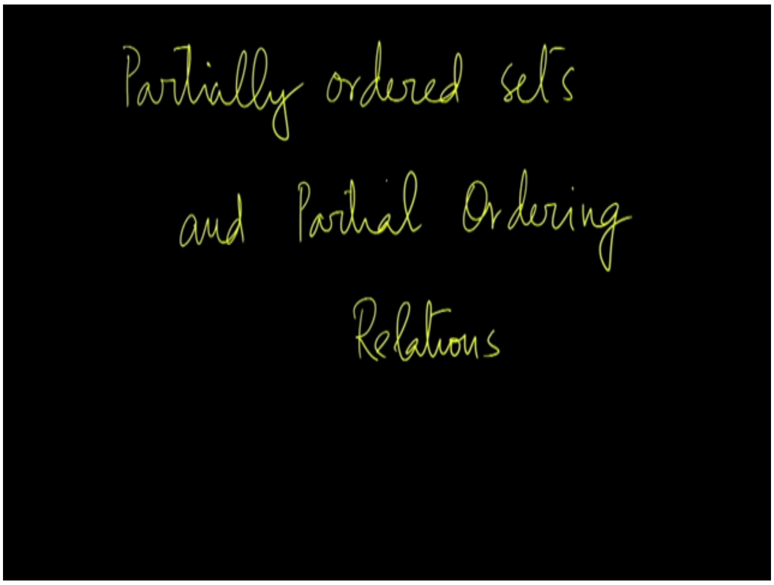

Today we shall study Partially ordered sets and Partial ordering relations.

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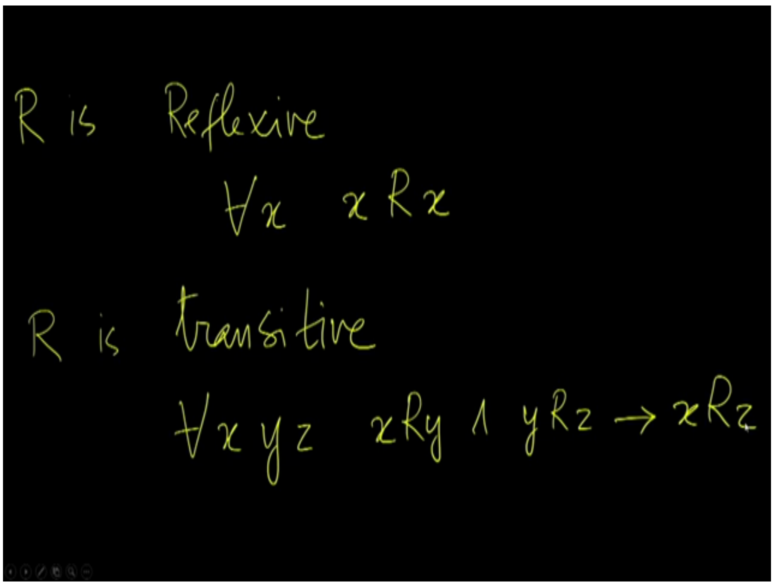

In a previous lecture we saw equivalence relations. An equivalence relation is one which is reflexive, symmetric and transitive. Here, we consider a relation which is reflexive which means for every x, if R is a relation that we are considering we say that R is reflexive if for every x in the domain it is the case that x R x and R is transitive, if for every x, y and z, x R y and y R z implies x R z. These are definitions that we have seen before.

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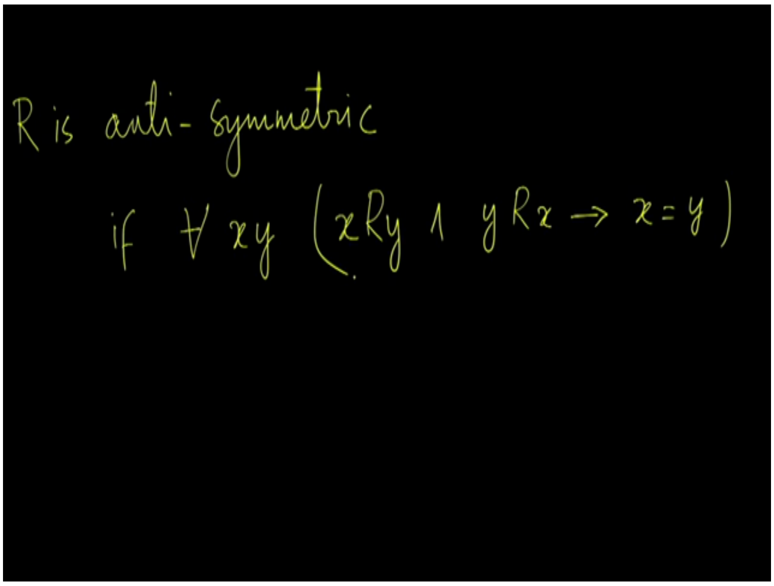

We say that a relation is anti-symmetric. We have seen symmetric relations before, they say that relation R is anti-symmetric, if for every x and y it is the case that x R y and y R x implies x equal to y, that is if the relation holds both ways between x and y then x must be equal to y. Which means for distinct x and y the relation can hold only in one direction either from x to y or from y to x, not both.

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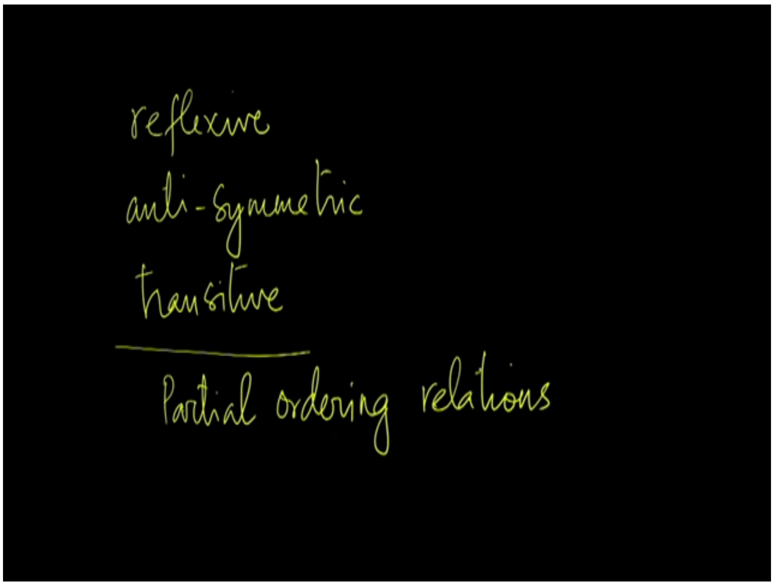

So, here we consider relations that are reflexive, anti-symmetric and transitive. These relations are called Partial Ordering Relations.

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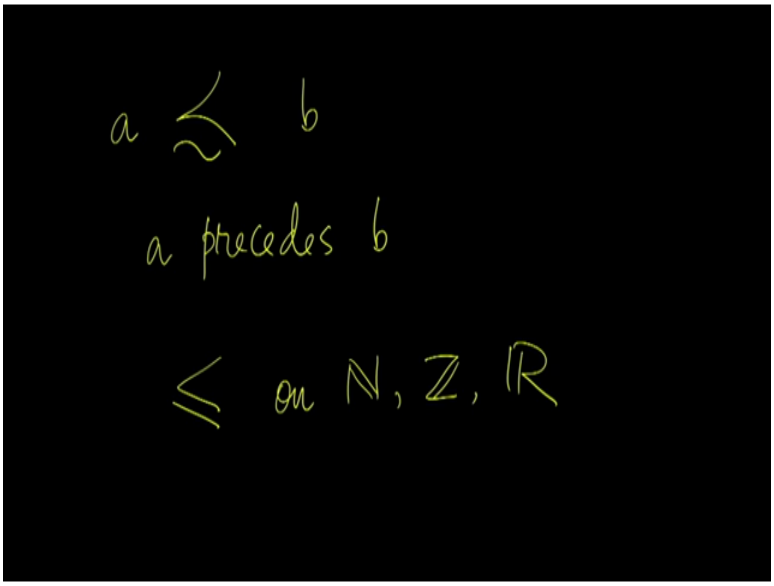

A generic symbol that we use for denoting partial ordering relations is this. We could write in this fashion and read this as A precedes B, of course this symbol is similar to the less than or equal to relation that we use on natural numbers or real numbers or integers which is not accidental because the less than or equal to relation on natural numbers integers, real's etcetera are also partial ordering relations.

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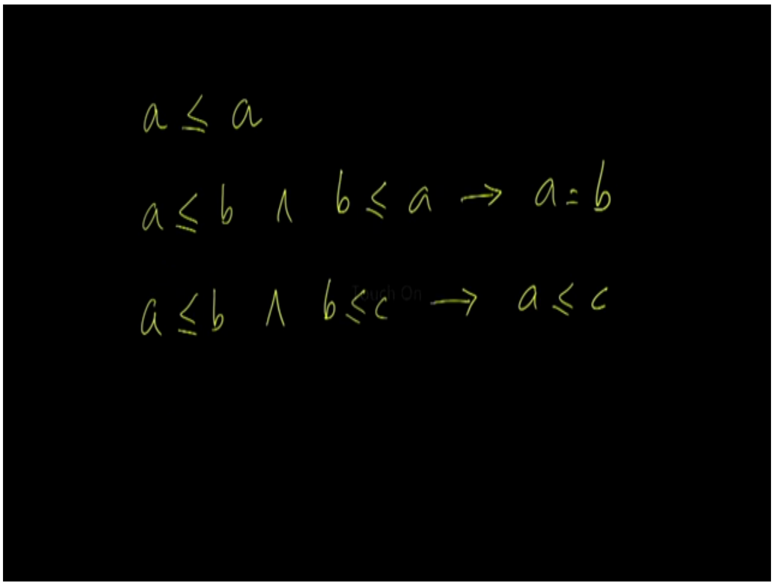

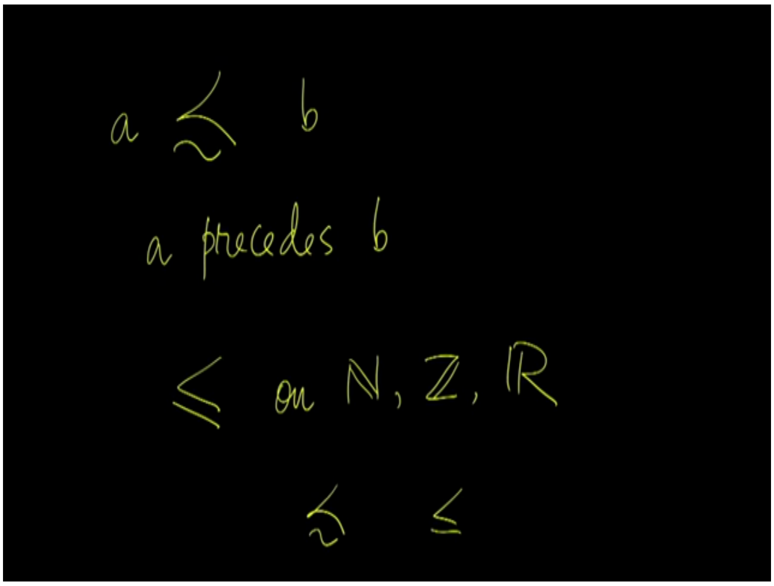

Because we know that for every number A less than or equal to A, if A less than or equal to B and B less than or equal to A then, A is equal to B which is the anti-symmetric relationship and if A less than or equal to B and B less than or equal to C then A less than or equal to C. Therefore, all three properties are satisfied by the less than or equal to relationship. Therefore, the less than or equal to relationship is a partial ordering relation, that is why the symbol that we use is similar to the less than or equal to relation.

Since this symbol is rather difficult to write, I will interchange it with the less than or equal to. So, depending on the context, you must realize that the less than or equal to relation might refer to another partial ordering relation.

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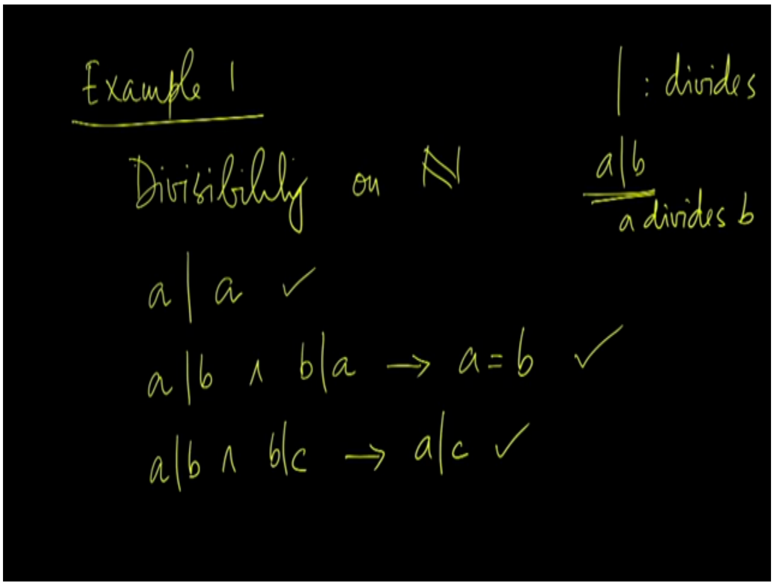

So, let us see some examples of partial ordering relations. Let us consider, the divisibility relation on natural numbers. For any natural number A, we know that A divides A, therefore the divisibility relation is reflexive. If A divides B and B divides A then A is equal to B, if A is a multiple of B and B is a multiple of A then A is equal to B. Therefore, the anti-symmetry relation also holds. By the way the vertical bar translates as divides.

So, when we write like this, what we mean is that A divides B. And thirdly if A divides B and B divides C then A divides C. The transitivity relation also holds. Therefore, the divisibility relation on natural numbers is a partial ordering relation.

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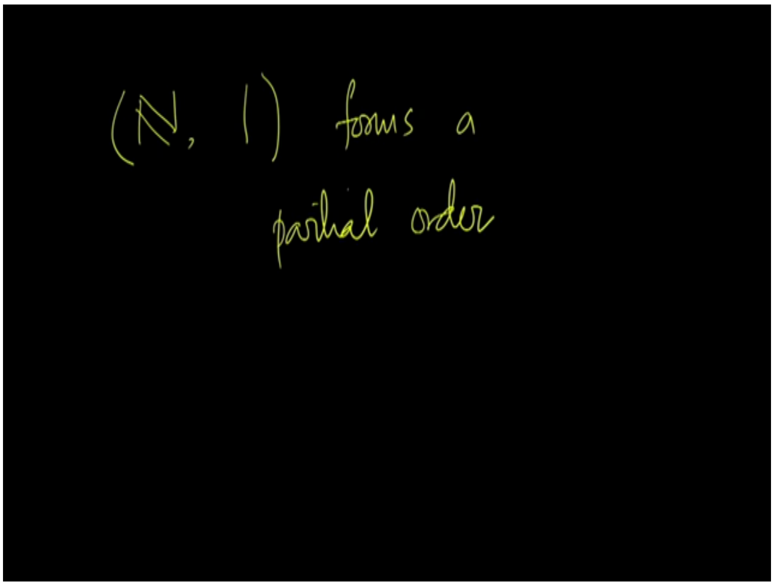

We say that, the set of natural numbers along with the divisibility relation forms a partial order, why is it called a partial order that will become clear soon.

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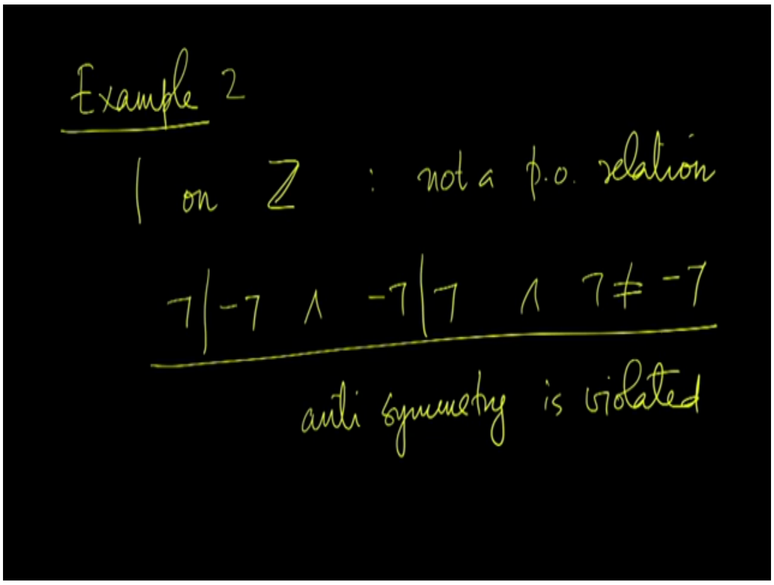

Our second example is, the divisibility relation on the set of integers. Here we find that this is not a partial order, not a partially ordering relation, why is this? This is because, we know that 7 divides minus 7 and minus 7 divides 7, yet 7 and minus 7 are not the same. Therefore, the anti-symmetry relation is violated, the anti-symmetry property is violated by this relation. Therefore, when we consider the same divisibility relation for integers instead of natural numbers we find that we do not get a partially ordered set or a PO-set.

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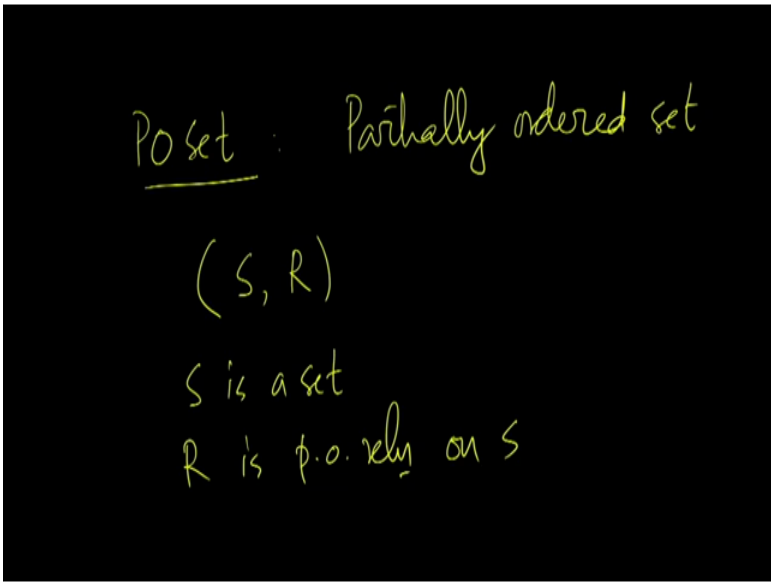

A PO-set for short stands for a Partially Ordered Set. A PO-set is an ordered pair S, R where S is a set and R is a (partially) partial ordering relation on S, this ordered pair is what is called a PO-set.

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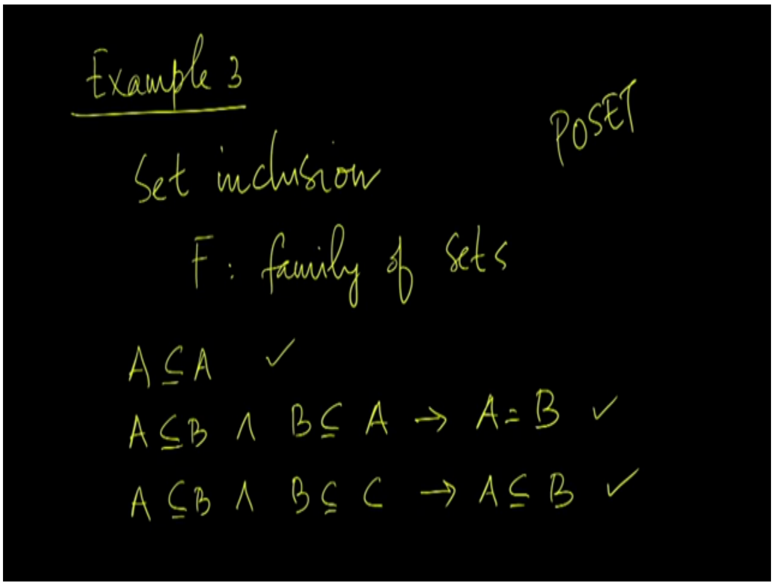

Another example, the example of set inclusion. Let us say, we have a family F of sets then for any A, we know that A is a subset of A therefore the reflexive property holds for the subset relation. If A is a subset of B and B is a subset of A then, A is equal to B, mind you, we do not use the proper subset relation here, we use the subset or equal relation and transitivity also holds, if A is a subset of B and B is a subset of C then A is a subset of C. Therefore, all three properties hold here, therefore this is a PO-set. The family of sets F along with the subset or equal relation forms a PO-set.

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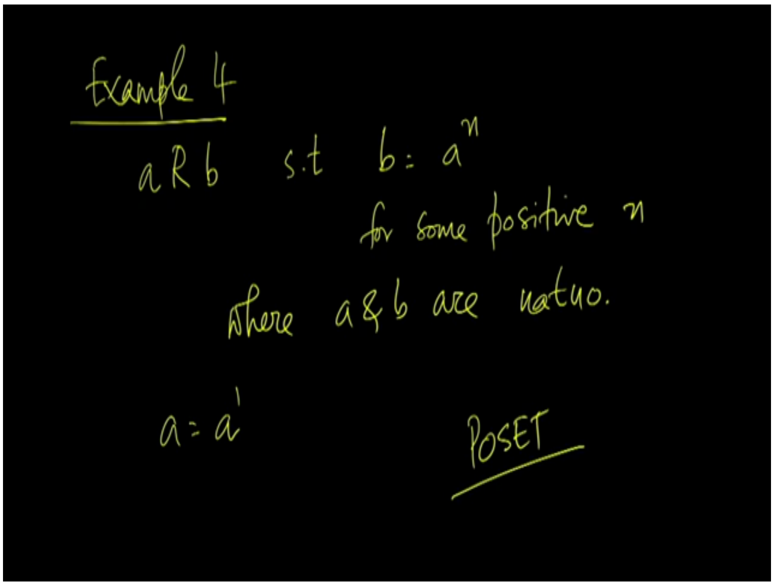

Another example is a relation a R b such that b equal to a power n for some positive integer n where, a and b are natural numbers. So, we are considering a relation on natural numbers, we say that a R b if b equal to a power n, this is also a PO-set because a is a power 1, if b is a power n and a is b power m then a is equal to b, where n and m are positive numbers. If b equal to a power n and c equal to b power m, then c can be expressed as an integer power of n for a positive integer therefore this is also a PO-set.

So, those are some examples of partial ordering relations and the corresponding PO-sets.

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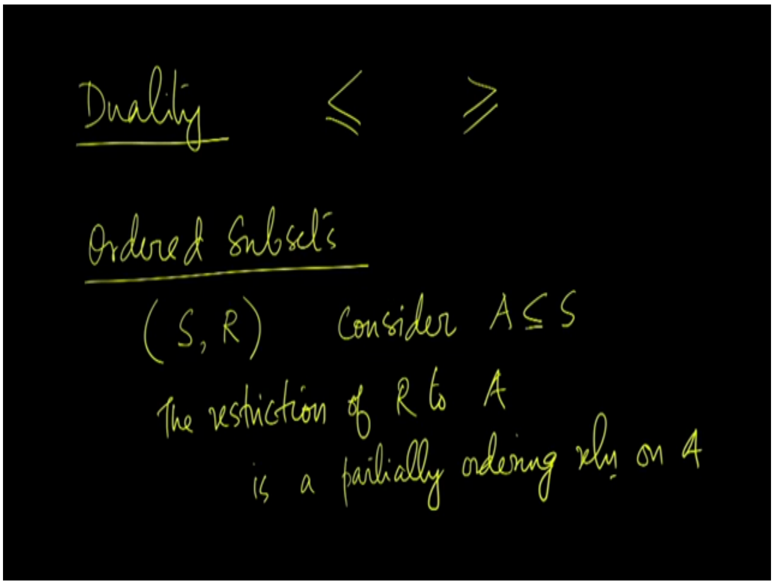

So, you would observe the duality here. The less than or equal to relation and the greater than or equal to relation not duals of each other, they are inverses of each other. We can consider subsets of PO-set. Let us say we have a relation R on a set S. Suppose, this is a PO-set, then consider a (set) subset A of S, then the restriction of R to A as you can verify is a partial ordering relation, so A is an ordered subset of S.

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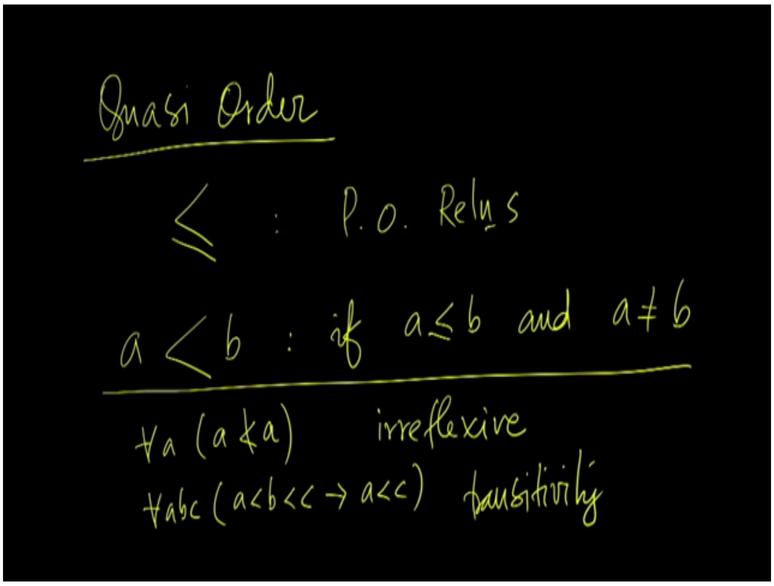

So, we have so far been talking about the less than or equal to relation or the greater than or equal to relation which is a dual of it. These we know are partially ordering relations but what about the less than relation? We say that a less than b or a strictly precedes b, if a precedes b and a not equal to b, we find that this is not a partially ordering relation because the anti-symmetry property does not hold and moreover the reflexive property also does not hold because it is not the case that a less than a.

In fact, for every a we can say that is not less than a. Therefore, this is irreflexive and for every abc, we know that a less than b less than c implies that a less than c. So, transitivity holds here. So, the less than relation has these two properties reflexivity and transitivity, when these two properties hold, then we have what is called a ‘Quasi-Order’.

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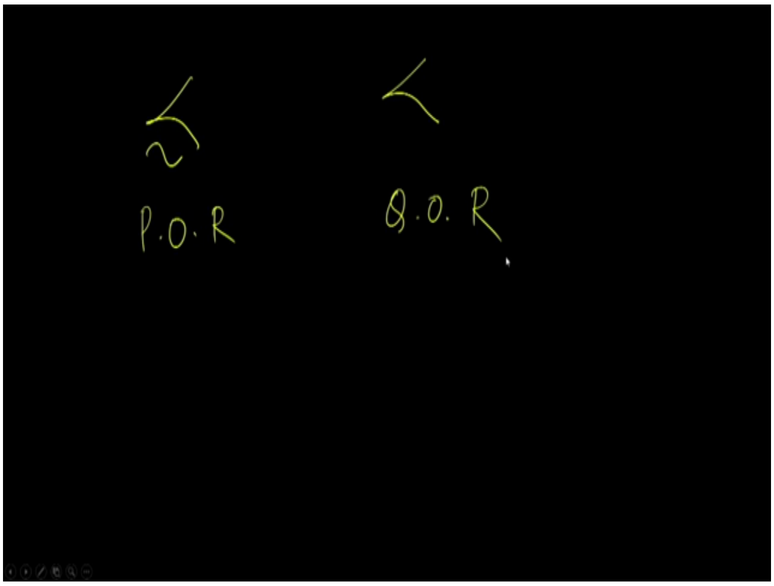

So, there is always a quasi-order which is associated with a partially ordering relation. When the precedes relation is a partially ordering relation, the strictly precedes relation is a quasi-ordering relation.

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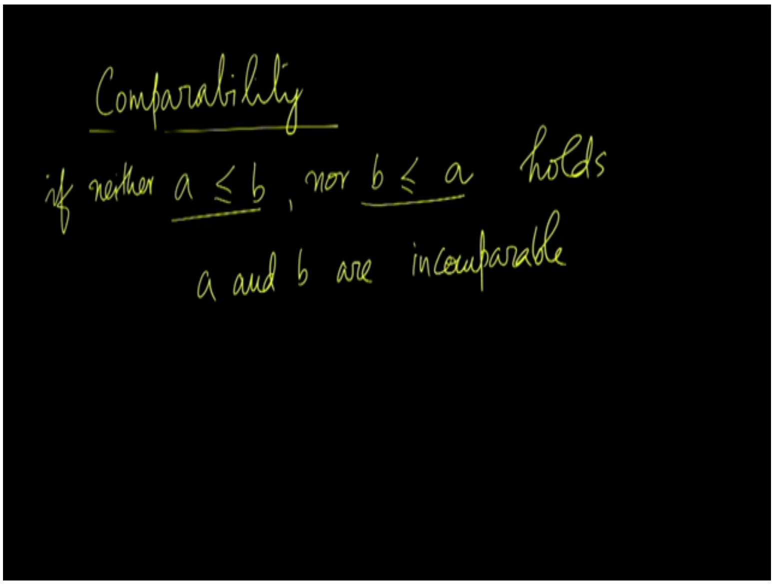

Now, let us consider the issue of comparability. It could be that a less than or equal to b for a pair of elements a and b or it could be that b less than or equal to a for the same pair or it is possible that neither may hold, if neither this nor this holds we said that a and b are incomparable. Remember the less than or equal to symbol here in fact stands for the precedence relation, if neither a precedes b nor b precedes a then we say that a and b are incomparable.

Of course you will not find such a pair when you consider the less than or equal to relation on natural numbers.

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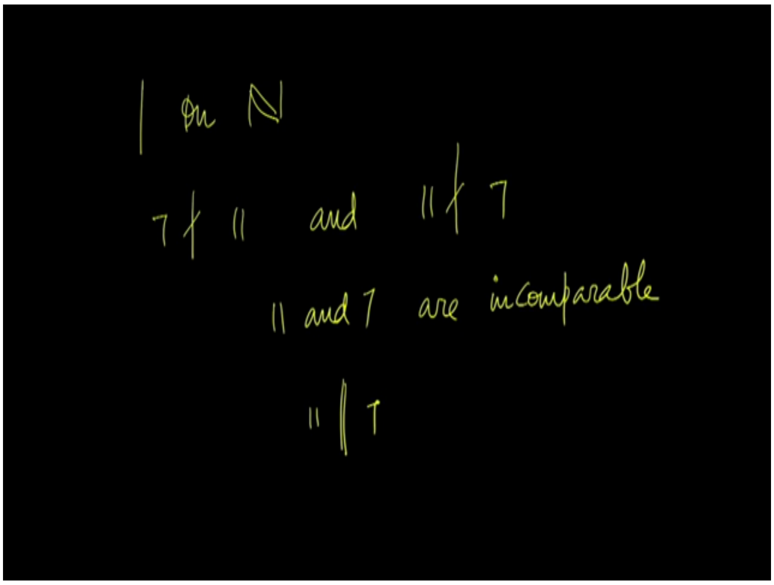

But then in some other cases you would be able to find incomparable pairs. For example, consider the divisibility relation on natural numbers. We find that 7 does not divide 11 and 11 does not divide 7, which means 11 and 7 are incomparable. Symbolically, we write using two bars 11 is incomparable to 7. So, it is possible for incomparable pairs to be there in some partially ordering relations, that is precisely why it is called a partially ordering relation.

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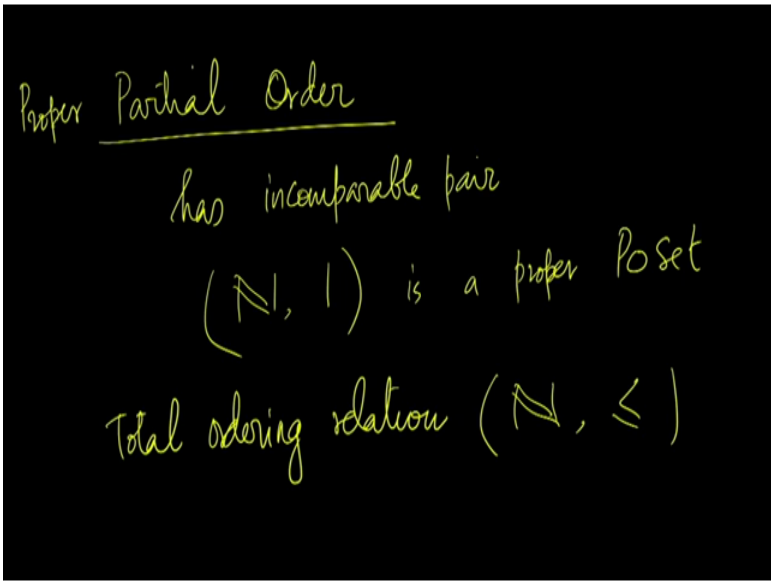

A partial order, a proper partial order has incomparable pairs. Therefore, the set of natural numbers with the (defective) divisibility relation is a proper PO-set. As opposed to a proper PO-set is a total ordering relation. In a total ordering relation, you will not be able to find a pair of elements that are incomparable to each other. For example, if you consider the set of natural numbers along with the less than or equal to relation.

You find that for every pair of natural numbers the less than or equal to relation holds, you take any pair of natural numbers a and b either a less than or equal to b or b less than or equal to a. It is not possible that these two neither relation holds between the pair. Therefore, the less than or equal to relation is a total ordering relation.

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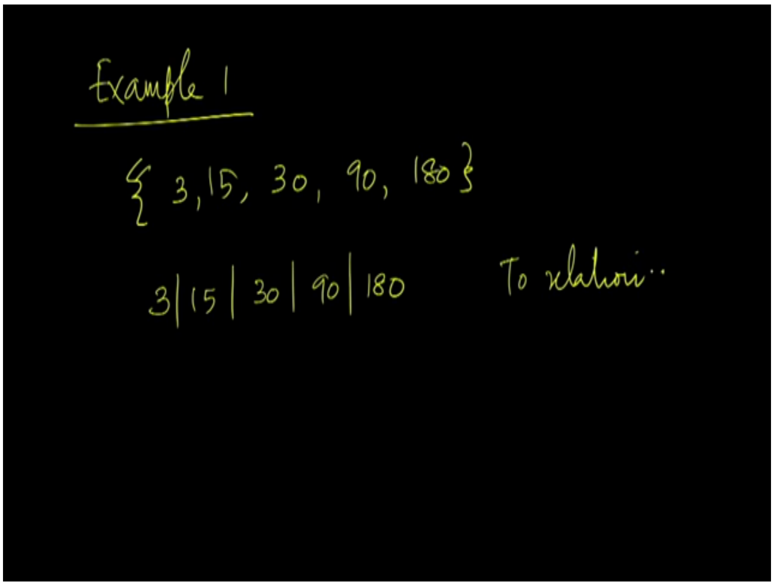

So, let us consider some examples. Consider this set that consists of 3, 5, 30, 90 and 180. We find that 3 divides let us make this 15, we find that 3 divides 15, which divides 30, which divides 90, which divides 180. Therefore, you take any two members in this, you find that there is a relation, the divisibility relation between them.

For example, you take 30 and 180 there is the divisibility relation between them, 30 divides 180, so the divisibility relation holds one way. Therefore, this is a totally ordering relation.

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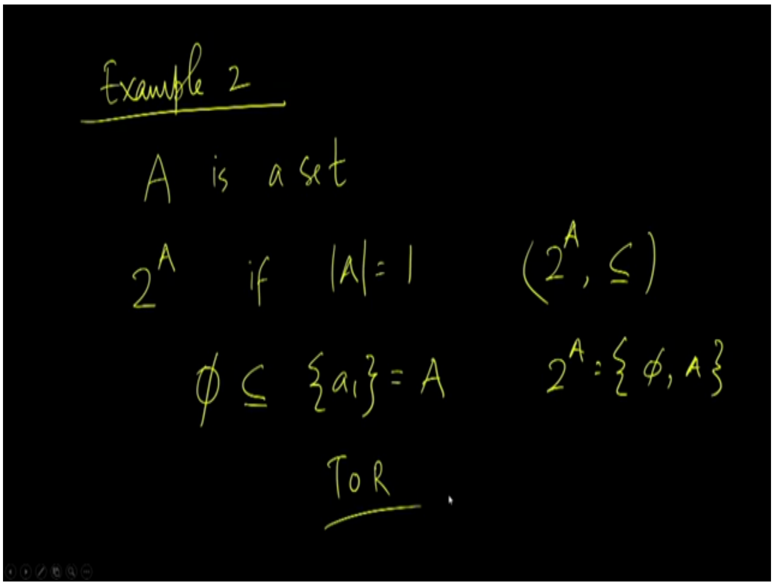

Another example, suppose A is a set consider the power set of A, if mod A equal to 1, then the power set of A with the subset or equal to relation is a totally ordering relation because there are only two subsets here, if A happens to be the singleton containing just A1 then there are only two subsets in 2 power A, 2 power A consists of just the empty set and A itself and the empty set is a subset of A, therefore, we have a total ordering relation.

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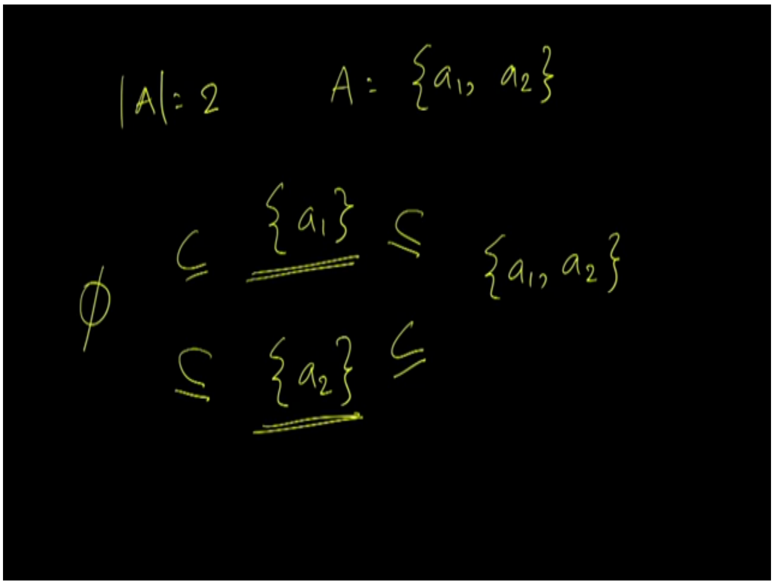

On the other hand, if the size of A is 2. Let us say A is made up of two elements a 1 and a 2, then we find that the empty set is a subset of the singleton containing a 1. It is also a subset of the singleton containing a 2 and these two are subsets of a 1 and a 2. But we find that these two singletons are not comparable to each other. The singleton a 1 is not a subset of the singleton a 2 and the singleton a 2 is not a subset of the singleton a 1. So, these two are incomparable.

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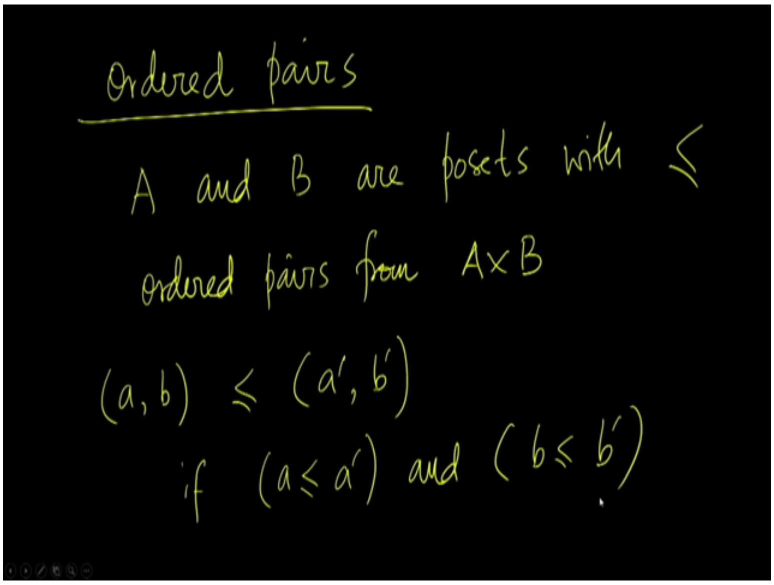

Considering various orderings, let us (considered) consider ordered tuples. First, let us consider ordered pairs. Let us say A and B are PO-sets with the less than or equal to relation or any generic precedence relation, so we have two PO-sets A and B. Let us consider ordered pairs from A cross B. So, we consider any relation between A and B. We say that ordered pair a, b is less than or equal to or precedes ordered pair a prime b prime, if a is less than or equal to a prime and b is less than or equal to b prime. So, this is one possible ordering.

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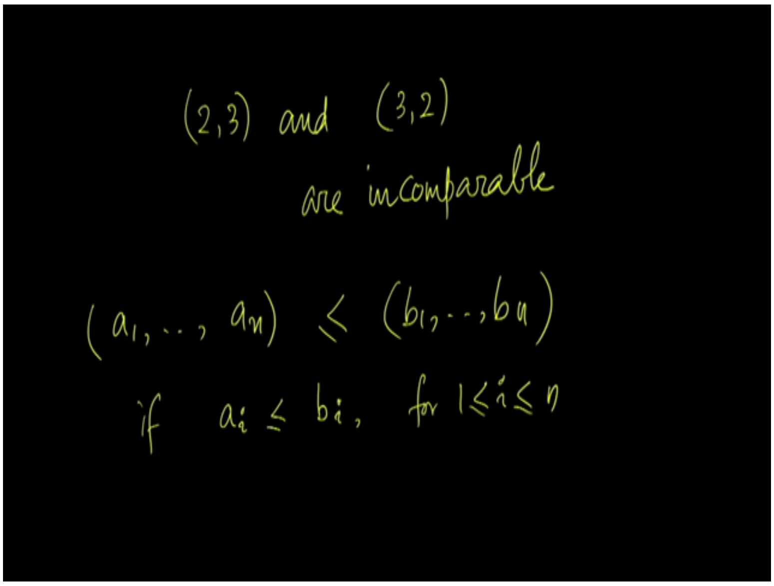

So, in this case we can say that 2, 3 and 3, 2 are incomparable according to this ordering. Extending this, we can consider ordered n tuples. We can say that ordered n tuple a 1 through a n is less than or equal to or precedes the ordered n tuple b 1 through b n, if a i is less than or equal to b i so, 1 less than or equal to i less than or equal to n. For every i it is the case that a i precedes b i that is when we say that the ordered n tuple a 1 through a n precedes the ordered n tuple b 1 to b n.

So, this is a straightforward generalization of the earlier ordering that we saw for ordered tuple, ordered pairs.

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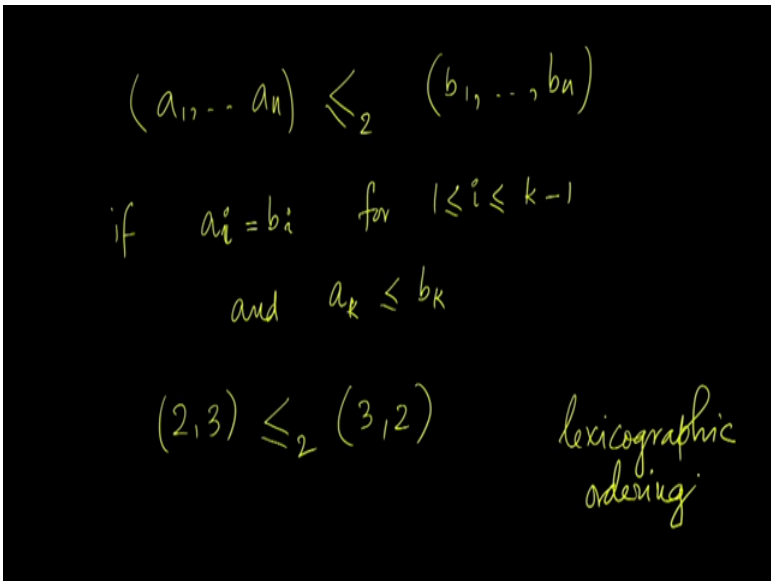

Now, the order tuple a 1 through a n can be thought to be less than or equal to b 1 through b n. Here, we are considering another order in relation, let me denote as less than or equal to 2. So, according to this ordering relation, we said that a 1 through a n is less than or equal to b 1 through b n, if a i equal to b i for 1 less than or equal to i less than or equal to k minus 1 and a k is less than or equal to b k.

So, according to this if you consider, ordered pair 2, 3 and ordered pair 3, 2 since 2 is less than or equal to 3 (in on) in the first component we can say that, this relation holds between them. 2, 3 comes before 3, 2 in this ordering. So, this is Lexicographic ordering.

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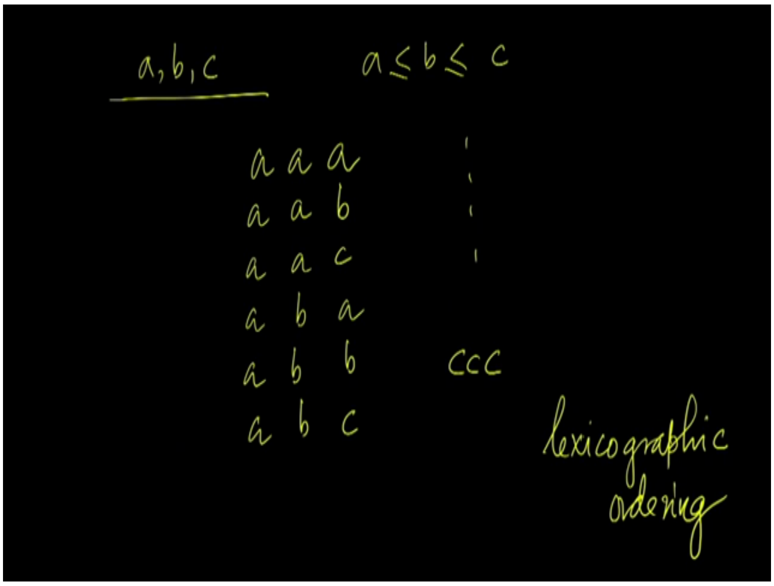

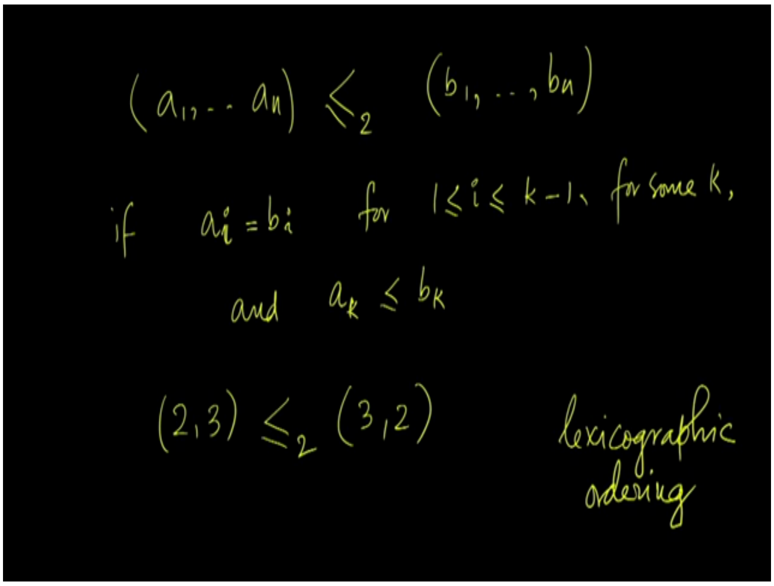

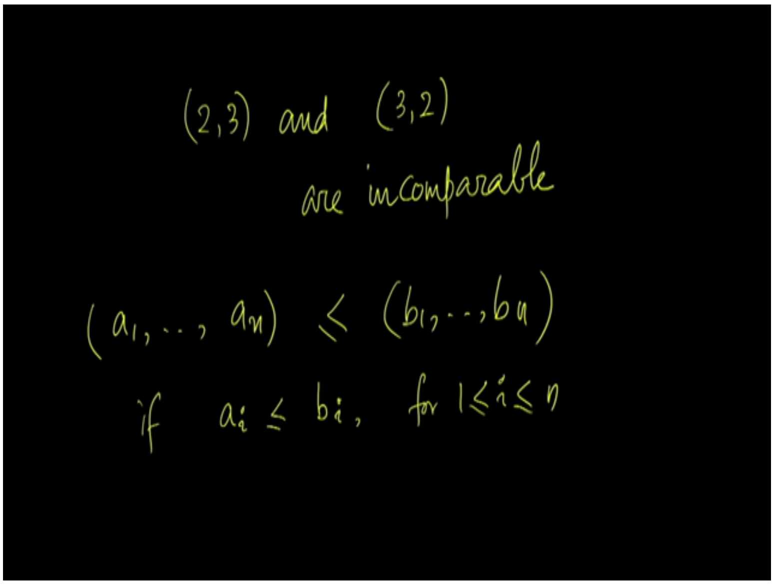

For example, if you were to consider all strings of length three, made up of three symbols a, b, c. If a less than or equal to b less than or equal to c. Then in dictionary order you would enumerate them like this. This is the first string of length three, the smallest string of length three. Then this would be the next string of length three, this would be the next. Now, you have a change in the second position and so on, ending with c, c, c. So, this is a lexicographic ordering of all strings of length 3.

So, that is precisely what we do here. We say that an ordered n tuple a1 through a n is less than or equal to an ordered n tuple b 1 through b n. According to this ordering, if a i equal to b i for 1 less than or equal to i less than or equal to k minus 1, for some k and it is the case that for that particular k, a k less than or equal to b k. So, we do not look at the positions which are further to the right of k that is k plus 1 through n could be anything.

So, these two ordered pairs order tuples match in the first k minus 1 positions and when you look at the kth position, a1 through a n has a smaller value. Therefore, we said that a 1 through a n precedes b 1 through b n. This is opposed to the previous ordering that we saw, so these two are different orderings of order tuples.

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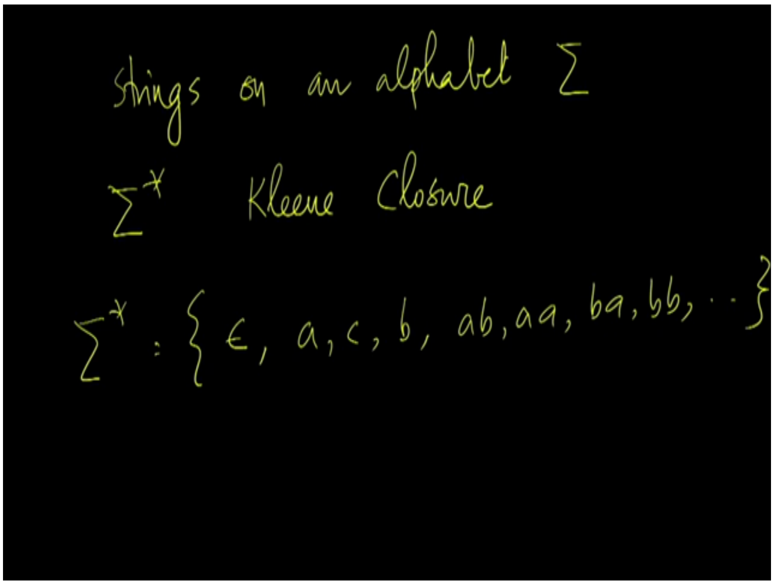

And then we can consider strings in general. We can consider an alphabet sigma and we can talk about all strings from alphabet sigma, which is denoted by sigma star. This is the Kleene Closure of sigma. This would of course consider the null string, the string of no length, which is made up of no character. Then it will have all strings of length one, all strings of length two and so on.

You can form an infinite set of strings from sigma even if sigma is finite. So, this set is sigma star.

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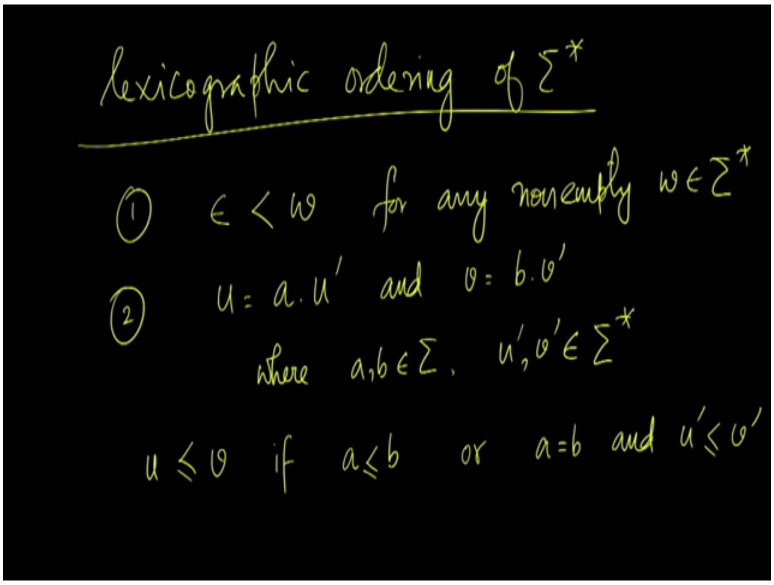

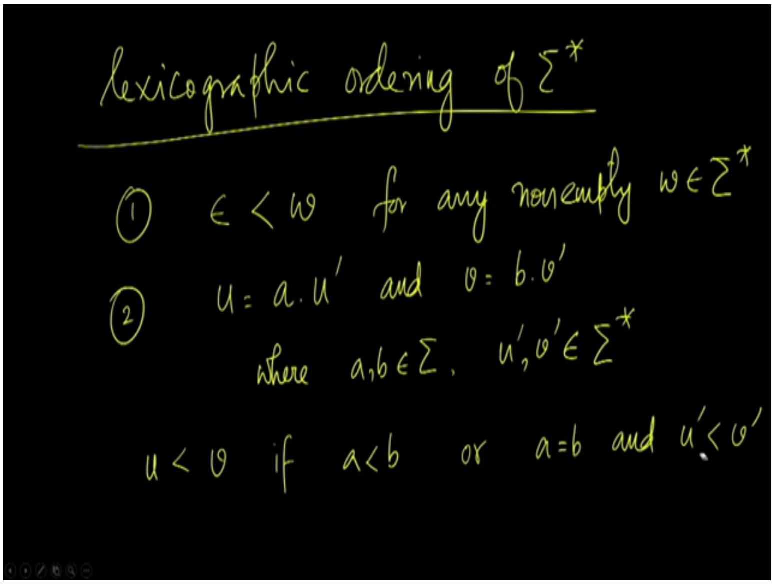

Sigma star could be lexicographically ordered, which means in dictionary order, you would prefer to order them in this way. We will say that the null string epsilon is less than w for any non-empty w, any non-empty string w from sigma star will have this property, it will come after the null string. Secondly, if u is character symbol a followed by u prime and v is symbol b followed by v prime where, a and b are symbols, u prime and v prime are strings from sigma star, then we would say that u is less than or equal to v or u precedes b in this ordering, if either a less than or equal to b or a equal to b and u prime precedes v prime but these are strict precedence.

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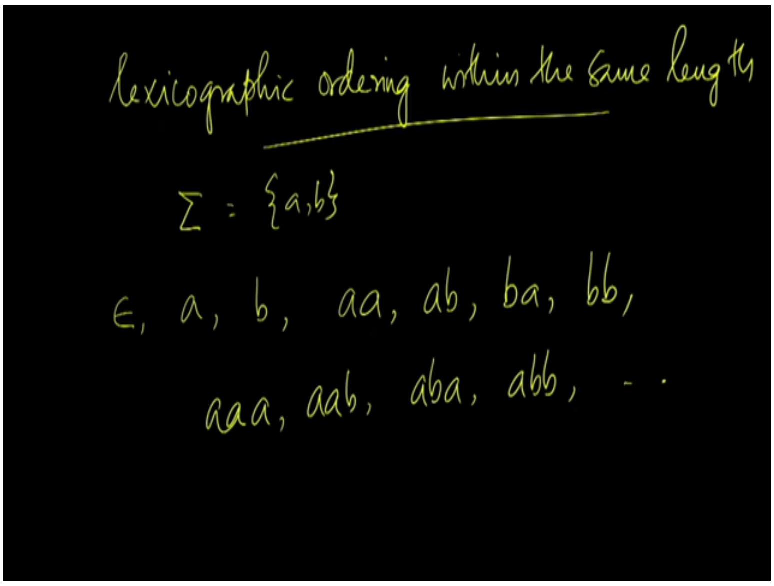

Of course, often we find a variant of lexicographic ordering, which would be lexicographic ordering within the same length. So, this ordering would be like this if sigma happens to be a and b just two symbols, then sigma star would be ordered thus first you enumerate the null string, then we have strings of length one a and b.

Then we have strings of length two aa, ab, ba and bb. Then we enumerate all strings of length three and so on. That is within the same length, we enumerate the strings in lexicographic ordering, but strings will be enumerated in the monotonic order of increasing length.

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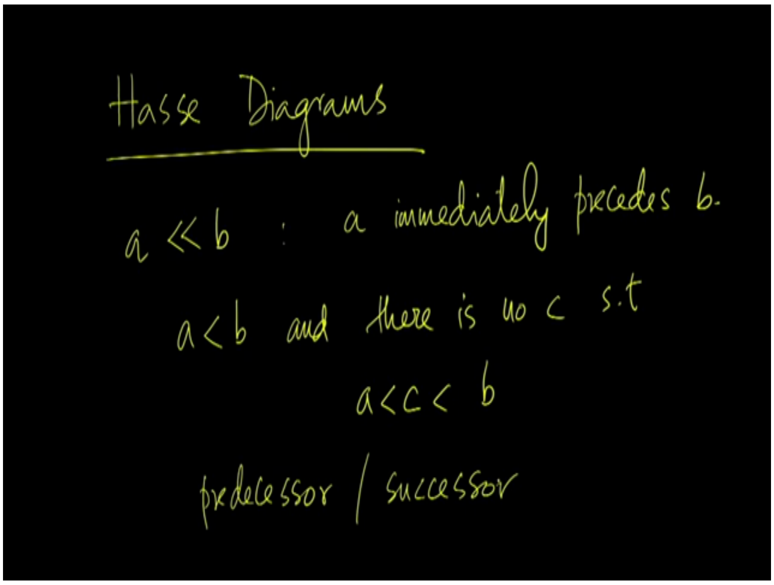

Now, we study Hasse diagrams. We say that a immediately precedes b, this notation says a immediately precedes c or precedes b. What it means is that, a precedes b and there is no c such that a precedes c, which precedes b. So, there is no intermediate element between a and b. So, in this case we say that a is an immediate predecessor of b or conversely b is an immediate successor of a.

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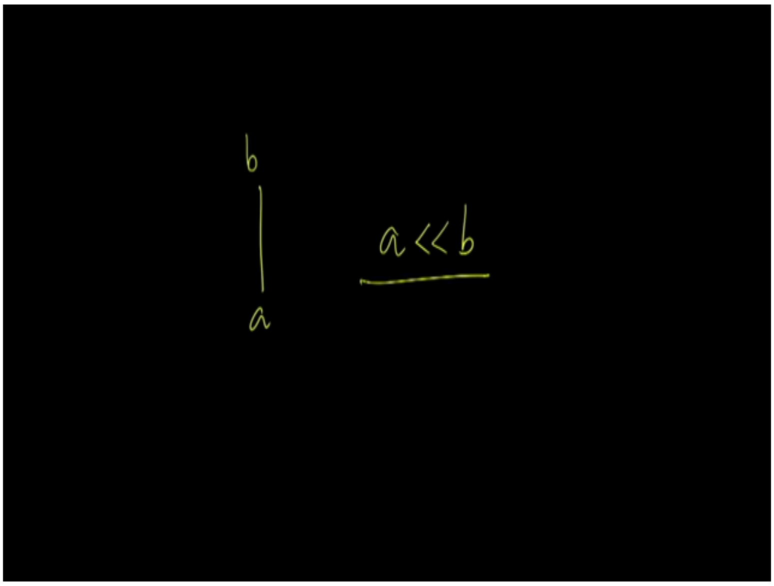

In a Hasse diagram, we draw the partial order using lines going from below to above. We do not usually place an arrow, when we place a and b in this manner, what we mean is that a is an immediate predecessor of b. So, the less than or equal to relation now flows from below to above.

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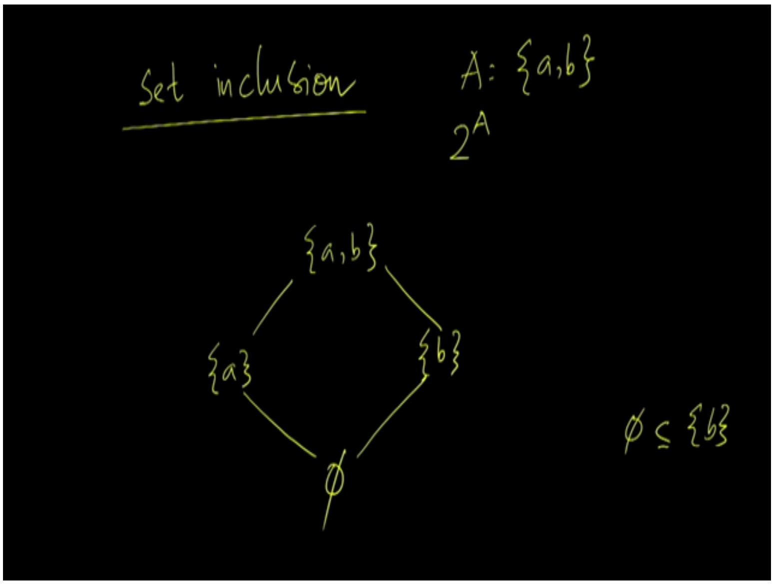

So, let us consider an example. Let us consider the example of Set Inclusion. We consider set A that consists of two elements a and b and let us look at the power set of A which is 2 power A. The members of 2 power A are a itself, singletons a and b and the empty set as we saw earlier. So, in the Hasse diagram we find that, the empty set is included in the singleton b, the empty set is included in the singleton a as well.

So, we draw lines in this manner to show that phi is less than or equal to b. Here, less than or equal to stands for the subset relation and similarly, we have lines of the sort too. So, this is the set inclusion partial order for two member elements.

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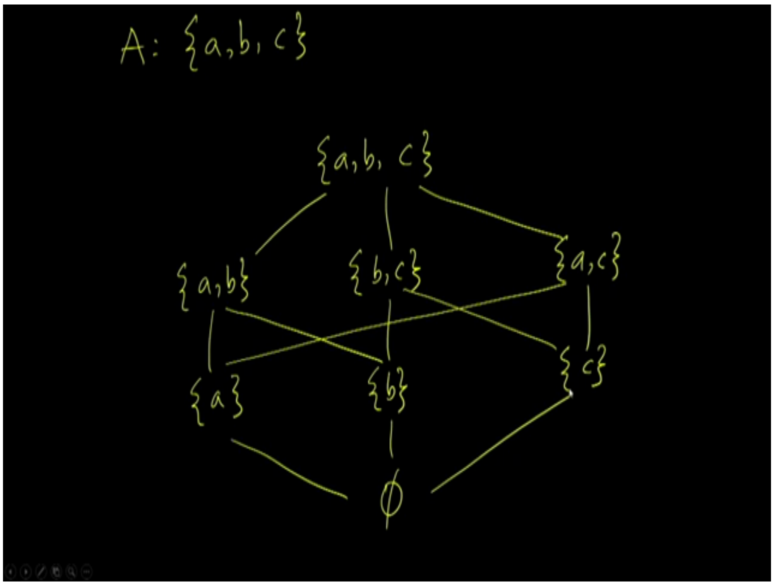

If A had three members, we would have elements arranged in this manner. There is only one 3 member subset. There are three 2 member subset and we have immediate predecessor relation between a, b and a, b, c. For example, a, b is an immediate predecessor of a, b, c. Similarly, b, c is also an immediate predecessor of a, b, c then we have singletons a, b and c. We have, edges of the sort, yes and then we have the empty set.

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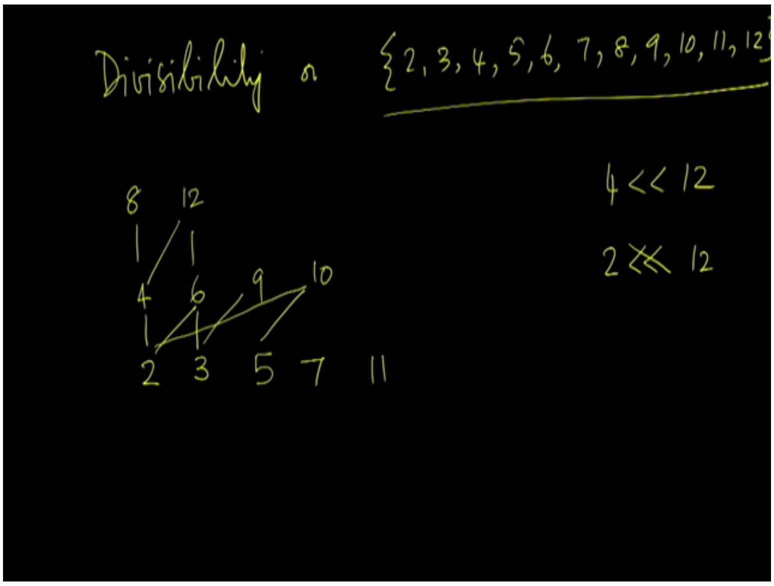

Let us consider the divisibility relation on the set 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12. The diagram will be drawn like this. 2 and 3 are incomparable, so you cannot have an edge between them, so they are placed at the same level. 2 does not divide 3 and 3 does not divide 2. Now, when we come to 4, we know the 2 divides 4. So, we have an edge from 2 to 4 in the Hasse diagram.

Now, 5 does not divide any of the previous numbers, it is a prime and the previous numbers do not divide 5. Then we have 6, 6 is divided by both 2 and 3, so we have edges from 2 and 3 to 6. 7 is a prime. 8 is a multiple of 4. 9 is a multiple of 3. 10 is a multiple of 5 and 2. 11 is a prime. 12 is a multiple of 6 as well as 4 that is there is an immediate predecessor relation between 4 and 12. We do not draw an edge from 2 to 12 even though 2 is a divisor of 12 that is because 2 is not an immediate predecessor of 12 because 4 is in between 2 divides 4 and 4 divides 12, so there is a chain of divisibility is from 2 to 12, a chain of length more than 1.

Therefore, there is an edge from 4 to 12 but there is no edge from 2 to 12. So, this would be the Hasse diagram for the divisibility on this set. The resultant partial order will be drawn like this.

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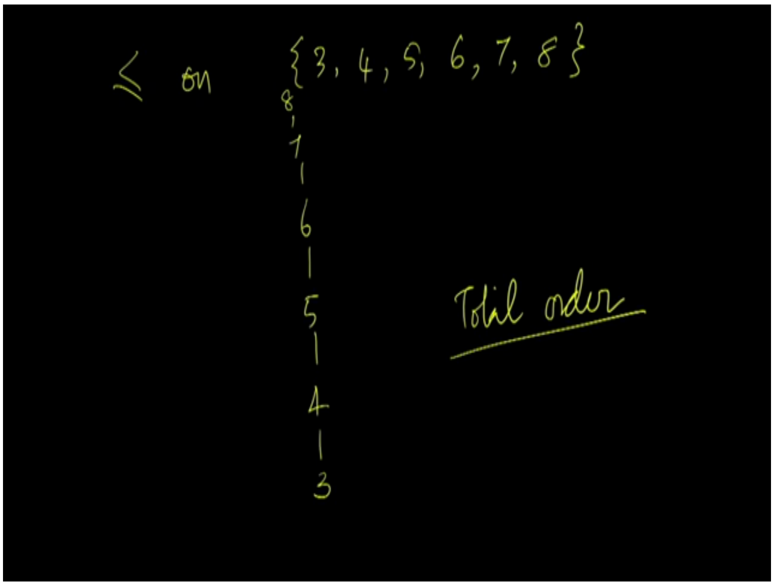

Then, let us consider the less than or equal to relation on the set 3, 4, 5, 6, 7, 8. Here, we find that the Hasse diagram has a simple form, that is because this is a total order, which means between any pair of elements the less than or equal to relation holds. But in the Hasse diagram, we do not draw every possible line, we show only the immediate predecessor. So, the immediate predecessor of 4 is 3, the immediate predecessor of 5 is 4 the preceding number.

In fact, to get the partial order you should take the, you should apply transitivity on this, on this predecessor relation that is shown here. For example, there is a change from, there is a path from 4 to 6 here in this diagram, therefore we know that 4 is actually a predecessor of 6, 4 is less than or equal to 6.

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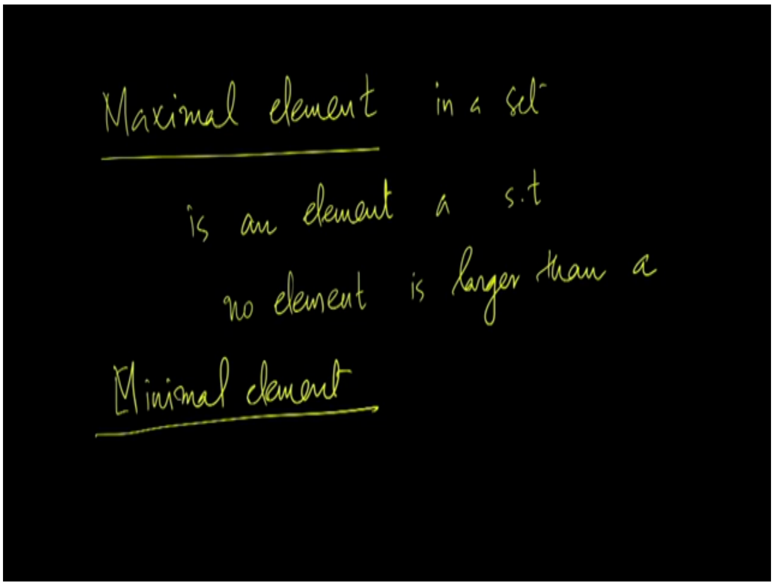

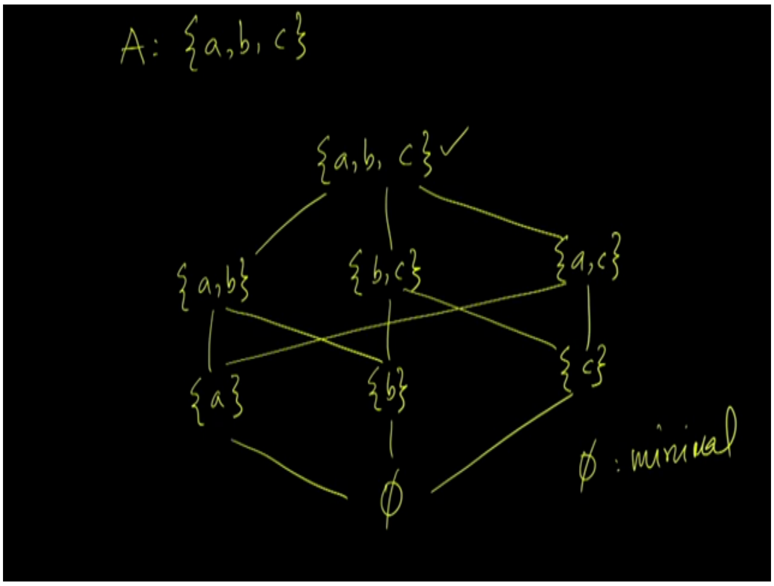

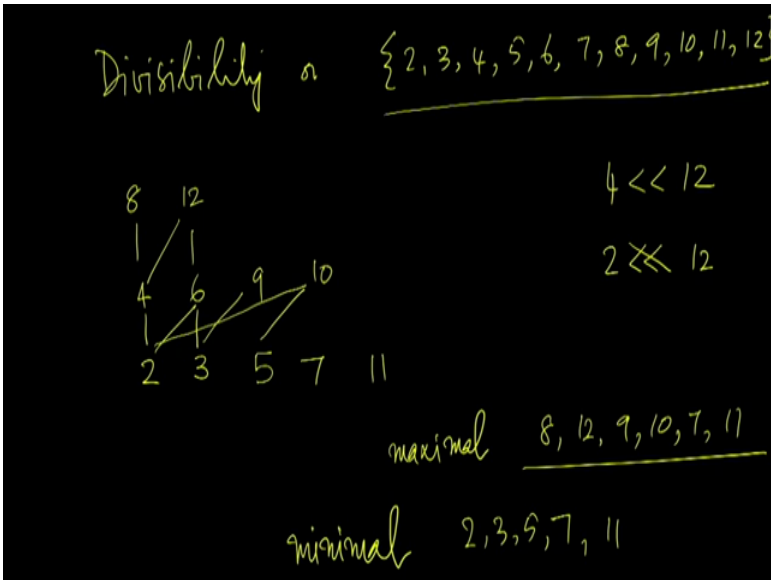

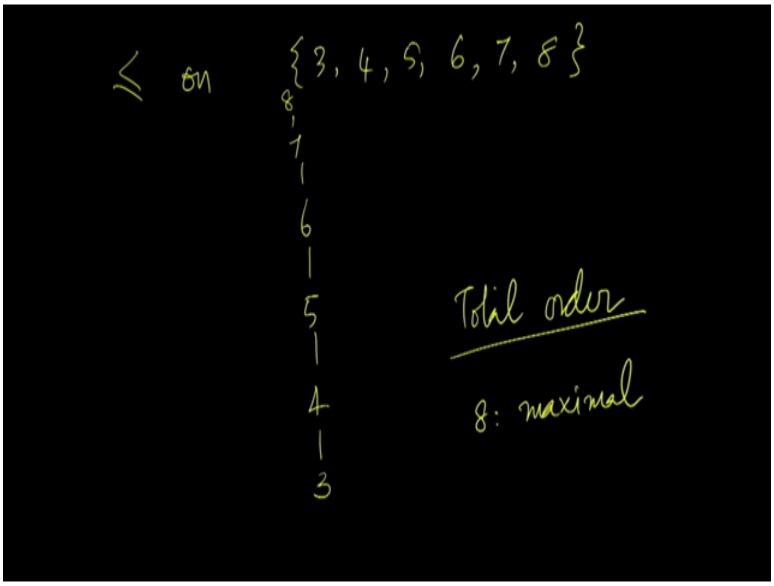

A maximal element in a PO-set, it is an element a such that no element is greater than or equal to it or no element is larger than it, strictly larger than it. So, if you look at these diagrams, here the set a, b, c is a maximal element, because no member in the diagram is above a, b, c. Here, we find that 8, 12, 9, 10, 7, 11 these are all maximal elements and here we find that 8 is a maximal element.

Analogously, a minimal element is an element such that no element is less than that. So, if you look at these diagrams, you find that phi is a minimal element, here 2, 3, 5, 7, 11 are all minimal elements. In this diagram, 3 is a minimal element nothing has below 3. So, those are the maximal and minimal elements.

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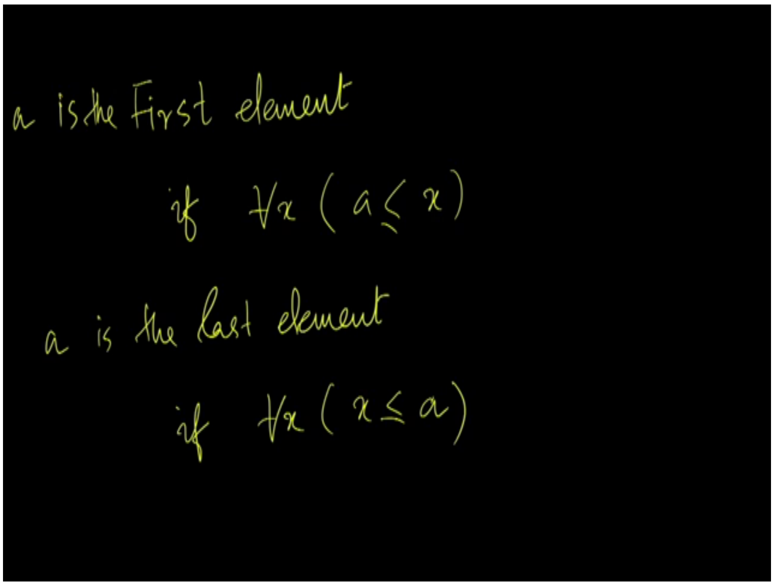
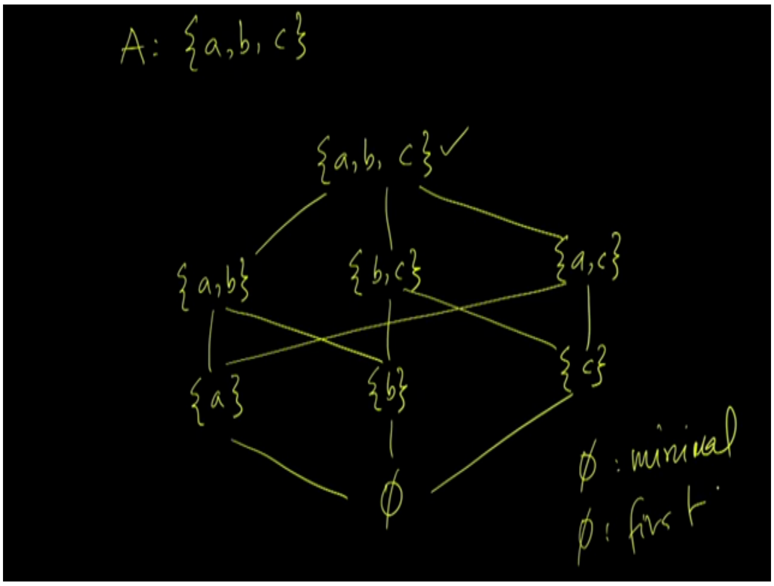
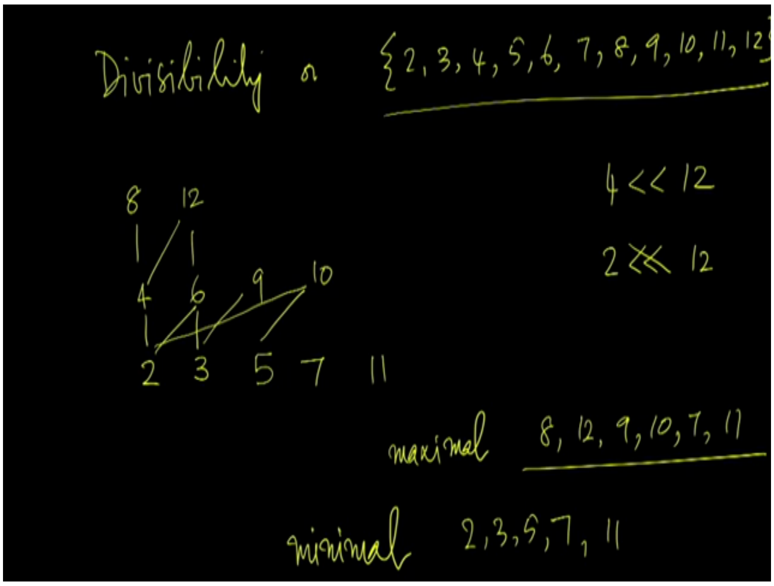
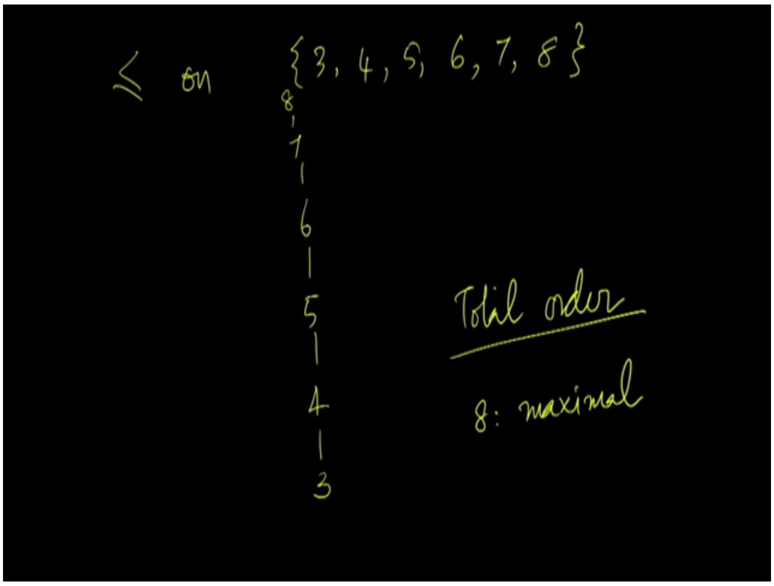
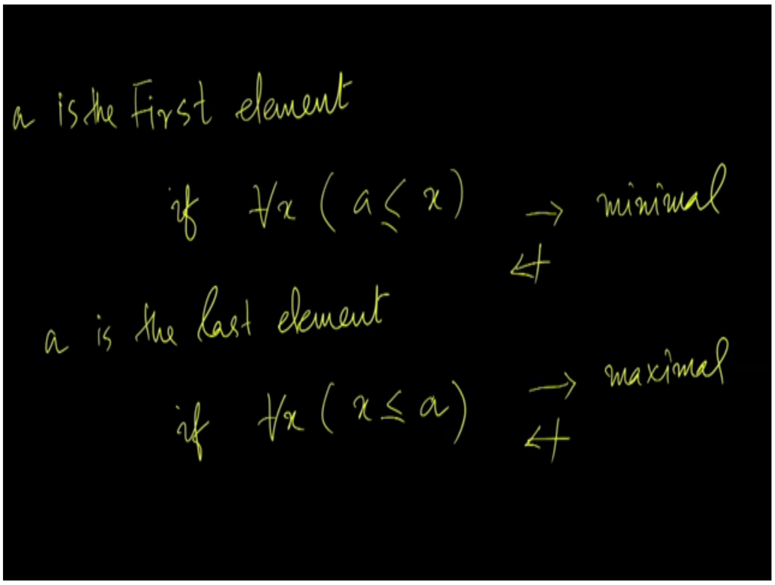



A: {a, b, c}
{a, b, c}
{a, b}
{b, c}
{a, c}
{a}
{b}
{c}
∅
∅ : minimal
∅ : first


Divisibility on {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}
8
12
4
6
9
10
2
3
5
7
11
4 << 12
2 ≪̸ 12
maximal 8, 12, 9, 10, 7, 11
minimal 2, 3, 5, 7, 11


≤ on {3, 4, 5, 6, 7, 8}
8
7
6
5
4
3
Total order
8: maximal



We say that, an element is the first element, a is the first element, if for every x it is the case that a is less than x or a is less than or equal to x. We say that, a is the last element of the PO-set, if it is the dual of this, which means for every x, x is less than or equal to a, x precedes a. So, if you look at the diagrams here, here we find that phi is a minimal element and phi is also the first element. Set a, b, c is a maximal element and also the last element.

But when we come to this, we find that it does not have a first element or a last element. 8, 12, 9, 10, 11 are all maximal elements, but they are incomparable to each other. So, there is no element which is a successor of everybody. Similarly, there is no element which is a predecessor of everybody. 2, 3, 5, 7, 11 are all minimal elements but there is no first element here.

Here 8 is a maximal element as well as a last element, 3 is a minimal element as well as the first element. So, from this we know that a first element is always a minimal element not necessarily vice versa. A last element is always a maximal element and not necessarily vice versa.

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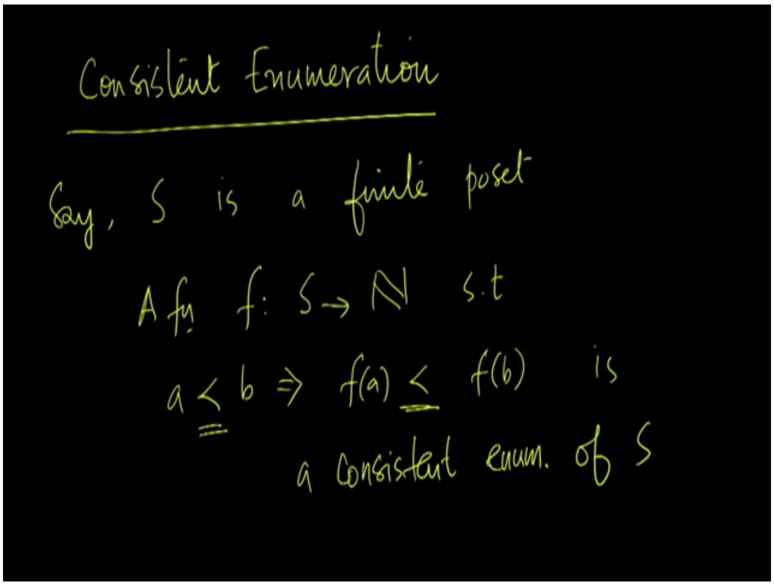

So, let us considered consistent enumerations. Say S is a finite PO-set, a function f from S to the set of natural numbers, such that a precedes b implies that f of a is less than f of b is a consistent enumeration of S. So, this is the precedence relation, whereas this is the less than relation over natural numbers. So, for a function to be a consistent enumeration of S, it should be a mapping from S to the set of natural numbers and it should be show that a precedes b implies f of a is less than f of b. So, we have effectively numbering the members of the PO-set.

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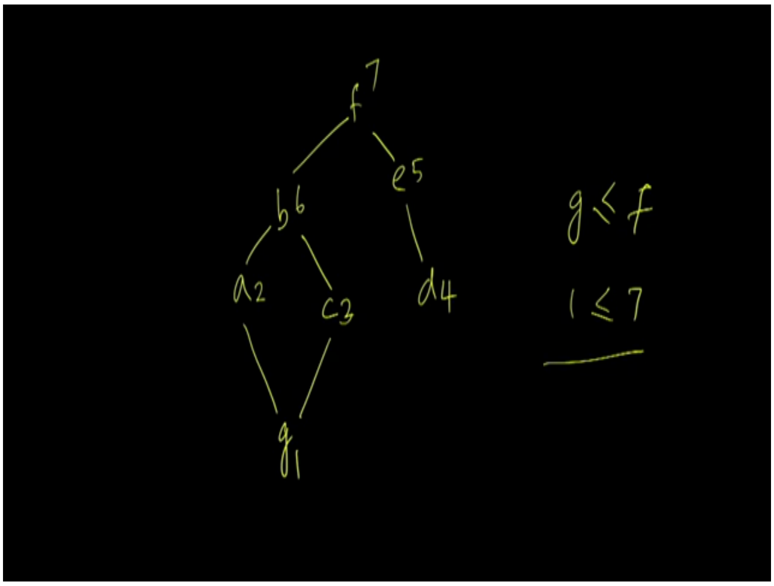

So, let us consider the Hasse diagram of a PO-set. Let us say, we have set of elements like this. So, a numbering show that, g gets number 1, a, c, d gets numbers 2, 3, 4; b, e gets a 6 and 5 and f gets 7 it is a consistent enumeration. g is a predecessor of a, g gets 1 and a gets 2 which is consistent and g gets 1 and c gets 3 and g is a predecessor of c which is consistent and g is less than or equal to f and 1 is less than or equal to 7, so once again it is consistent.

So, you can check that the precedence, every residence is satisfied for any pair of elements x and y. So, that x precedes y the number given to x is less than or equal to the number given to y.

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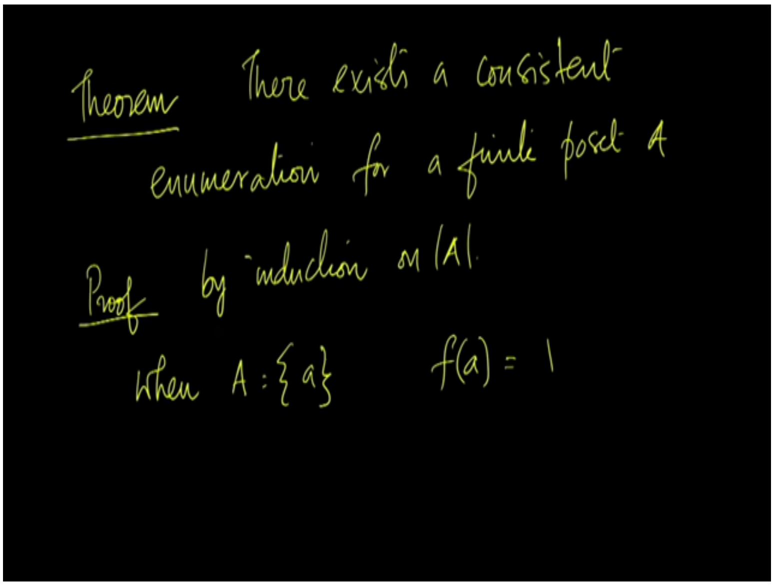

So, here we have a theorem, which says that there exists a consistent enumeration for any finite PO-set a. So, the proof of this theorem is by induction on the size of A, when A is a singleton, we will define f thus f of a equal to 1 and there is only one element here, so there is no conflict, the enumeration is a consistent one.

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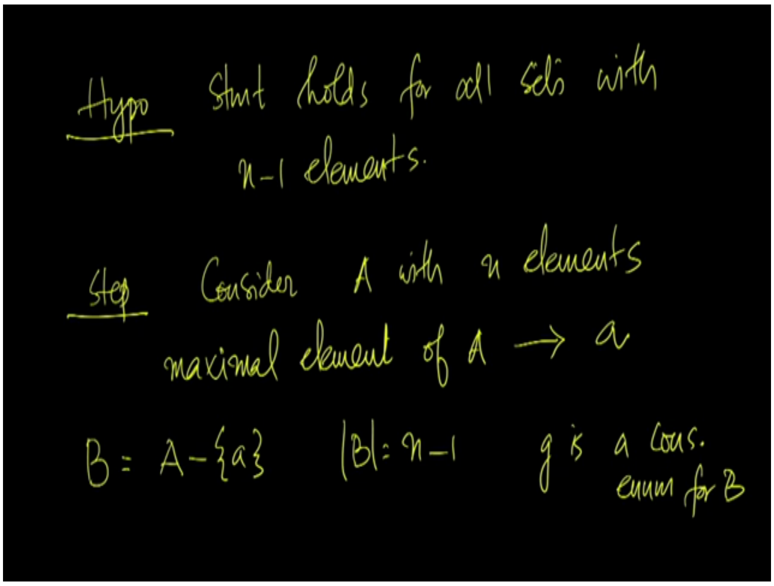

Now, by Induction hypothesis, let us assume that the statement holds for all sets with n minus 1 element. Now, for the induction step, consider a set A with n elements. Then imagine the Hasse diagram for A, when you look at the Hasse diagram you will be able to find a maximal element of A. There could be many maximal elements, let us pick one out suppose that is small a.

Then, if you consider A minus a which I call set B. We find that the size of B is n minus 1, therefore B should have a consistent enumeration. So, let us say g is a consistent enumeration for B then a consistent enumeration for A can be constructed very easily.

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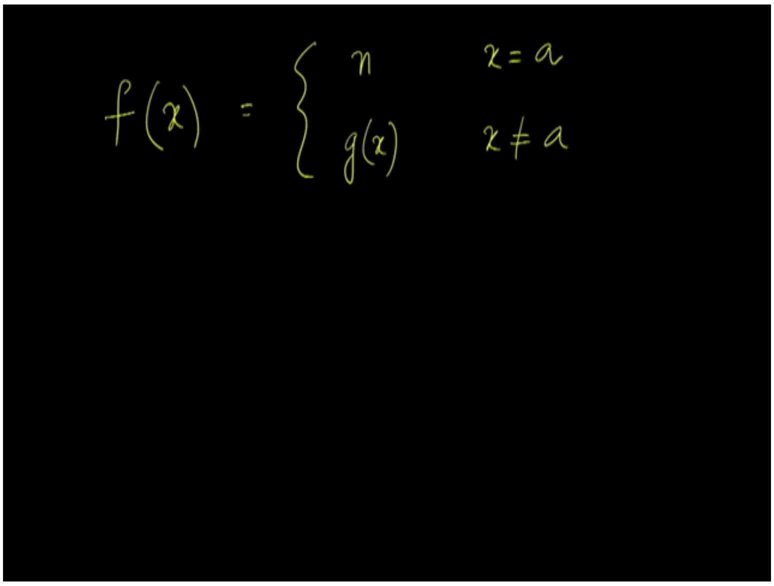

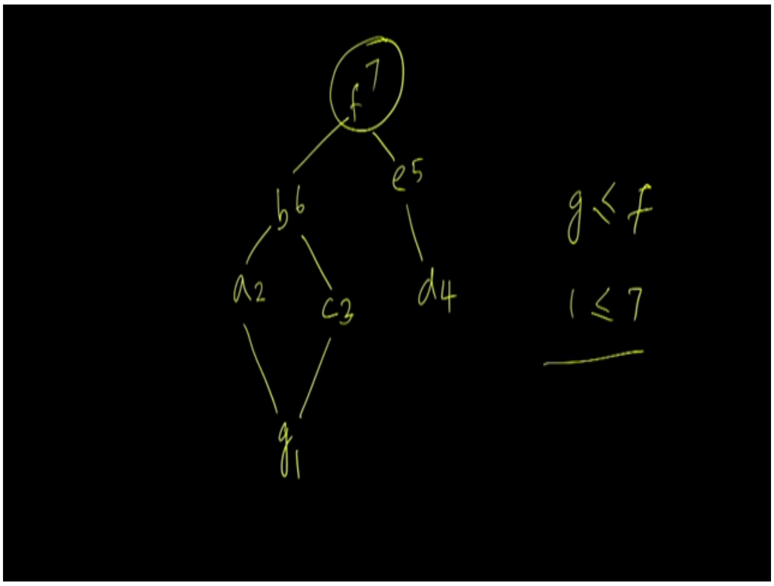

Let us define a function f, f of x will be defined like this, if x equal to a, we will put f of x as n, if x is not a, then x is a member of b as well. Therefore, g is defined for x, so we will define f of x as g of x. So, we are using the same enumeration as we got before, that is we got an enumeration for B. We assumed B x is inductively, we take that B or if we take that enumeration and then extend that enumeration by setting the function value for a to n. So, A will now be the nth element.

So, in the Hasse diagram, we pick out one maximal element. So, in this case f is the maximal element and then inductively we number the rest of the diagram, after that we put f back in place and give f the largest number. So, in this case that largest number is 7, 7 is the number of elements in the original set. So, the numbering that you obtain in this fashion is going to be a consistent enumeration.

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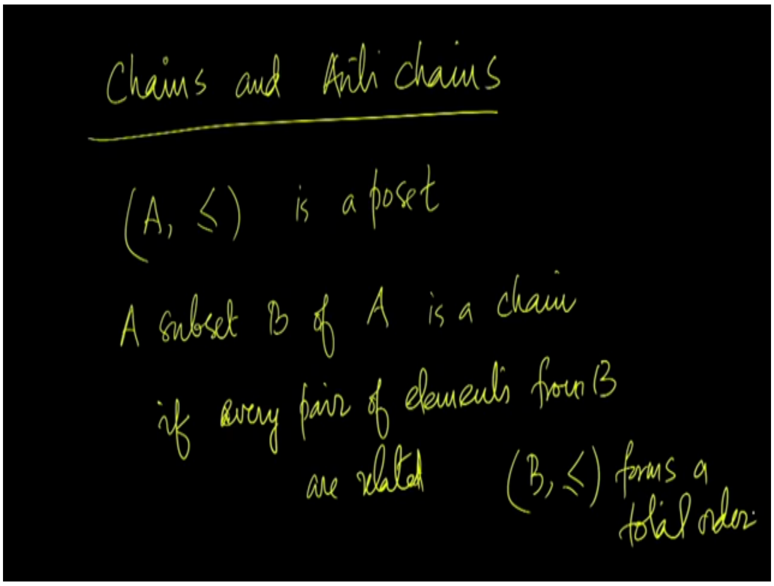

Now, let us study Chains and Anti chains. So, let us consider a PO-set A and less than or equal to a subset of A, a subset B of A is a chain if every pair of elements from B are relate, which means B along with the less than or equal to relation forms a total order.

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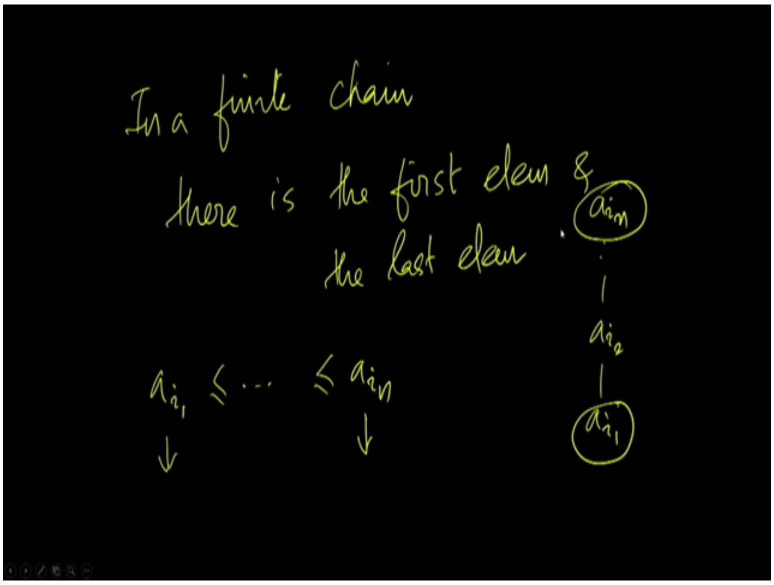

In a finite chain, there is the first element and the last element. So, a chain is a sequence of elements of the sort. So, you can always find the first element and the last element. If you look at the Hasse diagram of a chain, it would look like this. So, the bottom most element is the first element and the topmost element is the last element.

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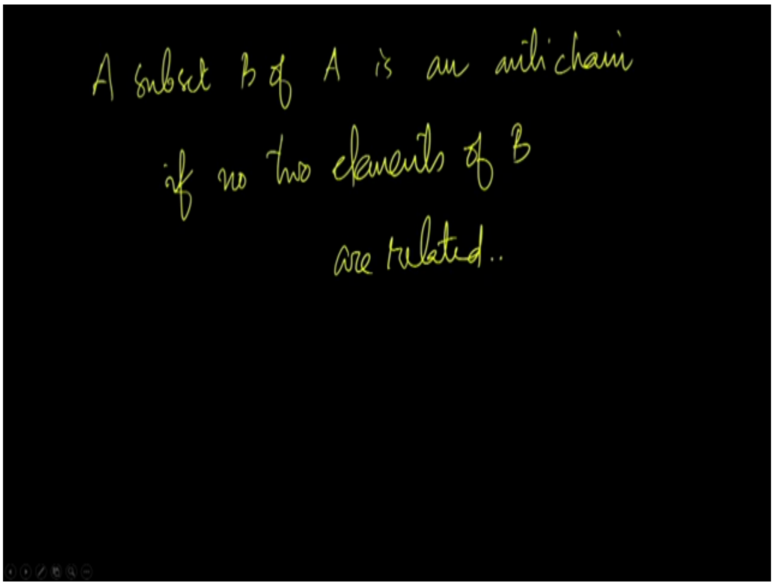

A subset B of A is an anti-chain, if no two elements of B are related. So, that is the definition of a chain and anti-chain.

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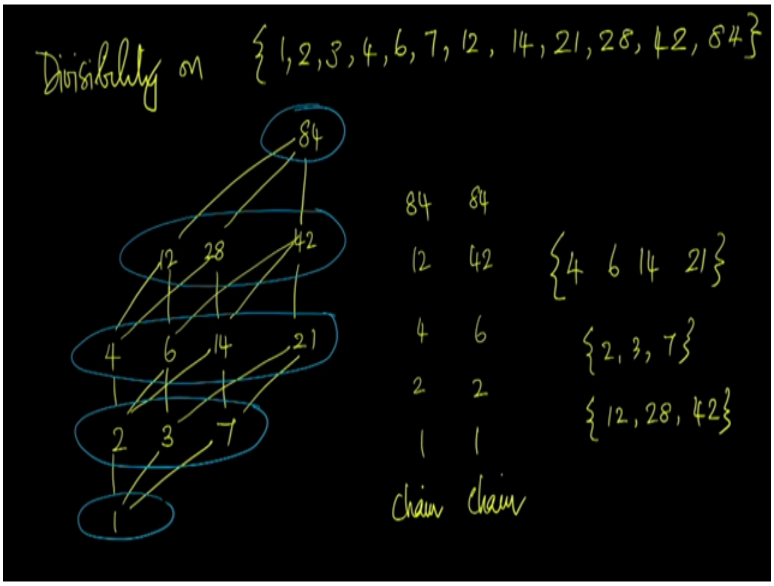

So, let us take an example now. Let us consider the divisibility relation, the set 1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42 and 84. This is the set of devisors of 84. Then we have starting from the bottom. We have 2, 3, 7 are all primes and 1 is a divisor of 2, 3 and 7. So, 1 will be a predecessor of 2, 3 and 7. Now, 4 as a successor of 2, 6 is a successor of both 2, 3; 12 as a successor of 6 and also a successor of 4. 14 is a successor of 7 and also of 2; 21 is a successor of 7 and also of 3. 28 is the successor of 14; 28 is also a successor of 4 because 28 is 4 into 7.

So, there is no number, so that 4 divides that number and that number divides 28. So, there is no intermediate element between 4 and 28. So, you have to draw a line from 4 to 28. Then 42 is the successor of 21 which is also a successor of 14 and 42 is a successor of 6 as well and then we have 84 which is a successor of 42, 28 and 12. So, this would be the Hasse diagram for this PO-set.

So, in this if you consider 1, 2, 4, 12 and 28 this is a chain. Similarly, 1, 2, 6, 1, 2, 4, 12 and 84 sorry 1, 2, 6, 42 and 84 is another chain. What would be an anti-chain? 4, 6, 14, 21 will form an anti-chain, that is because no two of these are divisors of each other. Similarly, 2, 3, 7 will also form an anti-chain. 12, 28, 42 also will form an anti-chain. In fact, 1 alone will form an anti-chain, 2, 3, 7 will form an anti-chain, 4, 6, 14, 21 will form an anti-chain, this is also anti-chain. We saw some of the anti-chain.

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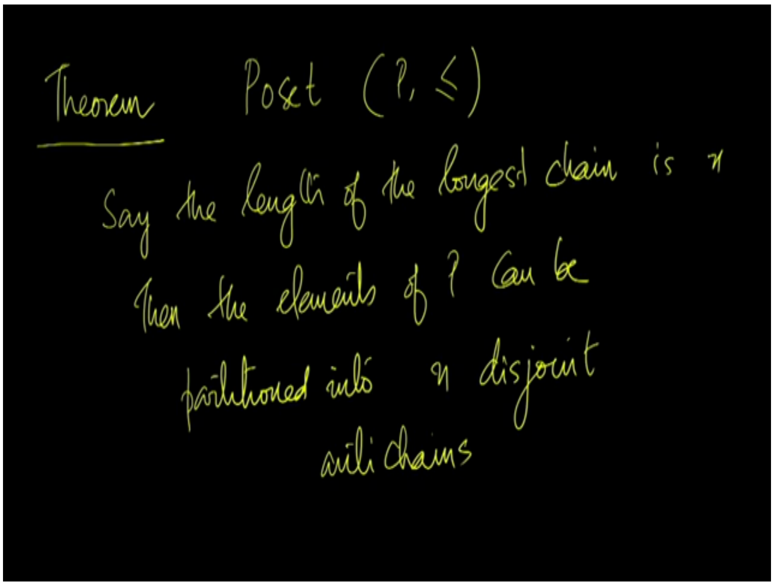

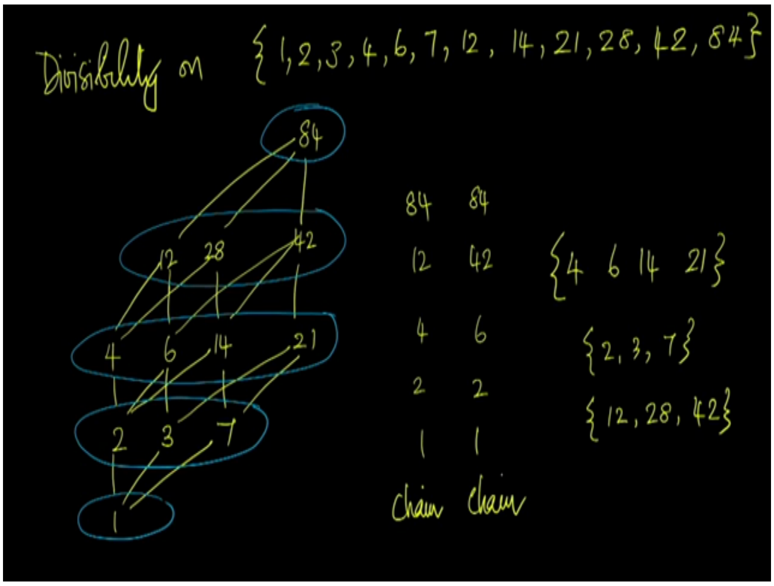

So, I will state this theorem without proof. Consider PO-set p with the less than or equal to relationship, say the length of the longest chain, this n. Then the elements of p can be partitioned into n disjoint anti-chains, that is the length of the longest chain is n and the elements of p can be partitioned (())(55:45) n disjoint anti-chains. As we have running this case, the longest chain has a length of 5, so we have managed to partition this into five disjoint anti-chains. These are not the only anti-chains by any means but this is one set of five disjoint anti-chains.

So, the proof is easy, I am leaving it as an exercise to you but to give a hint you can look at the maximal elements here. In this Hasse diagram there is only one maximal element which is 84, so you can take that out then the rest of the partial order has chains of smaller length and we can apply the statement to that inductively. So, once you partition the rest of the diagram into n minus 1 disjoint partitions then you can add the maximal elements that you have removed in the first step.

And once again the bases will be formed by the singleton set or the partial orders in which there is a chain of length at most one.

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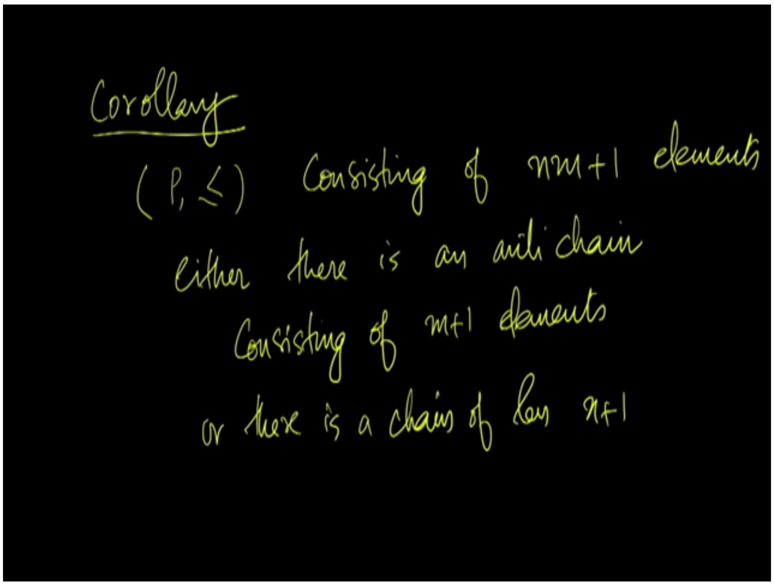

As a corollary, we find that for a PO-set p n less than or equal to consisting of nm plus one elements, either there is an anti-chain consisting of m plus 1 elements or there is a chain of length n plus 1. So, let us contradict the assumption that there is a chain of length n plus 1.

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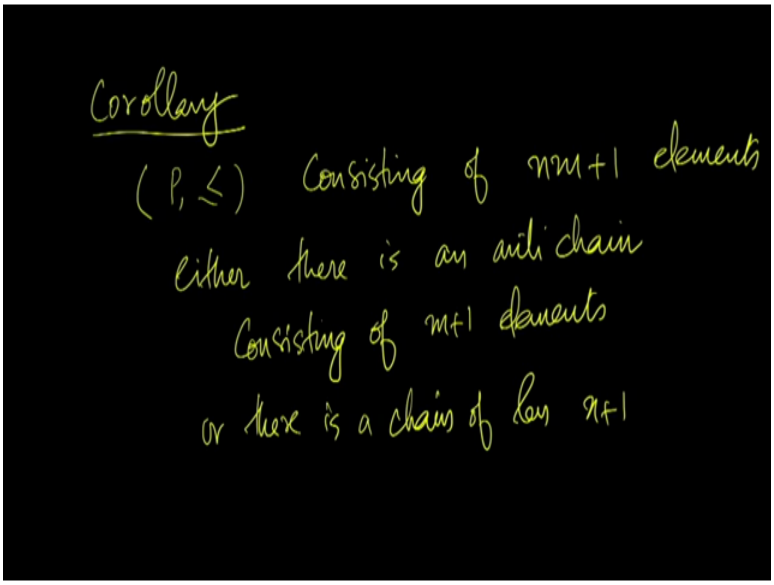

So, let us say the longest chain has a length of n at the most, then we can partition this into at most n disjoint anti-chains. So, this is a (partial) partition of the given set, we are partitioning it into at most n disjoint anti-chains. If no anti-chain has n plus 1 elements then each anti-chain has at most m elements, so the number of elements would be less than or equal to m into n, which is a contradiction because we have assumed that we have mn plus 1 elements.

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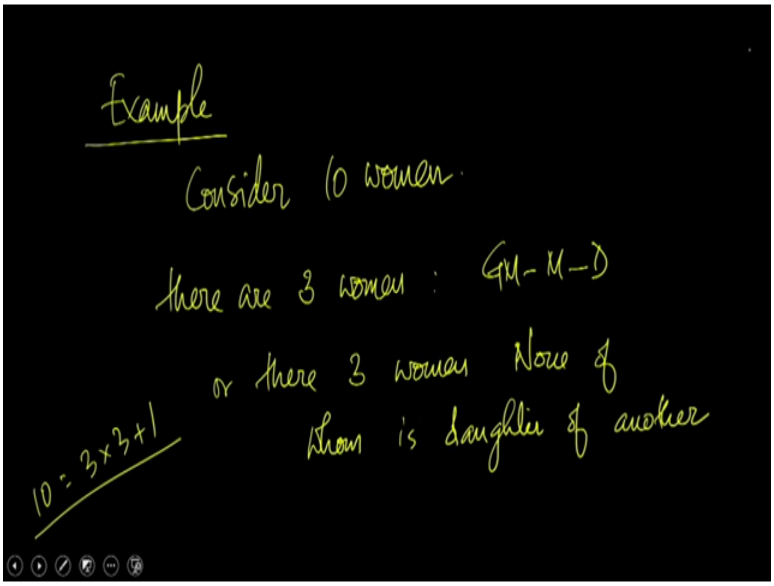

So, as an example, consider 10 women among them there are 3 women, who form a grandmother-mother-daughter triplet. So, that is a chain of descendants of length three. So, among these three women either there are 3 women that form a grandmother-mother-daughter triplet or there are 3 women none of whom is a daughter of another. There are 3 women none of whom is a descendent of another among those 3, that is because 10 equal to 3 into 3 plus 1.

So, we take n equal to 1 and n equal to 3 and m equal to 3 here and from the corollary we find that among 10 women, one of these two possibilities must hold. So, that is it from this lecture, hope to see you in the next, thank you.