Discrete Mathematics

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Lecture 5
Set Theory

Welcome to the NPTEL MOOC on Discrete Mathematics.; t

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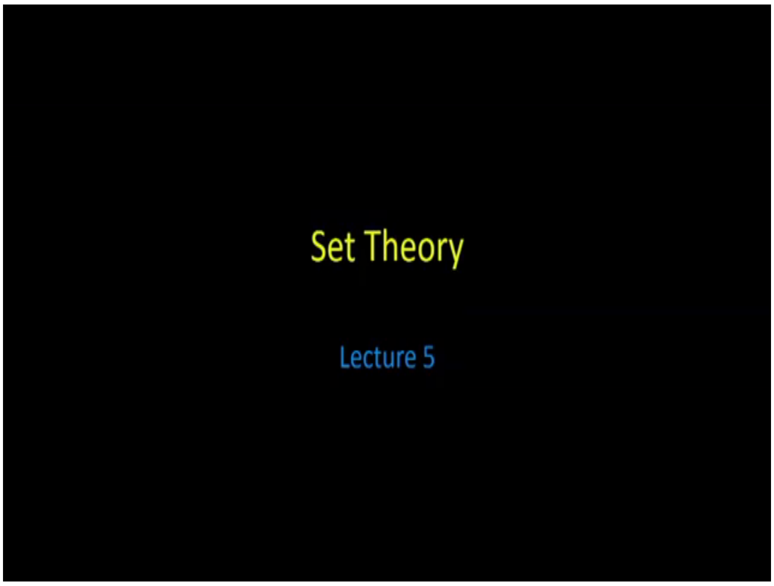

This is the fifth lecture on Set Theory. In the previous lecture, we saw that Naive set theory has problems with Paradoxes for example Russell's Paradox. So, when we (()) (00:46)axiomatize nNaive set theory we have to pay a great deal of attention, one such axiomatization is the one by Zermelo and Fraenkel. So, we shall take a peek at the Zermelo–Fraenkel axiomatization today.

Long discussion about this is not within the scope of this course, we may have occasion only to look at the axioms.

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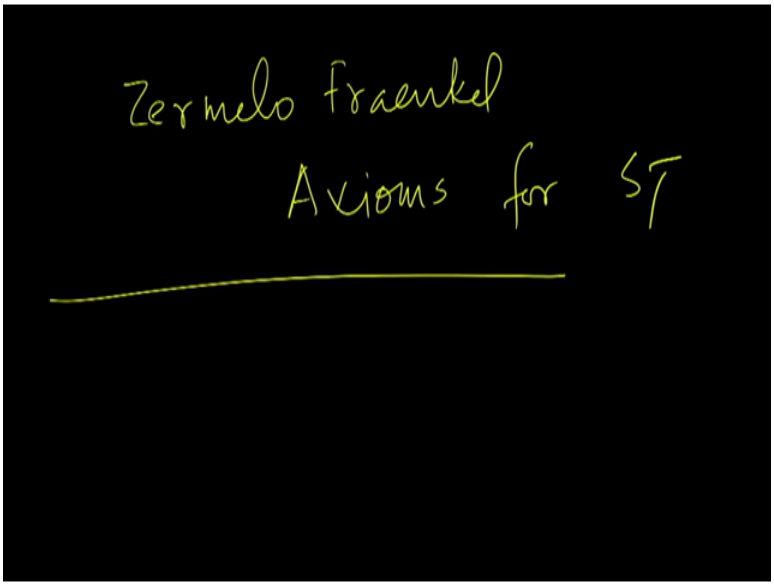

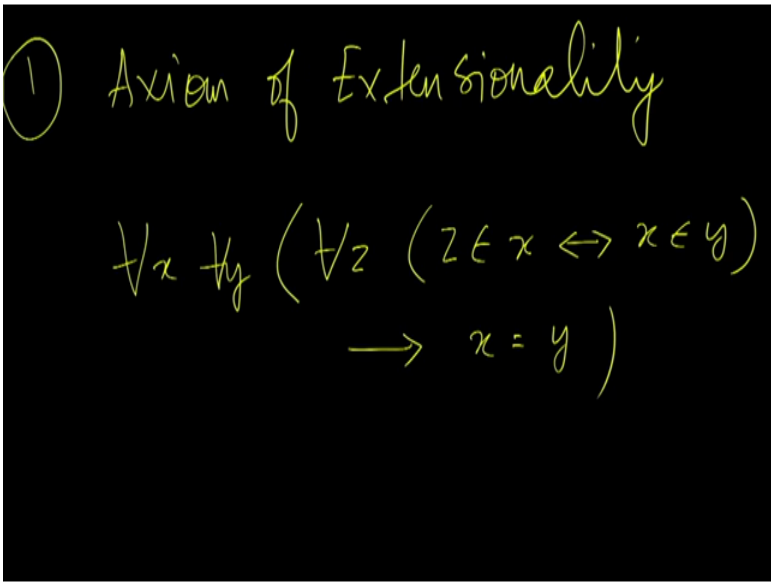

The first axiom is the ~~a~~Axiom of Extensionality. Axiom of extensionality says that, two sets are equal if and only if they have the same extensions. In other words, two sets have exactly the same members if and only if they are equal. Formally for all x, for all y, for all z, z belongs to x if and only if z belongs to y, implies that x equal y. For any two sets x and y, x and y have exactly the same extensions that is the same set belongs to both of them.

For every z, in which case x is equal to y, this is the first axiom.

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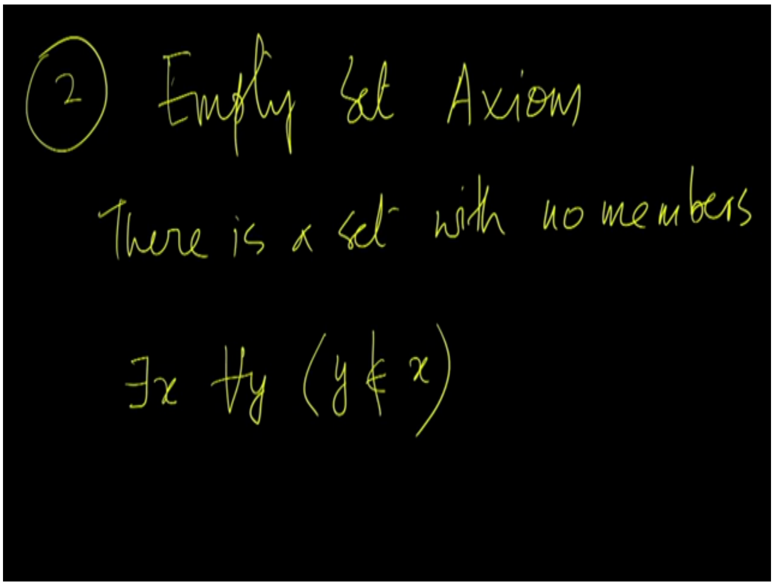

The second axiom is the Empty ~~s~~Set ~~a~~Axiom. The empty se~~a~~t axiom says that, there is a set with no members. ~~In~~ ~~o~~Other words, there exists~~there exists~~ an x so, that for all y, y is not a member of x. ~~t~~There is a set x so, that for every y, y is not a member of x, in other words x does not have a member.~~;~~ ~~s~~So, x is the empty set. So, this axiom asserts the existence of an empty set.

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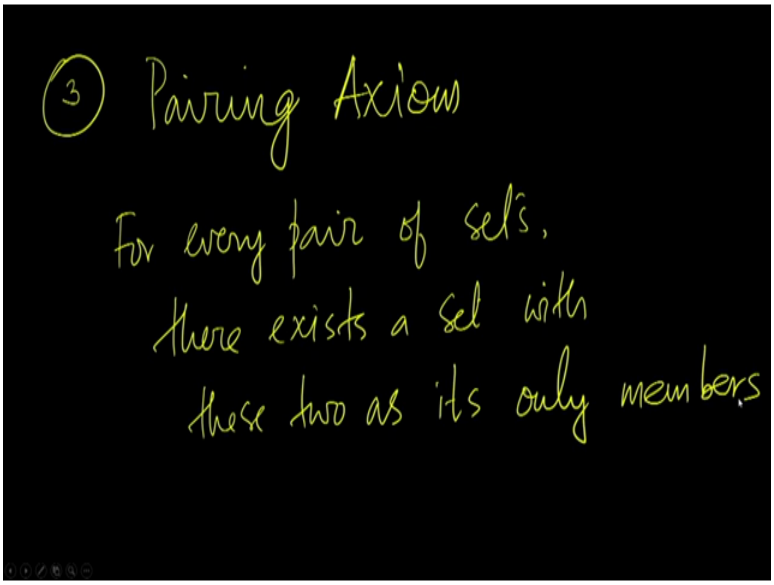

The third axiom is the Pairing ~~a~~Axiom. Pairing axiom says that, for every pair of sets there exists a set with these two as ~~se~~its only members.

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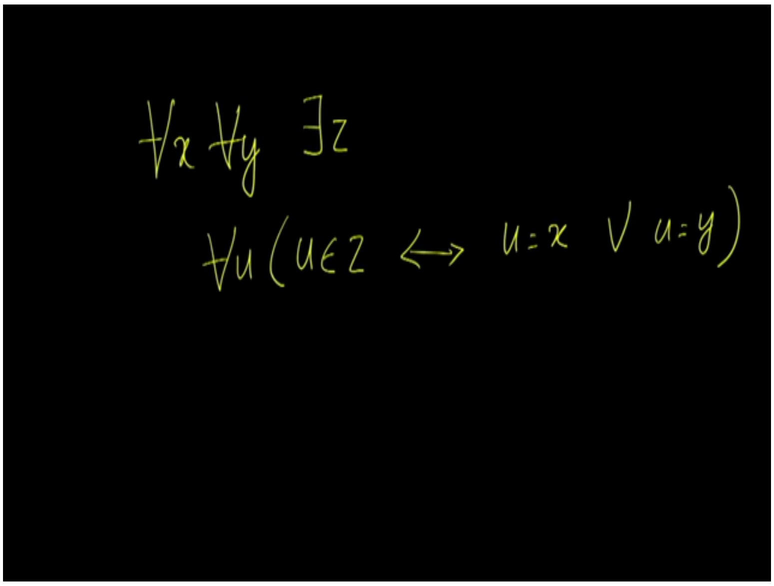

In other words, for every x and for every y, for a pair x, y of sets, there exists z. So, that for every u, u belongs to z if and only if, u equal to x or u equal to y. In other words, for every pair x, y of sets, there is a set z, so that something is a member of z precisely when that something happens to be either x or y. In other words, z contains exactly x and y and nothing else.

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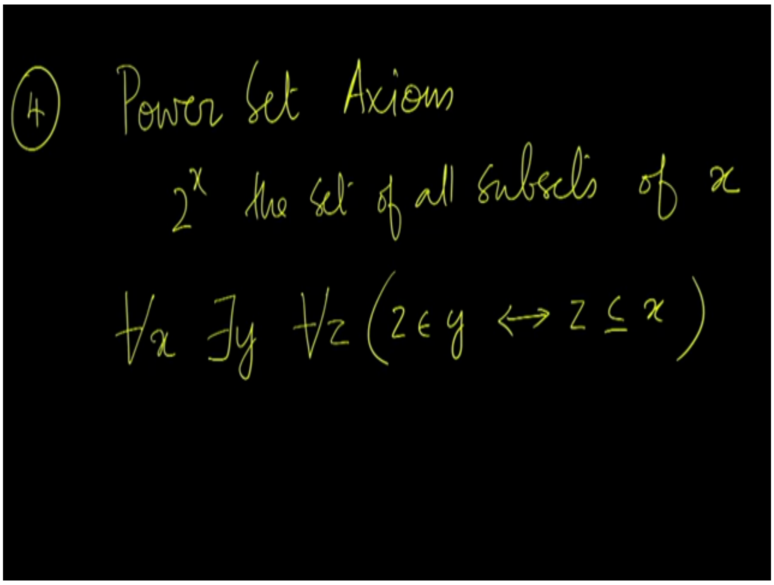

The fourth axiom is the Power Set Axiom. Power set of x we know, is the set of all subsets of x. So, the power set axiom asserts the existence of a power set, for every set x, there is a set which happens to be the power set of x. In other words, for all x there exists y, which happens to be the power set of x, so how do we state that? We have to say that for every z, z belongs to y, precisely when z is a subset of x.

In other words, z is a member of y, precisely when z happens to be a subset of x or y will contain precisely the subsets of x or y is the power set of x. So, axiom for asserts the existence of the power set for every set.

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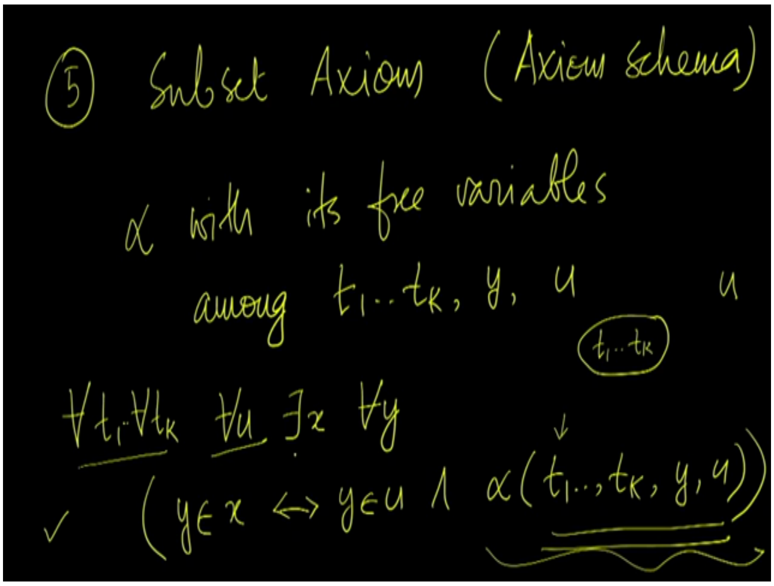

The fifth axiom is the Subset aAxiom.; tThis is in fact an aAxiom sSchema. For a formula alpha with its is free variables among t 1 through t k, y and u so, alpha is a formula with thfree variables among these, we have this following axiom. For all t 1 through t k, for every tuple t 1 through t k and for every u, there exists an x so, that for every y, y belongs to x if and only if y belongs to u and alpha of t1 through t k, y, u is true.

This will be an axiom for every formula alpha with its is free variables among t 1 through t k, y, u. So, when t 1 through t k, y, u are supplied as arguments to alpha, you have to perform the substitution if one of them happens to be the free variable. For example, if t 1 is not a free variable, then the argument which is applied here will not be substituted. So, what does it say? What it says is that, given any k topple t 1 through t k and a set u then we can pick out the members of u, which satisfy the formula alpha along with y and u with t 1 through t k.

In other words, given the tuples, tuple t 1 through t k and u there exists an x or set which we can synthesize from u and t 1 through t k so, that membership in x of y will be precisely when y is the number of u, in other words, x will be a subset of u. sSo, we are forming a subset of u, x is a subset of u and moreover u and y will have to satisfy the condition alpha along with t 1 through t k, this is the way of forming subsets of a given set u.

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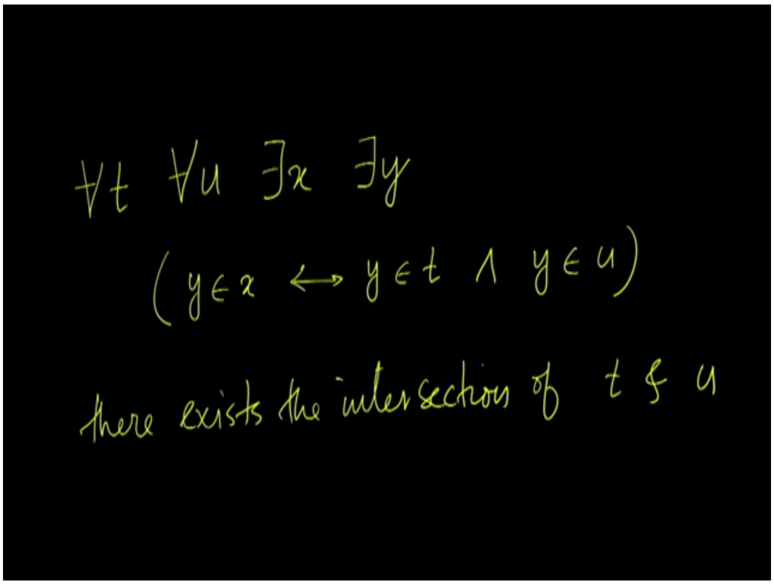

An example would be for all t, for all u there exists an x, so that for every y, y belongs to x precisely when y belongs to t and y belongs to u. Now, what does this assert? There exists the intersection of t and u. So, for every set u, when t is supplied we can form the intersection of t and u. In other words, from u we can form the subset of members of u which are also members of t.

So, this is a way of forming subsets of u.

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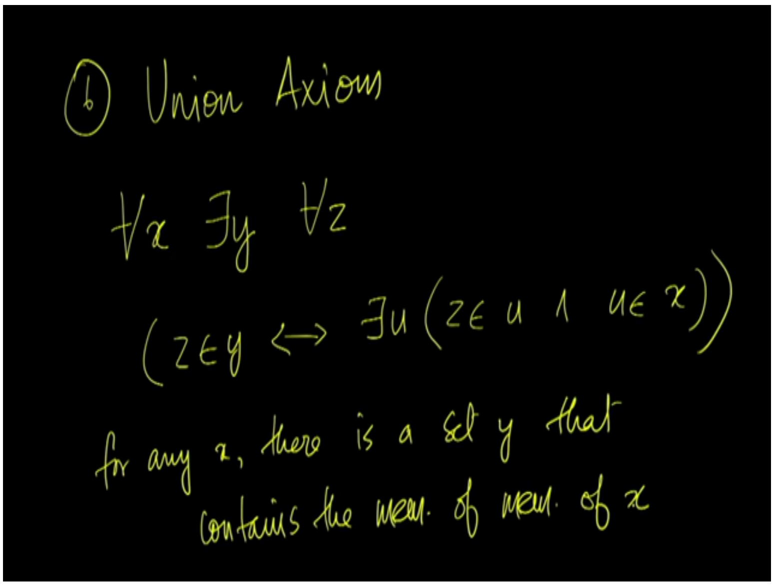

The sixth axiom is the Union ~~a~~Axiom. Union axiom says that, for every x there exists of y so, that for every z, z belongs to y precisely when there exists ~~a~~ u so that z belongs to u and u belongs to x or~~. Or~~ in other words, given any set x, we can construct a set y which will contain precisely the members of members of x, that is~~as~~ for~~ar~~ as z to belong to y that will have to belong to some u which in turn belongs to x.

So, for any x, there is a set y that contains precisely the members of members of x.

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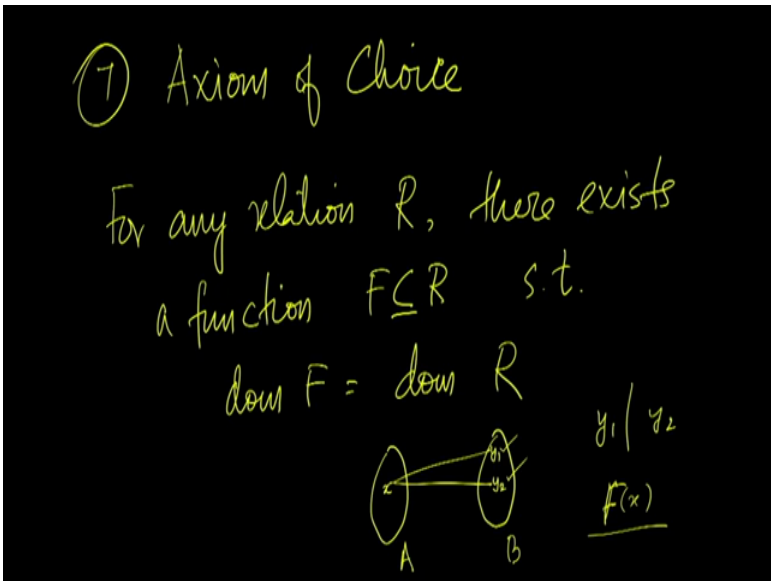

The seventh axiom in our list is the ~~a~~Axiom of Choice. Axiom of choice is that, for any relation R, there exists a function F, which is a subset of R, such that the domain of F is the domain of R.~~;~~ ~~w~~Why is this called the axiom of choice? Given a relation R, let us say from A to B therefore, this is a subset of A cross B, then consider some member x of A under the relation R, x may have two images.~~;~~ ~~l~~Let us say y 1 and y 2 but what we construct here is a function, the domain of which is identical to the domain of R.

Therefore, x will have to have an image under F as well. ~~b~~But then x has two images under R to form F, you will have to pick one of them, that is you have to exercise the choice y 1 or y 2.~~;~~ ~~o~~One of them will be f of x for the function F that we are going to construct. Therefore, we are exercising a choice, when we construct function F. So, what axiom of choice is that, for any relation R, there exists a function satisfying this condition that is the domain of the function is identical to the domain of R.

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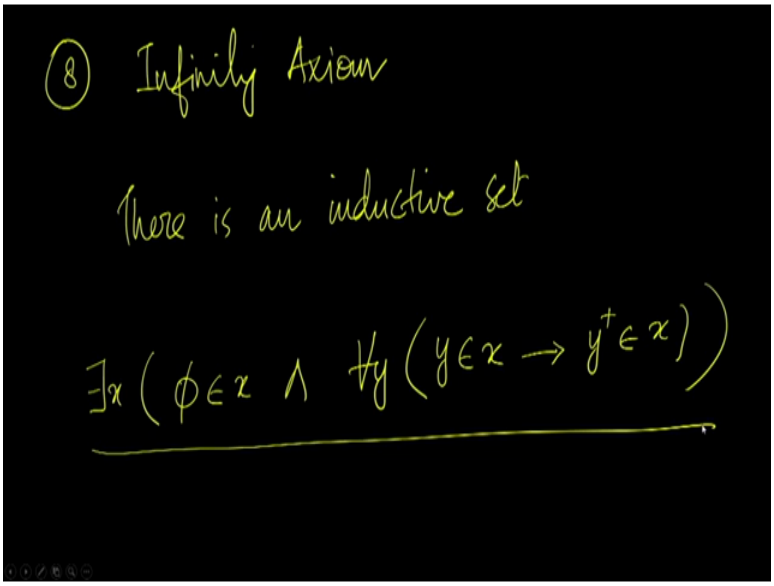

The eighth axiom is the Infinity ~~a~~Axiom, what this says is that there is an inductive set or formally there exists an x, so that the empty set belongs to x and x is closed under the successor operator. For every y if y belongs to x then the successor of y also belongs to x that is when x is closed under the successor operator. So, the infinity axiom says that there is an inductive set.

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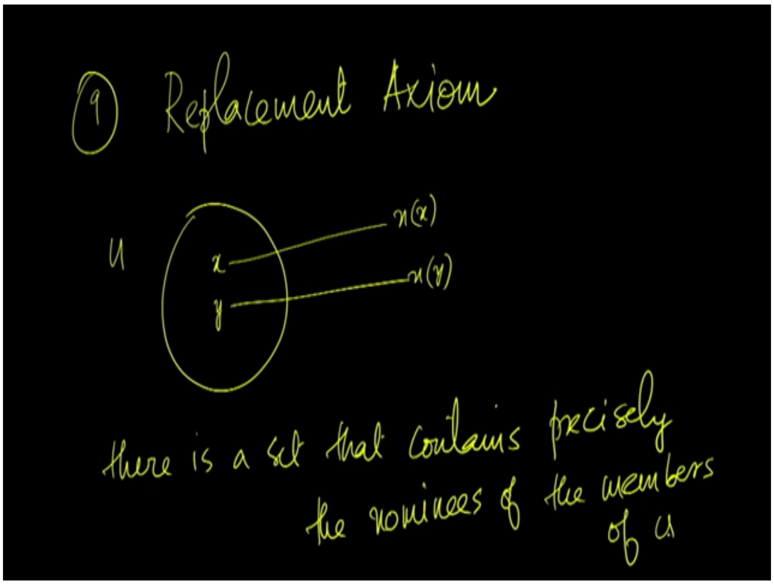

And the ninth axiom is the Replacement ~~a~~Axiom. Consider set u, suppose every member of u has a nominee ~~an f~~ n of x, ~~o~~for y the nominee is ~~a~~n of y. So, the nominee function defines a unique nominee for every x, then what does axiom says is that, if every member of u has a nominee then there is a set that contains precisely the nominees of the members of u.

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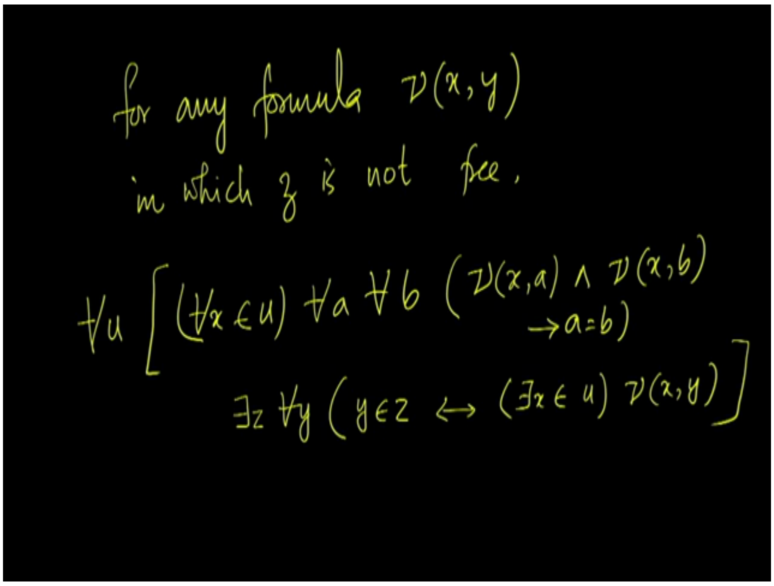

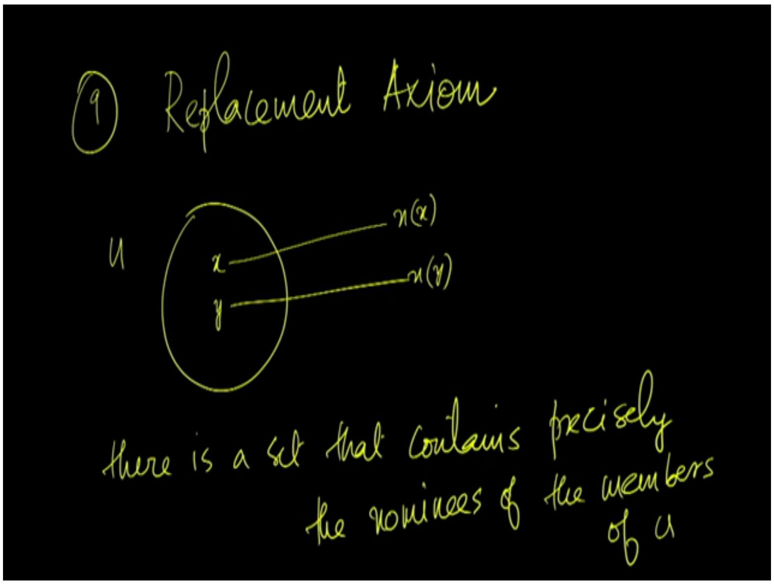

Or in other words, for any formula nu of x, y which in fact asserts that y is the nominee of x, in which z is not free. The following is an axiom for every u, for all x belonging to u, for all a and b nu of x, a and nuew of x, b implies that a equal to b. So, this is the antecedent of an implication what this says is that for every a, b nu of x, a and nuew of x, b implies that a equal to b.

In other words, there is exactly one a for every x, so that nuew of x, a is satisfied. In other words, x has a unique nominee. Then there exists z, so that for all y, y belongs to z if and only if there exists x belongs to you, so that nu of x, y. In other words, there exists a z which contains exactly the nominees of the members of u, that is for every y, y belongs to z precisely when y is the nominee of some x which belongs to u.

So, the assertion is exactly that we had in mind. There is a set that contains precisely the nominees of the members of u.

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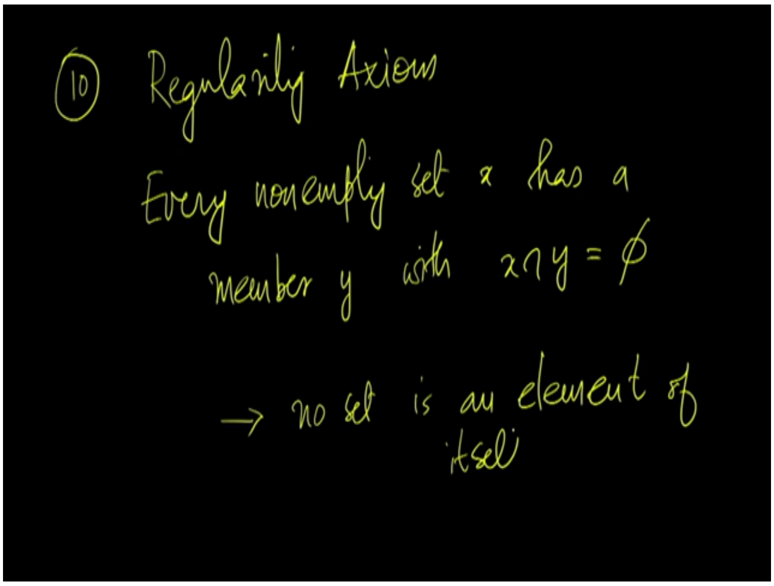

And the final axiom is the Regularity ~~a~~Axiom. To say is that every non-empty set x has a member y with x intersection y equal to phi. You can show that this implies no set is an element of itself. The paradoxes that are known within ~~n~~Naive set theory will not arise within Zermelo–Fraenkel set theory.

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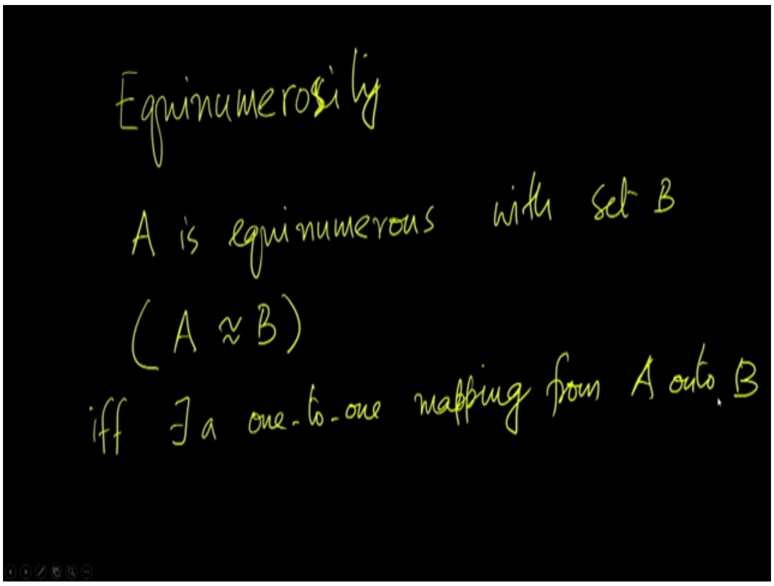

Now, let us consider the notion of Equinumerosity. We say that set A is equinumerous with set B denoted in this fashion, A is equinumerous with set B if and only if there exists a one-to-one mapping from A onto B.

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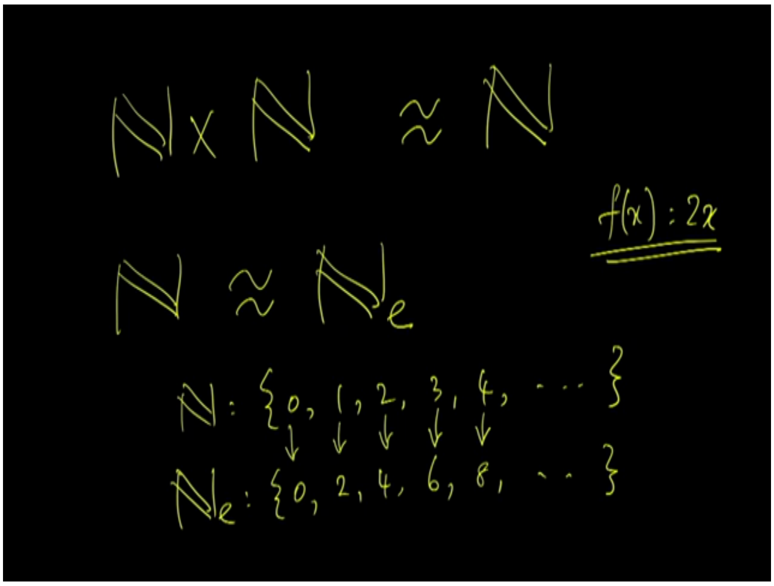

We have seen that N cross N is equinumerous with N. The set of all ordered pairs obtained from N is equinumerous with N itself. Similarly, N is equinumerous with the set of all even natural numbers. N is the set of all natural numbers. Set of all even natural numbers is Ne. So, there is a mapping from N to Ne which is 1 to 1 and onto, in which we map x to 2x.

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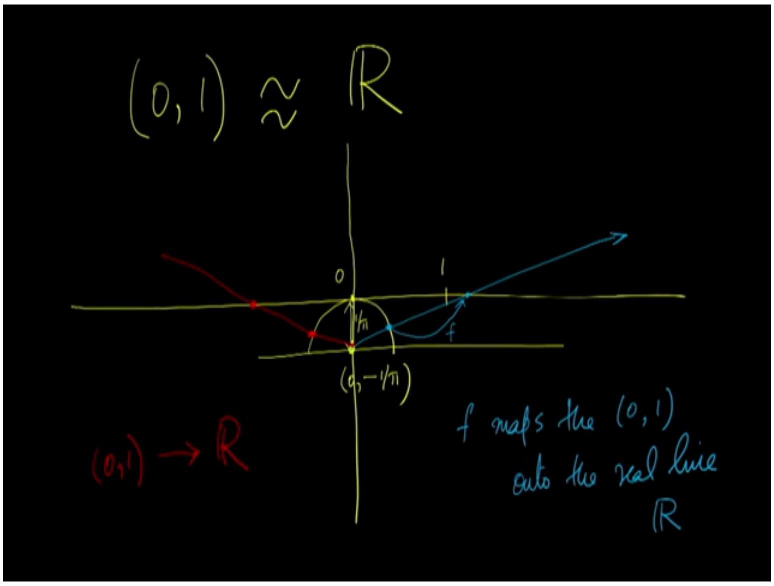

Coming to real numbers, we can show that the interval from 0 to 1 ~~th~~is equinumerous with the set of all real numbers.~~,~~ ~~h~~How do we show this? To show this, we consider the real line, let us say this is the origin and this is 1. So, we are considering the set of all points from 0 to 1 on the real line, we want to show that this set ~~a~~is equinumerous with the points from the real line itself.

To prove this what we do is this. Consider the portion of the real line from 0 to 1, that is a line segment from 0 to 1.~~,~~ ~~w~~We take it and bend it, so that it forms a semicircle and arrange the semicircle, so that the real line is a tangent to the semicircle. So, the length of the semicircle is 1 because this has been obtained from the interval 0 to 1 by bending the interval 0 to 1, so this has a radius of 1 by pi.

So, the origin of the circle would be 0 minus 1 by pi,~~3~~ ~~o~~on the real plane. So, this is what the origin is and then let us say we draw a line passing through the origin of the circle and some point on the real line.~~,~~ ~~t~~

This ray, the ray with the origin as its~~ is~~ vertex will intersect the semicircle at some point and it will intersect the real line at exactly one point. Then, let us define a function f, which maps the circle point onto the real line point. ~~s~~So, f is defined in this manner. So, this function f maps the interval 0 to 1 onto the real line, which is the set R.

Consider another line for example, this will pass through these two points. ~~s~~So, this point is mapped to a negative real number. So, it shows that the interval 0 to 1 has a 1 to 1 onto mapping to the set of all real numbers. Therefore, the interval 0 to 1 is equinumerous with the set of all real numbers.

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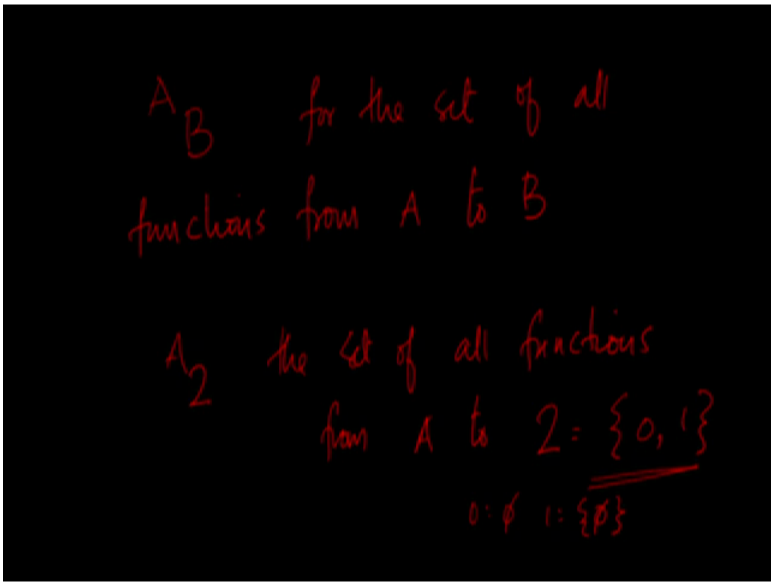

Let this be the notation, for the set of all functions from A to B. In particular, A 2 will denote the set of all functions from A to the set 2 which according to our definition is this. We are considering the natural number 2. In the embedding of the theory of natural numbers in set theory we had defined natural number 2 as the set 0, 1, where 0 is the empty set and 1 is the single term containing the empty set.

So, by superscript A 2 we denote the set of all functions from A to 2.

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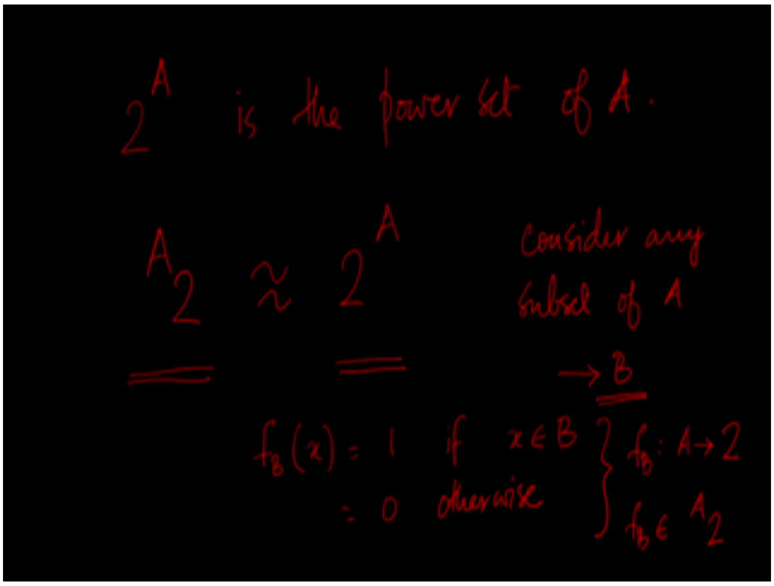

Recall 2 power A is the power set of A. We claim that superscript A 2 is equinumerous with 2 superscript A. This is the set of all functions from A to 2 and this is the set of all subsets of

A. These two sets are equinumerous but how do we show that they are equinumerous? Let us consider any subsets of A. Suppose B is a subset then B has a characteristic function. The characteristic function of B, f B is defined in this manner, f B of x is 1 if and only if x belongs to B or in other words it is 1 if x belongs to B it is 0 otherwise.

The characteristic function is a binary function. So, f B happens to be a mapping from A to 2. So, f B is a member of the set of all functions from A to 2. So, what we find is this, corresponding to any subset B of A there exists a unique function f B of superscript A 2 and this is a unique function. The characteristic function of B happens to be a member of superscript A 2, therefore these two sets are equinumerous.

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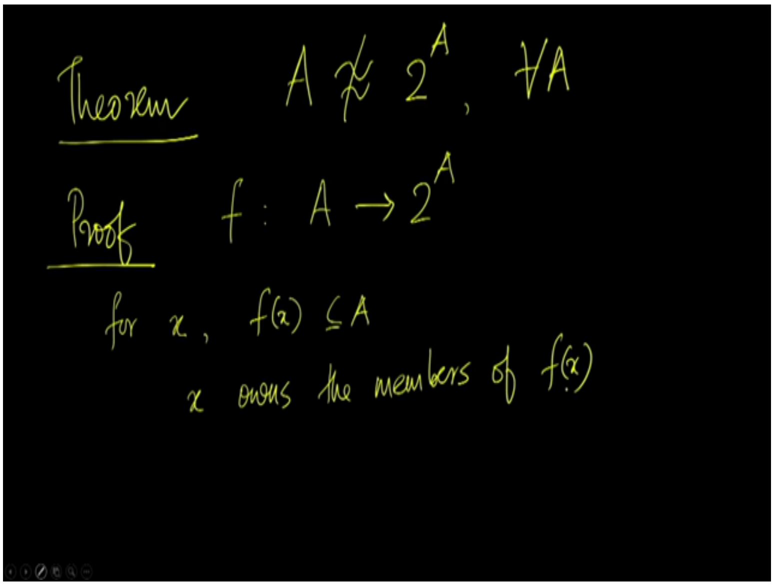

It is a theorem. It says that A is not equinumerous with its own power set, no set is equinumerous with its own power set. How do we prove this? Consider any mapping f, consider an arbitrary mapping f from A to 2 power A so it maps the members of A to subsets of A. So, for x, f of x is a subset of A when x is a member of A. So, let us say that x owns the members of f of x.

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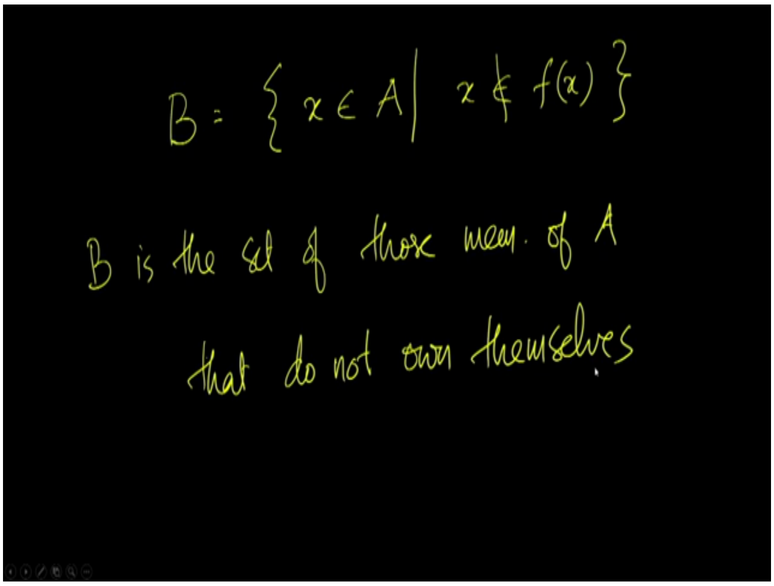

Then let us define a set B, B is the set of precisely those members of A, so that x does not belong to f of x. In other words, B is the set of those members of A that do not own themselves.

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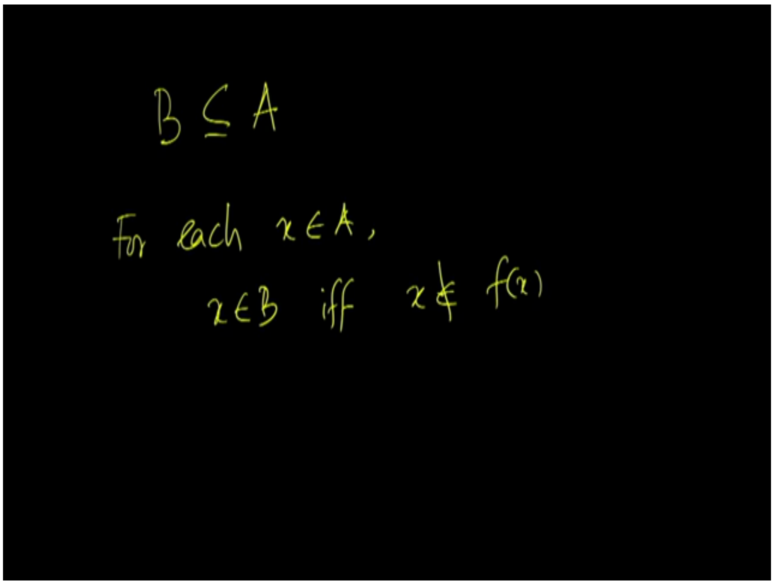

Then by definition B is a subset of A. For each x belongs to A, x belongs to B if and only if x does not belong to f of x by definition.

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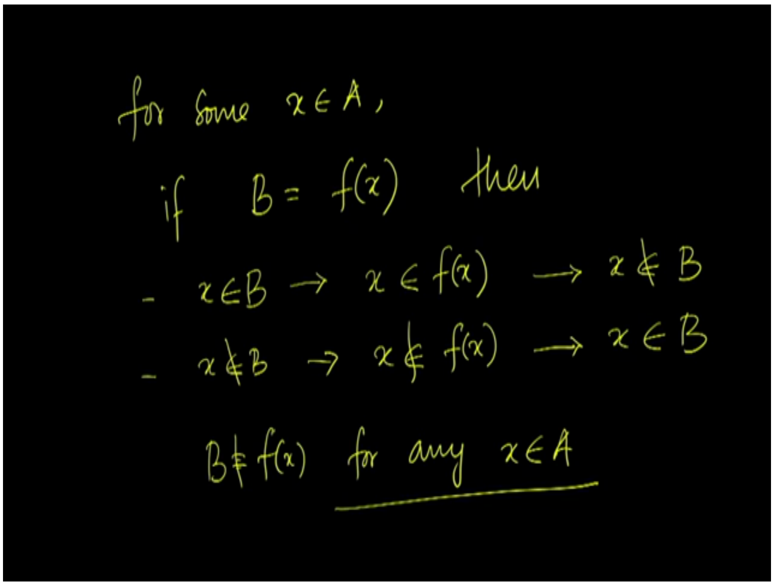

For some x belonging to A, if B~~b~~ equal to f of x. Suppose, B happens to be the image of some x under f, then let us consider the possibilities, one possibility is that x belongs to B but then B is the same as f of x, then x belongs to f of x, if x belongs to f of x then x should not belong to B because B happens to be the set of precisely those that do not own themselves. So, if x belongs to f of x then x owns itself, so x should not belong to B.

On the other hand, if x does not belong to B then x does not own itself this implies that x belongs to B so either way we get a contradiction. Therefore, what we have is that B is not equal to f of x for any x.

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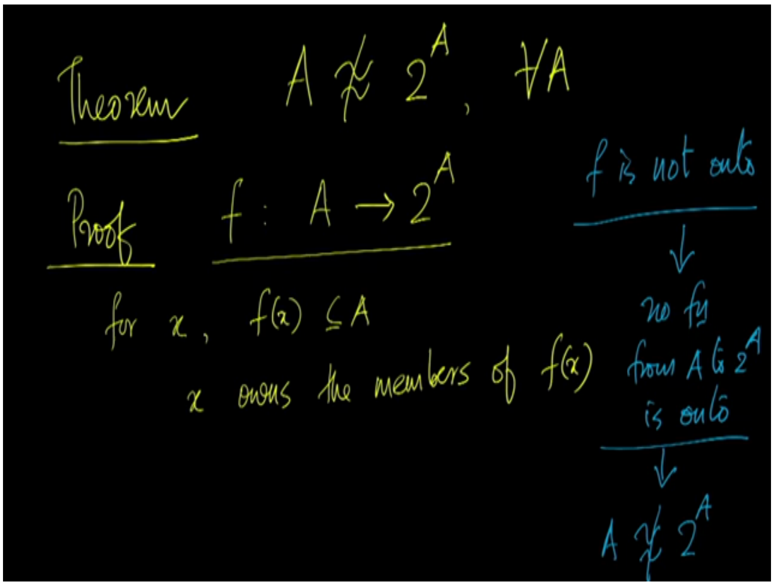

But B is of course a subset of A which means B is a member of 2 power A. Therefore, there is a member of 2 power A that is not an image under the function f or in other words f is not onto, remind you the function f that we considered as an arbitrary one, we have considered an arbitrary function f here and what we have shown is that this function is not onto. So, any function f from A to 2 power A is onto.

So, what we have established is that f is not onto. Since f is arbitrary we have that no function from A to 2 power A is onto which means A is not equinumerous to 2 power A. In other words, no set can be equinumerous with it is own power set.

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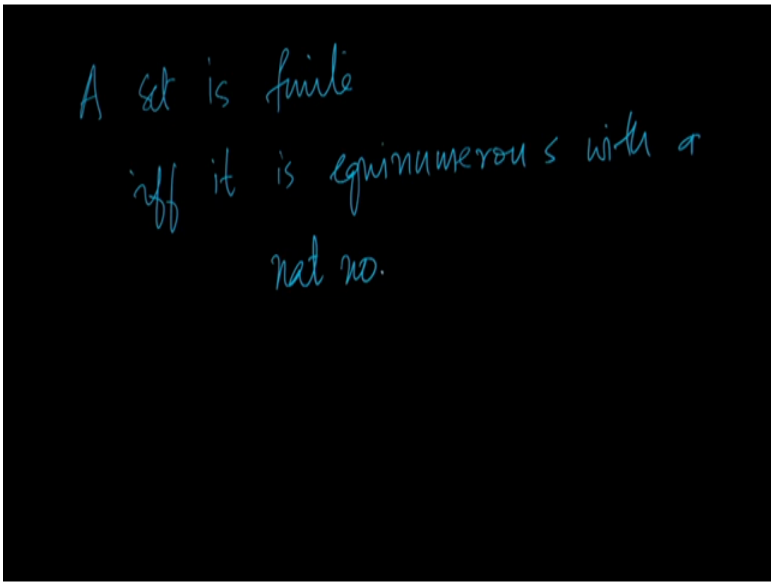

We say that a set is finite if it is equinumerous with a natural number.

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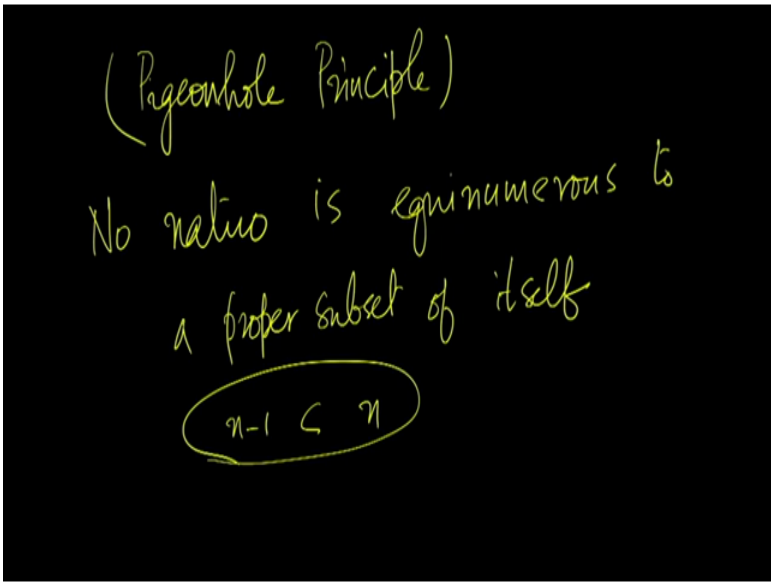

Now, let us consider a theorem which is famous under the name Pigeonhole principle. For pigeonhole principle says is that no natural number is equinumerous with a proper subset of itself, remember n minus 1 is a subset of n under the definition of our natural numbers. So, what it says is that no natural number is equinumerous with any smaller natural number.

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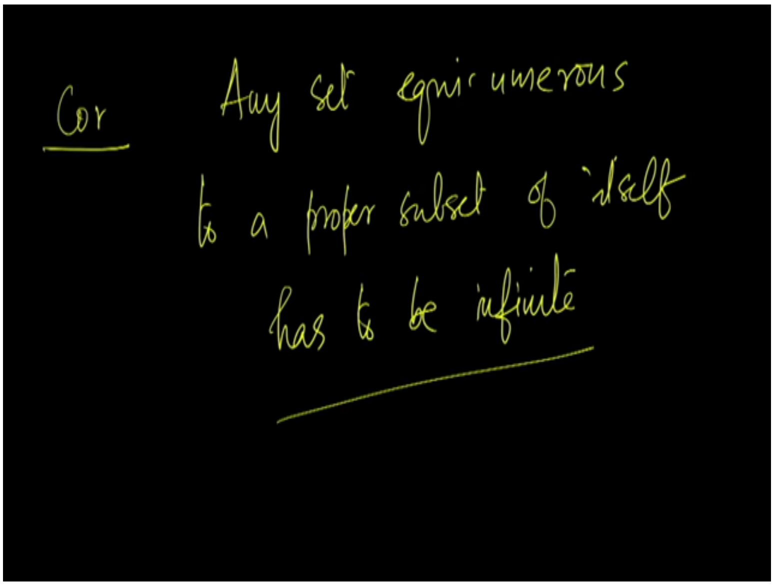

Therefore, as a corollary we can argue that any set equinumerous with a proper subset of itself has to be infinite. In other words, it is not equinumerous with any natural number.

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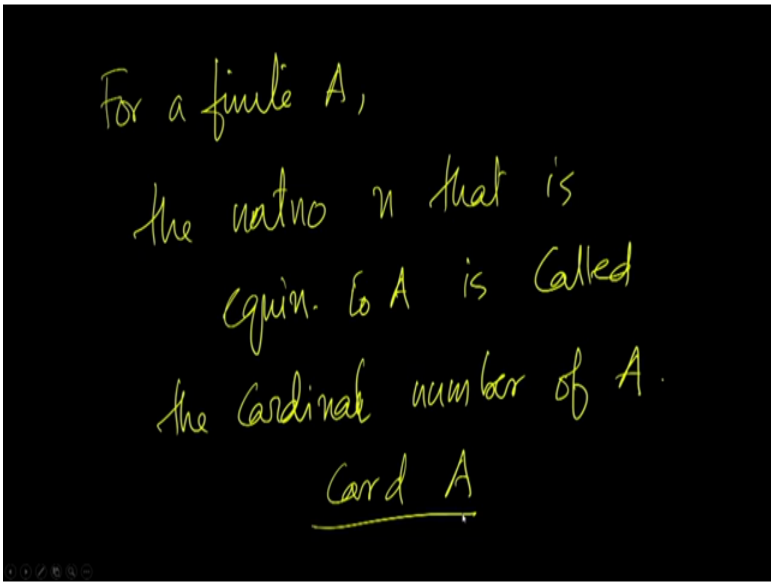

For a finite set A, the natural number that is equinumerous with A is called the Cardinal number of A, this is denoted as Card a.

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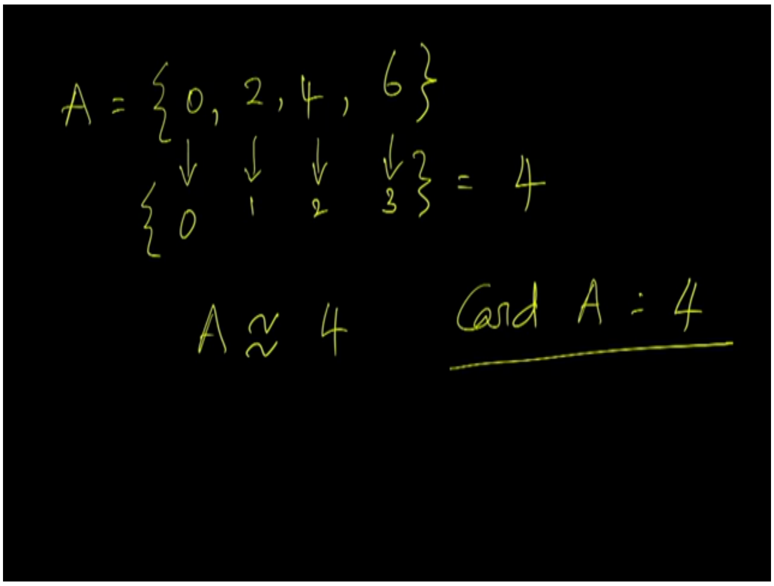

For example, consider this set, this is equinumerous with 4 there is a one-to-one mapping from the given set A onto the natural number 4 therefore A is equanimous with 4. Or in other words, the cardinal number of A is 4 or we say the cardinality of A is 4. So, for every finite set there is a natural number that forms it is cardinality. Two finite sets are equinumerous means they have exactly the same cardinal numbers.

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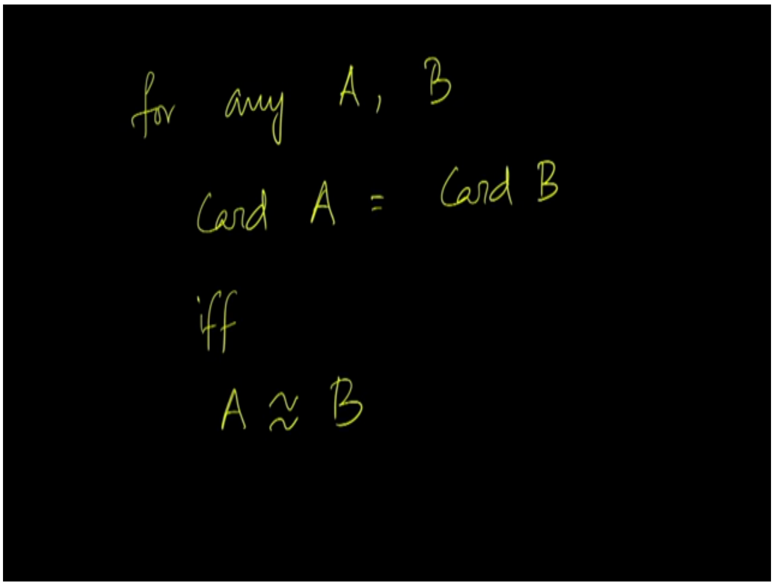

Or in general, for any two sets A and B finite or infinite, we say that the cardinality of A is equal to cardinality of B by the definition of cardinality this is if and only if A is equinumerous with B. So, there is a one-to-one on to mapping from A to B that is precisely when the cardinality of A and B are identical.

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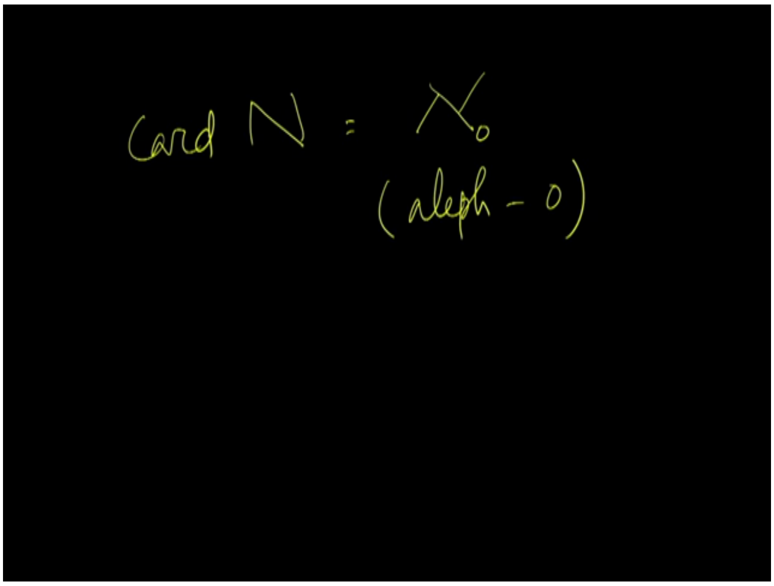

The cardinality of the set of natural numbers is denoted as aleph naught using the Hebrew letter aleph.

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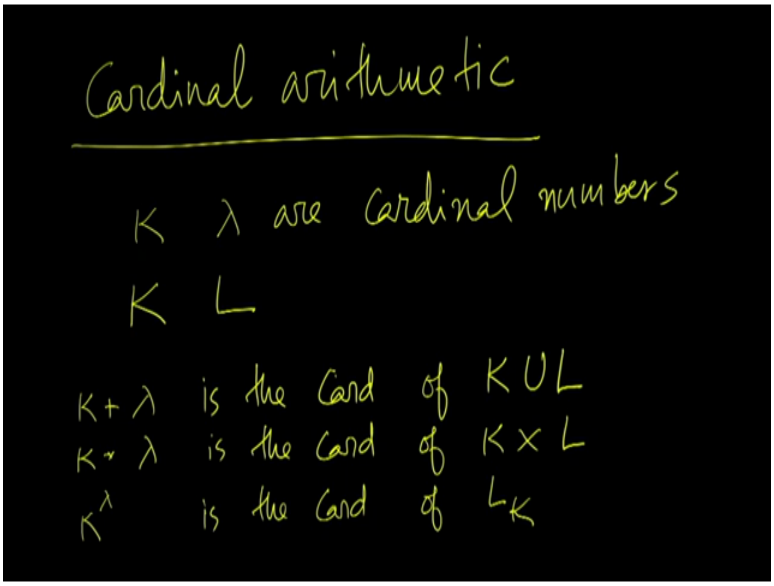

Using cardinal numbers we can form what is called cardinal arithmetic, cardinal arithmetic has several interesting properties. If kappa and lambda are cardinal numbers which means there is a set a with cardinality kappa and there is a set b with cardinality lambda. Or let us say using matching letters K and L with cardinalities kappa and lambda respectively then kappa plus lambda is the cardinal number of K union L, kappa into lambda is the cardinal number of K cross L, kappa power lambda it is the cardinal number of the set of all functions from L to K when k and lambda are finite these of course function the way we expect.

For example, if K and L are finite sets K union L has K plus lambda elements at the most if K and L are disjoint sets, k into lambda is the cardinality of K cross L which is indeed the case, K has kappa elements and L has lambda elements. So, kappa into lambda is the cardinality of K cross L, kappa power lambda is the cardinality of L to K there is a set of all functions from L to K.

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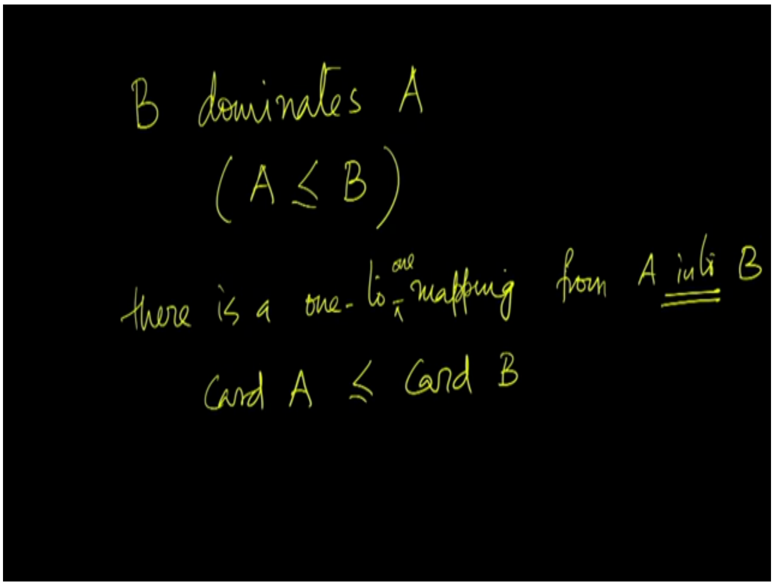

We say that set B dominates A which was denoted as in this fashion for two sets A and B we write A less than or equal to B, to indicate that B dominates A we say those precisely when there is a one-to-one mapping from A into B remind you this is into mapping which means the cardinality of A is less than or equal to the cardinality of B.

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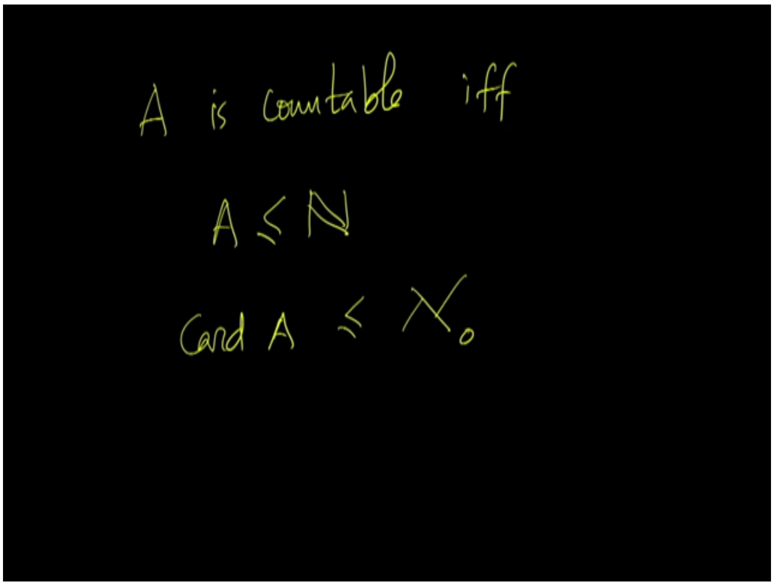

So, using this definition we can say that A is countable if and only if A is dominated by the set of natural numbers. In other words, the cardinality of A is less than or equal to the cardinality of aleph naught.

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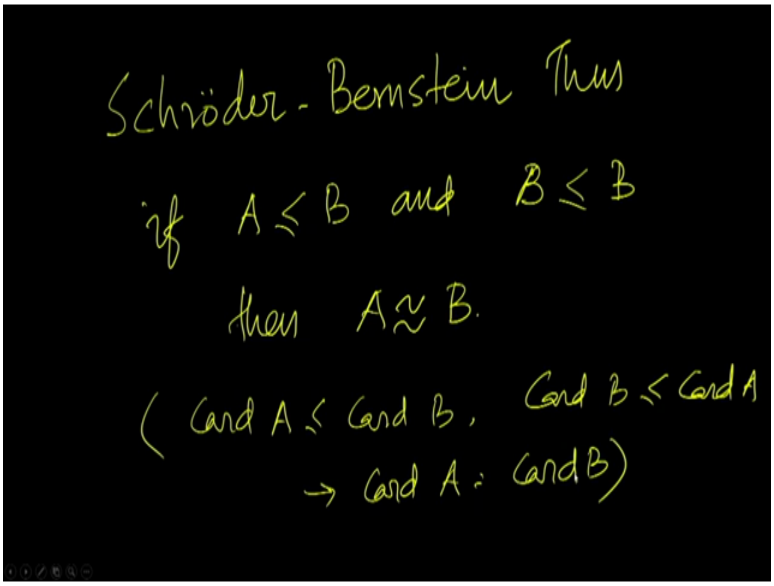

A famous theorem of set theory called Schröder–Bernstein theorem says that if B dominates A and A dominates B then A and B are equinumerous. In other words if the cardinality of A and B have this property that the cardinality of A is less than or equal to the cardinality of B and the cardinality of B is less than or equal to the cardinality of A then the two have identical cardinalities.

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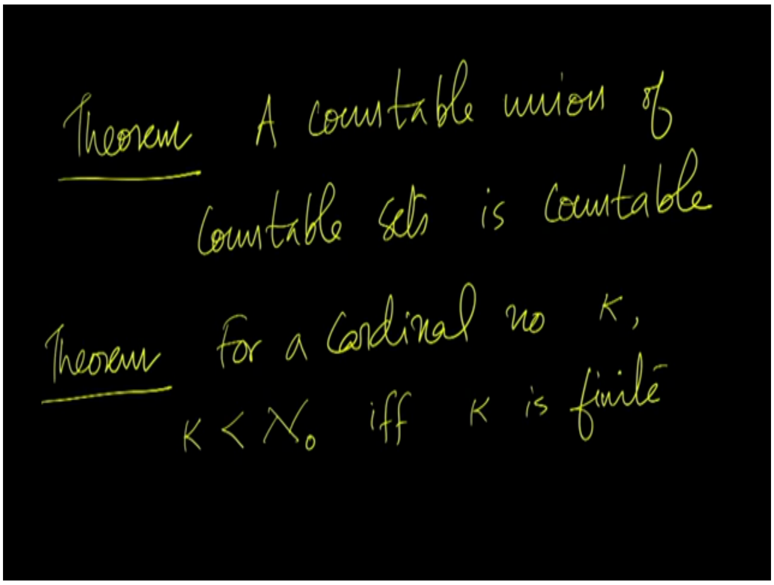

Related theorem is that A countable union of countable sets this count. Another theorem asserts that for a cardinal number kappa, kappa is less than aleph naught if and only if kappa is finite. In other words, in a sense the set of natural numbers is the smallest infinite set.

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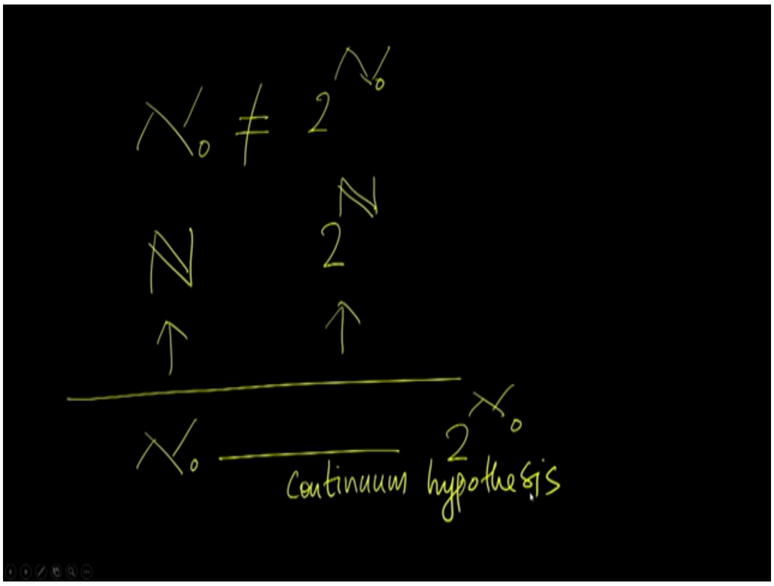

So, we know that aleph naught and 2 power aleph naught are not identical, aleph naught is the cardinality of the set of natural numbers and 2 power aleph naught is the cardinality of the set of all real numbers. We know that this is a countable set and this is not a countable set. So, the two have different cardinalities but can they have a cardinality between these two? Cantor conjecture that, there is no set of cardinality between aleph naught and 2 power aleph naught, this conjecture was called the Continuum hypothesis.

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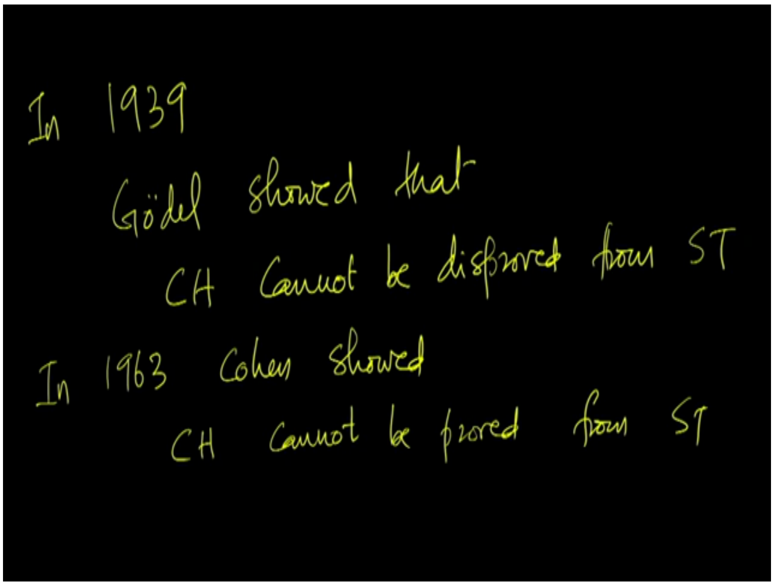

In 1939 Gödel showed that, the Continuum hypothesis cannot be disproved from the Zermelo–Fraenkel axioms of set theory. 1939, Gödel showed that Continuum hypothesis cannot be disproved from the axioms of set theory. In other words, the contradiction of continuum hypothesis cannot be proved. Many years later, in 1963 Cohen showed that Continuum hypothesis cannot be proved either.

So, the statement that is Continuum hypothesis that is there is no set with cardinality between aleph naught and 2 power aleph naught can neither be proved not be disproved from set theory.

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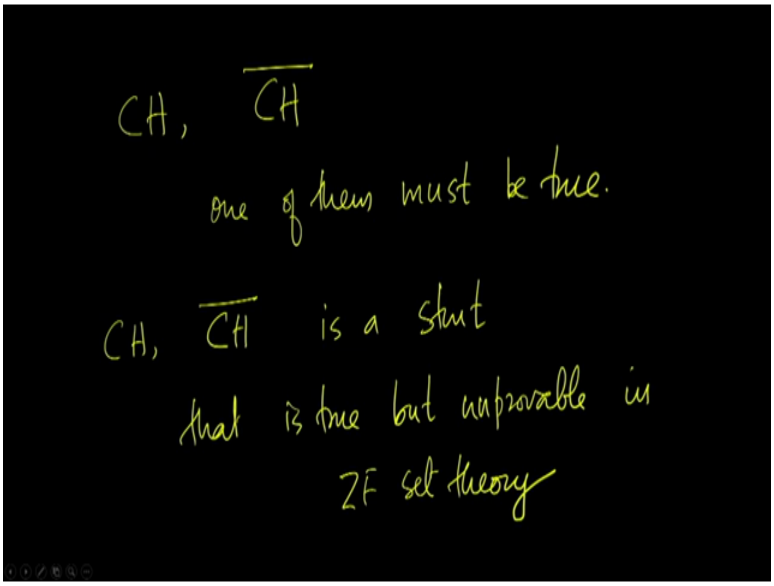

But if you consider the two statements, the Continuums hypothesis and it is contradiction, one of them must be true. Therefore either CH or CH bar is a statement that is true but unprovable in Zermelo–Fraenkel axiomatization of set theory. So, this is a statement which is true but unprovable in set theory. Hope to see you in the other lectures, thank you.