Discrete Mathematics

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Lecture 19

Embedding of the theory of rational numbers in set theory Paradoxes 2

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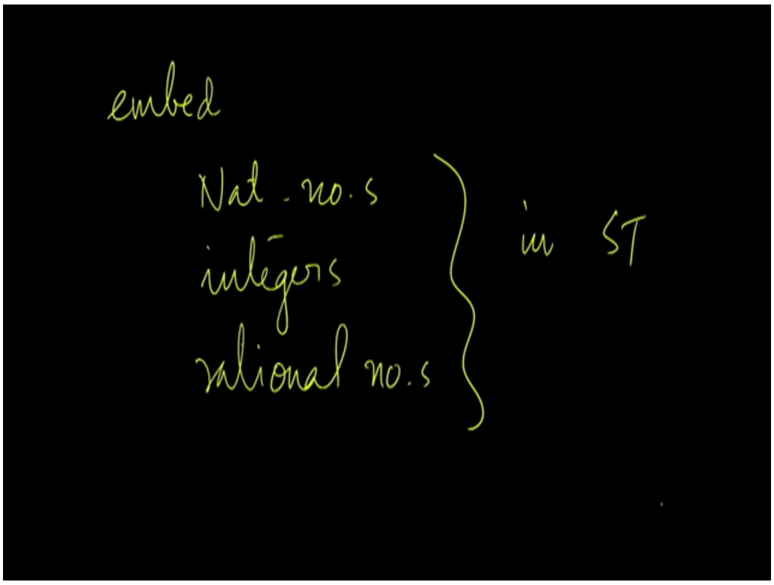

Welcome to the NPTEL mooc on Discrete Mathematics. This is the fourth lecture on Set Theory. In the previous lecture we have seen, how to embed the theories of natural numbers, integers and rational numbers in set theory.

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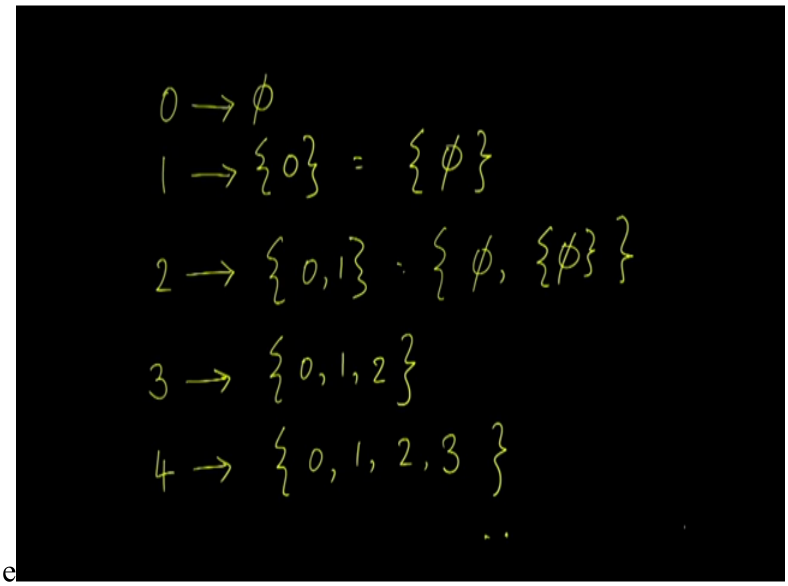

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In particular, we define the natural numbers, thus 0 was defined as the empty set, then 1 was defined as the successor of the empty set which turns out to be the singleton containing the empty set or in another words a singleton containing just 0.

The successor of 1 is 2 which turns out to be the two members set containing 0 and 1 or containing the empty set and the singleton containing the empty set and then the successor of 2 is 3 which is the set containing just three members 0, 1 and 2 and the successor of 3 is 4 which contains just four members 0, 1, 2, 3 in general the natural number N can be represented using the set containing 0, 1, 2, 3 etcetera up to N minus 1, that is natural number 0 to N minus 1 will form the set which has named natural number N.

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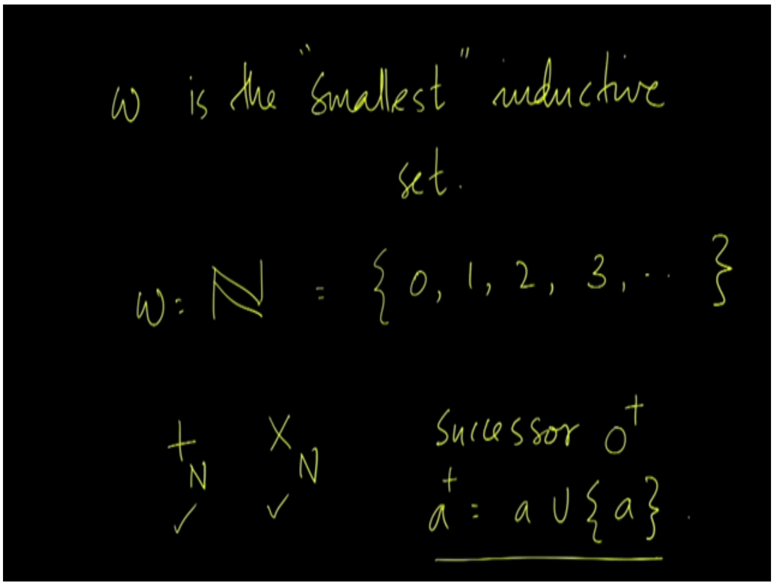

So, every natural number in the sense is defined as a set. We define the notion of inductive sets, an inductive set is a set which contains the empty set and is closed under the successor operator. The successor operator on set A gives us the set containing the which has a union of A and the singleton containing A. Then omega turns out with the smallest inductive set, omega is an inductive set, in itself and this omega is defined as the set of all natural numbers.

So, as you can see omega contains the empty set which is 0, its successor 1, its successor 2 and so on. So, by necessity because it should be closed under the successor operator it should contain all these 0, 1, 2, 3 etcetera. So, omega is defined as the set of natural numbers. So, every natural number is now a set and then we define operations on natural numbers as Set Theoretic Operations.

For example, addition on natural numbers is defined in terms of the successor. For example, the sum of N and M plus 1 is the successor of the sum of N and M, so we are defining the addition, the sum on N and M plus 1, recursively use in the sum of N and M a smaller number and the successor operator.

So, affectively addition is defined in terms of the successor operator and then multiplication in the same way can be defined using addition the product of N and M plus 1 is the product of N and M plus N, so we define multiplication using addition. So, this way the operators on natural numbers, operations on natural numbers could be define as Set Theoretic Operations.

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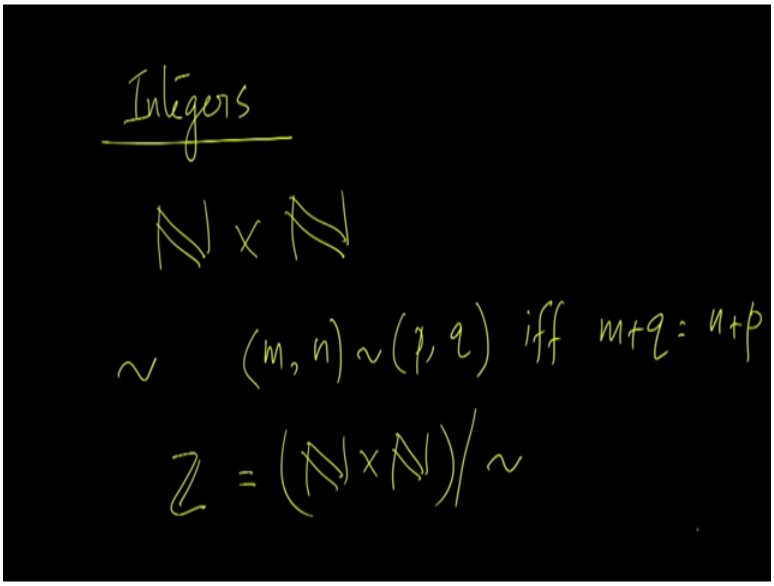

Then we went on to embed the theory of integers on the set of; in set theory, we consider the cross product N cross N where N is the set of natural numbers. We consider as a subset of this, in particular we consider a relation tilde which is define in this manner. Ordered pair M, N is in relation till day with P, Q if and only if M plus Q equals N plus P. The idea is that M mins N should equal P minus Q and then the set of integers Z is defined as the set of equivalent classes under this operation, this relation tilde that is the quotient of N cross N with tilde is what the set of integers is.

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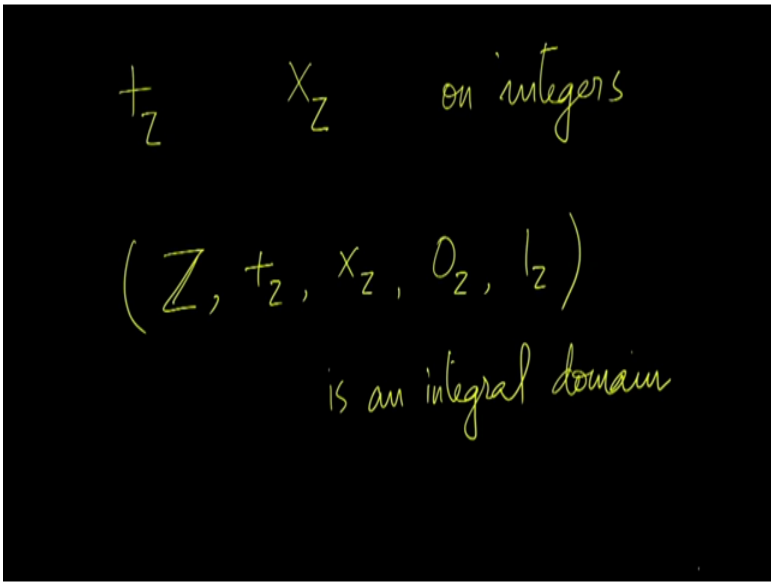

And then we define operations on integers as operations on such equivalence classes, such an equivalence classes in a integer, so operations on integers should be translate into operations on such sets. So, we define plus Z and into Z appropriately. So, along with these operations, Z with integers 0 and 1 forms an integral domain.

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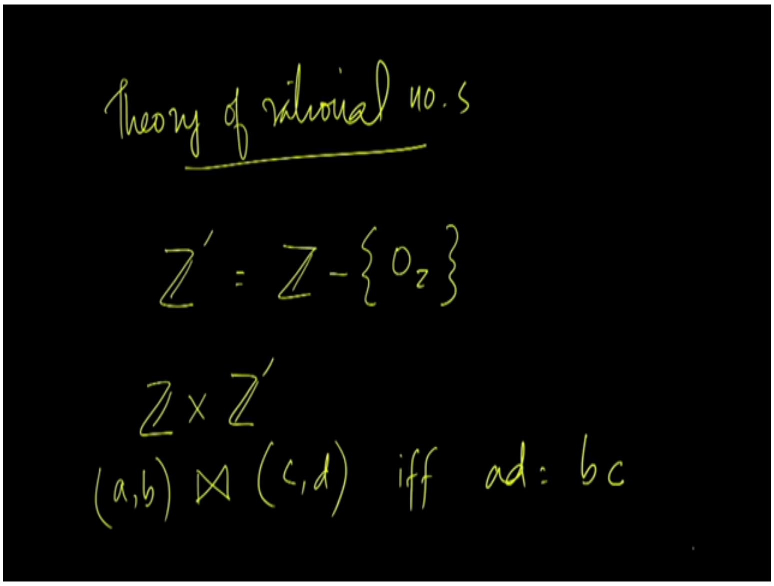

Then we went on to the theory of rational numbers and so, how to embed the theory of rational numbers in set theory. In particular, we consider the set Z prime which is Z with (()) (04:41) 0 and then we consider the cross product Z cross Z prime and we define a relation ordered pair A, B join ordered pair C, D if and only if AD equal to BC. The idea is that A by

B should be the same as C by D. That is fraction A by B should be the same as the fraction C by D.

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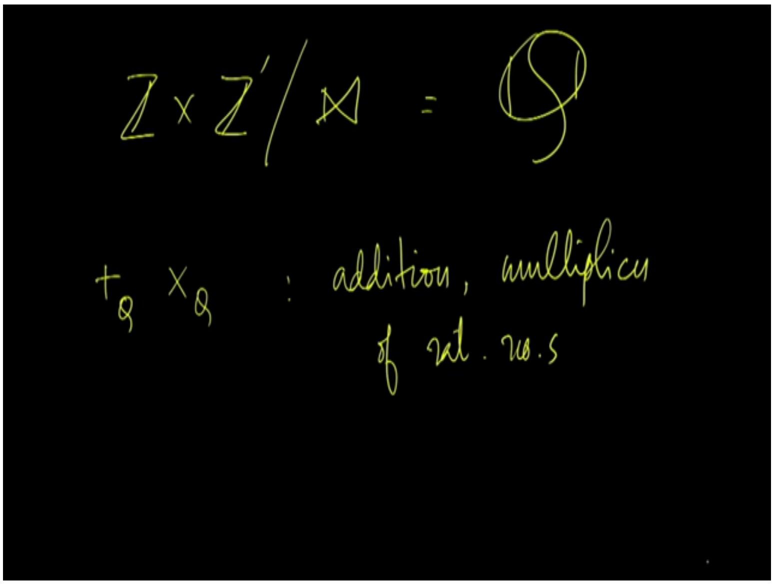

Then the set of all equivalence classes of this relation is defined as the set of all rational numbers. In other words Z cross Z primes questioned with the joint operation is called the set of all rational numbers. So, a rational number is a equivalence class under this relation and then the addition operation and the multiplication operation were appropriately defined consisting with our notions of rational numbers addition and multiplication.

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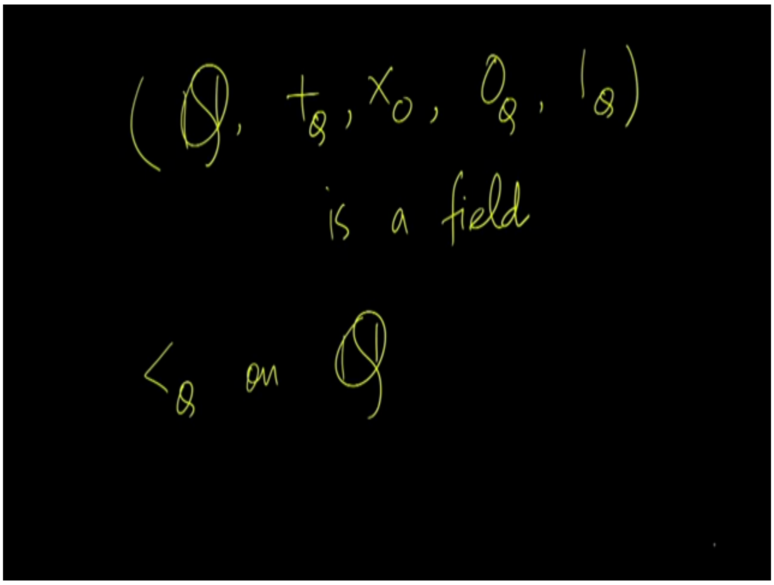

And then we found that the set of rational numbers along with these two operations addition and multiplication thus define and this Q here and 0 and 1 is a filed and we define the less than relation as a linear order on Q.

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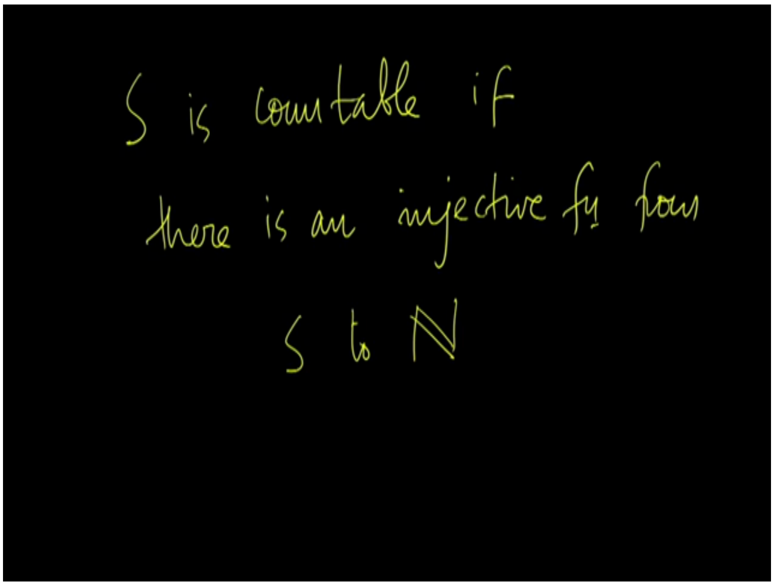

We say that the set S is countable, if there is an (injection) injective function from S to N that is the members of S could be counted using natural number. So, you could say this is the 0th member, this is the first member, this is the second member and so on.

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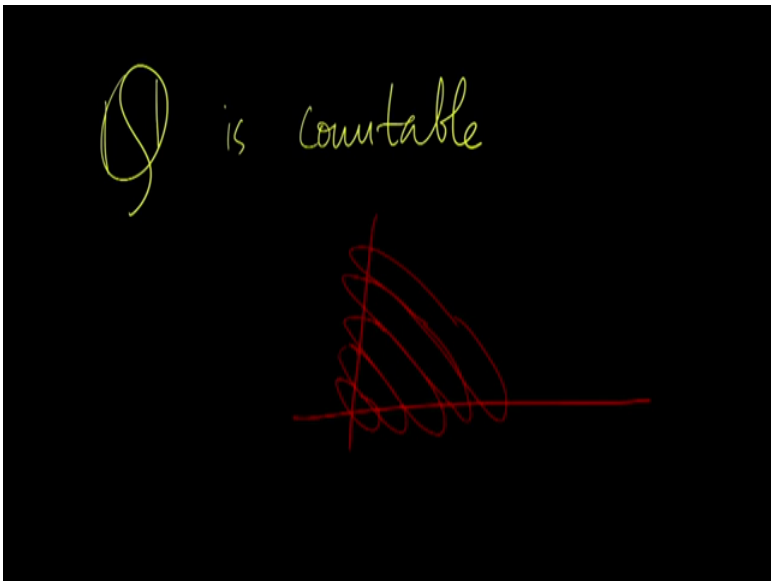

We saw that the set of rational numbers is countable, in particular when you consider all ordered pairs of natural numbers this is countable, that is if you consider the first quadrant then you could count the natural numbers in this order. That is counting (them) order pairs belonging to one diagonal at a time, you can, you can count all of them. Extending this notion you can show that every rational number, the set of all rational numbers is countable.

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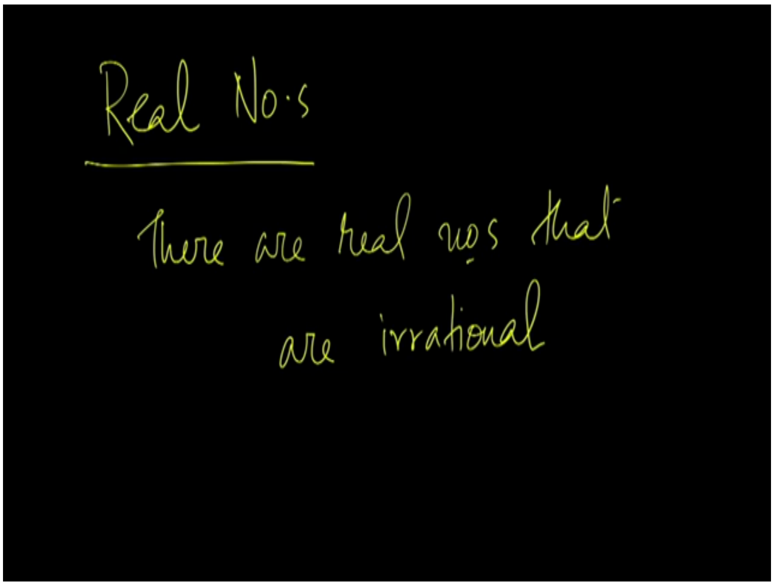

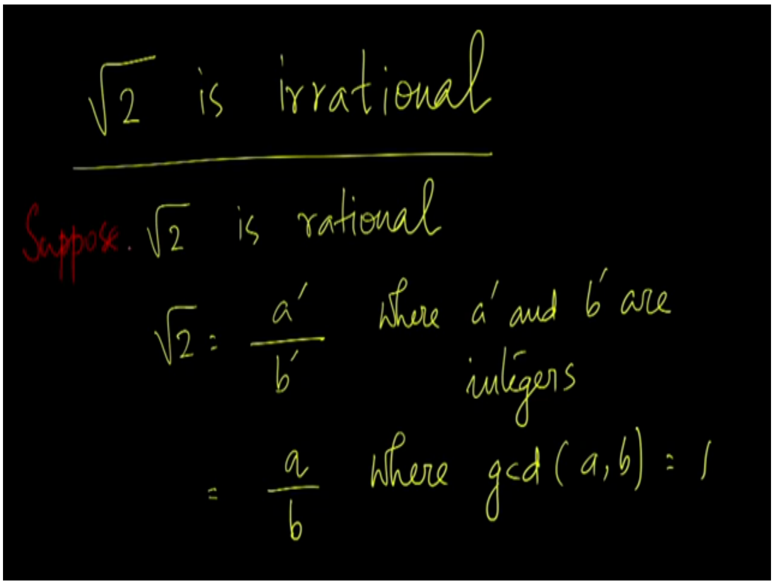

Now, we come to real numbers. Of course we know that there are real numbers that are not rational or irrational. In particular, root 2 is irrational. It has been known for long that root 2 is irrational, the proof goes thus, assume the contrary, so suppose root 2 is rational, so here we suppose is that root 2 is rational. If root 2 is rational then we would be able to represent root 2 as a fraction A prime by B prime, where A prime and B prime are integers.

So, consider this fraction A prime by B prime, out of A prime and B prime we can remove the common factors and get this fraction in reduced form. Let us say A by B is the reduced form. What I mean is that A and B do not have a common factor or that gcd of A and B is one.

So, if root 2 is a rational number then the (())(07:24) A and B so, that root 2 is A and B and gcd of AB equal to 1

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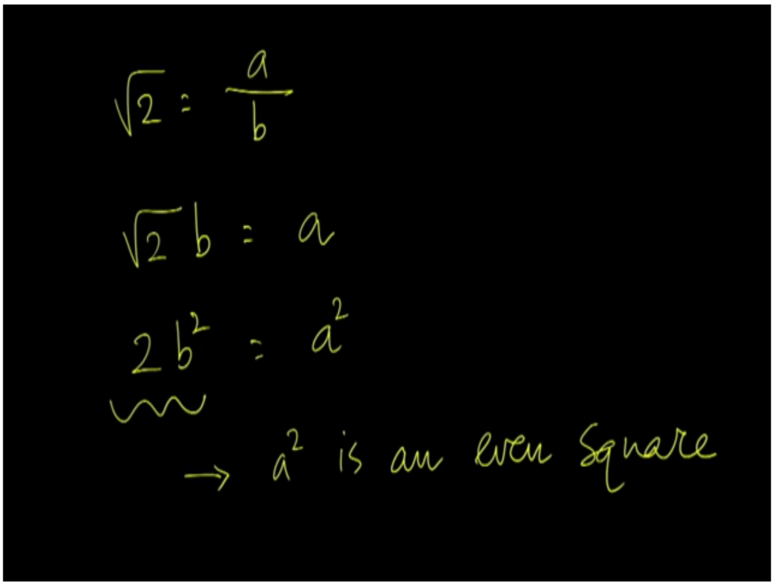

So, root 2 is A by B. If root 2 is A by B then root 2 B is A, squaring both sides we have 2 B square equal to A square. So, A square is the square that is what we have on the right hand side. On the left hand side, we have 2B square which is an even number. It is 2 multiplied by something, which means A square is an even square.

We know, that an even square is the square of an even number. The square of an odd number is always odd. Therefore, if A square is an even number then A is also an even number.

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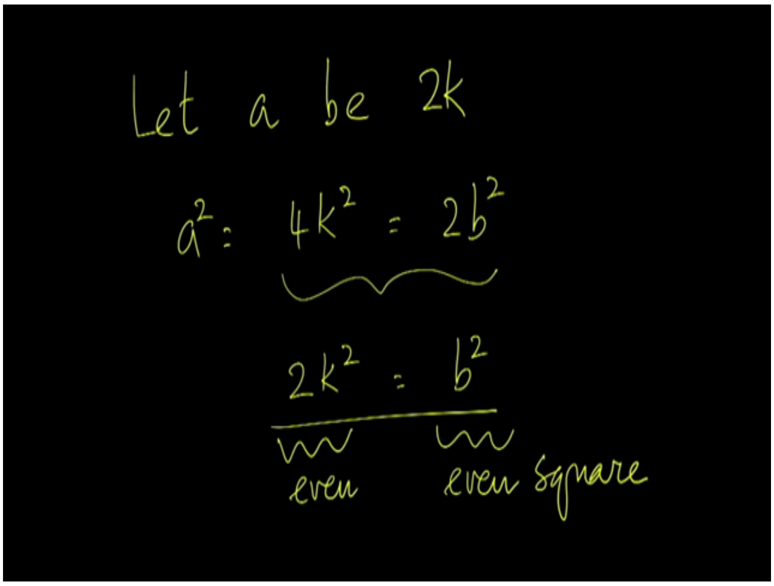

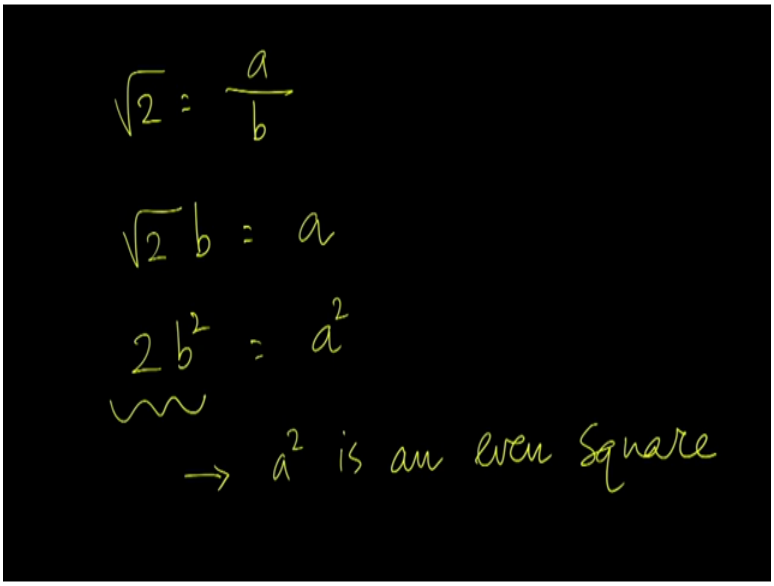

So, A is even. So, let A be 2K. Then substitute it in this equation. We have A square equal to 2B square. So, A squared equal to 4K square, which is equal to 2b square. So in particular, let us consider this equation 4K square is equal to 2B square. So, 2 is common factor on both sides. So, let us cancel 2 from both sides, so we have 2K square equal to B square. Now, 2K square is even therefore, B square is even as well, which means B square is an even square.

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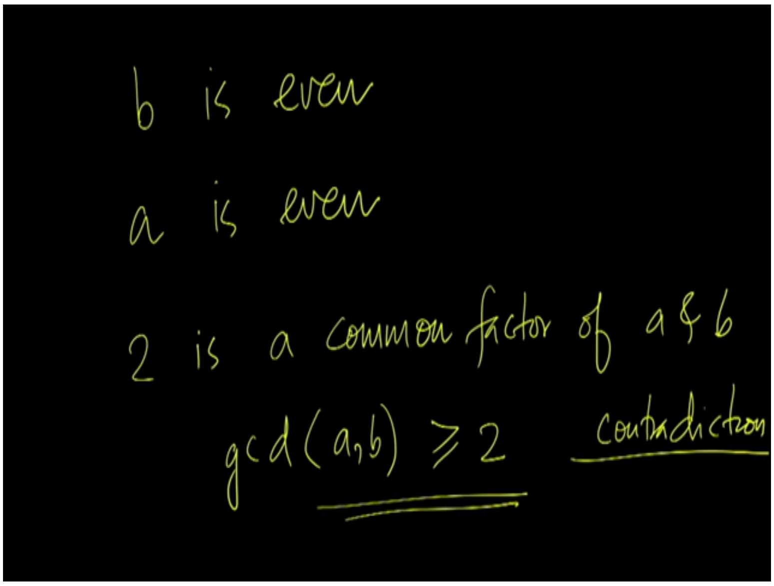

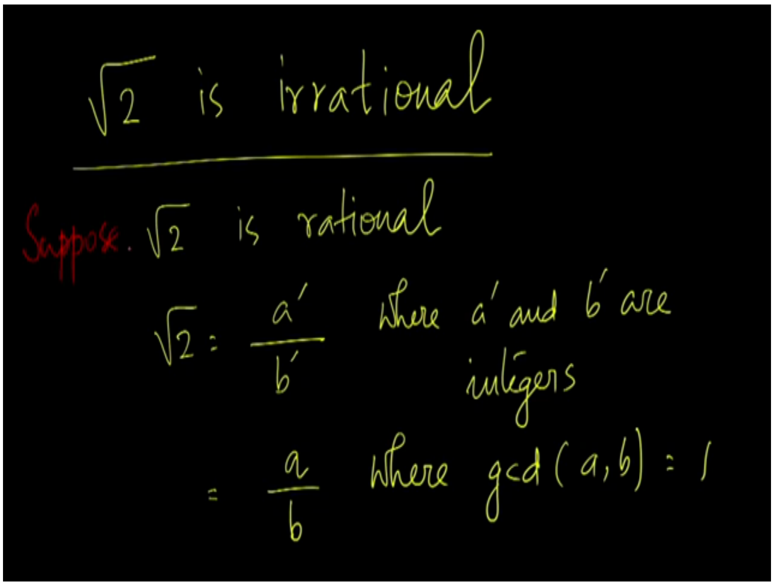

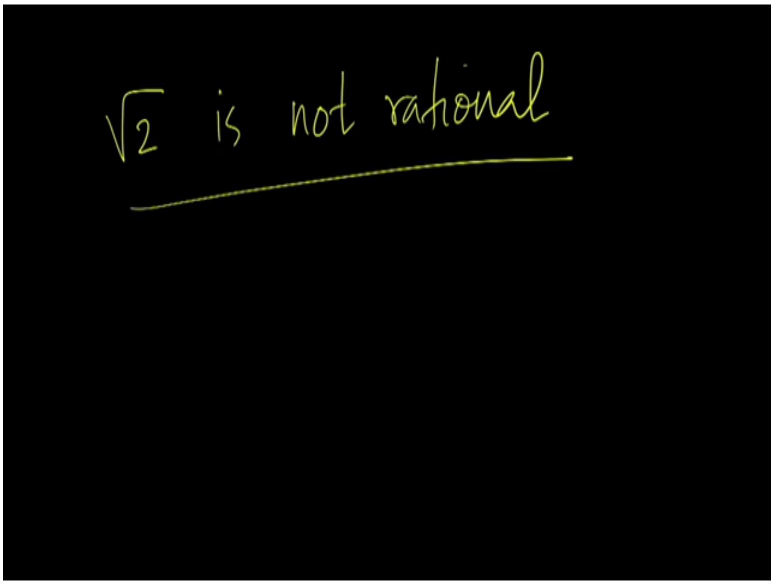

Which means B is even, but then we had earlier found that A is even, so A is even and B is even which means 2 is common factor of A and B or in other words gcd if A B is greater or not equal to 2, at least 2 is a common factor which is a contradiction because here, we assume that gcd of A B is equal to 1. Root 2 has been written in the reduced form A by B. So, A and B do not have a common factor, but here we find the 2 is a common factor, which is a contradiction. Therefore, root 2 cannot be rational, root 2 is irrational. So, there are irrational numbers.

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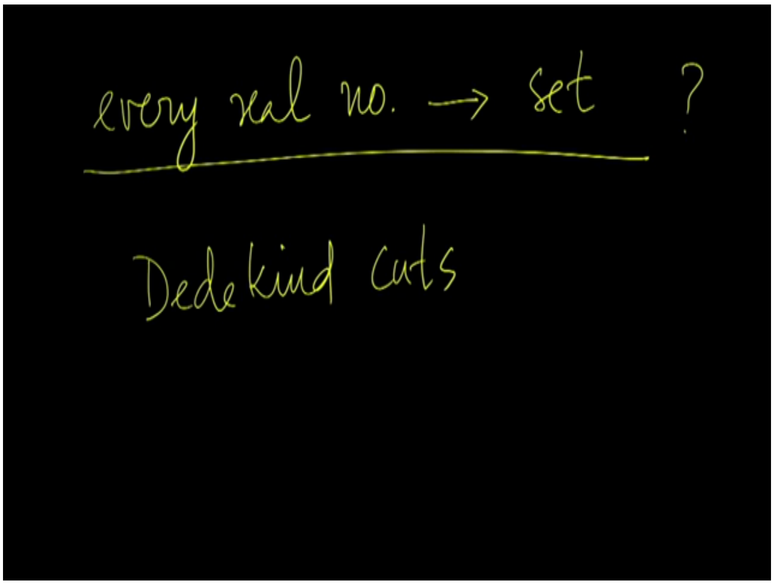

So, there are real numbers which are not rational. Now, we are trying to embed the theory of real numbers in set theory. So, what we need is that every real number should be constructed as a set. The sets that we construct are called 'Dedekind Cuts'.

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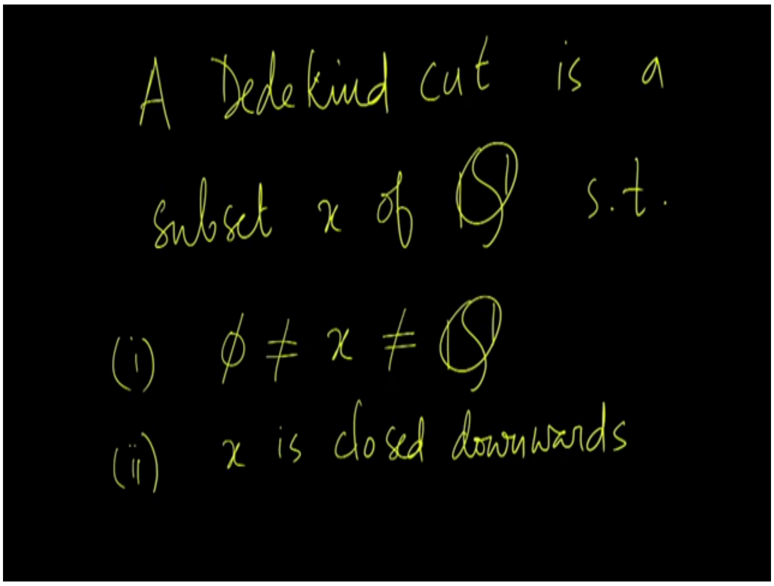

We define a Dedekind cut thus a subset X of Q. So, X is a set of rational numbers such that X is not empty and X is not Q. X is closed downwards and there is one more condition which is that X has no largest member. Such a set is called a 'Dedekind cut'. So, once again a Dedekind cut is a set of rational numbers but not every set of rational numbers is a Dedekind cut. In particular, a Dedekind cut cannot be the empty set, it cannot be Q either and a Dedekind cut has to be closed downwards.

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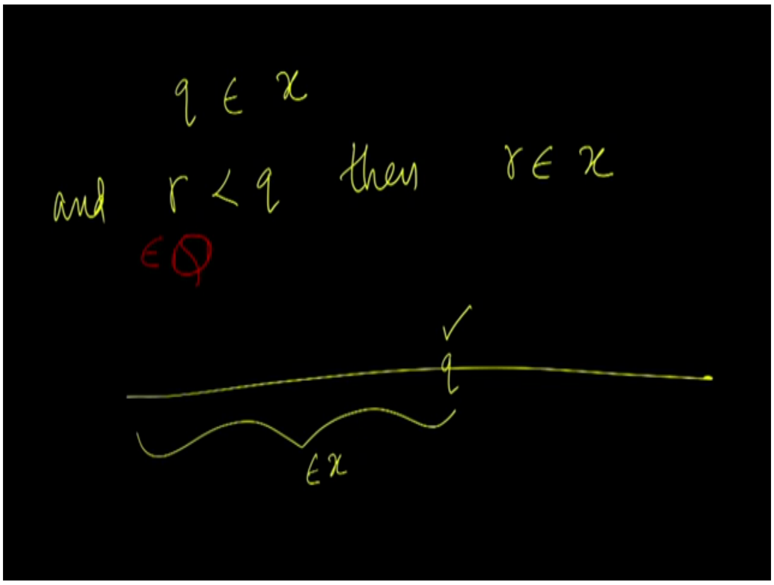

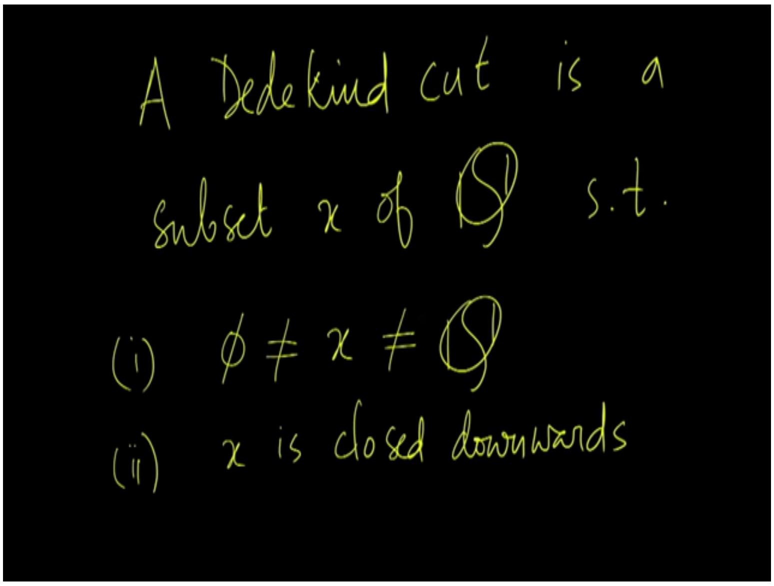

What it means is this, if X is a Dedekind cut then X is a set of rational numbers. Let us say Q is a member of X, so Q is a rational number which belongs to X and let us say R is less than Q and R is rational number which is less than Q. So, let us say R is a rational number which is less than Q. In that case R belongs to X.

So, what we mean is this. When we say that X is closed downwards, we mean that for every rational number Q which belongs to X and for every rational numbers R which is less than Q, if Q belongs to X then R belongs to X.

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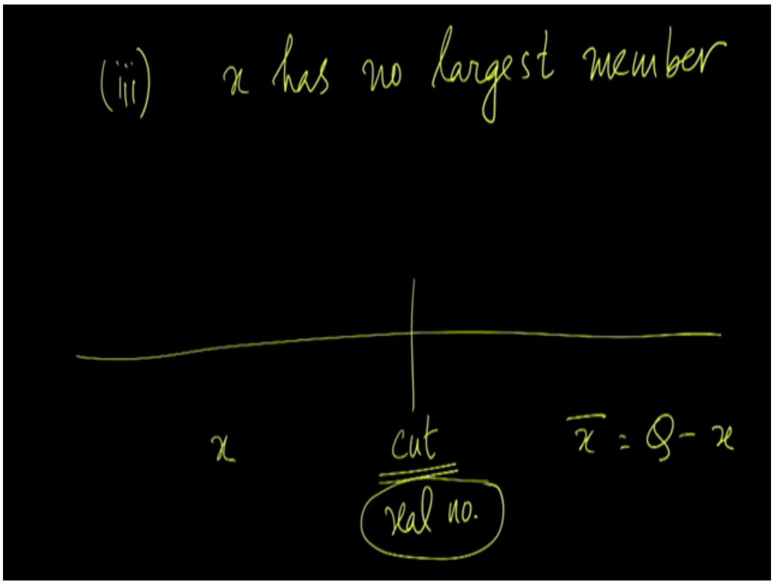

And we say that X has no largest member. So, such a set is a Dedekind cut. So, we can say that, when you consider a set of all rational numbers, the Dedekind cut X divides it into two sets. X is one set and the complement of it X bar is the other set, that is Q minus X is X bar, the relative complement. It partitions the set of all rational numbers into 2.

So, on the smaller side we have X, on the higher side we have X bar. Every rational number belonging to X bar is larger than every rational number belonging to X. X has no largest member, X bar may or may not have a smallest member. Such a partition of the set of rational numbers is affected by a Dedekind cut.

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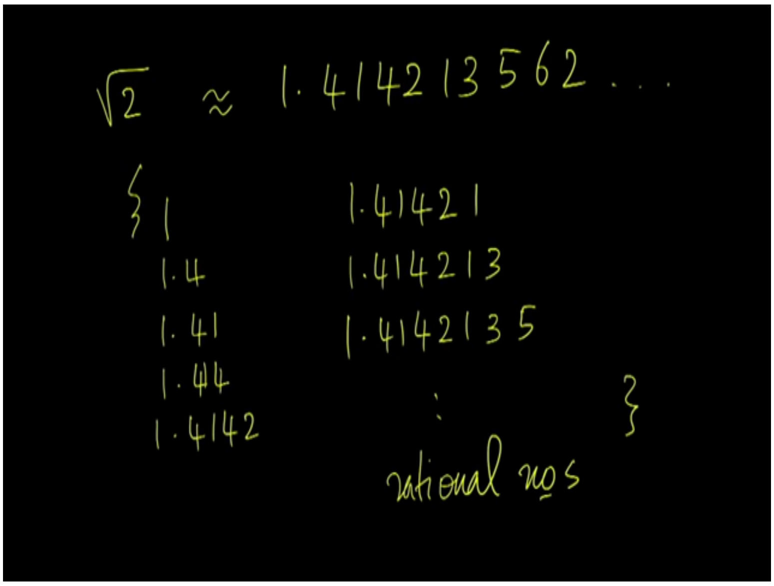

In a particular, let us consider the (rational number) irrational number root 2, which we have just shown to be rational. So, a decimal approximation for root 2 is this 1.414213562 etcetera. In particular, if I consider a set of rational numbers containing 1, 1.4, 1.41, 1.414 etcetera. These are all rational numbers that are smaller than root 2. So, the Dedekind cut corresponding to root 2 will contain all this rational numbers. It is a super set of all these.

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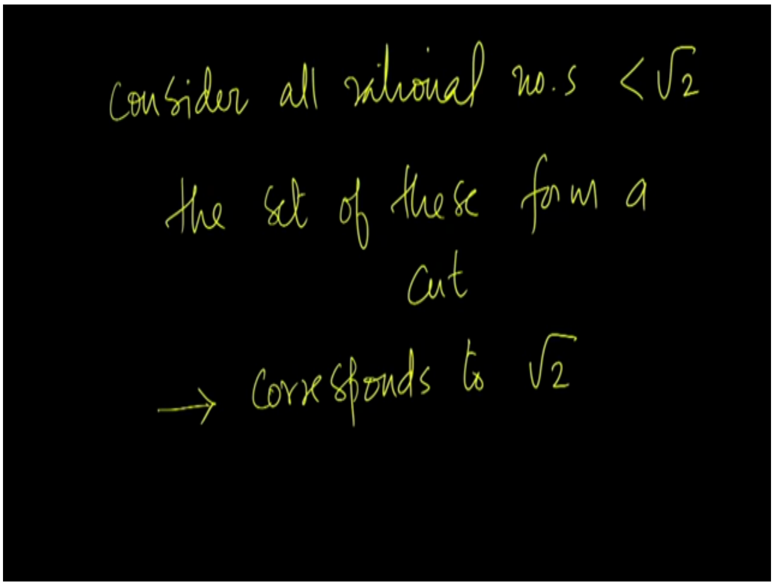

We consider all rational numbers that are less than root 2. The set of these form a cut. This cut is what we equate with root 2, that is the real number root 2 is equated to this cut. So now, every real number becomes a set. In particular, it becomes a Dedekind cut

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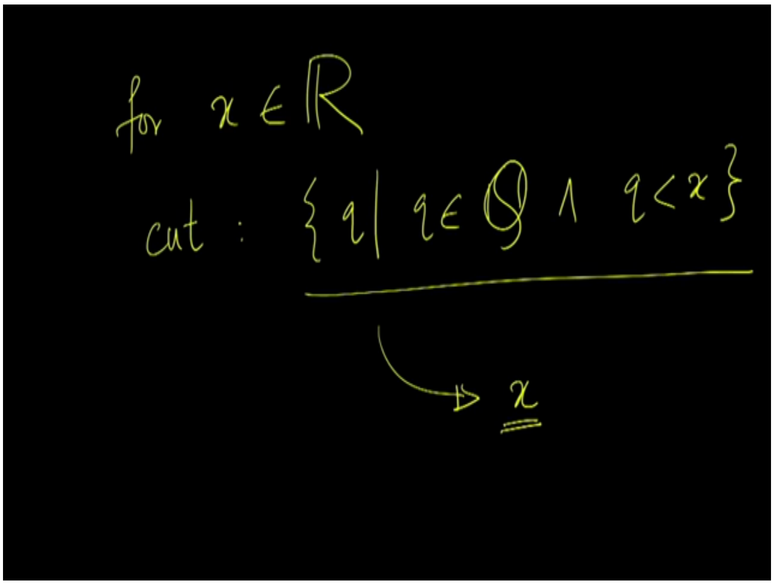

For every real number X the cut that is the associated to X is a set of all rational numbers, that are less than X but of course the cut cannot be defined in this manner because here, X is a real number. So, the cut is define only using rational number as we saw earlier.

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So, the set of all real numbers is now, the set of all Dedekind cuts. Every real number is Dedekind cut.

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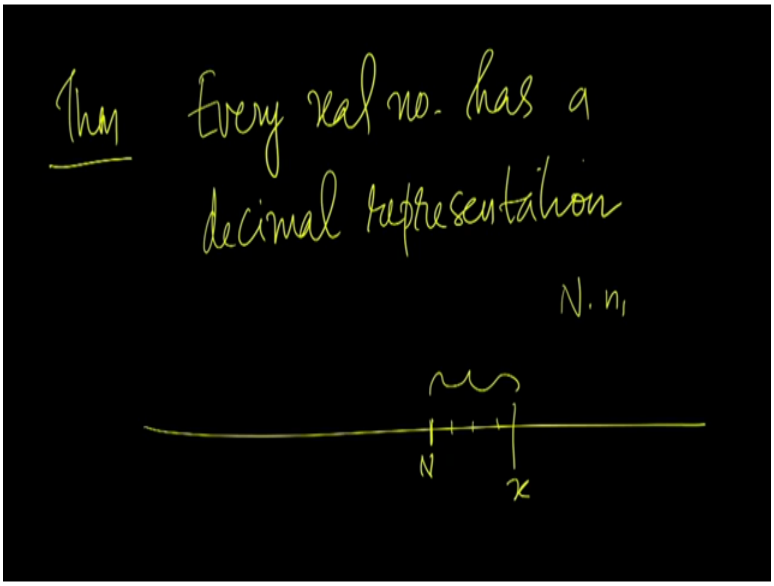

It can be shown that every real number has a decimal representation. For example, when you are given real number X, consider the X on the real line and then consider the largest integers smaller than X. Suppose that is N then N is the integer approximation to X that is X floor will be N.

Then consider the portion from (N)the portion on the real line from N to X. This segment has a length less than 1 divide this into 1 tens. The number of 1 tens from N to X will be the next digit, suppose that is N1. So, N1 will be the digit in the tenth place and so on. So, you can construct the decimal representation of the real number in this manner.

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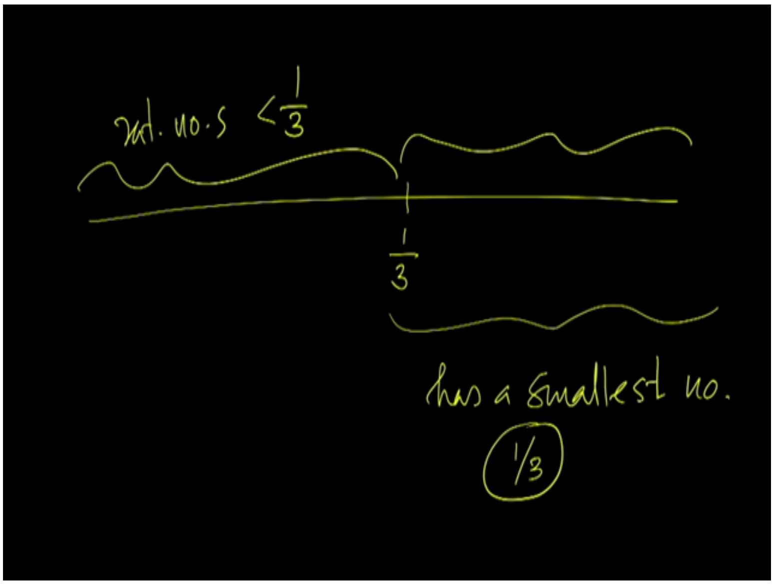

Consider the rational number 1 by 3, a rational number is also a real number. Therefore, we have a Dedekind cut associated with 1 by 3 as well. The set of all rational numbers less than 1 by 3 will form the Dedekind cut that is associated with the real number 1 by 3, then its compliment X bar in this case has a smallest number which happens to be 1 by 3 itself.

Therefore, if the complement of a Dedekind cut has a smallest member then that Dedekind cut corresponds to a rational number, if it does not have a smallest member then it corresponds to an irrational number.

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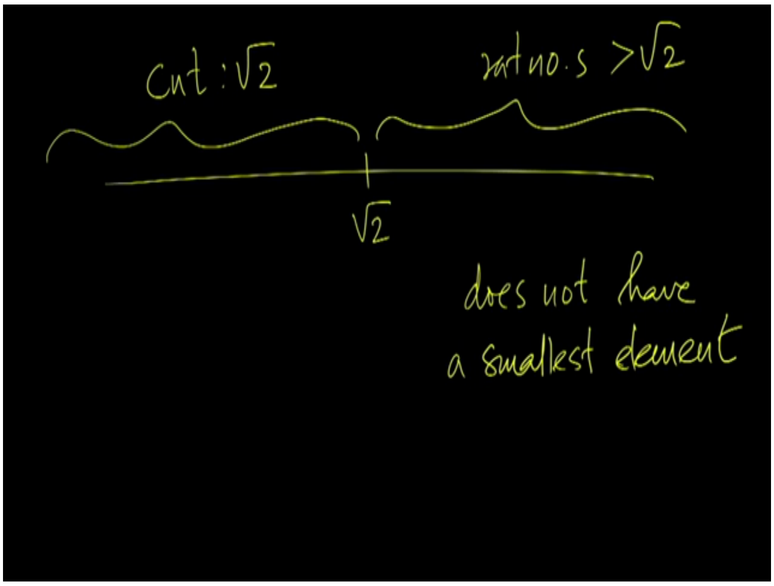

In particular, for root 2, the cut corresponding to root 2 is the set of all rational numbers less than root 2 and its compliment is a set of all rational numbers greater than root 2, since root 2 is not rational it will not belong to either set. So, the compliment in this case we see does not have a smallest element.

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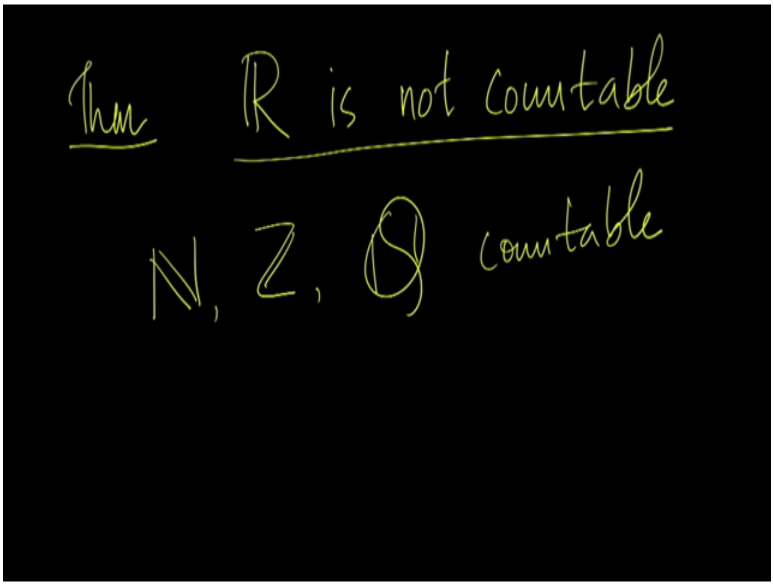

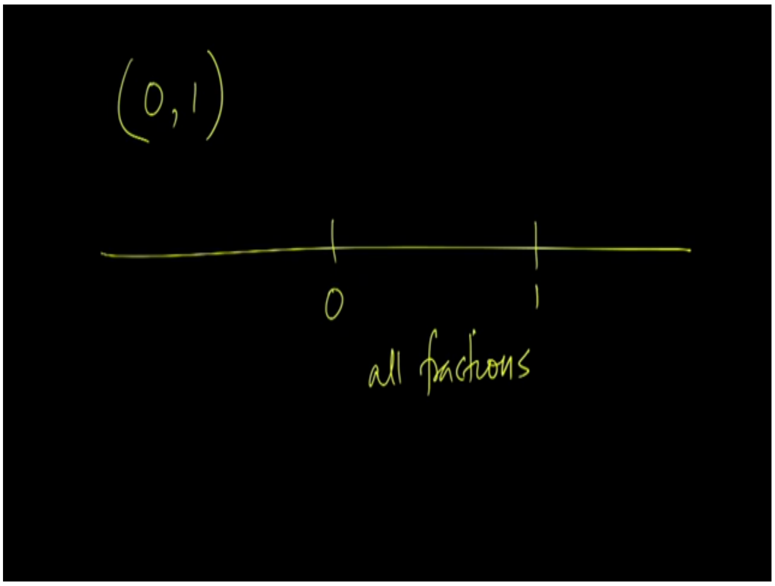

We can show this is a set of all real numbers, R is not countable. We saw earlier that the set of all natural numbers, the set of all integers, the set of all rational numbers are all countable, but the set of all real numbers is not countable. So, let us prove this now.

In particular, let us consider the part of the real line that is with 0 and 1 excluded. So, we consider the interval 0, 1 opened at both ends, that is we are considering all real fractions.

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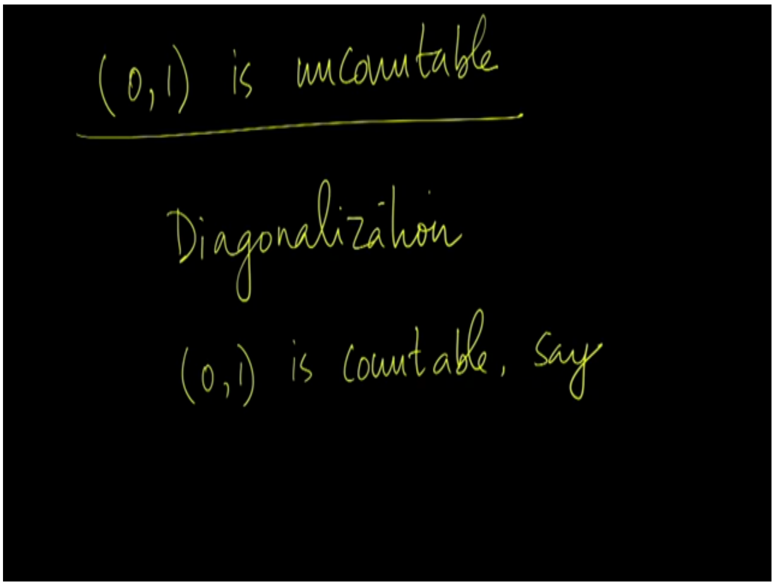

We will show that 0, 1 is uncountable. You see a technique called 'Diagonalization'. If 0, 1 is uncountable, then its super set are also should be uncountable. So, let us assume that 0, 1 is countable, that is the set of all real fractions is countable let us say and we will derive a contradiction.

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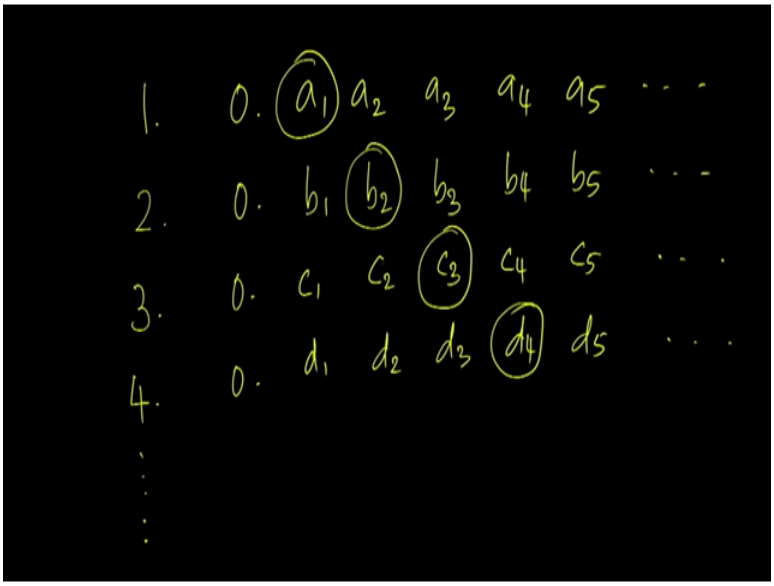

If this is countable then the (())(16:54) 121 into mapping from the set of real fractions to the set of natural numbers. So, you could say this is the first fraction, this is the second fraction, this is the third fraction, this is the fourth fraction and so on. So, there is a enumeration of fractions. So, let us say we have this enumeration. So, let us say 0. A1, A2, A3, A4 etcetera is the first fraction. So, A1, A2, A3 etcetera are all digits. This is the decimal representation and the second fraction is 0. B1, B2, B3 etcetera.

Then in the diagonalization technique we pick the diagonal digits, from the first fraction we pick A1, from the second fraction we pick B2, form the third fraction we pick C3 and so on. In general, from the ith fraction we pick the ith digit.

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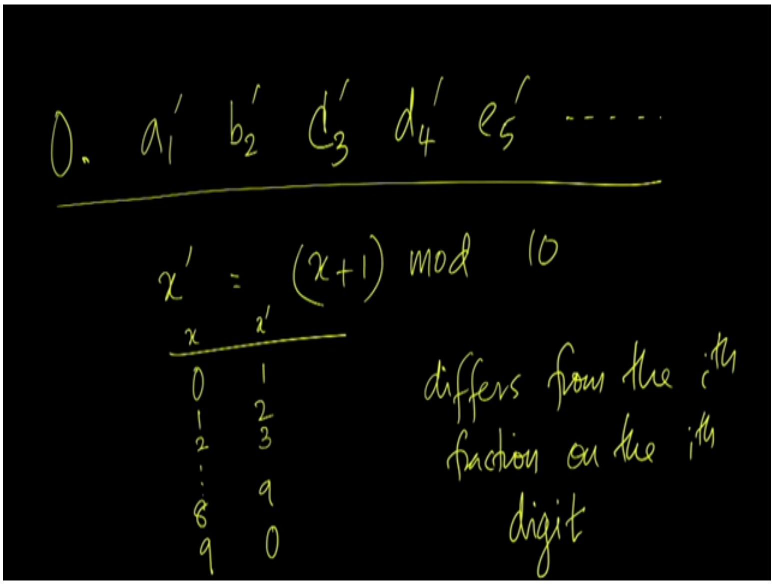

So, after picking digits in this fashion, we form a new fraction, which we write thus 0. A1 prime, B2 prime, C3 prime, D4 prime etcetera. Here X prime is defined as X plus 1 mod 10. For example, if X is 0, X prime is 1, if X is 1 then X prime is 2 and so on and when X is 9, X prime is 0.

So, we can see that X prime certainly differs from X. So, here what we do is this, we construct a new fraction which we write in this manner 0. A1 prime, B2 prime, C3 prime etcetera. So, this fraction differs from every single fraction in enumeration.

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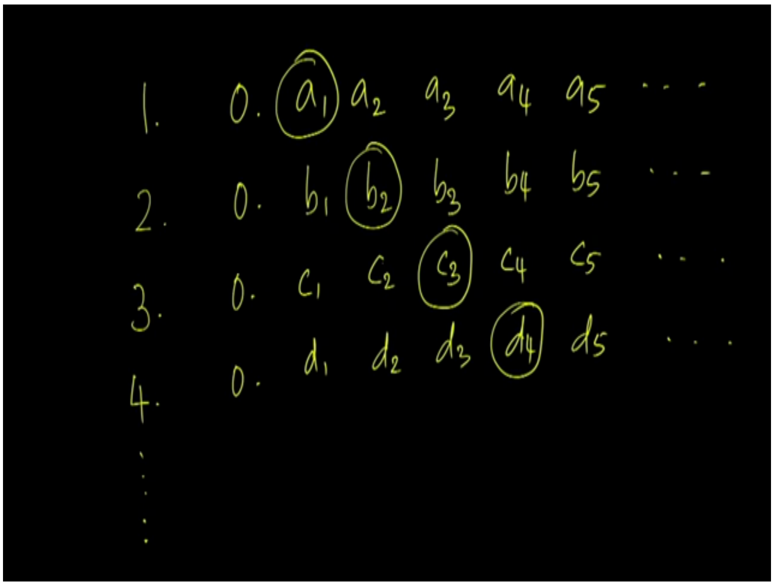

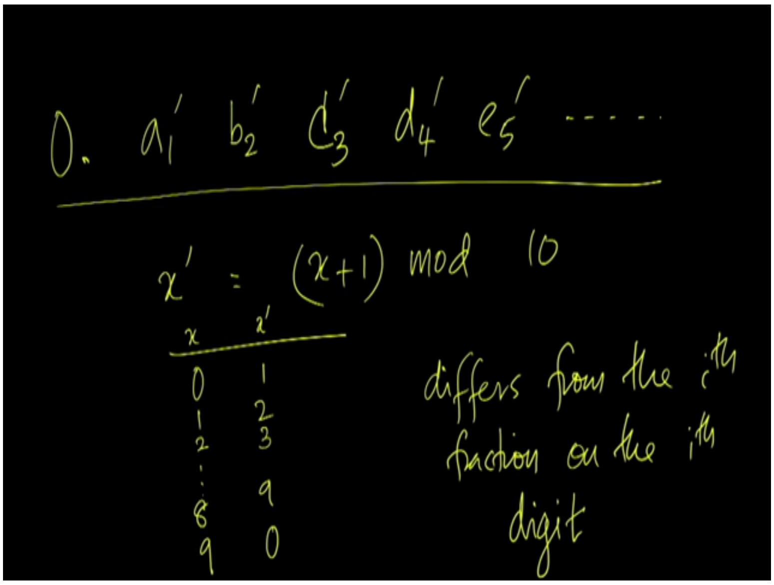

It differs from the first fraction. In the first fraction, we have A1 in the first position, whereas in the new fraction that we have constructed we have A1 prime at the first position. It differs from the second one because in the second one we have B2 in the second position but we have B2 prime in the new fraction. It differs from the third fraction and the third position. In general, it will differ from the ith fraction in the ith digit.

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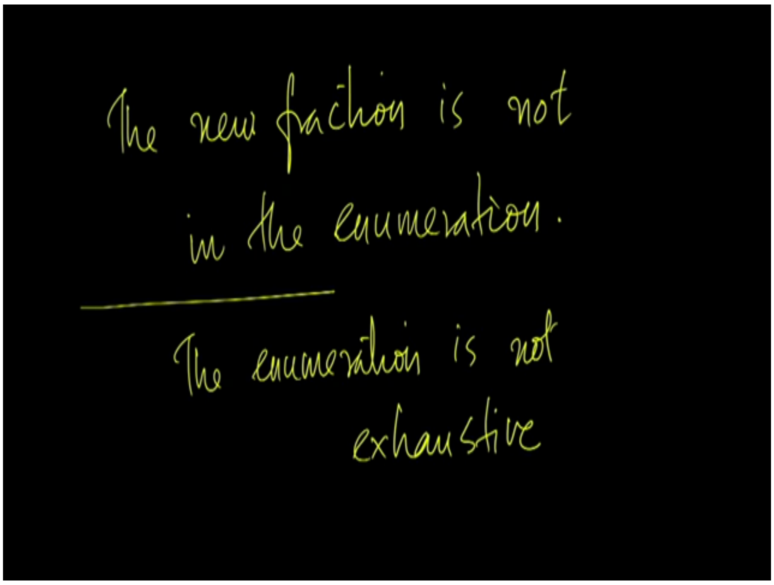

So, the new fraction is not in the enumeration. In other words, the enumeration that we had, the hypothetical enumeration that we had is not exhaustive. So, that is the contradiction. We assume that the set of all real fractions was enumerable and therefore we assume this a numeration was exhaustive.

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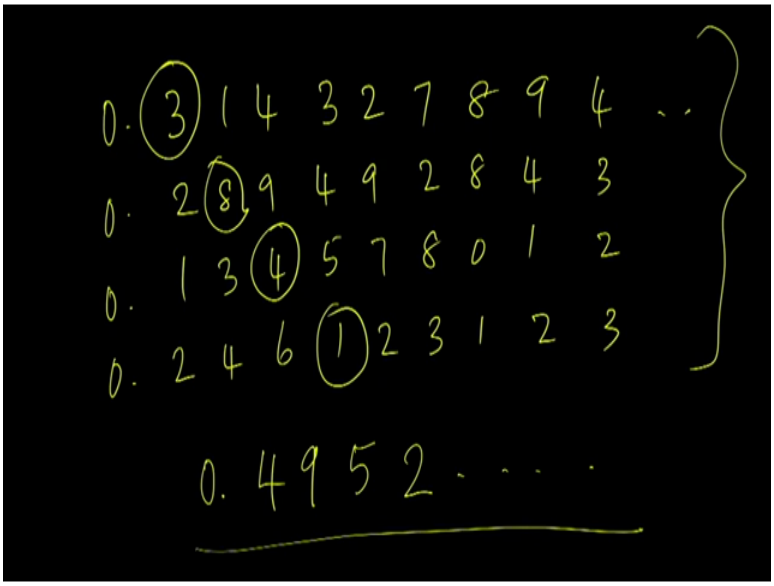

So, here is an example of the construction. So, if the fraction said, we had were like this then the first fraction here has 3 in the first position, so in the new fraction that we construct we will write 3 plus 1, 4. We had 8 in the second position of this second fraction. So, we would write 9 in the second position of the new fraction. We had 4 in the third position of the third fraction, so we will write 5 in the third position of the new fraction and so on.

So, the new fraction that we construct does not match any of the existing fractions. So, this technique is called the ‘Diagonalization technique’.

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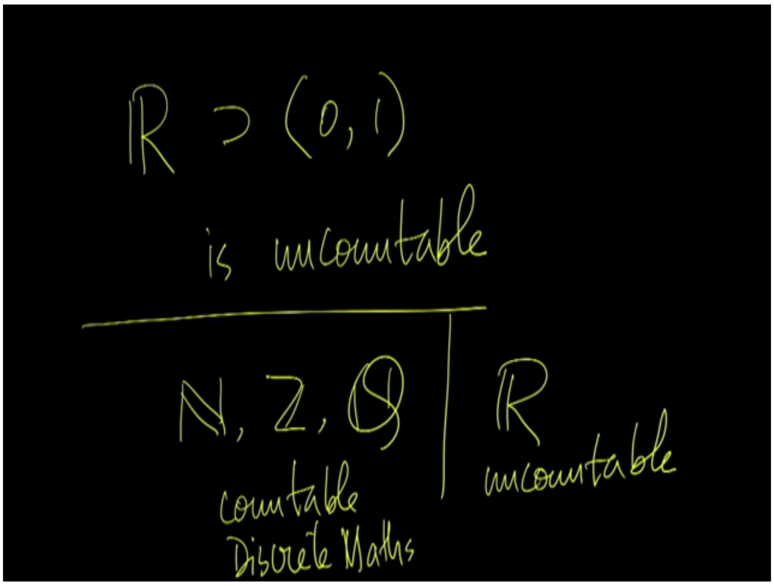

Since, the set of all real numbers is a super set of the set of real fractions 0, 1 R is also uncountable. So, what we find this that is the set of natural numbers N, the set of integers Z, the set of rational numbers Q are all countable, but the set of al real numbers is uncountable set. Discrete mathematics deals with the countable sets.

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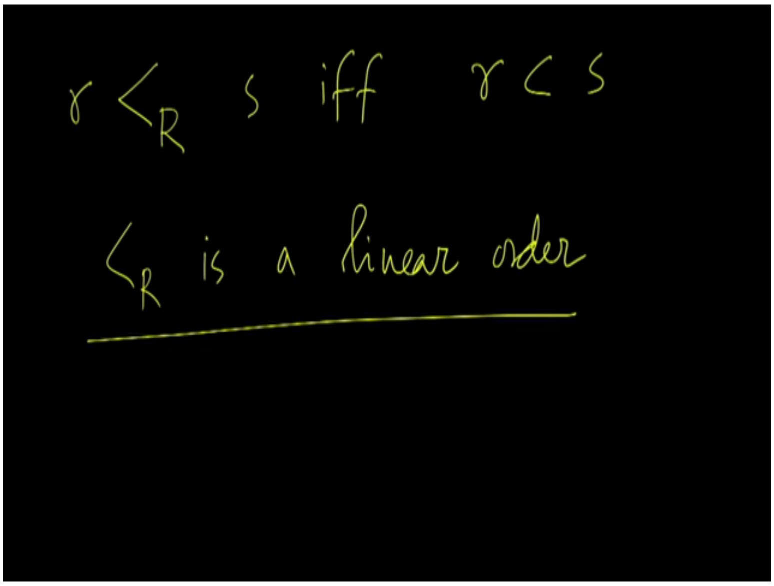

Now, we define a linear order less than relation between real numbers in this manner, we said that a real number R is less than real number S, if and only if R is the subset of S. So, R is a Dedekind cut here, which is a set and S is also a Dedekind cut. We say that R is the proper subset of S that is precisely when R is less than S. So, when the less than relation is defined in this manner, we can say it is a linear order.

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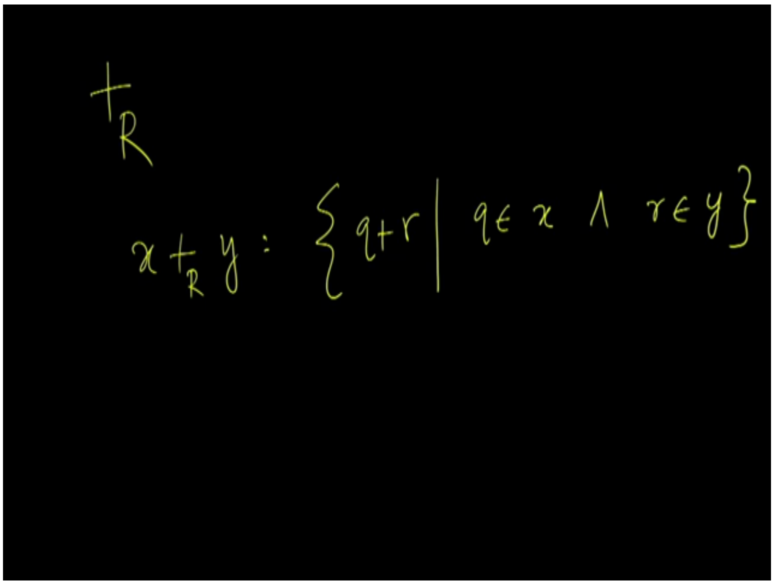

Then, we define the addition operator. The real number addition operation is defined this, for real numbers X and Y, X plus Y is defined as the set of all rational numbers Q plus R so that Q belongs to X and R belongs to Y. Here, X and Y are treated as Dedekind cuts.

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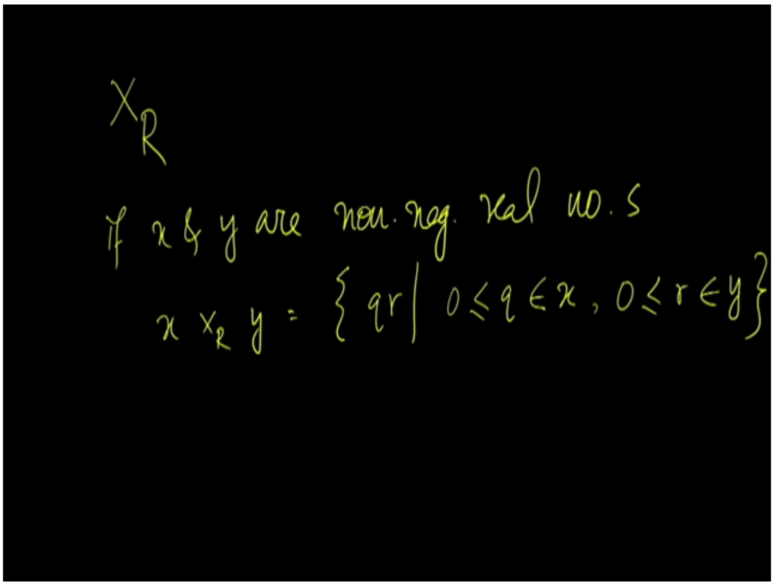

Similarly, the multiplication operation is defined like this for non negative real numbers X and Y. The product of X and Y is defined as the product of all rational numbers Q and R. So that Q is greater than or equal to 0 and belongs to X and R is greater than or equal to 0 and belongs to Y.

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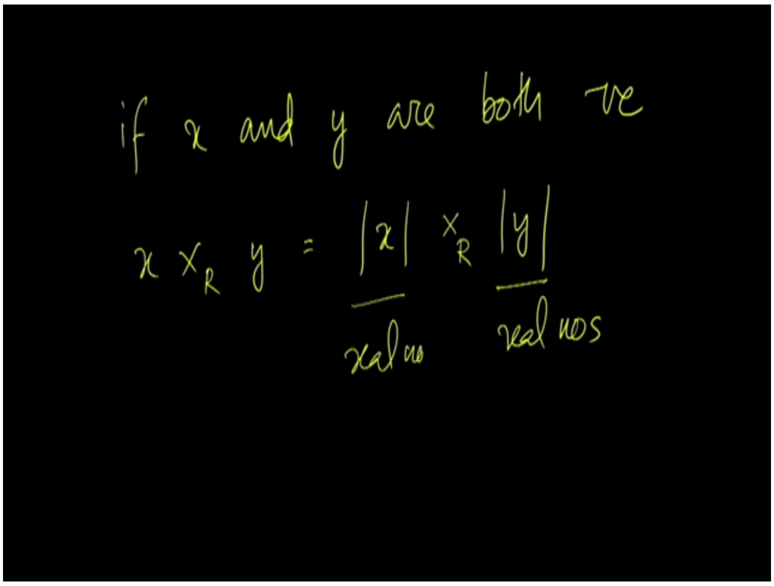

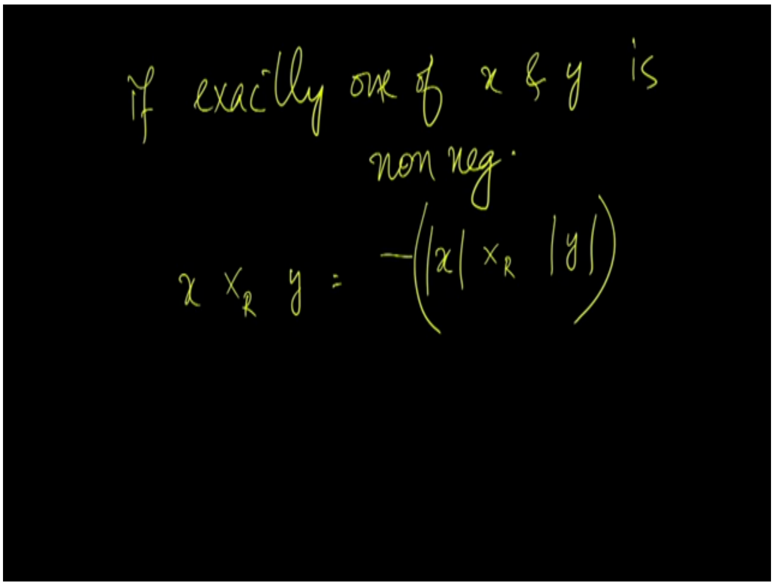

If, X and Y are both negative, then X into Y is defined as the absolute value of X multiplied by the absolute value of Y. We do a real number multiplication here. If exactly one of X and Y is non negative, then X into Y is defined as the negative of the magnitude of X multiplied by the magnitude of Y. The multiplication here is the real number multiplication again.

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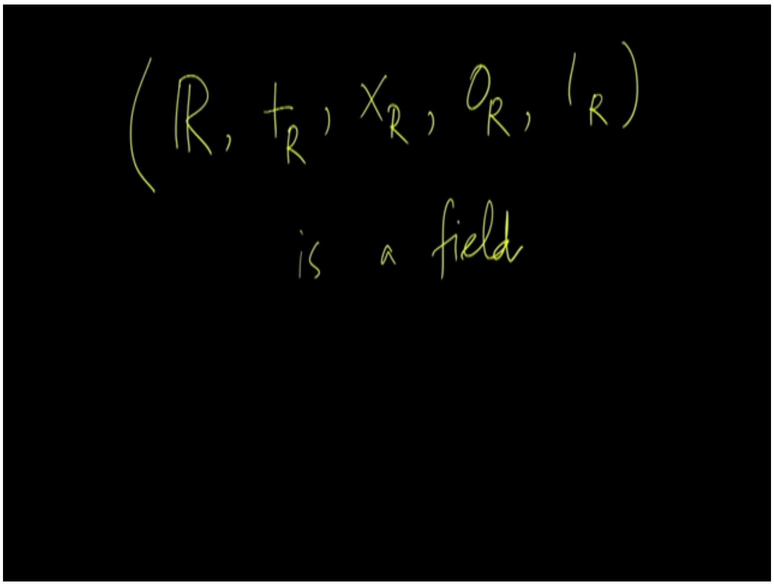

So, we can see that the set of real numbers along with these operations addition and multiplication and the real 0 and real 1 is a field.

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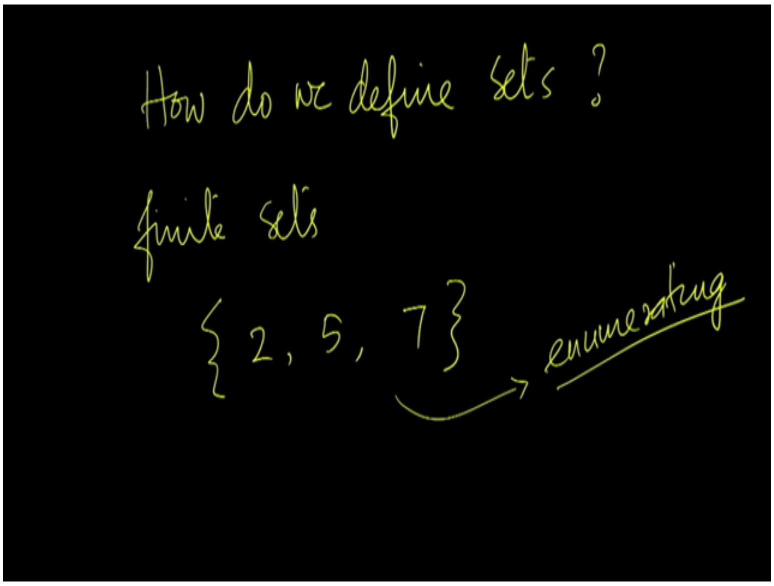

Now, let us see how we define sets. If what we have as a finite set. We can just enumerate the members of the set. For example, the set 2, 5, 7 has three members, we can explicitly list three members and enclosed them within the bracers and this is one representation of the set. This is what is called an enumeration of the members of the set. So, the set can be represented using an enumeration or we can represent the set using an abstraction.

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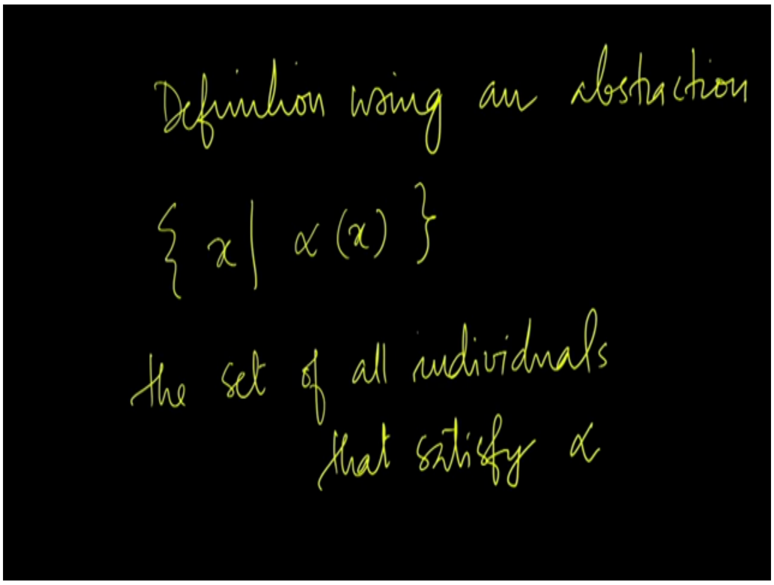

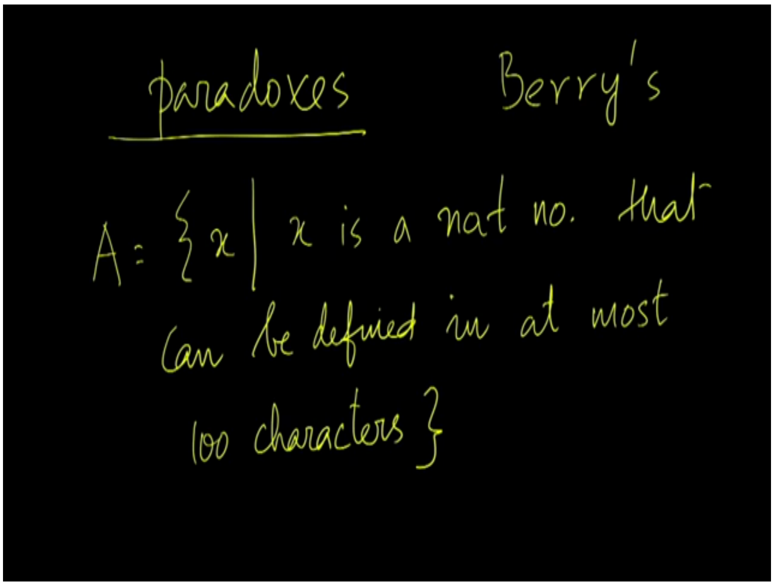

So, let us say alpha is a first order formula with a free variable X. Then we could write the set of all individual X, so that X satisfies alpha so this is an abstract representation of the set, but an abstract representation can lead to paradoxes. One such paradox is this. This is called Berry's Paradox. Let A be the set of all numbers X, such that X is a natural number that can be defined in at most 100 characters.

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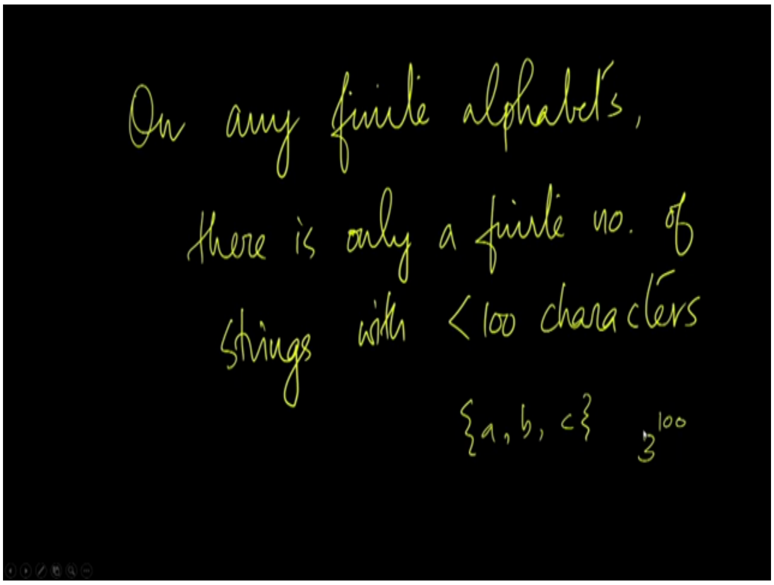

On any finite alphabet there is only a finite number of strings with less than 100 characters. For example, if you have three members in the alphabet, if your alphabet is A, B, C then how many strings can have exactly 100 characters? You consider a string of 100 characters, so there are 100 positions to fill. At each position we have three choices, so we have 3 power 100 ways in which 100 character strings can be constructed.

So, here we are talking about strings of at most 100 characters. So, we can count the total number of such strings, so that is the finite number.

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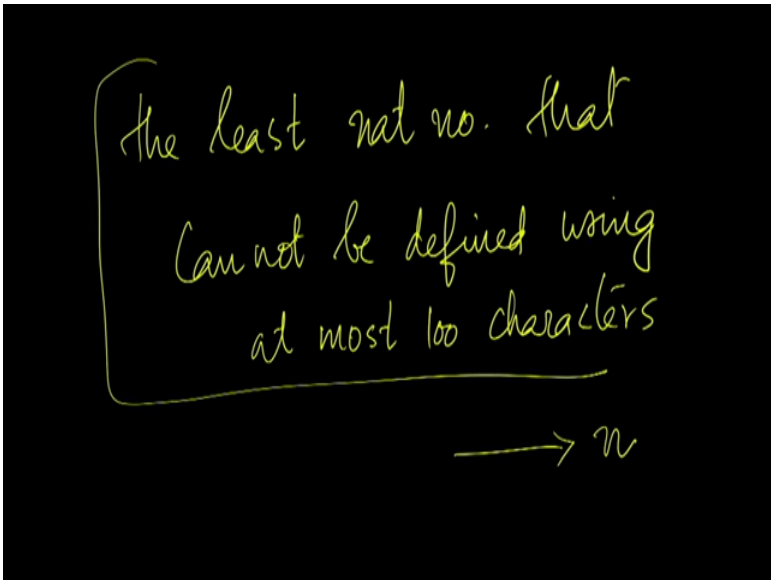

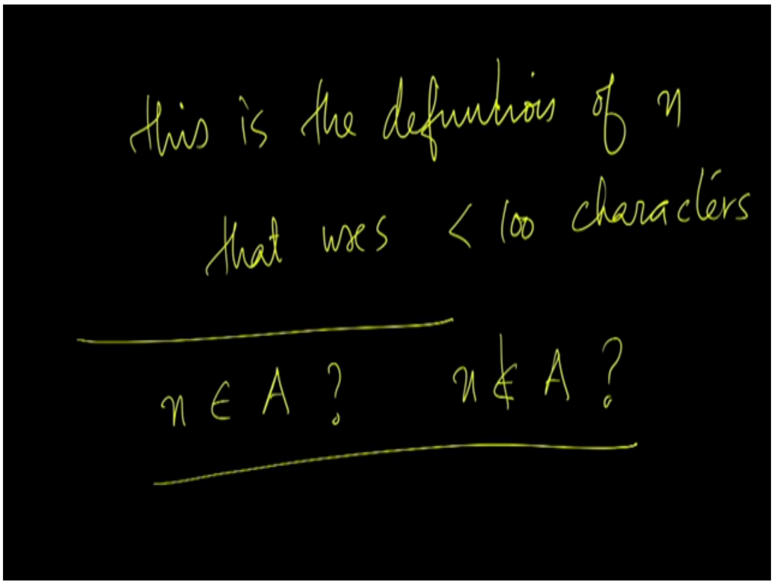

Since, there is only a finite number of strings with less than 100 characters. We can talk about the least natural number that can not be defined using at most 100 characters. We can, cannot we? Let that number be N, but then this is the definition of N, that uses at most 100 characters, then we have this question asked will N belong to A or does it not? We have a paradox, but then we can get rid of such paradoxes, if we are precise with the definition. So, here the problem was with the use of the word definition here. So, there is an (())(24:34) here, either by using multilayered notions of definitions we can avoid such paradoxes.

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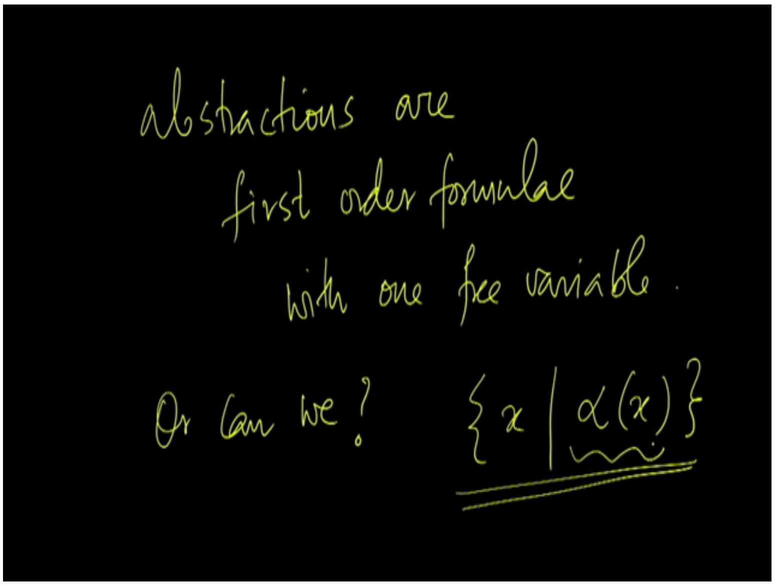

But does it allow us to get rid of all paradoxes, infact not. Even if (you) we use a precise first order formula for representing alpha here, we can still have paradoxes.

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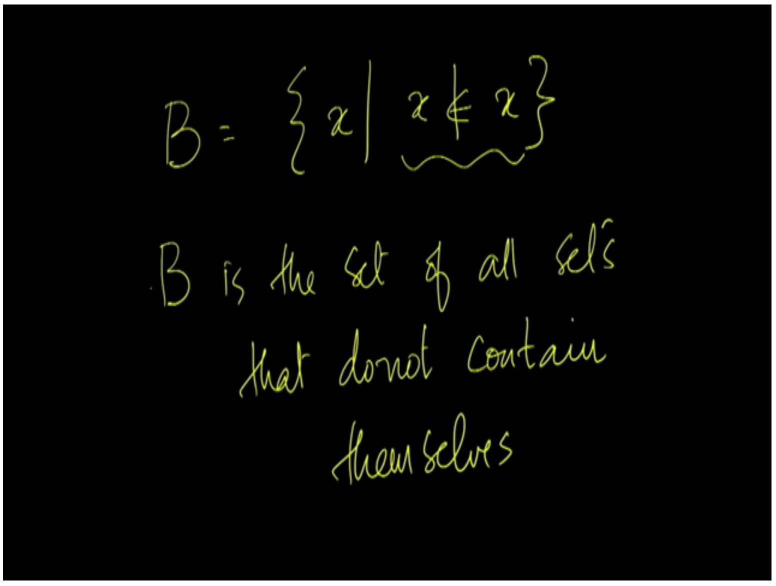

Consider the set B define in this manner. B is defined as the set of all X such that X does not belong to X. So, here the abstraction is very precise. It is defined using the set membership notion. So, B is defined as a set of all sets that do not contain themselves.

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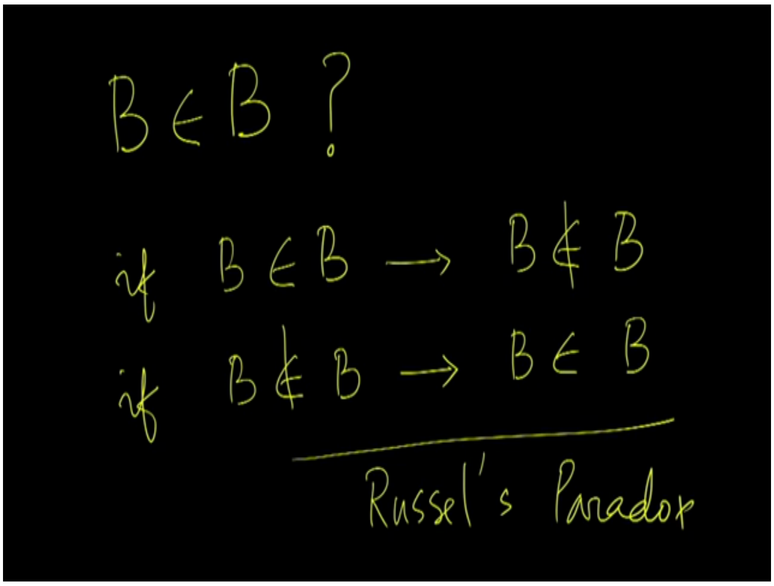

But then does B belong to B? If B belongs to B, then B should not belong to B. On the other hand, if B does not belong to B, then B should belong to B. This paradox is called 'Russel's Paradox'.

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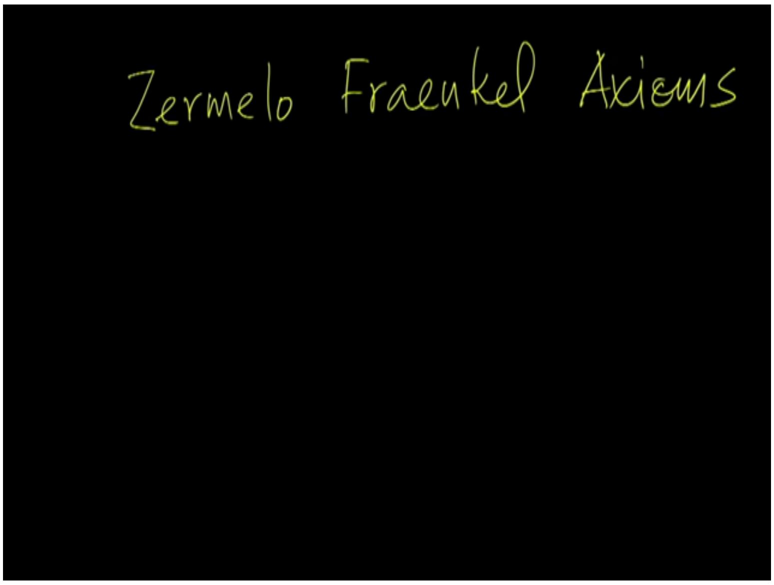

So, this is the fundamental paradox. This is because not every collection of objects can be dimed a legal set. What it means is that, the axioms of set theory have to be carefully formulated. One such formulation is Zermelo Fraenkel Axioms. We shall study about the set axioms in the next class that is it from this lecture, hope to see you in the next, thank you.