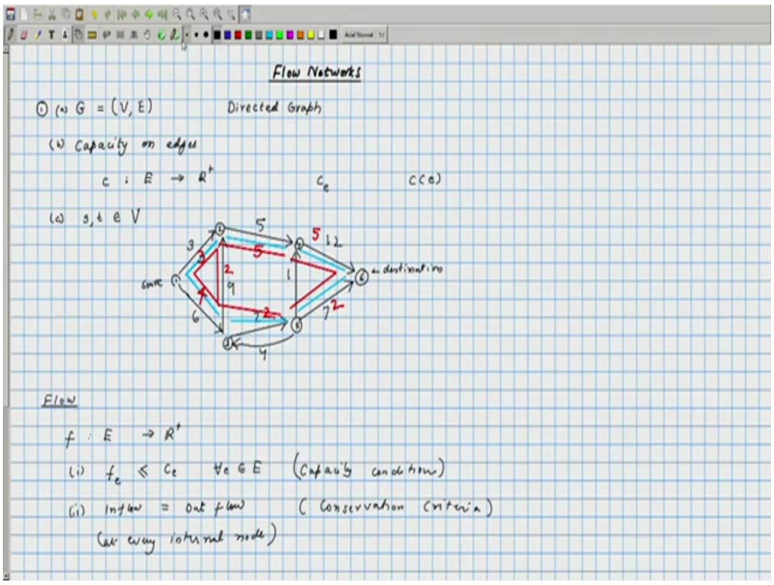
Discrete Mathematics
Professor Sajith Gopalan
Professor Benny George
Department of Computer Science & Engineering
Indian Institute of Technology, Guwahati
Lecture No. 17
Network Flows

In this lecture we will learn about flow networks. So flow networks are nothing but capacitated directed graphs, let us formally define what are flow networks.

(Refer Slide Time: 0:45)



So flow networks has two components; first is a directed graph so G we will denote it by the graph by G V, E. v is a set of vertices and e is a set of edges and this is a directed graph. This would mean that the edges if we denote rise by U, v that is the direction on the edge, it goes from U to v and not v to U. And the second component of flow network is a capacity on edges, so we can think of that as a function that is say C from the set of edges to positive reals that would mean for every edge there is a number associated with it, a nonnegative number which we will refer to as the capacity of the edge.

So we will use the notation Ce or say C of e to indicate the capacity of the edge e. So let us see an example of such a network, so this network has 6 vertices numbered 1 to 6, there are some number of edges. The edges have given the direction so 3-5 is an edge whereas 5-3 is not an edge in this particular graph, but you could have edges going in both directions so since we are looking at directed edges, we could have an edge going in the backward direction as well so this would be a flow network. And there are there is a third component, third component are the source and destination, there are two vertices S, t belonging to v which we will call as a source and destination, so in this if our s is a vertex 1 then this is what we will call as a source and 6 is what we will call as a destination.

So flow networks contains three parts; first is a directed graph, second is the capacity on the edges and the third is two designated vertices which we will refer to as source and destination. We will further assume that all the edges that involve the (edge) vertex s are outgoing edges. So here this there will be a small trouble in this particular diagram, in particular the 3-1 edge, it is an incoming edge, it is an incoming edge at the vertex 1 so what we will assume is that there are no such edges. So these edges would be an outgoing edge. And at the destination all the edges are incoming edges, so these are the restriction. Capacities has to be positive, at the source all the edges are outgoing edges, at the destination all the edges are incoming edges. And that the internal nodes that is the nodes that are neither source nor destination the edges could be in either directions, so this is the definition of a flow network.

Here we have been putting in edge weight so we could put them so we put some numbers on the edges those denote its capacity. So C of the edge 2-4 is going to be 5, the interpretation that we are going to give to this is we can think of this as pipes which can carry a certain amount of fluid and the amount maximum amount of fluid that it can carry is bounded bounded by the capacity of the edge. And the problem that we are interested in solving is what is the maximum amount of flow that we can take from source to destination. So the additional assumptions what are the conditions that should be met since we are thinking of this as fluid flow, we can say that there are some natural conservation criteria that each node should satisfy.

In particular, at any of the internal nodes the incoming flow must be equal to the outgoing flow and on any edge the maximum amount of flow that can be there is going to be bounded by the capacity of that edge. So we will formally define what a flow is and the objective of this lecture would be to have a procedure or characterise what is a maximum flow that is possible in a flow network. So let us formally define what a flow is, flow we will denote it by f and that is a function from edge to reals. So at each edge we are assigning certain amount of flow, and this will satisfy some conditions. The first condition is the capacity condition which states that fe should be less than or equal to Ce for all e belonging to the edge set. The maximum amount of flow that can be routed through any particular edge is bounded by the capacity of that particular edge.

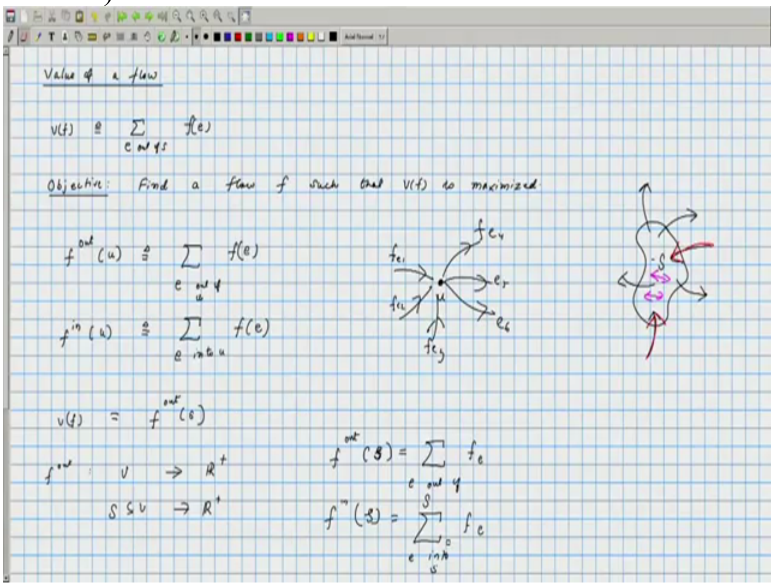
The second condition is the conservation criteria, which says that the amount of flow that comes into an internal node should be equal to the amount of flow that leaves that particular node, this should be satisfied for all nodes except source and destination. So we will just write this as inflow should be equal to outflow, we will formally define what these terms are but this is what the conservation criterion says; inflow is equal to outflow. So at every internal node, so nodes other than s and t are what we refer to as internal nodes, at every internal node the incoming flow is equal to the (out) outgoing flow. So let us try an example, and construct a flow on this particular flow network.

So let us say I have 1 unit flowing from 1 to 2, and this is also 1 so all my flows are going to be of unit size that can be indicated by blue line. So if I look at this flow you can verify that at every node the incoming flow is equal to the outgoing flow because 1 unit is coming in and 1 unit is going out. So this is a network where the flow is bounded by, when so if you take these blue edges we get a valid flow, and at any node if you look at it carefully, you can see that when at any edge the maximum amount of flow is only one and therefore it naturally means the capacity criterion.

Now we could look at increasing the flows, so let me just indicate it by numbers, so let us say if I had sent 3 units through this and 4 units through this, so there should be 3 flowing through this, the edge 2-4, and the 4 that comes here it could either routed through the 3-2 edge or through 3-5 edge, 3-5 edge I can send only 2 and the remaining 2 goes here and 5 comes here and 5 goes here and 2 comes here. So this is a little more involved flow in the sense that more edges are participating in this flow, the flow still respects the capacity conditions and the conservation criterion. But the amount that is flowing from the source to destination is equal to 7 units, which is better than the previous 2. But is this the best possible? Can we find anything significantly better than this?.

In fact, in this we can argue that it is not possible to find any flow which does better than this, so that is a theorem that we will learn later on in this lecture it is called as a Max flow min cut theorem. So if we you use the Max flow min cut theorem, we can say that in this particular network we cannot expect to get a flow of value greater than 7, we have not yet defined what is the value of a flow we will do that, this is just a glimpse of what we will be saying in the remainder of this lecture.

(Refer Slide Time: 11:25)



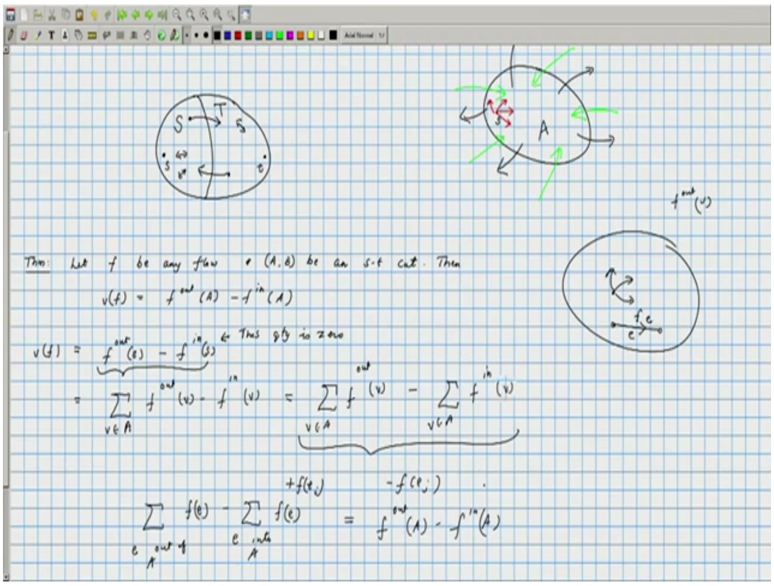
So, so far we have defined what is a capacitated network and we have defined what a flow is. A flow is some assignment of nonnegative integers to the edges in such a way that the capacity criterion are met at every edge and the conservation criterion are met at every node. Now we will define what is the value of a flow. So value for flow is defined as summation or all the edges e out of s fe, so the flow naturally associates nonnegative integers to every edge. Now look at all the edges at S, all the edges are outgoing edges if you sum up the flows of all of them what (what) we will get is referred to as the value of the flow. Our objective is to maximize a value find a flow which maximises v of f. Find a flow f such that v of f is maximized. Let us understand this quantity v of f carefully.

So we will define some additional quantities, so f is defined for each individual edge, we will define f out at a particular vertex, let us say u. So this is defined as the outgoing flow at vertex u so that would be summation over all edge e out of u fe and similarly we can define f in of u is summation e into u f of e. So if you have a particular vertex u and there is lots of incoming edges and each of them carry a flow, so let us say this is fe1, this is fe2 and this is fe3, if you sum up over this whatever you get is f in of u. And similarly if you have a lot of outgoing edges, so let us say as fe4, e 4 was this edge, e 5 and e 6, the flow on the edge 4 that is fe4 and you sum up over all those outgoing edges their flows, what you get is f out of u.

And therefore we can say that value of the flow is now simply f out of s because s was our starting vertex the source vertex, the flow out of s is defined as that was basically the definition of the value of the flow. So f out was the function from vertices to real numbers, now we can extend it as functions from subset of vertices, so s subset of v to Rplus, we can extend it in a natural fashion so f out of s would be so this is a set of vertices so let us call this as s and there are lots of edges which leave s and of course the edges that come into s so the incoming edges let us mark in red, outgoing in black itself and then there are other edges which are within s itself, so if you sum up over all the outgoing edges their some will be f out of S.

So this is summation e out of s that is summed over all the edges that are outgoing edges with respect to S, their flow if you sum it up fe that will be f out of s, and similarly you can have f in of s which will be equal to so this s is the capital S and not to be confused to the source vertex. Summation e in to s fe that will be f in of s and therefore, we can verify that value of the flow is also equal to value of a set which contains a vertex s and does not contains the vertex T. So if you sum up over any collection of vertices, which contains s but does not contain t that will also be the value of the flow so let us see why that is a case.

(Refer Slide Time: 16:51)



So we will first define what is a cut, or we will call this as s to t cut. And s-t cut in a graph G is partition of the vertices into two groups or two parts such that one part contains s and the other contains t, so these are the vertices. If we split it into two parts and this part contains s and this part contains t, then this split is called as an s-t cut. And if you look at the edges of the graph, the edges are 4 kinds, the edges which have both its endpoints in s itself. So there are these kinds of edges which starts and ends in s and there are these edges which starts and ends in t, and there are these other edges which starts from s and go to t and then there are these edges which starts from t and comes to s.

And these edges which go across the cut that is from s to t or t to s they are called as cut edges. So we will now prove the theorem, so let f be any flow from s to t and A, B be an s t cut, when we say A, B is an s-t cut what we mean is A is a collection of vertices which contains s, and B is the complement of A which contains vertex t. Then value of the flow is equal to f out A minus f in of A, so we had defined what is f out and f in for a collection of vertices so f out of A is a sum over the flow over every edge that leaves A and f in A is a sum over the flows of every edge that comes into A, and this difference will be equal to value of the flow, the proof is very simple. Basically put in all the definitions together, so A is a set, it contains s.

So now we have all these outgoing edges and what we want is the sum over all these outgoing edges that will be your f out A. So outgoing edges we are marking in red, and incoming edges let us mark in green, your RHS is going to be sum over the black edges minus sum over the green edges, whatever is the flow on these edges they have to be added up appropriately, the black edges you add up, the green edges you add up, subtract them what you get is the RHS. Whereas, the LHS is going to be the sum of the outgoing edges at s, s has only outgoing edges so you look at all red edges, their sum is going to be your left hand side, now we need to show that these two quantities are equal, so how do we do that?

So note that value of the flow is equal to f out of s minus f in of S, okay so this is so because f out of s is the real value of f of the value of the flow, f in of s is 0. As this is 0 we can write this equation, so f out minus f in for the particular node s if you do that what you will get is the value of the flow. Now for every other node in this graph other than t, f out s minus f in s is going to be 0. The conservation law says that since f is a flow, if you take this quantity out minus in for any other vertex that is going to be 0, so we can also write this as summation over all v belonging to A, f out v minus f in v is going to be equal to this quantity. We can write this because the vertex t is not there in this collection A, the vertex t belongs to the part B so let us look at this summation.

Every vertex belonging to A, it is either the vertex s or it is a vertex for which f out v minus f in v is 0 because the only vertex for which it was nonzero over s and t, s we have already considered, t is guaranteed to be not in A because A-B is an s-t cut, so this summation is going to be value of the flow. So this we can just simply write it as summation v belonging to A, f out v minus summation v belonging to A, f in v. Now if you look at this expression, what this does is for each vertex in A look at all the outgoing edges and for them you add f out v.

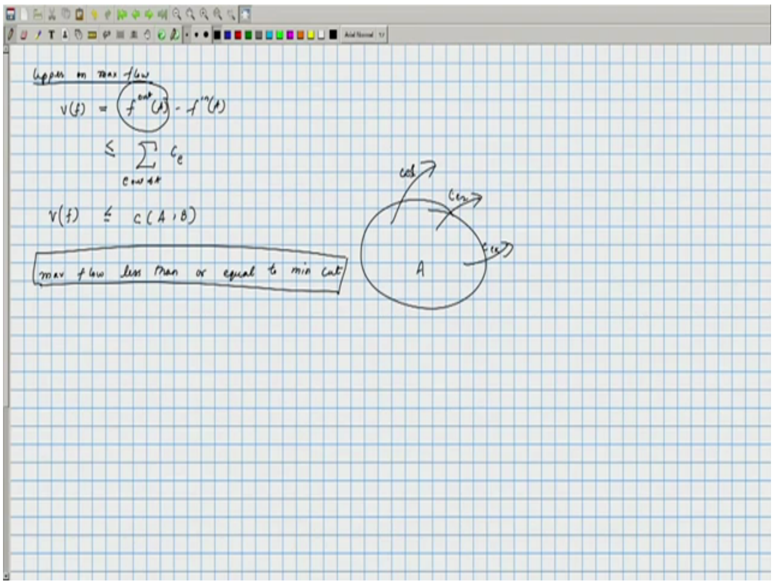
So this is a particular edge, we are going to add fe while we compute the first term of the summation.

Now the edges can be of 3 kinds; the internal edges, the edges that lies within A, the edges that go from A to B or the edges that comes from B to A, these are the only kind of edges possible. The edges which are within B to B are not even considered so A to A edges what happens to those edges? For them fe is counted once as positive and once as negative, so if you look at this entire summation the internal edges are going to cancel out and the external edges are the only thing that are going to remain. So if you look at this term, there will be lot of terms of the form fe i, some of them are going to appear as plus, some of them are going to appear as negative, so let us understand which one appears as positive and which one appears as negative.

All the edges which lie within A, they come in both positive and negative and therefore they cancel out each other. Those edges which originate in A and ends up in B are going to come only in this is a positive format, so one of them you give as positive and the other as negative, all the outgoing ones are the ones which are going to be positive. So this summation we can write as sum over so here the summation was over vertices, some over edges e such that e is out of A, fe those are the edges that are going to remain. And all those vertices which comes from B to A they are also going to remain because they are going to come in as part of f in of A so that will be summation e into Afe.

And this is going to be summation, and this by definition is going to be f out A minus f in A, so that means the value of the flow is equal to the quantity that we are interested in. So from this what we can say is, the maximum value of any flow is bounded by the positive quantity in this term.

(Refer Slide Time: 26:31)

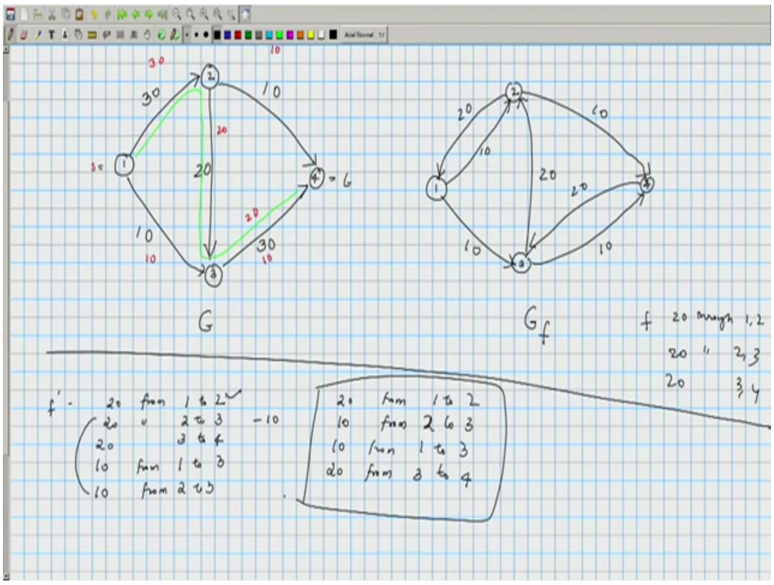


So we can get an upper bound on the flow, so we know that the value of the flow of any flow is going to be equal to f out of A minus f in of A . And if you look at any particular cut A, the f out is going to be at most equal to the sum over the capacities, so there are lots of outgoing edges let us see their capacities are Ce, Ce1, Ce2 and so on, Ce K are the outgoing edges. So f out is going to be less than, so this quantity is going to be less than summation e out of A Ce and minus some other quantity so we will just ignore that quantity because we are interested in only upper bound, and this quantity is what we call as the size of the cut, so we call that as C A B.

So if we call this value as a cut size, what we have shown right now is v f is going to be less than or equal to the size of the cut. What we will later on see is that there will be a flow such that it matches the size of the cut. There will exist for any capacitated network we can find a flow such that the value of that flow is going to be equal to the capacity of some particular cut and that would be the max flow min cut theorem.

Here what it says is, you take any cut, the size of the cut is a natural upper bound on the flow that is possible, so in particularly we can write that maximum flow is going to be less than or equal to min cut, find the cut whose capacity is the least and this theorem says that any flows value cannot exceed the value of the min cut. How do we show that these quantities the min cut will be equal to the max flow that is what we will do in the remaining part of this lecture, so we need some additional concepts.

(Refer Slide Time: 29:12)



So we will define the concept of what is called as the residual graph. So we do that, let us motivate this concept through an example, so let me look at the capacitated network with 4 nodes. The 1-2 edge can carry let us say 30 units and the capacity of 2-3 is 20 and 3-4 is again 30 that is 1-3 is 10 and 2-4 is also 10, what is the maximum amount that can be, what is the maximum possible flow?. So let us look at this particular path indicated by the green line or green curve. We can route 20 units through this particular path without violating any of the conditions but once we have routed 20 units of flow we cannot send any additional flow without affecting the already existing flow.

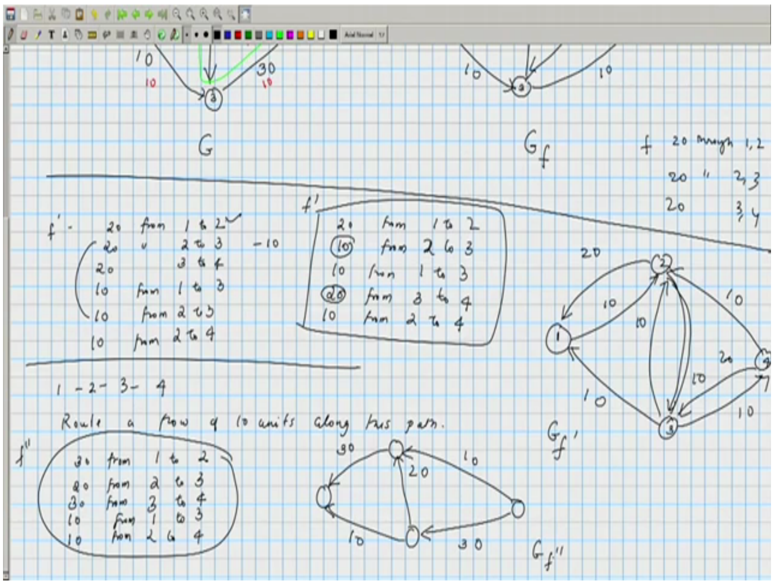
But clearly we could have let us say sent 30 units from 1 to 2 and split 10 to 2 to 3, so we could have done the following, 30 starts from here, 10 goes here, 20 goes here, and this 20 goes here, additional 10 comes from here and that is sent on this. So you could send 40 units from 1 to 4, this is a source and this is the destination and you can send 40 units and that is going to be the maximum possible. So let us look at this case where we were only sending we were only sending 20 units of flow from 1 to 4 using the green edges. In that case we will have a residual graph, we will define the concept of residual graph, it will look something something like this. 1 to 2 the capacity of this edge is completely minutes used to the extended of if you look at 1-2 edge there was already 20 units flowing through that so we can reverse that so that will be indicated by a back edge which can carry 20 units.

And forward capacity has been used to some extent, 20 units have been used so there is a remaining 10 units that can be routed. And 2 to 3 this going to be there was a forward edge which will saturate it so we can only send back 20 units along this. And if you look at the vertex 4 this edge remains because all the 10 units is available, you can route 10 additional flow through this and 3 to 4 there was 20 units going in the forward direction so there is 20 units in the backward direction and there is a residual capacity of 10 that can be used, and here there is an additional 10 units of flow capacity that we can still use, so this is the residual graph. So this if we call as the graph G and this is the residual graph for the flow f.

So f was routing 20 through 1-2, and 20 through 2-3, and 20 through 3-4, so corresponding to that flow we will get this particular residual graph. Now look at the residual graph, in the residual graph there are many paths from let us say 1 to 4, and those residual paths can be used to route additional flow. For example, you could take the 1 to 3, 3 to 2 and 2 to 4 and you could route an additional 10 units, once you do that the flow changes so in that case the already existing flow of 20 units was there, now we are routing an additional 10 units. So the new flow becomes so let us call this as f prime, this is equal to think of it as a function, 20 units from 1 to 2, 20 units from 2 to 3 and 20 units from 3 to 4 was already existing, and 10 from 1 to 3 and 10 from 3 to 2, this we are routing in the residual graph but we want a flow in the original graph so routing it from 3 to 2 is essentially amounting to decreasing the flow because here we are using the back edge.

1 to 3 was a forward edge, 3 to 2 was a back edge, and on that back edge if you are sending 10 that means we are decrementing the original flow, so I will just write it as minus 10 from 2 to 3 and 10 from 2 to 3. So now the new network would essentially be, the new flow of the new network the new flow would be 20 from 1 to 2, and 10 from 1 to sorry 2 to 3, 10 from 1 to 3, and 20 from 3 to 4. So this is yeah so these two combine and gets us 10 from 2 to 3, so this is the new flow that we can this is an additional flow of 10 units that we can get by looking at the residual graph.

(Refer Slide Time: 37:27)



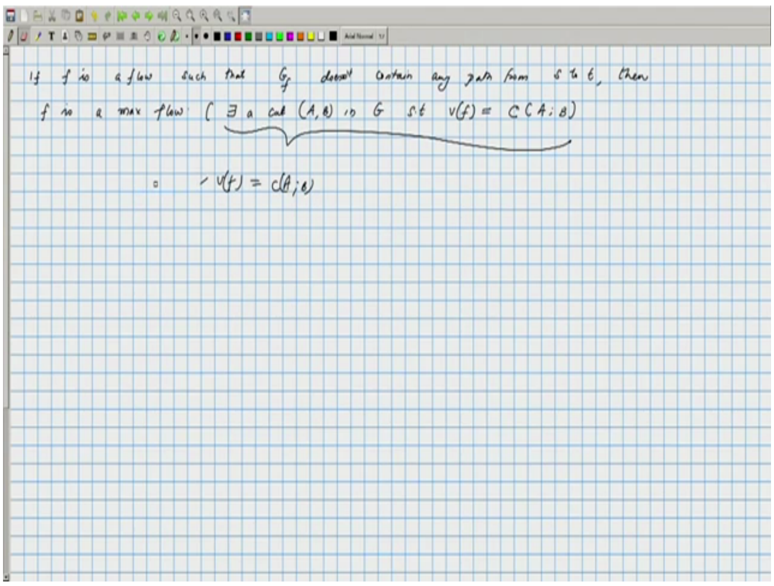
Now corresponding to this new flow f prime we can again compute the residual graph that will essentially be of the following form, from 1 to 2, so 1 to 2 there was a flow of 20, so there is still back flow possible of value 20 and forward there is still a residual capacity of 10 and 2 to 3, we have missed one particular edge namely 10 from 1 to 3, 10 from 2 to 3, and we had some flow of 10 from 2 to 4. So that will be the changed the flow. So there will be a back flow of size 10, there is no forward capacity and 1 to 3 is going to be a backward edge of capacity 10 and from 2 to 3 there was a capacity of 10 that is used so you can have 10 in the forward direction and 10 in the backward direction. And 3 to 4 there was 20 units used so there is a 20 units back flow and 10 units in the forward direction, so this is the residual graph corresponding to the new flow.

If you look at this particular residual graph what you can see is, it is still possible to route another 10 units of flow, so that means there is a path from 1 to 2, 3 to 4 and you can route a flow of 10 units along this path and that would give rise to a new flow and that flow essentially will be as follows, so 10 units from 1 to 2, this is augmenting the already existing 20 units of flow. So you can say 30 from 1 to 2, and from 2 to 3 there is already a flow of 10 we are increasing, so this was a 10 units of flow that was already there from 2 to 3 in the original graph, and we are using this particular edge to route an additional 10 so that would amount to 20 from 2 to 3. And 3 to 4 is the other one that is affected, there was a flow of 20 from 3 to 4 we are increasing it by 10, so 30 from 3 to 4, and the other edges they remain untouched.

10 from 1 to 3 that is going to be remaining as such and 10 from 2 to 4 is also going to remain as such. So now if you look at the original graph, this is a valid flow in the original graph and corresponding to this so let us call this as f double prime, if you compute the residual graph that will be as follows. Let us look at edge 1-2, there is already a flow of 30 1 to 2 is a flow of 30 and that is the maximum possible and therefore there is no residual capacity, the only thing that will remain is back edge of capacity 30. And from 2 to 3 there is a flow of 20 and that edge is also saturated so only thing that will remain as a single back edge of size 20. And 1 to 3 is also 3 to 4 is 30 so that is also saturated so you will have a back edge of capacity 30.

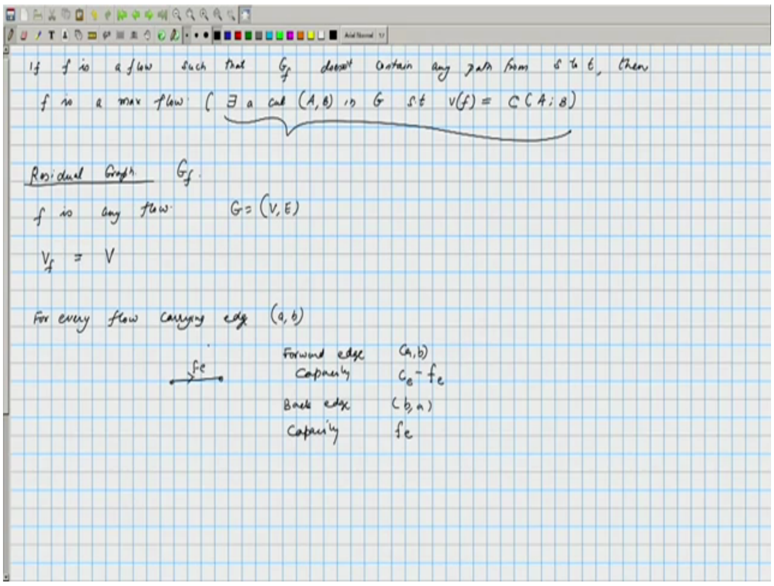
And 1 to 3 that is also saturated and there will be back edge of capacity 10, and 2 to 4 there is a flow of 10 so that is also saturated you will get 10 units. Now if you look at this residual graph G f double prime, you will see that there are no paths from s to t and therefore it is not at least in this situation it does not look that we can really increase the flow any further. We will prove that as a mathematical theorem in particular if you have a particular flow, so you take any arbitrary flow, corresponding to that flow we can compute whatever is known as its residual graph. And if the residual graph does not have any path from s to t then we will show that there is a cut which is saturated, so let us write this down.

(Refer Slide Time: 41:24)



If f is a flow such that the residual graph G f does not contain any path from s to t then f is a max flow that is what we will prove. In order to show that f is a max flow what we will prove is there exist a cut A-B in G such that value of f is going to be equal to the capacity of this cut A-B. When the value of this flow is equal to the capacity of A-B, it is impossible to increase the value if there is any other flow whose value so if g was another flow so is it value of g is greater than value of f then automatically the value of g will be greater than c AB but we have argued earlier that value of g can be utmost c AB. So if you show that there exist a cut whose capacity is equal to the value of the flow then we have enough reason to conclude that that flow that we have found is a max flow.

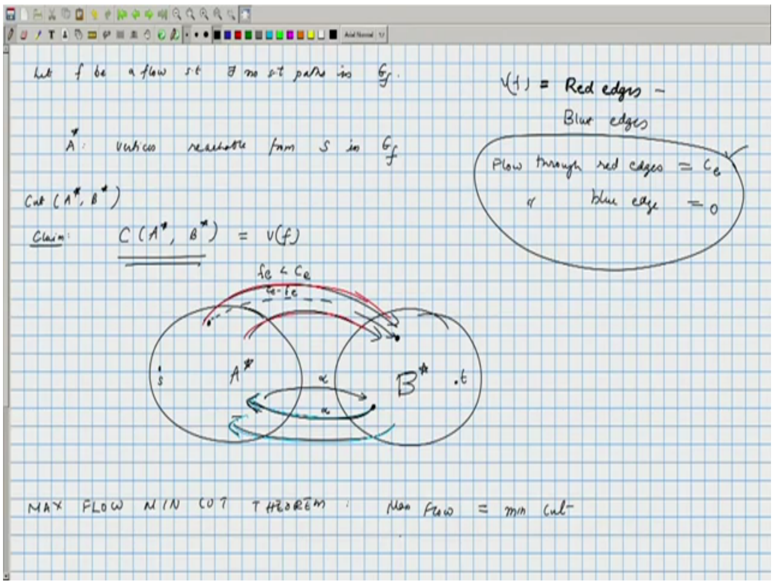
(Refer Slide Time: 43:01)



So let us formally define what is G f the residual graph, so f is any particular flow, the vertex at V f will be equal to V, vertex sets are the same, the edges we are going to have additional edges and the capacity of the edges are also going to be changing. So for every flow carrying edge A, B so this is some particular edge A, B and if it carries a flow fe, then we will have two edges; first is the forward edge, so forward edge will be from A to B and its capacity will be capacity of the original edge minus whatever is the current flow. And the backward edge will be B, A and its capacity will be equal to fe, whatever was the flow that is going to be the capacity, so this is how the residual graph is defined.

Vertices are the same as the original graphs, if a particular edge is carrying a flow of value fe then you have forwarded whose capacity is bounded by the difference of the existing flow and the maximum capacity and there will be a back edge whose capacity would be equal to whatever is the value of the flow, so this is the formal definition of the residual graph. And now what we need to show is that if the residual graph does not have any s to t connection, if it does not contain a path from s to t then we can find a cut which is saturated by this particular flow.

(Refer Slide Time: 45:15)



So we will define the cut, so let f be a flow such that there exists no s-t paths in G f, so now let A be the vertices reachable from s in G f, so in the residual graph look at all the vertices reachable from s. Clearly since s-t path is not there, A does not contains t and the cut consists of this A and its complement. So the cut that we have we will call it as let us say let us call this as A star, so cut A star B star is defined in the following way, vertices reachable from s in G f forms A star, and not reachable from s was remaining vertices forms B star. Our claim is the value of this cut A star B star is equal to the capacity of this cut is equal to the value of the flow, how do you prove this.

So let us look at these vertices, this is A star and the vertex s is sitting somewhere inside that and then there is v star which are the unreachable vertices and t is sitting somewhere inside. And of course there are these edges which go from A to B there are edges from B to A as well. The value of the flow we have argued so v f we can say is equal to the red edges minus blue edges, the flow through the red edges minus the flow through the blue edges. What we will argue is that the flow through red edges is equal to the capacity of the red edges, and flow through the blue edges is equal to 0, so this is a cut which is saturated. So if you look at the forward edges in the cut, they have all be in saturated so this is going to be the maximum possible flow.

So if we prove these two statements, our conclusion follows because value of the flow was flow through the red edges minus flow through the blue edges, and if you show that the value of the flow is equal to C AB, in order to show that we just need to show that red edges are operating at full capacity, and blue edges are operating at 0 capacity. So now let us show that red edges are operating at the full capacity it is straight forward because suppose there was a red edge which is operating at capacity less than C.

So let us say this edge has capacity ce and the flow was anything less than fe, that would mean that in the residual graph there is a forward edge from A star to B star with capacity so there is going to be some edge from this path to this path whose capacity is ce minus fe, and that would make this particular vertex over here in B star come to A star, but we argued that B star are all the unreachable vertices and therefore there cannot be forward edges so therefore every edge from A to B is operating at its full capacity so this part is done.

For the second part, if any of these edges was carrying some particular capacity, if the blue edge had some capacity had some flow through it some nonzero amount let us say this has a flow of Alpha then in the residual graph there is going to be a back edge from here to here with the same capacity Alpha, and that would make this particular vertex here B reachable but we have argued that B star, now definition said B star was all the unreachable vertices and therefore cannot be any such edge going from A star to B star.

So in particular that would mean that the value of the flow through blue edge should be 0. And therefore now we have found a particular cut, if there is no path from s to t in the residual graph then using that information we can compute a cut in the original graph and argue that that cuts capacity is reached by this particular flow. So since we have found the flow which matches the value of the cut, we know that, that is going to be the maximum flow and that is basically is the proof of Max flow min cut theorem. So this states, the value of the maximum flow will be equal to the capacity of the min cut, one direction we had already proven and this basically says that if the residual graph does not have a path, there is no way we can increase the flow any further.

So if we have taken an arbitrary graph and computed the residual graph and kept on doing that, still we can do so the flow keeps on increasing and it should, and if you assume that the weights are all integers then at some point the process must stop, and when the process stops we know that there are no s to t paths. And when there are no s to t paths, there is a cut which is being saturated so that is basically the proof of Max flow min cut theorem.