Discrete Mathematics
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Lecture No. 15

Planar Graphs

So, in this lecture we will learn about planar graph.

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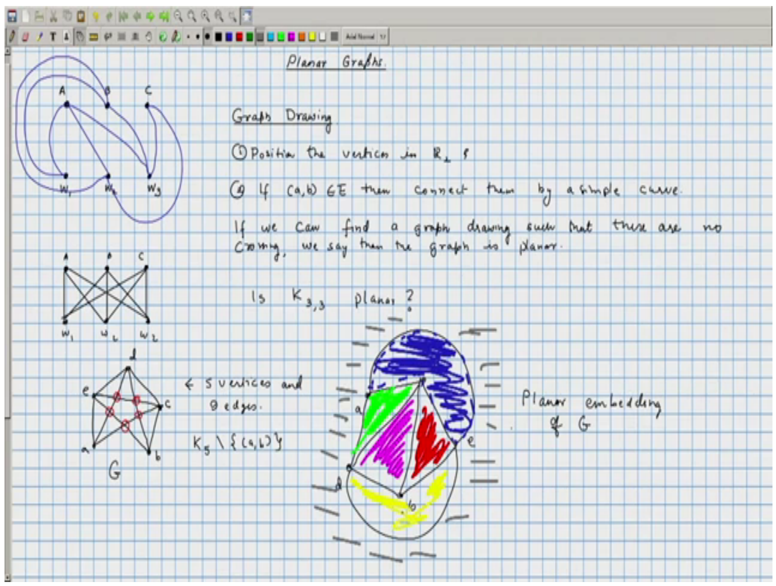

We will start with a puzzle, so imagine that there are 3 houses let us say A, B and C and there are 3 wells, W1, W2 and W3. What we want is to construct paths from each house to each well, but these people A, B and C do not get along well with each other, they do not get along with each other, so they want their own individual paths, they do not want their paths to ever cross.

So, maybe for A1 we can try it like this, there can be a direct so, we can have paths of this kind A to W1, A to W2 and A to W3 and similarly we can draw a path from B toW1, B to W2 and B to W3 and C to W3, C to W2 but once this has been drawn somehow at least in this drawing W1 is inside a region and there is no way we can go from C to W1 without intersecting one of the already existing paths, now, is this an artifact of our drawing? Did it happen because we drew it in a certain kind or is it the case that no matter how complicated we draw the paths, we still cannot manage to get non-intersecting paths.

So, in that case, so we want to answer this question so, we can reformulate this in a graph theoretic fashion by asking the following graph is planar or non-planar. We could think of this as A, B and C W1, W2, W3 and we have the following graph with all these edges present, we are interested in knowing whether this graph can be embedded on the plane. So, we can talk about what is called as a graph drawing. So, when we say a graph drawing we mean, we have to find so, position the vertices in the plane and, and if A, B is an edge then connect them by a simple curve.

And if we can find a graph drawing in such a way that the edges, the curves corresponding to the edges do not intersect then we say that the graph is planar. If we can find a graph drawing says there are no crossing, then we say that the graph is planar. So, now the puzzle can be reformulated as is K33 that is the graph that we are interested in is it planar. So, we will develop some tools by which we can answer this question, we will, what we will show is that K33 is not planar and what will help us do that is something known as the Euler's theorem.

Let us see one more example of planarity so, let us say we look at this particular graph on 5 vertices. So this is 5 vertices and 9 adjacent it is the complete graph from which exactly 1 edge has been removed. So you can write this is K5 minus 1 particular edge, let us say if we call that a edge AB. That is the graph that we are looking at, is this a planar graph? Now if you look at it, there are a lot of crossings in this as well, so here there is a crossing this is another crossing, so there are lots of crossings in this graph, but can we redraw it in such a way that all the crossings can be avoided? In fact, so, what I am trying to emphasize is that if 1 drawing involves a lot of crossing, that does not mean that we cannot redraw it to get another drawing where there are no crossing. In this case, we can do that and that can be seen in as follows.

So, there is an inner star which we can think of it as a Pentagon, so A is connected to C, is connected to E and E is connected to B, B is connected to D and D is connected to A. So, that takes care of the inner pentagram, so 5 edges are taken care of the other edges that are missing as B, C, C, D, D, E and E, A so those also we can draw BC is an edge and CD is an edge and DE is an edge, we can write in this way and EA is an edge. So this is an, this is a redrawing of the same graph in such a way that there are no crossings and once we draw a graph that in this particular fashion, that is called as a planar embedding. So, what we see here is a planar embedding of the graph G.

Now, once we have a planar embedding let us imagine that so, we have the entire plane and we have these particular embedding, if you just remove the edges and the vertices from this picture, what happens is the plane may get split into disjoint portions. So, basically you can think of this as we are removing this particular region and we could ask this question how many geographically connected regions would remain. So, we will give a name to those regions that will basically be called as a face. So, here in this diagram, there are going to be this is one face and the green colored region is another face, pink is going to be another face, this is yet another face.So in all you can see that there are going to be 1, 2, 3, 4, 5, 6 faces, the outside is also going to be a face, the outer region, the entire outer region that can be an infinite region, that is going to be the sixth face. So Euler's theorem basically relates the number of faces, number of edges and the number of vertices.

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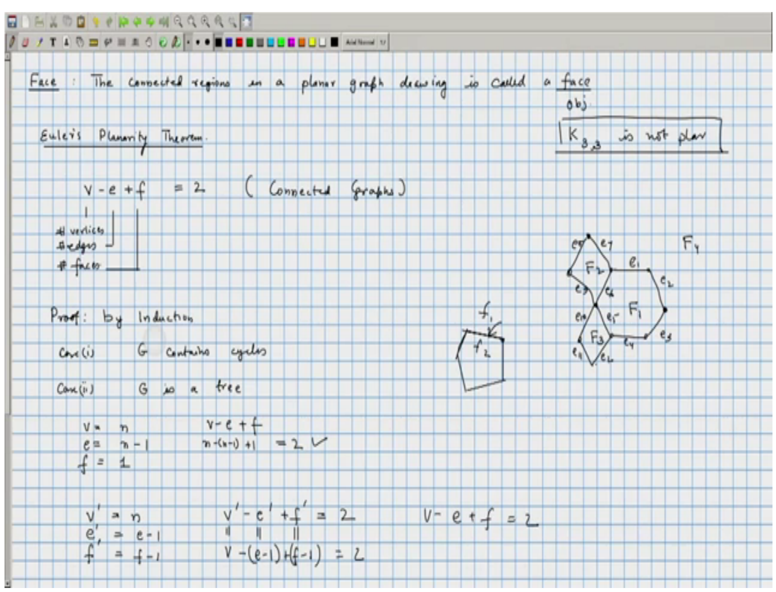

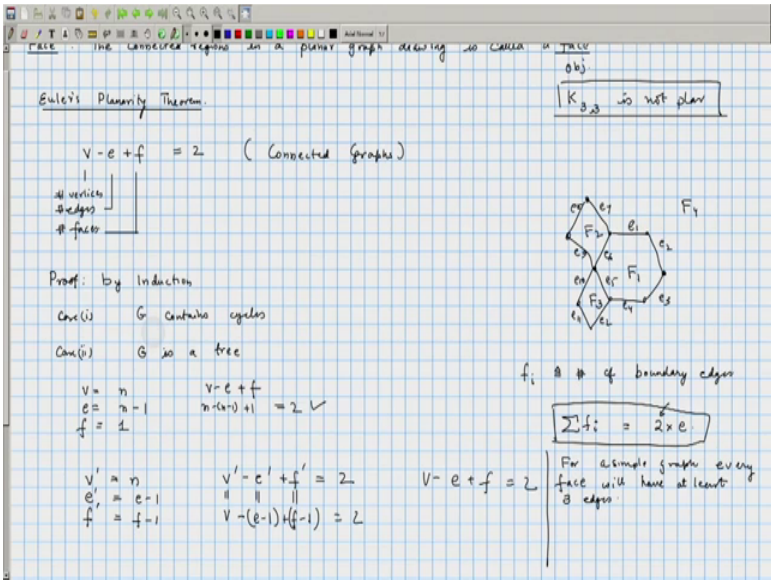

So, let us state Euler's theorem, before that, let us write down the definition of a face, so we will informally think of it as the connected regions in a planar graph drawing is called a face. So now, we are in a position to state the Euler's Planarity theorem so, it say that the number of vertices so V is a number of vertices and e is the number of edges and the f is the number of faces, and V minus e plus f will be equal to 2. So how do we prove this statement? The proof is fairly simple, we can use an induction on the number of edges so, this will be true for connected graphs.

So take a connected graph, it can either be a cycle free graph, or it could contain cycles So we will split the proof into two parts, so case one, let us say G it is our graph, the only other cases when G does not contain cycle so, the connected graph without cycles, that is called a tree, so G is a tree, when you have a tree and if you embed the tree into the plane then there is, does not create any regions, and therefore, the number of faces will be 1. So case two proof is easy, V will be equal to number of vertices, if you call it as n, and e will be equal to n minus 1 and f will be equal to 1. So V minus e plus f will be n minus n minus 1 plus 1, and that is going to equal to 2.

That is a easy case, other cases are also fairly easy. So if you have cycles, then one edge, there is at least. So if you have cycles, and if you embed it, you are going to get some particular region, which is a face could have any number of edges, we do not bother about how many edges are there in a particular face, but of course, what we can conclude is that there is going to be some particular edge which is part of some cycle and there are 2 faces on the either side of this edge, so let us say that its f1 and f2 if you remove this particular edge from the graph, you will get a smaller graph and in that smaller graph, what we can say is a number of vertices do not change, number of edges of reduced by 1, the faces f1 and f2 is going to coalesce into a single face.

So, V will now be n and e will be let us say the e prime, as if you think of this as a reduced graph, the number of vertices in the reduced graph is the same as the original one and e prime will be equal to e minus 1 and f prime will be f minus 1. And by induction hypothesis, we can say that V prime minus e prime plus f must be equal to, plus f prime must be equal to 2. So, this is equal to V and this is equal to e minus 1, and this is equal to f minus 1. So that can be rewritten as V minus e plus f is equal to 2, so that is a straightforward proof of Euler's Planarity theorem.

Let us try and show that K33 is not planar, this is what we had, this is our main objective, let us try and prove this. In order to do that, what we will, we will try and derive a corollary of Euler's Planarity theorem, so now, if you look at simple graph, that is graph without self loops or parallel edges, so let us look at any planar graph and any planar graph will have a drawing or an embedding in the plane. Now, if you look at a particular edge, any edge is shared by at most 2 faces, so let us look at the face and if you look at a new face, there are some number of edges that is bounding this particular face. So, if this is a planar embedding of a graph, the number of faces here are, let us say this is f1, this is f2, this is f3 and the outer face is f4.

Now, if you look at f1 these edges e1, e2, e3, e4, e5 and e6 are the boundary edges and you can do this for each particular face. For f3, the boundary edges are going to be e5, e10, e11, D2 note that any edge can act or serve as the boundary edge for at most 2 faces so, so associated with each face, we can have this number fi, this denotes the number of boundary edges. So if you sum up these fis over all the faces, what you will get is 2 times the number of edges. Now, if you take a simple graph, each face is going to have a boundary with at least 3 edges.

For a simple graph, every face will have at least 3 edges. So here and in this, we are assuming that every edges while writing this particular formula, we are assuming that there are no vertices of degree 1 we can, we can argue the summation of fis 1 times the number of edges, because if you have vertices with degree 1, the edge corresponding to that is path of only 1 face, so, the 2 term does not come.

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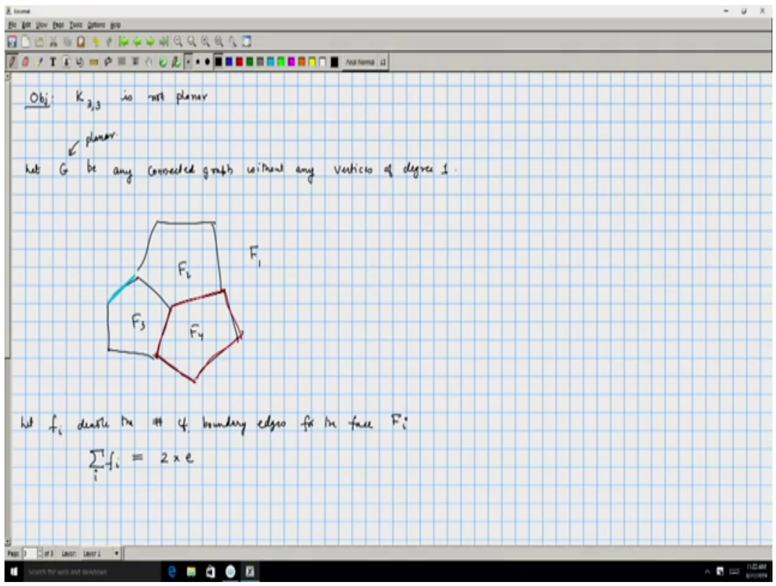

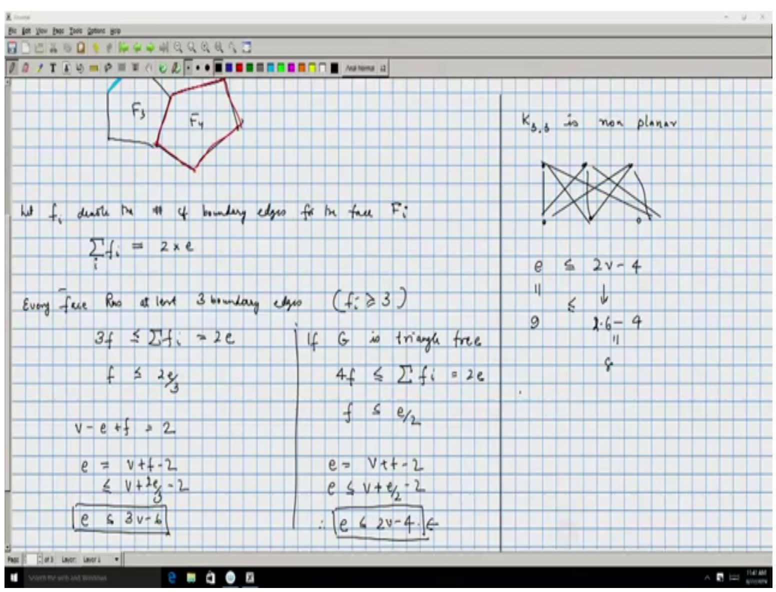

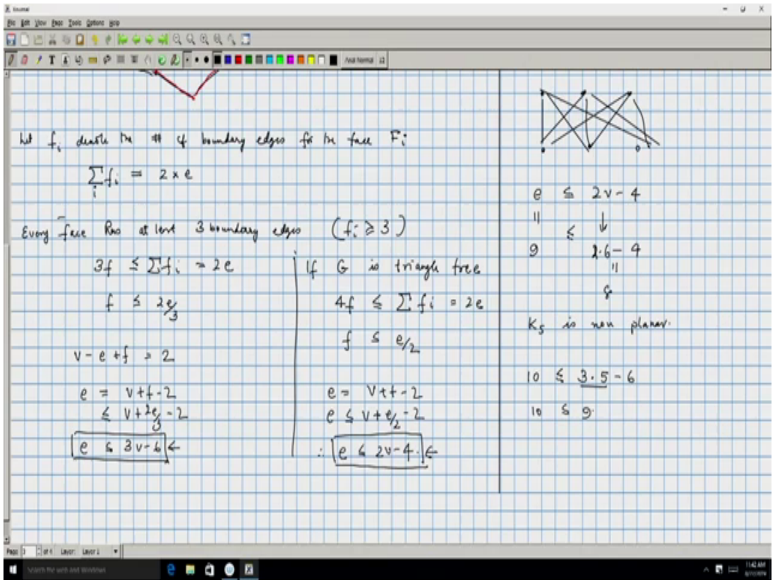

Our objective is to show that K33 is not planar. In order to do this, we will obtain a corollary of the Euler's Planarity theorem. Now, let G be any graph with, so consider a connected graph without any vertices of degree 1, so this would mean that every vertex is part of some particular cycle. Now, if you further assume that this planar graph that is an embedding. So, consider a particular embedding on the graph and if you look at the faces, they denote them by f1, f2, f3, f4 and so on, we can associate with each face its boundary edges, for example, these red edges are the boundary edges of the face f4.

Now, let small f5 denote the number of boundary edges for the face fi. So note that every edge is part of precisely 2 faces. For example, if you look at this blue edge, that is a boundary edge of f3 and f1 and therefore, we can conclude that summation fi over all the faces, that is going to be equal to twice the number of edges, because each boundary edge is being counted twice, once for each of the face that it belongs to. So now, if you take a simple connected graph, we can argue that every face will have at least 3 boundary edges, just 3 boundary edges, then each fi is greater than or equal to 3. Therefore, if f as the total number of faces, 3f is going to be definitely, so summation fi is at least 3f this is equal to 2e, so we can conclude that f is less than 2e by 3.

Now, if we substitute this in the Euler's formula, V minus e plus f is equal to 2, what we get is, e is equal to V plus f minus 2, and that is going to be less than V plus 2e by 3 minus 2, therefore, e is less than 3V minus 6. So this is a corollary that we can use. And if we assume that graph does not have any cycles of length greater than, length less than 3. So then we can conclude that every face has at least 4 boundary edges. So if the graph is triangle free, then what we have is every face must contain at least 4 edges. So 4f will be less than summation fi overall faces, and this is equal to 2e, so f is less than e by 2. And in that case, if you substitute in the formula e is equal to V plus f minus 2, we will get e is less than or equal to V plus e by 2 minus 2, so there we can conclude that e is going to be less than 2V minus 4. So these are the two things that we can use in order to show that various graphs are non-planar.

So let us do that, so the first thing we will show is K33 is non-planar. So this is 6 vertices and 9 edges. If we substituting e is less than 2V minus 4, e is equal to 9, and should be less than or equal to 2 times V, that is 2 into 6 minus 4 that is equal to 8. So 9 must be less than 8, if K33 was planar, it must satisfy this particular relation. Because K33 is a bipartite graph, it is triangle free, and therefore, this condition applies. And if it applies, the 9 should be less than 8. And that is not the case, so we have our contradiction.

If we look at K5, that is another graph that we can show to be non-planar. We showed that a particular graph obtained by removing just 1 edge from K5 was planar. But if we include that edge into it, we have K5 and K5 we will show is non-planar graph. Again, it is simple application of this particular corollary, number of edges in K5 is 10, because there were 2 pentagons, and 10 should be less than 3 times number of vertices, number of vertices was 5 minus 6. So this is 15 minus 6, the 10 should be less than 9 if K5 were planar, so that is a proof of non-planarity of K5.

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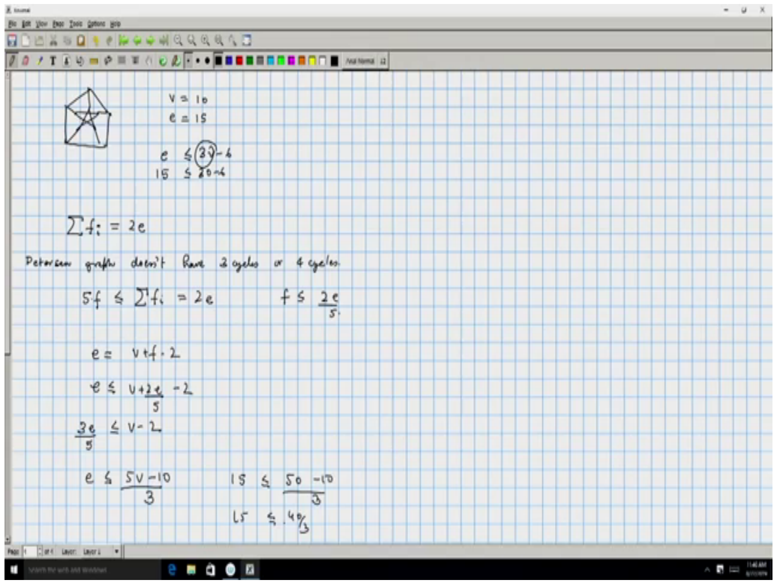

We look at a slightly more complicated example. But essentially the same principle if you look at our Peterson graph, this is a graph with10 vertices. So V is 10 and number of edges is 15 and we can show that this is certainly not a bipartite graph, because it is not cycle. So the only condition that we can hope to apply from the (coral), the 2 corollaries that we have drawn is e is less than 3V minus 6. But that condition unfortunately, it does not lead to a contradiction because e is 15 and 3V is 30 minus 6, so the condition holds, but the condition holds does not necessarily mean that the graph is planar.

But here, what we can do is we can look at the original condition that says summation f of overall faces is equal to 2e. So here, when there are no 4 cycles, or 3 cycles, so if you so Peterson graph does not have 3 cycles or 4 cycles, is something that you can easily check. And because of this, every face must contain at least 5 edges. So 5 times f is going to be less than summation fi that is equal to 2e.

So, we can conclude from this that f is less than 2e by 5, if we look at e is equal to V plus f minus 2, we can conclude that e should be less than and if it were planar, then e must be equal to V plus f minus 2 and e should be less than or equal to V plus 2e by 5 minus 2. So, 3 times e by 5 should be less than V minus 2, or in other words, e should be less than 5V minus 10 by 3, now if you substitute the values in case of Peterson graph's e is equal to 15. And that is less than 50 because V is 10 minus 10 by 3, you will get 15 is less than 40 by 3, that is clearly a contradiction.

So we have seen planarity and we have seen multiple applications of Euler's Planarity theorem to prove that various graphs are not planar to prove the non-planarity of various graph, we will stop here.