Discrete Mathematics
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Lecture No. 14
Bipartite Graphs: Edge Coloring and Matching

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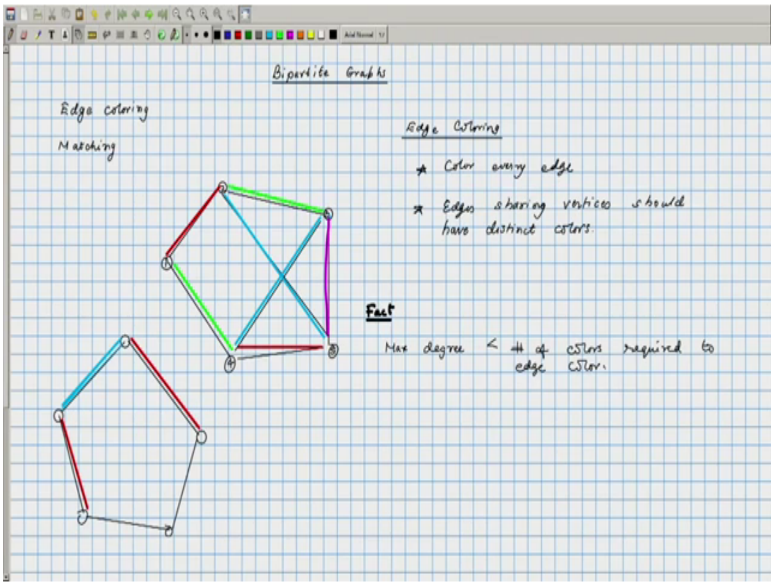

In this lecture we will learn more about bipartite graphs. So first we will look at some notion known as edge coloring and then we will look at something called as matching. We will look at these properties with respect to bipartite graphs. So let us see what is edge coloring in a graph. Let us take an example, this is a graph on 5 vertices. There are some number of edges. The objective is to color the edges, that is give each edge some color. The restriction is adjacent edges should have different colors.

So write in the following way, so we require two things, first color every edge and second edges sharing a vertex they should have distinct colors. So any such coloring will be called as a valid edge coloring. We want to find an edge coloring which minimizes the number of colors used. We could of course give different colors, distinct colors to each edge and surely all the properties would be met but that does not minimize the number of colors used.

One thing that we can see is for this particular graph we would require at least 3 colors because there are some vertices whose degree is 3. Now the question is can we do it in exactly 3? 2 is impossible because since there is a vertex of degree 3, all the 3 edges should get distinct colors. So let us see if we can do that. So this is red. This is given blue color, and the third edge is given green color.

Now the 1 4 edge could be given green color and 4 5 could now be given, say red color and 4 3 must be given blue color. And now if you look at vertex 3, there are 2 colors already being used here, (red) blue and green. So the 3 5 edge cannot be blue or green. And because the 4 5 edge is using red color, this cannot be red either. So maybe you will have to use a different color. Let us say if we use a pink color then this is a valid coloring. So here we required 4.

But is 4 really required? Those 4 colors used, can we do with 3 colors is a question and we want to answer this question for bipartite graph. So the specific question that we will look at is, is there number of colors required to edge color a graph equal to its maximum degree? So every graph will have maximum degree and will the number of colors required to edge color be equal to the max degree? Clearly in general graphs this is not the case.

For example, we could take just the 5 cycle. So in the example that we have considered it is not clear that we require 4. May be there is a coloring which requires just 3, we can think about that. And we can construct examples where the max degree of the vertex is not sufficient. For example, if we took this particular graph, this is a 5 cycle, the degree of any vertex is only 2.

But we can argue that 2 colors will not suffice to edge color this graph because, we can without loss of generality assumed it that the one of the edges is colored blue and then its neighboring edges have to be colored using a different color, so let us say it is red. And its neighboring has to be colored using blue. If we use any other color we are exceeding the number of colors but there are two neighbors and both of them cannot be given blue color because of the construction of this graph.

So in general graph, the you can construct examples where max degree is not equal to the number of colors required to edge color. So you can summarize that the following way. Max degree can be less than the number of colors required to edge color. In the general graph this could be the case. The C5 or any odd cycle is an example where this max degree is less than the number of colors required to edge color. Now let us focus on just bipartite graph. If we restrict our attention to bipartite graph, can every bipartite graph be colored with number of colors which is equal to the max degree? We will show that this is the case.

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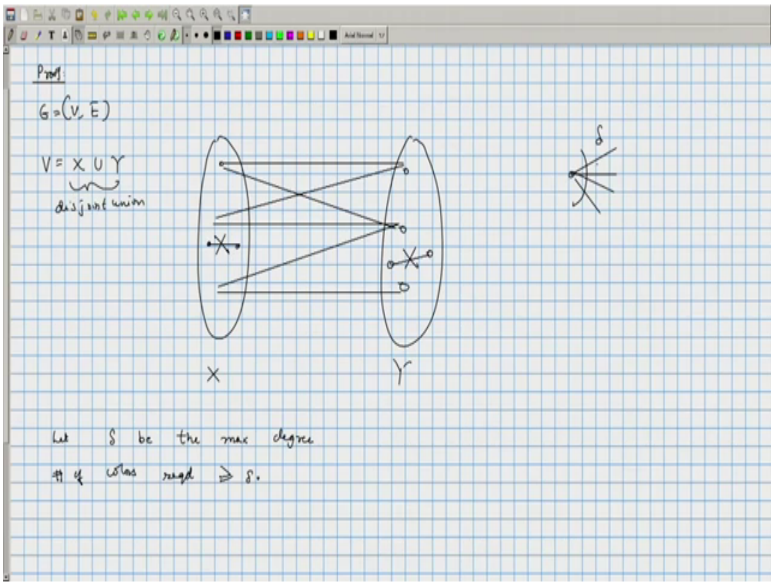

So this is a theorem that we will prove. So let G be a bipartite graph. Number of colors required to edge color G is equal to max degree of G. Why is this so? What is the proof? So first of all what is the bipartite graph? You can split the vertices into 2 parts, say X and Y. So if think of G as V. E where V is the set of vertices and E is the set of edges, V can be written as X union Y. So this is the disjoint union. They do not share any vertices so V can be the vertex that can be split as X union Y such that all edges are between X and Y.

So you cannot have edge of this kind, from X to itself. So this is forbidden. And you cannot have an edge from Y to itself. This is also forbidden. So all the edges go from x to y. That is the definition of a bipartite graph. So if we look at a bipartite graph we want to show that the max degree is equal to the number of colors required to edge color the graph. So clearly we can see one one direction that is you require at least those many colors.

So let delta be the max degree. Number of colors required is going to be greater than or equal to delta because at least delta is required. This is because look at that particular vertex with degree delta. Each of these edges must get distinct colors. So number of colors required definitely is greater than or equal to delta. Now if you can show that there exists a coloring which uses no more than delta colors then it means the minimum number, so this is the min number of colors required, so that is going to be equal to delta.

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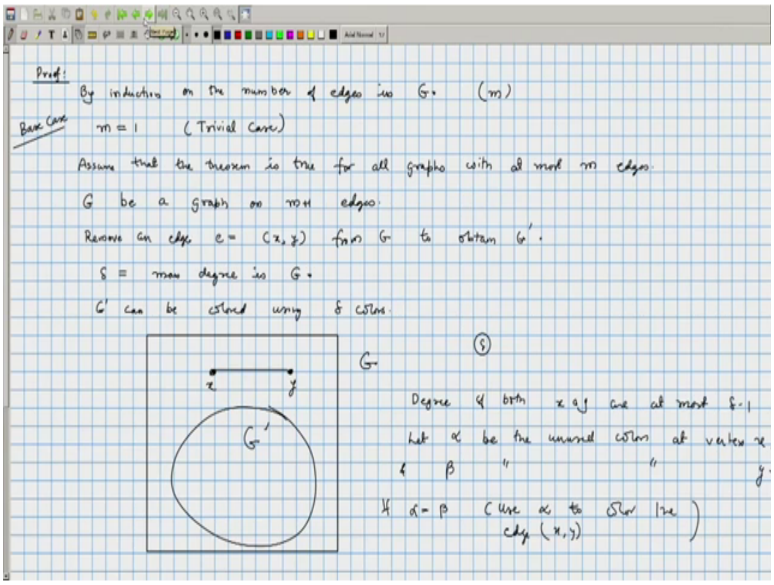

So how do we show that? That is the case. So proof will be by induction. We will induct on the number of edges in the graph. Suppose we denote the number of edges by m. Then m is equal to 1 so this is the base case. m equals 1 would mean that there is bipartite graph with a single edge and you require only one color. So you can think of that as a trivial case. To color it requires only one color. So we will assume that statement is true for all graphs or bipartite graphs with at most m edges.

Now we will look at a graph with m plus 1 edges and show that it can be colored using delta colors. So let G be a graph on m plus 1 edges. So now, this is some bipartite graph we can just remove one edge from G. So remove a particular edge, any arbitrary edge. So let us call that as the edge x, y from G to obtain G prime. So we just removed one particular edge, some arbitrary edge and we will get a smaller graph.

Clearly by our induction hypothesis, on the smaller graph G prime you can color it with delta colors. So delta is the max degree in G. So clearly we can obtain a coloring of G prime which uses no more than delta colors. G prime can be colored using delta colors. Now let us look at these vertices x and y. This corresponds to the edge which we have removed. This is the edge that has not been colored. So G is the main graph and G prime is the residual graph and if we add this edge x, y we will get this is the complete graph G.

Now we need to color this and if we manage to color it without using any additional color then we are done. We will see how that can be done. Now look at the vertex x. Its degree is at most delta in the original graph. Since one edge has been removed the degree of x in G prime is at most delta minus 1; degree of both x and y, so there is an unused color left at x and y. So if you look at the vertex x and y in the graph G prime there is some color that is unused.

So let alpha be the unused color at vertex x and beta be the unused color at vertex y. So whatever are those unused colors there should be at least 1 because degree of x is at most delta and since we are removing one edge at x, we would have used only delta minus 1 colors, maximum of delta minus 1 colors, the left-behind color is what we will call as alpha. Now the simple case is when these alpha and beta were the same.

If alpha and beta were both, let us say, equal then we can simply use that color to color the edge x, y. So x y there is unused color, we can just give that unused color to that particular edge. But when alpha and beta are different we are in a little bit of trouble. But we will see that case can also be dealt with. What we will do is we will look at the coloring given by G prime. G prime was the smaller graph which resumed, which by resumption can be colored because G prime has at most m edges.

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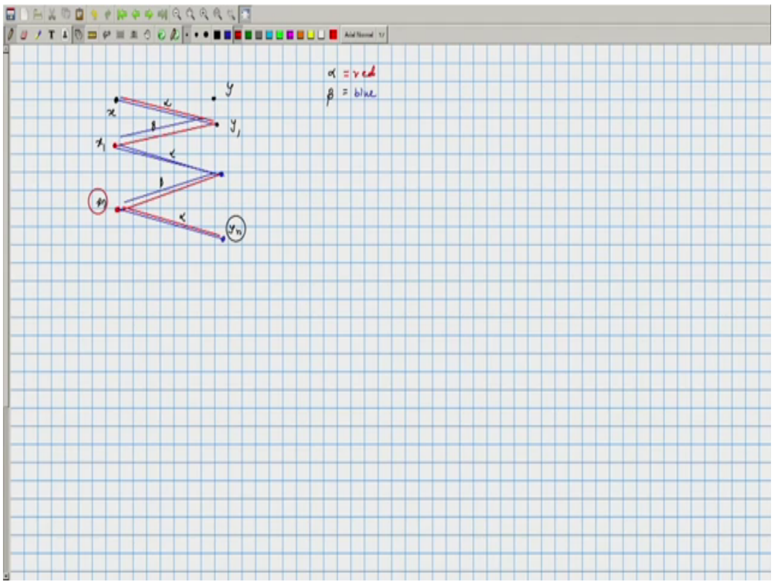

So now let us look at this vertex, so x and y are 2 vertices. The color that was unused at x was alpha and the color that was unused at y was beta. Now, so alpha we will just denote it by red and beta we will denote by blue. Now the red color was not used at x and blue color should have been used at x. So let us look at the blue colored edge from x to the other side. So that we will go to some particular vertex which we will call as y1. Now if you look at vertex y1 there are 2 situations.

There is either a red edge back to the other side or the red edge is not used. If the red edge is not used then we will stop. If the red edge is used we will take the red edge. So the red edge goes to some other place, some other vertex. Let us call that as x1 and then from here we will go back via blue edge and we will continue this process, so all that we are doing is the following.

Start at the vertex x and you go to the other side via the blue edge. If there is no blue edge then we will stop. And when we go to the other side you will come back via the red edge. If you cannot find the red edge you will stop. So you will get an alternate path of red and blue edges. The claim is this process has to stop at some point. And once this process stops we can recolor this particular path without requiring additional colors. So let us see how that can be done.

At any point when we stop it means there is no edge to be taken for the other color. So what it means is we could have changed the blues to red and the reds to blue and still things would work fine. Why is that so? So let us say this is yn and xn. The process stopped at yn because there was no red edge to take to go to the other side. That is guaranteed because this is a bipartite graph and because all the edges go to the other side. So you can change this back to red.

But if you change that back to red there is a violation at xn but we can change that back to blue. So we can alternate in this particular manner and get another valid coloring. So now what happens is, initially the color that was used at vertex x was blue and now that blue has changed to red and since we initially assumed that red color was not used and the blue was now a free color. So when blue is a free color, that is the same color that is free for vertex y and therefore we can color the edge x y using the blue color. So that is the proof.

So let us just quickly recap the proof. We looked at the graph and we removed one particular edge. The smaller graph can be colored using delta colors. Now the larger graph that you get by introducing x, y into the graph, into the residual graph can also be colored using delta colors because we could start at the vertex x and keep on alternating using the two colors that were left unused at the vertices x and y and once you get these alternating path of colors you can swap the colors on the path to recolor the edges. Once you recolor the edges the vertices x and y both will have the same unused color and that color can be used for coloring the edge x, y. So we will now move on.

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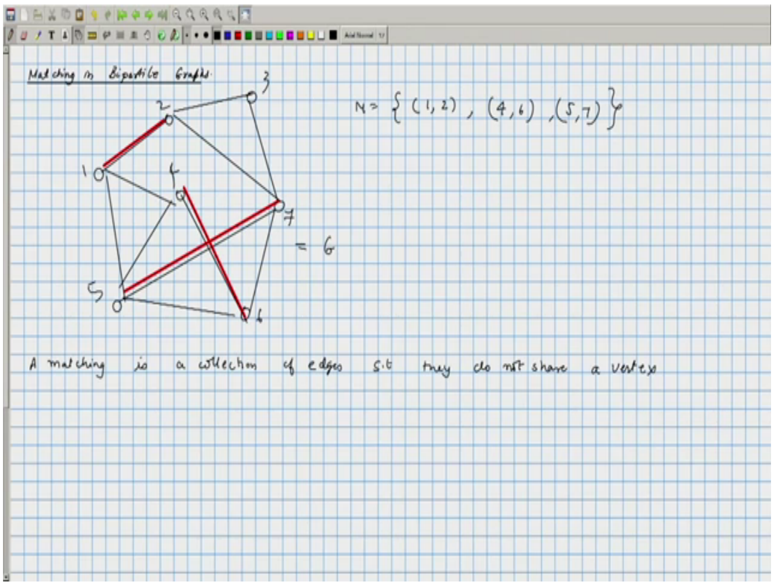

We will look at another property known as matching. So matching is, you can find matching in general graph but here we are going to restrict our attention to bipartite graphs. Let us understand what is a matching. We will start with the general graph itself. This is a graph on 5 vertices, let us say 7 vertices. A matching is simply a collection of edges such that they do not share a vertex. So the red colored edges will basically form matching, that is an example of matching, so I have drawn 3 red edges in this graph and this forms the matching.

If you number vertices as, if you number the vertices as 1 to 7 the matching will consist of the following edge, the matching which is described here using the red edges will consist of the following edges; 1, 2, 4, 6 and 5, 7. Look at any 2 pair of edges inside this collection. They do not share a vertex. This is also a maximal matching in the sense this particular graph G, if we call this graph G, this graph G cannot have a matching of size greater than 3. It has only 7 vertices and each matching edge will take 2 vertices with it. So the maximum number of such pairs that you can obtain is certainly less than or equal to 7 by 2 and 3.5 being a non-integer, the maximum possible is only 3. So 3 is the maximal matching as well as the maximum matching. So let us see few other examples.

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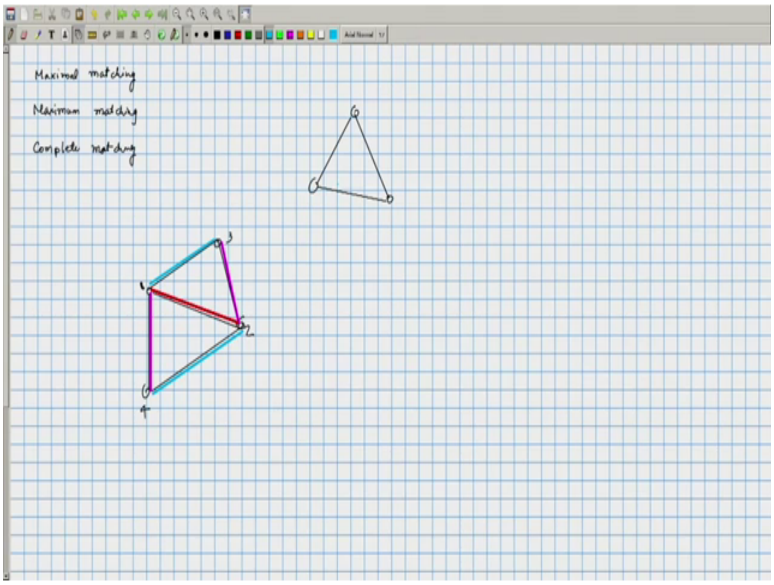

So we will look at these concepts more carefully; maximal matching, maximum matching, complete matching or perfect matchings. So maximal means there is no additional edge that one can add to this collection that is given to you to get a matching. Let us see an example. So if you look at this particular graph G and if you just look at the red edge, if you look the red edge that forms a maximal matching because you cannot add any more edges to this collection without violating the matching property because all the other edges must share a vertex with either vertex 1 or vertex 2.

But this is clearly not the maximum matching in the sense there are larger matchings that can be obtained. For example, if you look at the blue colored edges that is an example of a maximum matching in the sense this cannot be further extended. But this is not unique maximum matching. The pink edges will also form a maximum matching. In this case this is also a complete matching or a perfect matching because all the vertices have been matched.

If we had looked at a different graph namely the triangle, the maximal matching as well as the maximum matching will consist of precisely one edge. If you take any particular edge that is going to be a maximum matching you cannot further extend it. So you cannot get a matching which cover all the vertices. So there is no, are no complete matchings in this particular graph. What we will see in the remainder of this lecture is a characterization for when a bipartite graph has a complete matching?

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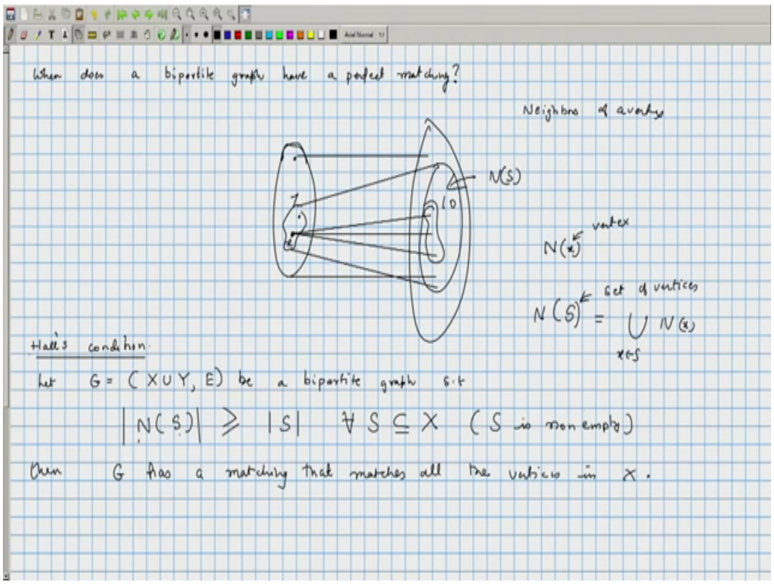

So this is the question we will answer. When does a bipartite graph have a perfect matching? So we will focus our (attention) we will restrict our attention to bipartite graphs which have equal number of vertices when we talk about perfect matchings because if one side had, let us say 7 vertices and the other side had 10 vertices we cannot clearly match all the all the vertices but we can hope to match 7 of them.

So if we match 7 of them we could still call that as a complete matching although it is not a perfect matching. So we will look at one side of the graph and we will try and figure out when can all the vertices on one side be completely matched and we will call that as a complete matching. The perfect matching case can be handled by this because all that you have to check is the other side should contain an equal number of vertices. So the other side contains 7 vertices, our (cri) our criteria will basically help us figure out whether it has a perfect matching or not.

So condition is given by Hall's theorem. This is known as Hall's condition. We will require the notion of what is the neighbors of the vertex to state this condition. So you look at one particular vertex. It is connected to many other vertices. These vertices would be called as neighbors of x and we will denote it by Nx, okay so here x is a vertex and Nx is a set of neighbors of x. If instead of vertex, this is a set of vertices, let us say S, so this is a set of vertices which we will assume is not empty, NS would be basically union of x belonging to S Nx.

So this is your set S and all the neighbors of this together would be called as NS. So clearly one can imagine that the number of neighbors is strictly less than the number of elements in a set then there is no possibility of a complete matching because those vertices cannot be matched. It does not have enough number of counterparts on the other side. If this condition is met for every subset, then there is perfect matching. That is what Halls condition says. So let us write it down formally.

So let G is equal to X union Y. So the vertex set I am just writing it as X union Y where X is one side and Y is the other side, E be a bipartite graph such that the size of neighbors of S, if you look at the size of this, that is greater than or equal to size of S for all S subset of X then G has a matching that matches all the vertices in X. So everything in X can be matched and if the number of vertices on the other side is equal to the size of X then we know that it is a perfect matching.

So let us look at the condition carefully. The number denotes the set of neighbors of S and the size of that should be greater than the size of S for every subset of X. If this condition is met, this theorem or Hall's Theorem guarantees that there will be a perfect matching in the bipartite graph G. But one direction is very easy if, for some set S if size of that set is greater than the number of neighbors then clearly that set at least cannot be matched. So what we will prove is when NS is going to be greater than or equal to S for every subset S, G will contain a perfect matching.

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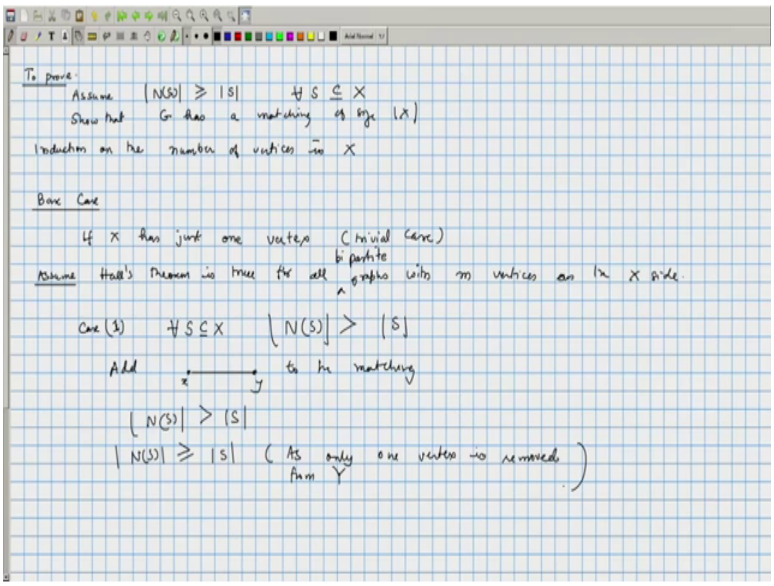

So this is what we have to prove. We will assume neighbors of S is a set of size larger than the size of S for all S subset of X. So with this assumption show that G has a matching of size X. So we will split the proof into 2 cases and we will do the proof based on induction, so proof method is again using induction on the number of vertices in X. If X had just one, if X had just one vertex this is a trivial case because that one vertex should be connected to some other vertex in Y because otherwise the neighbors is not going to be greater than or equal to size of S.

If it is one neighbor, then of course we can choose that particular edge and that will be a matching whose size is equal to the size of X. Now let us take a graph with, we will assume that statement is true for all graphs with m vertices on the X side, all bipartite graphs. So there are 2 cases, for all subset S of X, non-empty subsets, neighbors of S is greater than the size of this is greater than the size of S.

So now in this case we can just take any particular edge. So let us say we choose one particular edge x y, add it to the matching and now look at the remaining graph that is the graph obtained by removing vertex, vertices x and y from the from the original graph. So when you have removed this the degree is going to reduce by at most 1. We already had the condition that NS was greater than size of S for all S, so now after removal of vertex x and vertex y, NS will be guaranteed to be greater than or equal to size of S. As only one vertex is removed from Y we know that for every set S the number of neighbors is greater than or equal to size of S.

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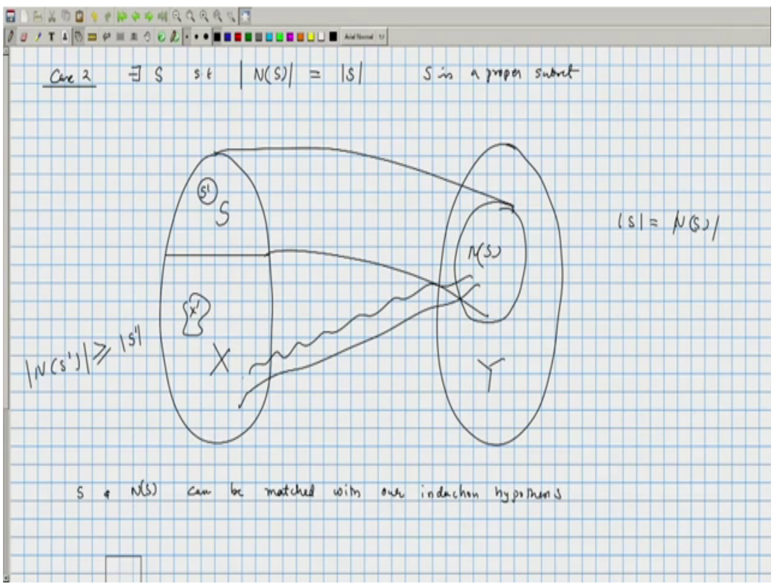

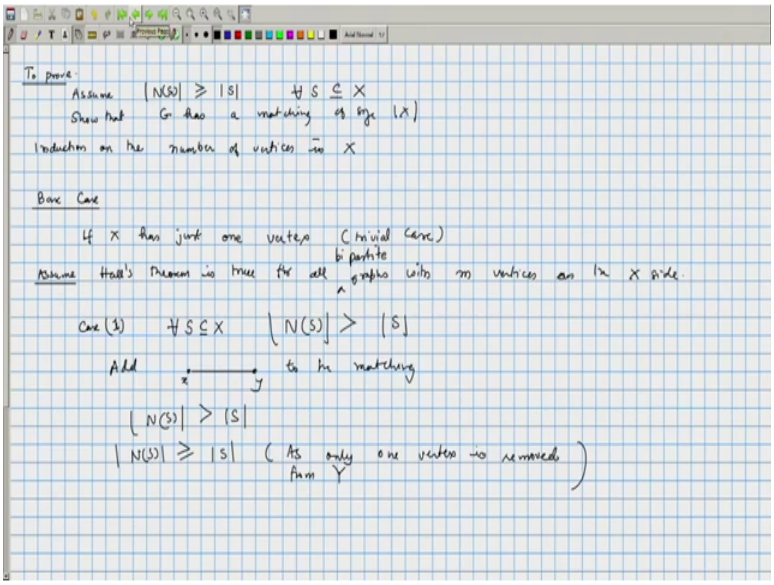

So in case 2 there exists a set S such that neighbors of S, its size is equal to the size of S. This is a slightly tricky case, be careful about this. So let us say this is our set X and this is our set Y. And we want to find a perfect matching between these. And we have a set S here, so let us say that is a top portion of this part X and number of neighbors is exactly equal to the size of S.

So let us say this is mapped to this particular region. This is neighbors of S and we have size of S is equal to size of neighbors of S. Now if you look at, so this subset should be there exist a proper subset, we can assume that S is a proper subset. So our earlier case was, I mean, for every subset N of S was strictly greater and now we are going to assume that there is proper subset for which the size is exactly same. Now this is the case, let us restrict our attention the set S.

We can apply the induction hypothesis on this set S and get a perfect matching for S involving just neighbors of S. So S and NS, now since S is a proper subset we know that the number of elements in S is strictly less than the number of elements in X. So S and NS can be matched with our induction hypothesis. Why is this so? Look at S as a set, I mean, so look at the induced graph that you obtain by considering just the vertices in S.

Now that induced graph, that is also going to be a bipartite graph and that bipartite graph is going to be completely restricted to this portion because all the neighbors were in NS. Of course there could be edges which come from neighbors of S to the remaining portion but those edges we are ignoring by just looking at S and NS. If you look at any subset of S, its number of neighbors, when all its neighbors are going to lie in NS NS because NS was the neighbors of the entire set.

So any subset's neighbor should basically be in NS and because of our condition that every subset had at least as many neighbors as its size. We know that every subset will also satisfy this property. So if you look at a subset S prime, neighbors of S prime, that set size is going to be greater than or equal to size of S prime for every subset of S, even while we are restricting to just the graph formed by vertices S and NS. So this portion involving the top half of X, involving S and NS, they can be perfectly matched.

Now what about the remaining portion? Can the remaining set X be matched to Y? The only problem is if we look at subsets inside X, let us call it as X prime. They might have certain neighbors but some of those neighbors could be in NS but we will show that, that is even when some of the neighbors are in NS, our condition that NS is greater than, the neighbors of X prime is a set of size larger than the size of X prime will be a valid assumption. So what we already, what we have right now is S and NS can be matched.

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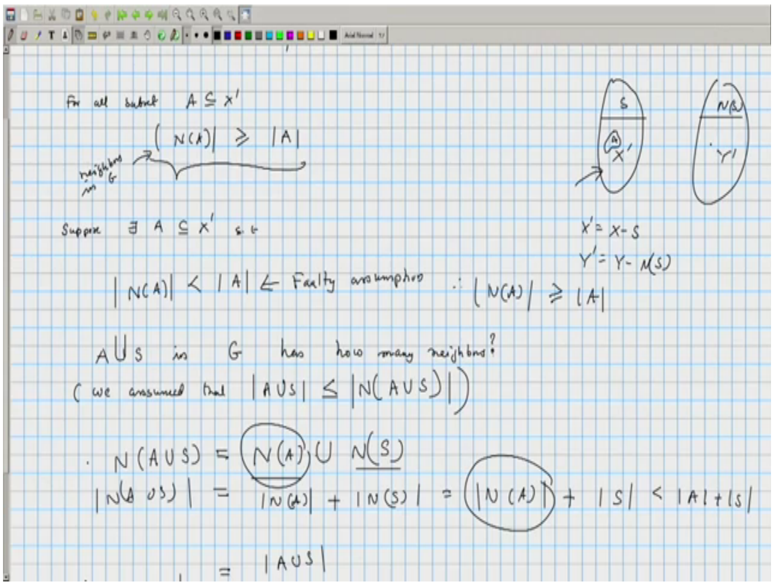

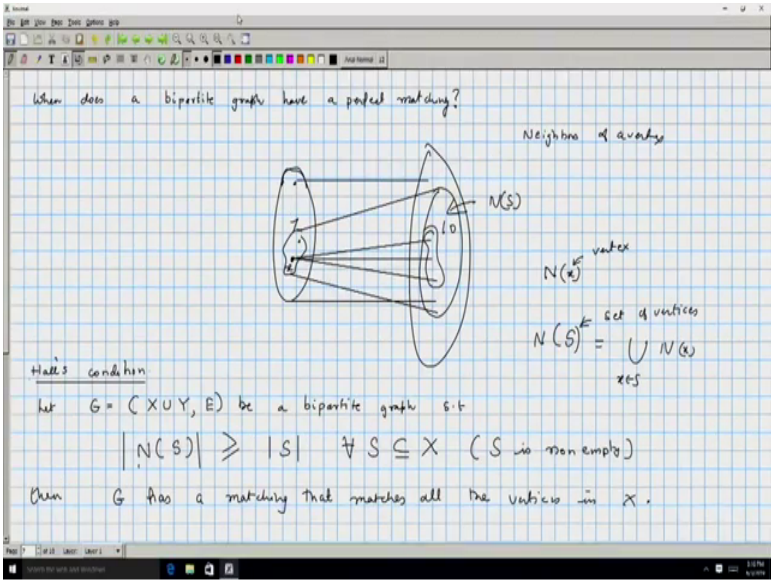

And what we need to show is X prime and N of X prime restricted to Y minus NS can be matched. This is the diagram. X is S was matched to neighbors of S and this portion we called it as X prime. So X prime is equal to X minus S and let us call the remaining portion as Y prime here. So Y prime is equal to Y minus NS. So we want to show that X prime and Y prime can be matched.

So we we can apply induction hypothesis provided we can show that for all subsets A of X prime N of A is greater than or equal to size of A. This condition was true earlier but at that time we would have been, when we look at a subset of X prime and when we count its neighbors, some of those neighbors could be in NS. But now we are allowed to count only those neighbors which belong to Y prime but we will show that this condition is still true, so why is that so?

So let us assume the contrary. Suppose there is a subset A of X prime such that this condition is violated. Suppose there exists a subset A of X prime such that N of A is strictly less than, the size of N of A is strictly less than size of A. Then we will show that there is a subset in the original graph where Hall's condition is not met. So the particular subset is easy to construct. Let us look at A union S. So A union S in G has how many neighbors?

By our assumption A union S in G had at least as many neighbors as the size of A union S. Neighbors of A union S, its size is greater than size of A union S. This is our assumption. We will show that condition is now violated, so here this, when I say NA, this stands for neighbors in G prime. So I am abusing the notation N a little bit. Sometimes we use that to denote the neighbors in G, sometimes we use it to denote the neighbors in G prime. But the context makes clear as to which is the meaning that we are giving to N of A.

So when you look at N of A, A is the subset of X prime so we are looking at the neighbors in X prime. So size of NA union S, this set is equal to neighbors of A union neighbors of S and this is a disjoint union. So clearly every element of N of S has to be there in neighbors of A union S and clearly neighbor of A also has to be in this particular collection, now neighbor of A is when it is restricted to Y prime.

So now N A union S, its size N A union S is equal to size of NA plus size of NS. But size of N S, we had it to be equal to size of S. So this equals to size of NA plus size of S. And size of NA, its size is less than size of A so this will be less than size of A plus size of S. So what we have here is, and size of A plus size of S, that is A and S does not have any common elements so that is size of A union S. So here we have the following inequality, N A union S is of size strictly smaller than A union S.

So that is the contradiction and therefore our assumption is wrong namely where we assume that N of A is less than A is a faulty assumption. So that would mean NA is going to be greater than or equal to A. And now we can apply Hall's Theorem to the smaller set X prime and we can get a perfect matching, we can get a matching of size X prime between X prime and Y prime.

Combine these two matchings and you will get a matching in the whole graph. So what we have accomplished is the following; in any bipartite graph if the numbers of neighbors of S is greater than the number of elements in S for every subset of X where X is one side of the graph then the graph will definitely have a perfect matching. Stop here for today.