**Discrete Mathematics**
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**Lecture 11**
**Introduction to graph theory**

Welcome to lecture on graph theory. In this section, we will learn about various kind of graphs. You can think of functions and there graphs but these graphs are very different graphs the only commonality between the two of them is that both are pictorial representations of we can think of them as pictorial representations of certain mathematical objects or graphical representation of certain mathematical objects. So let us formally define what is are graph?

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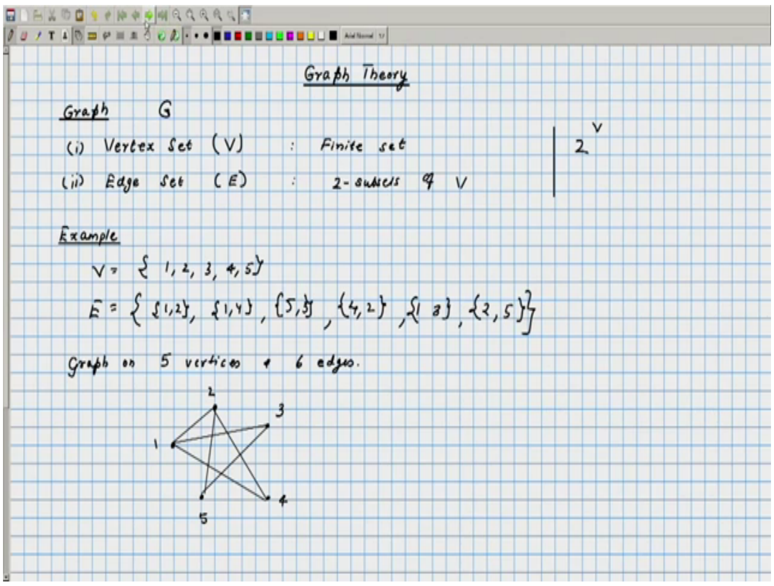


So it is usually denoted by letter G and the graph will have two components, the first is going to be call as vertex set which we will denote by the letter V. This is a finite set and this finite set we will regard it as the vertex set. The finiteness assumption is not sacrosanct we can have graph where the vertex set is infinite, but for time being we will restrict ourselves to finite sets. And then the second component of graphs is the edge set which we will denote by e. So this is again a set so we will think of them as two subsets of V.

In other words look at the power set 2 power V from this consider all the subsets of size 2, any subset of that will essentially be the edge set we will see an example. Suppose our vertex set is, let us say 1, 2, 3, 4, and 5 our edge set is going to be 2 elements of sets or edge set is going to be a subset consisting of two elements subsets. If you think of this particular graph it has two components, the first being the vertex and the second part being the edges.

So here the edge set consist of six elements, so we can say that this is a graph on 5 vertices and 6 edges. The usual diagrammatic representation of this graph is as follows, we will have 1 dot or 1 point for each vertex. So, 1, 2, 3, 4 and 5 were the vertices and for each spare of it we just connect those corresponding dots, so 1, 2 is an edge and 1, 4 is an edge 5, 3 is an edge 4, 2 is an edge 1, 3 is another edge and 2, 5 is an edge. So this is the graph that we have in mind.

So what we have drawn here is the diagrammatic representation of this particular graph. So graph you can think of as a network of nodes which are connected each other and which node is connected to which node is captured by the edge set. So where do we look at these things, let us first see a problem many different problems in mathematics can be modeled as graph theoretic questions and then can be solved.

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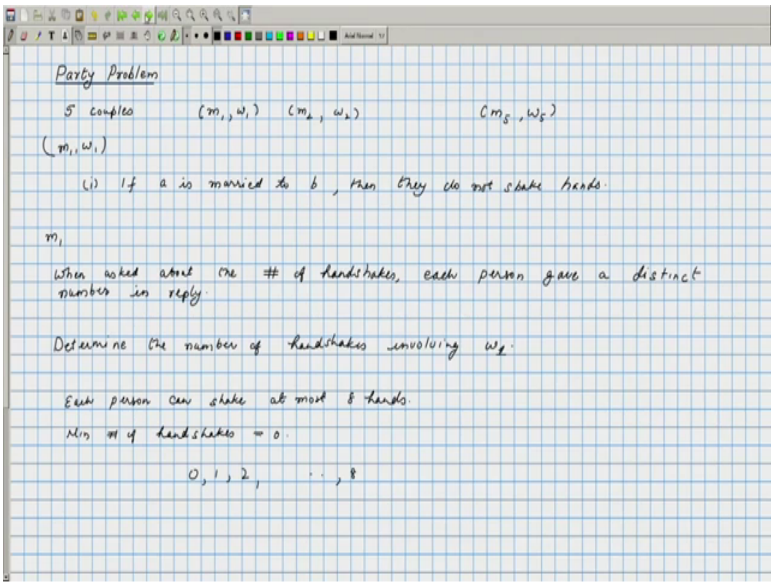

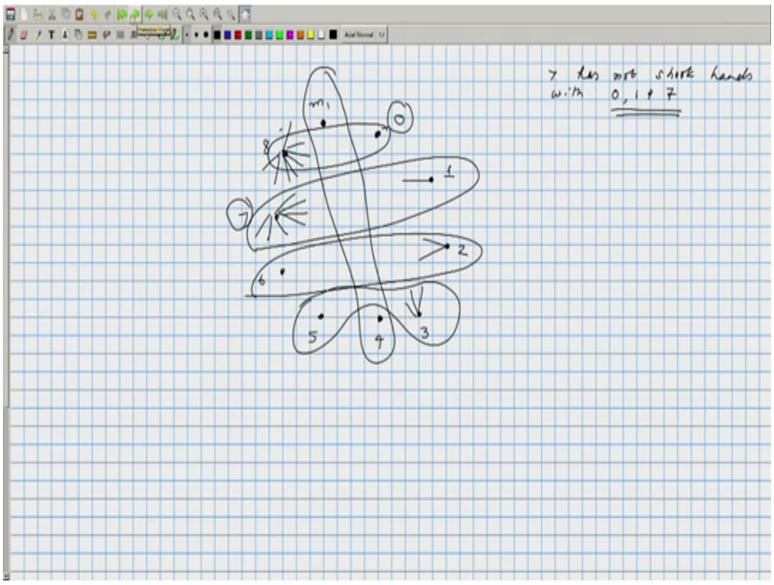

Let see one problem this is known as the party problem. So this is description of the problem, so there are let us say 5 couples, so let us call each couple as m1 w1, m2 w2, and so on m5 w5 are your couples and one of these couples let us say m1 w1 they decide to host a party. So there is a party which will consist of 10 people, and these 10 people when they meet some of them know each other before and some of them do not know, and people shake their hands. So each person may or may not shake hands with another person.

But the rules of this problem is that spouses do not shake each other’s hands. So if a married to b or if they are partners then they do not shake hands, this is the setting. And what happens after the party is over m1 asks everybody else, everyone ask all the other nine people as to how many hands did they shake? With how many people each of them exchanged the hand shake? And they give their answers and what turns out is that each one of these people had a distinct number of hand shake. So we will write that as the outcome.

When asked about the number of handshakes to each person gave a distinct number as reply. Now the party problem is the following, with this information can we determine how many people shook hands with w1. So determine the number of handshakes involving w1. So its looks surprising because there is hardly any information but we will see that the amount of information that is already present is enough to determine this number. So m1 was a person who had asked the people about the number of handshakes and the nine other people gave answers and all their answers were different.

Form this information we can figured out the exact number of handshakes involving w1. So how we solve this? I mean how do we determine this? So when we know that the number of handshakes were different for each person and each person can shake at most 8 hands, so that is a fact. And the least number of hand shake is 0. Minimum number of handshakes is equal to 0.

So since each of the nine people gave distinct answers, there answers has to be 1, 2, 3, up to 8. So each person the number of handshakes is some number distinct number between 0 to 8, so that is all we know. Let us draw this diagrammatically we will think of a graph, so each person is a vertex in this graph consist of 5, 6, 7, 8, 9, 10 these are the 10 people. Now m1 we will write it as this particular vertex and all the other vertices we will just number them by the number of handshakes.

So surely there is one person who has shook 0 hands, there is another person who has shook 1 hand, another person 2, another 3, 4 and so on 8. Now we could think of a graph in which these are our vertices and there is an edge between two vertices, if there is a handshake involved in two of them. So if knew this graph then of course we can determine and if we had this complete graph, then from that we hope to we know who is this spouse I mean how many handshakes are there involve in the person who is the spouse of m1.

But we do not know much about this graph other than the fact that if you look at the number of edges connecting 0 there are no hand shake involved with 0, so 0 is not going to be linked to any other vertex. One is going to be linked to some particular thing we do not know which among these is connected to 1. From 2 there are 2 such things where exactly (())(11:56) we have no clue, similarly for 3 and similarly for 8. So there are 8 edges out of 8, 7 edges out of 7 and so on.

And for m1 we do not know how many edges are there from m1, how many people shook hands with m1 we have no idea, but from this diagram itself we can infer some certain things 8 and 0 must be married to each other. The reason is as follows, any person can shake at most 8 hands he does not shake hands with himself, he does not shake hands with his spouse. There are 10 people if you take these two out there are 8 people. So 8 if somebody has shook 8 hands his spouse is a person who has shook no hands.

Because the only other person with whom he did not shake hands is himself. So 8 and 0 they are a couple. Now if you look at the person who has shook 7 hands and if you look at the person who has shook 1 hand. This 1 hand shake person is means has certainly shook hands with the $8^{th}$ number person, and he or she has not shook hands with any of the other people. 1 clearly has had a handshake involve in 8, and since all his handshakes are accounted for we know that he has not shook hands with anybody else.

Whereas, here we have a person who has had 7 handshakes. The 7 seven handshake person would have 3 people with whom he has not had a hand shake. 1 of those people is a 0 hand shake person. The other is himself or herself who is the third person. 0 is not the spouse of 7, there is a another person with whom 7 has not had a hand shake with. If you look at 1 is only handshake is with 8 and so 7 has not had a hand shake with 1, that is clear. More over 7 has had hand shake with every other people, in other words 7 has not shook hands with 0, 1 and 7.

So 0, 1 and 7 have not are not involved in handshake involve in 7, so the others cannot be 7's spouse. 0 is certainly not spouse of 7, 7 is also not a spouse of 7. So 7 and 1 are a couple. You can recall on this logic and you will we can conclude that 6 and 2 are a couple and 5 and 3 are also a couple. Now after all these 4 couple have been accounted for what is remaining is m1 and 4, they must be a couple, and the questioned that we started off with was how many hands did m1's partner shake?

And answer clearly is 4 because this particular person shook hands with 4 people. So in a nut shell what we looked at is we constructed a graph where there is an edge between two vertices, they had shook hands with each other, if they had a hand shake. Once we constructed this graph, we could pair of couples and once we paired up couples what remained was m1 and the person who has shook 4 hands, and therefore m1 that person is going to be m1's partner and that helps us determine the number of handshakes at m1's partner has exchanged. So we saw graphs and we saw an interesting problem involving graphs.

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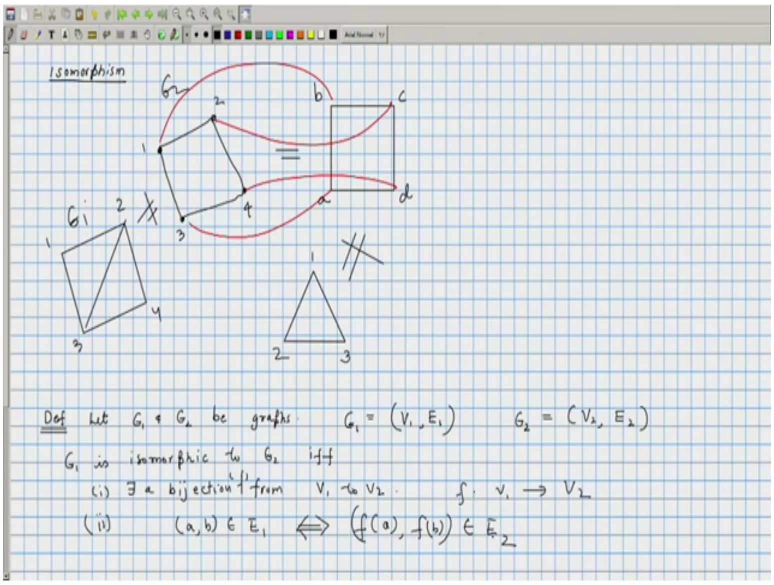


Now let us look at this problem of when are two graphs equal. Once we defines a mathematical structure we want tell when are this two mathematical structures same. And in the motion is that of isomorphism if we had a graph of this kind. So, this are vertex 1, 2, 3, and 4 and another graph where the vertices are named a, b, c and d they are essentially same object. So, this is an motion that we want to capture in terms of isomorphism. So, this are equal and that s is not equal to you can say 1 2 and 3.

So, this the motion that we want to capture via isomorphism so, let us provide a definition. So let g be the graph and let g1 and g2 be graph and we will assume that g1 is v1, e1. So, vertex set is v1 and edge set is e1 and for g2 we will assume that this equal to v2, e2. We will say that g1 is isomorphic to g2. If, certain conditions hold the first requirement is there should be bijection between vertex sets. For example in this example that we have taken we could map 1 to b, 2 to c, 3 to a, and 4 to d.

Not only that once we have map this the pair of vertices which had an edge. When you if look at any pair of vertices such that there is edge between them if you look at the image there is an edge as well. If this two conditions are met then, we will say that the graph are isomorphic. Let us quickly do 1 example if, we had this particular graph 1, 2, 3, 4 note that we could map vertices each other. The vertex 1 in g 1 could be map to vertex 1 and g to 1 and so on.

But, if you do that there is an edge 2, 3 which, is unaccounted for there is no, I mean if you are trying to map g1 to g2 there is an edge between 2 and 3 but there is no edge between the images of 2 and 3, oaky? So this is not an isomorphism so formerly g1 is isomorphic to g2 if there exist a bijection from v1 so, let us called this bijection f from v1 to v2. So, f maps v1 to v2 and the second condition is if a, b belongs to e1 implies and it is implied by f of a, f of b, belongs e2. This is the definition of isomorphism.

So basically we want to rename the vertices in one graph by just we want to rename the vertices in such a way that if there is an edge between two vertices then the renaming also preserves that edge. And further if there is an edge in renaming then when you precompute or look at the back image the graph should have an edge there as well. This condition is met then we will say that the graphs are isomorphic. So let us see couple of example of graphs. So determining whether 2 graphs are isomorphic that is a difficult question. Let us see some examples to understand why this might be difficult?

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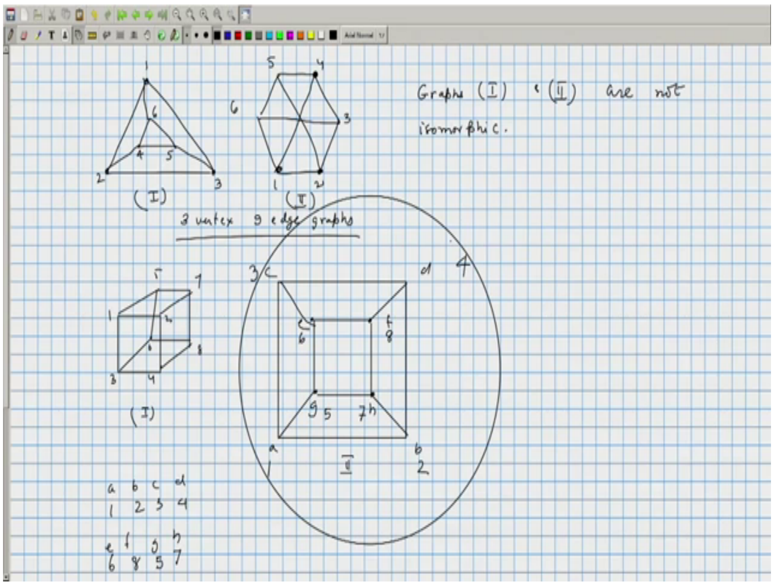


So, let me construct couple of graph and you can ponder about it whether these graphs are actually isomorphic or not. So this graphs has 6 vertices 1, 2, 3, 4, 5, 6 this is also a graph of 6 vertices. There are 9 edges here 3 for the outer triangle and 3 for the inner triangles and 3 connecting them and here are there also 9 edges. So, 3 vertex nine edge graph there are two of them it is drawn here. Now are these graphs isomorphic? You can argue that these are not isomorphic because, if you take there is in graph 1 there is triangle that means there is collection of vertices such that they are all adjust to each other there is collection of 3 vertices which are all adjust to each other. Whereas in the graph 2 no two vertices have, I mean if you take any two vertices which are connected they never have a common neighbour. So you look at 1 and 2 they are connected and they do not have a common neighbour and that takes care by cemetery if you look at 1 and 2 that takes care of lot of edges like 1 2, 2 3, etc. 3 4, 4 5, 5 6 and 6 1 are taken care of. If you look at 1 and 4 that is the other kind of edge they also do not have a common neighbour. So that takes care of the entire graph. So, if we can argue that these graphs are not isomorphic, so it is tricky to find the reason for why certain pair of graphs may be non-isomorphic. Let us take couple of other examples, so here is a graph you think of it is a cube with 8 vertices.

And here is an another graph with 8 vertices. So let us call this a graph 1 and graph 2, are these graphs isomorphic? In fact they are and you can think of them as same they are the same objects mathematically. So if you think of this as a, b, c, d, e, f, g, h you can map this numbers in such a way that the edges are preserved across the map. You can think of a, b, c, d as 1, 2, 3, 4 and e, f, g, h as, so, this is your a, b, c you have to be careful while you write the map a, g is an edge. So this is 5 a is 1, b is 2, c is 3, d is 4, g has to be 5, h has to be 7, f has to be 8 and, e has to be 6. So can look at this map so this map is an isomorphism between these two graphs.

So the cube if it is laid out flat on a plane you will get something like this and that placement essentially preserves the edges, and therefore we can argue that they are isomorphic. So these examples more or less they look like different graphs, let me give you yet another example where it is not straight forward that the graph are isomorphic or are not isomorphic.

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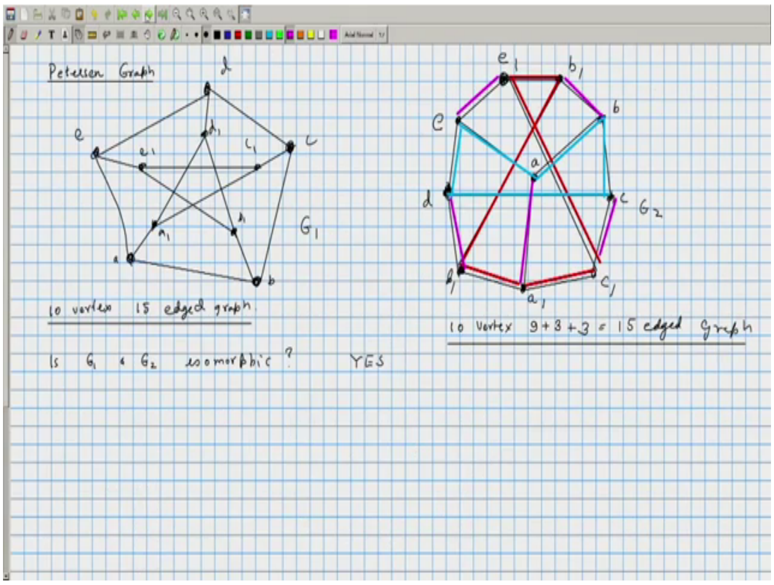


So, this is special graph called as Peterson graph. So this is a graph on 10 vertices. So, this is the Peterson graph there are 10 vertices and the outer pentagon has 5 edges and the inner star also has the 5 edges and then there are these connecting edges there are 5 of them. So this is a 10 vertex, 15 edge graph. Let us call this is g1 and I will draw an another graph think about whether these graphs are isomorphic. So I have 1, 2, 3, 4, 5, 6, 7, 8, 9 vertices. And there is 1 vertex in the centre this is also a 10 vertex graph and the connections are as follows.

So this is also nicely cemetery graph so between the centre vertex is there and then in between each of these spokes there are precisely 2 other vertices. And then this vertex is connected to this vertex, this vertices are connected and these vertices are also connected. So this is another graph again the outer circle cycle has 9 edges plus there are 3 spokes and then are 3 of these cross edges so there is total of 15 edges. Now are these graphs are isomorphic? Is g1 and g2 isomorphic? Can you somehow twist the graph g1 or g2 and get something when and get the other graph. Twist g1 and get g2, is it possible?

So we can show that they are isomorphic. Let us fix an isomorphism so we will name these vertices as a, b, c, d, e and a1, b1, c1, d1, e1. So let me just call this as called as a, b, c, d and e. Clearly there is a cycle involved in them and then so if you think of that is the outer circle the other vertices should form an inner cycle. So if I call it as a1 so a1 is connected to c1 and c1 is connected to so a c1 connected to e1 and e1 is connected to b1 and b1 is connected to d1 and d1 is connected to a1. So we can draw that as a we can think of that as the other cycle.

We need to have the other cycle a to cl, c1 to e1 and e1 to b1, b1 to d1 and d1 to a1 and other cycle was this and you can see that the corresponding edges e to e1, b to b1, c to c1, a to a1 and d to d1 they are also present. It is not easy to transform one of the others to give the other one, but, you can see that all this if you map it like this all the edge relations are essentially taken care of. So the drawings of the graphs may look different but they could still be isomorphic. So Peterson graph is an example of these are two examples two representations of the same graph these graphs are isomorphic.

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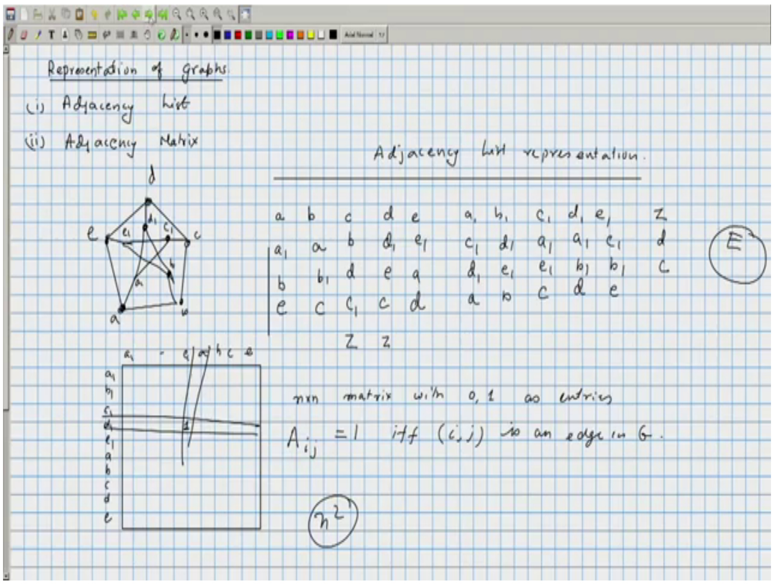


So the next thing that will learned about is, about representation of graphs. So how do we if we wanted to represent these graphs on a computer. If we had to write on a pen and paper these are all fine but we have to represent it on a computer how do we do it? There are two common ways one is called as the adjacency list and the second is called as the adjacency matrix. So let us take Peterson graph and see what is the adjacency matrix of presentation and adjacency list of presentation? So a, b, c, d, e so when you represent it as adjacency list for each vertex there is an entry.

So a is connected to if you look at vertex a it is connected to a1, b and e, b is connected to a, b1 and c, and c is connected to b, d, c1. And d is connected to d1, e and c, and e is connected to e1, a and d, a1 is connected to c1, d1 and a, b1 is connected to if you look at vertex b1 is connected d1 e1, and it is also connected to b. If you look at c1 is connected to a1 and e1 and it is connected to c, if you look at d1 is connected to a1 and b1and, it is also connected d, and e1 is also connected to if you look at e1 it is connected to c1 b1 and e.

So, this representation is called as the adjacency list representation, note that here this is a very nice cemetery graph each vertex has a corresponding list of length 3 but, if this is not a regular graph, so these kind of graphs is an example of what is a regular graph and, if you are not of this kind for example you hand a another vertex. So, let us call this as a z then there will be an entry corresponding to z and z is connected to only d and c and if you look at d and c there will be these entries corresponding to z.

So, this is not a symmetric one as the previous one. So this the adjacency list representation. So when you talk about adjacency list presentation for every vertex the u have a list telling which are all the other edges which have whichever edge with this particular vertex. Adjacency matrix representation is a slightly different one, so here you represent the entire information in terms of the matrix. So, a1, b1, c1, d1, e1 and a, b, c, d, e. So if you are looking at the Peterson graph the rows and columns are index by the vertices.

Further move if you look at the entry the d 8 entry this entry is 1 if an only if d1 a is an edge. In other words adjacency matrix is an n cross n matrix with 0, 1 as entries. So the ij entry so if you call the matrix as a, and a i j, as a i j entry this is equal to 1 if and only if i, j is an edge in g. So clearly every graph will have unique adjacency matrix unique up to the ordering of the vertices if your number the vertices as 1 to n and then use your rows and columns appropriately than there is a fixed matrix which contains all the information about the graph.

So that is the adjacency matrix representation. When we want to run on to algorithms on matrices on graphs we essentially convert it into one of these forms, these are the most standard forms used for graph algorithms and then our algorithm would process these objects. So adjacency matrix is going to be of size n square where as adjacency list is going to be of size proportional to the number of edges.

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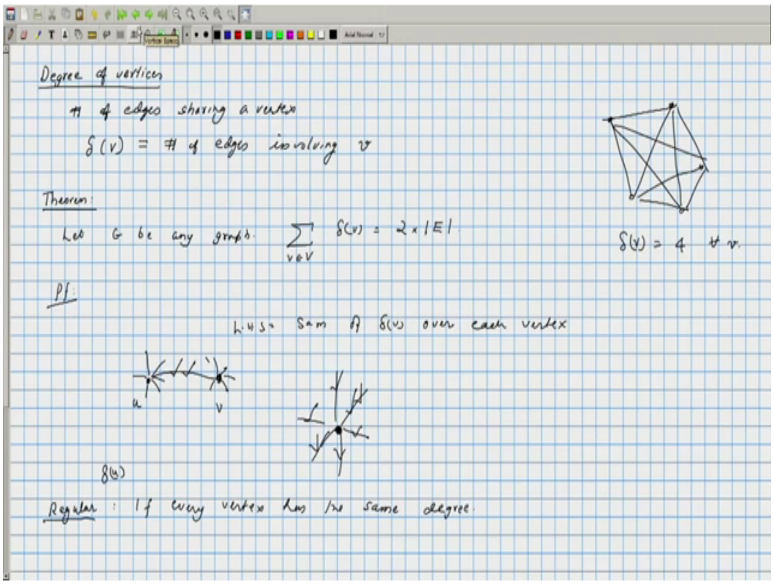


The next thing that we will learn about graph is something called as its degree, degree of vertices. So this is simply defined as the number of edges sharing a vertex. So degree is defined for a particular vertex. So it is denoted usually by delta. So delta v is equal to number of edges involving v. So if we look at the so if you look at this graph the degree of vertex 1 is going to be 1, 2 and 3 whereas degree of vertex 4 is 2. And degree of 2 is going to be 3 degree of 5 is going to be 2 and degree of 3 is again going to be 2.

If you add them up what you will get this case is 12. And this turns out to be two times the number of edges. So that we will state as theorem. So, it is not particular to this particular graph in any graph if you look at the degree of each vertex and sum it over all the vertices you will get twice the number of edges. So let g be at any graph submission over v belonging to the vertex set delta v is going to be equal to two times the number of edges. Proof is straight forward let us look at each vertex and look at how many outgoing edges are there add it up that is the quantity on LHS.

So, the left hand side is equal to sum over sum of delta v over each vertex. So each vertex you are looking at the number of edges at that particular vertex. Now if you think about this particular summation if you look at any particular edge each, edge is being accounted precisely twice. So if you look at any particular edge say between a vertex u and v when you are computing delta u this edge you a can take once and whenever you are counting delta v you can take it again. So, what we are suggesting a following when we are doing this counting over at vertex we just look at all out going edges at one particular vertex.

And whenever it is counted we just tick on those vertices. Now if you are summed up over every vertex you will know that each edge is ticked exactly twice. One for each of its n points. So every vertex would have been ticked exactly twice, I mean sorry every edge would have been ticked exactly twice. So total number of ticks that you put, by one count it is going to be submission over v delta v and since every edges stick exactly twice the total number of stick is just two times number of edges. So that is the proof.

A graph will have some definitions which will use later, we talked about regular or regularity when we talked about the Peterson graph, so we will call a graph as regular if every vertex has the same degree. So if you look at this particular graph the cycle there is a regular graph because every vertex has degree 2. It will also take the complete graph so here every vertex is connected to every other vertex and delta v is equal to 4 for all v take any vertex its degree is going to be 4. So that is the notion of regularity.

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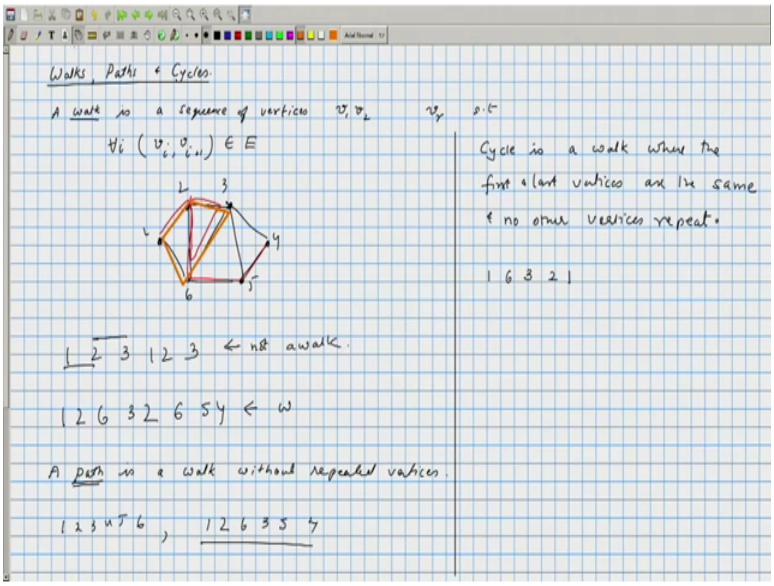


Now let us introduce some more terms some more basic terms in graph theory. We will learn about walks, paths, and cycles. So walk is the easiest to describe a walk is just a sequence of vertices, such that between the consecutive vertices there is an edge. So a walk is a sequence of vertices if you are given a particular graph a walk in that particular graph. So I have not written it down but that is what it mean. A walk in graph is a sequence of vertices v1, v2, vr some of this could be repeating does not matter, but the only requirement is for every valid i, vi, plus 1 is an edge in the graph.

So i can range from when we say valid i, i can range from 1 to r minus 1 so that is going to be called as a walk. A walk is just a sequence of vertices such that between adjacent vertices there is a between vertices adjacent in the sequence there is an edge. So if this was our graph if you take 1 to 3, 1 to 3 between 1 and 2 there is an edge and in between 3 and 2 there is an edge but between 3 and 1 there is no edge, so this is not a walk. Whereas if you take 1, 2, 6, 3, 2, 6, 5, 4 you can verify that that is walk but walk is nothing but the one indicated by particular red line.

So note that in this second walk what we have written there are repeated vertices a walk without repeated vertices is going to be called as a path. So that gives us a next definition a path is a walk without repeated vertices. So in a graph of n vertices the longest path can have at most n vertices in it. So if you take 1, 2, 3, 4, 5, 6 that is path if you take 1, 2, 6, 3, 5, 4 that is also a path and this path will have 5 edges in it.

So we have learned what is a walk and what is a path and the next thing is what is a cycle? So cycle is a walk where the first and last vertices are the same and no other vertices repeat. Natural to call it is a cycle if we have taken 1, 6, 3, 2, 1. This particular graph 1, 6, 3, 2, 1 so that is going to be a cycle. So no vertex repeats other than the first end last. So in some textbooks you will see that the last vertex is not explicitly mentioned and we will think about cycle involving those we will just assumed that the last vertex the first vertex are connected, but for this course when we say cycle we will just list out all the vertices path including the repeated ones. The first and the last one are repeated ones so, that is a cycle.

Now let us describe some more special kind of cycles.

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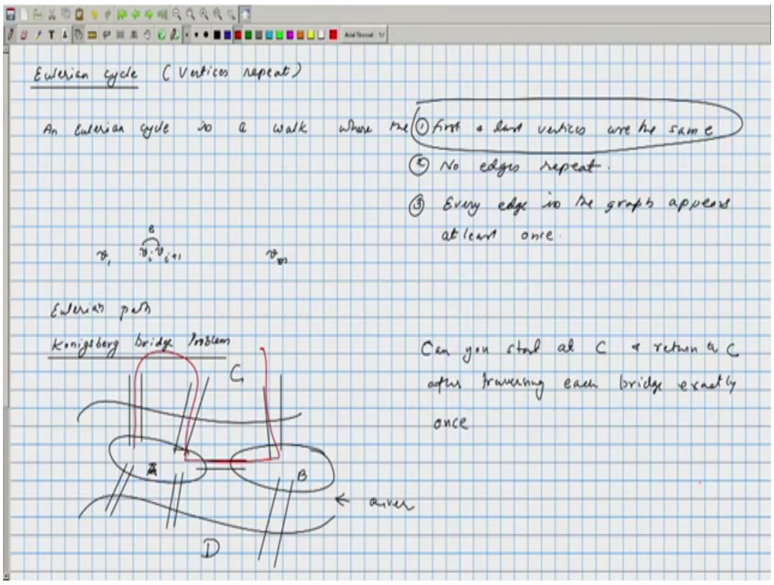


There are two types of cycle that are we will introduce one is something called as an Eulerian cycle and the other is called as Hamiltonian cycle. We can also refer to this as the Eulerian circuit. This is in the strict sense you cannot think of this is a cycle because vertices repeat. So Eulerian cycle is not a cycle in this sense of vertices not repeating, but it will be a walk Eulerian circuit is walk which starts at some particular vertex and ends back at a particular vertex. So we will formerly define a Eulerian cycle is a walk where the first and last vertices are the same.

So this the first requirement first and last vertices are the same that is notion of cycle. The second condition is no edges repeat, what does it means for edges to repeat after all we can think of a walk, we have all these sequence of vertices between any two vertices. So, vi and vi plus 1 we have an edge so you think of an edge appearing there and if you look at the sequence of edges that are being present in the walk, none of those edges repeat.

And the third requirement the crucial one is that, every edge in the graph appears at least once. So what it means for a graph to be Eulerian is the graph has an Eulerian cycle then we say the graph is Eulerian. There is also a notion of Eulerian path where all these conditions should be met but the first and the last vertices we will relax this condition that the first and the vertices are the same. So if just this condition 1 is extended than we will say that what is resulting as an Eulerian path.

So, now the question is a given a graph does the graph has an Eulerian cycle. So this problem was investigated by the great mathematician Euler. So Euler basically looked at what is now known as a Königsberg bridge problem. So this a problem was initially posed in the following way this is a river and there are two river islands and there is a bridge two either banks there are two bridges to either banks from this island 1 and from island and let us call this as a and from island b there are bridges to either of the banks.

So this is the river and there is a bridge connecting the islands. Now the question is can you start somewhere in one of the islands or on one of the banks and then come back to same place after visiting every after being on every bridge at least once. So can you start at a bank let us say this is a bank c and this is the other bank d. So can you start at c and return to c after traversing each bridge exactly once. It is also same as can you traverse this without lifting your pencil and without redrawing anything.

So one attempt would be start from here go like this but when you reach here their so no way you can complete that particular circuit. So that attempt does not work but that does not mean that does not mean that none of the attempt are not going to work. So Euler solved this problem and showed that this graph cannot be traverse, I mean this particular this is not yet graph but this can be converted into a graph and certain property of that graph would imply that there are no way of solving this particular problem.

So we will see that so when does a graph have an Eulerian cycle and when does a graph not have Eulerian cycle.

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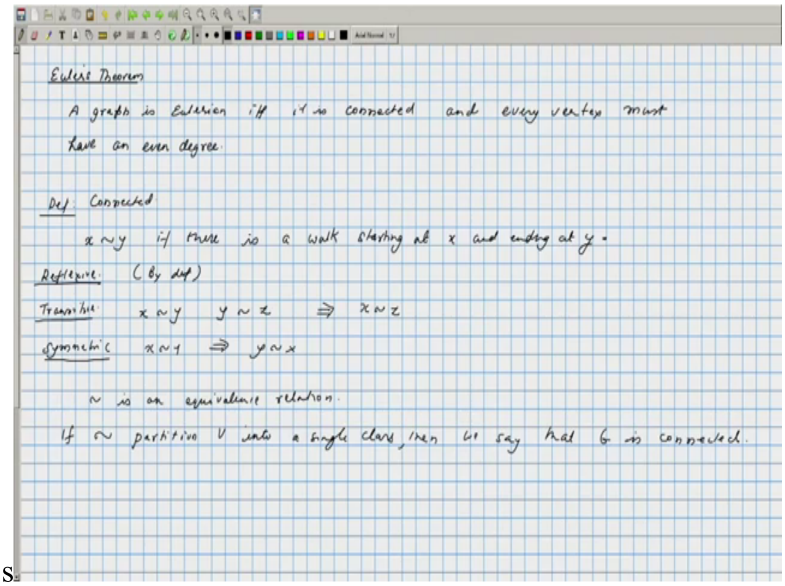


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So the following theorem characterise as it, so we will write as Euler theorem. One of the many many Euler theorems a graph is Eulerian that is it contains an Eulerian cycle if an only if it is connected, that is a requirement what does it mean to be connected we will see in a while and every vertex must have an even degree. So before we go into the proof let us define what is connected. So let us look at all the vertices on the graph and we can define a relationship.

So we will say x is related to y if there is a path starting at x and ending at y. For simplicity we can just think of I mean we can replay the notion of path with a walk makes a proof simpler. So walk mean that there could be a repeated vertices, path means there are no repeated vertices. If you had a walk you can easily, if there is a such a walk then you can just throw away the repeated vertices or the portion where the portion of walk where words are repeating and get a path.

But we will just stick with the notion of walk for defining connected. So this is a relationship between vertices. Now you can see that this a equivalence relation we will say that a vertex is automatically connected to itself. So it is reflexive by definition and it is transitive because if x is connected to you y and if y is connected z that mean there is a path starting from x or there is a walk starting from x to y and there is another walk starting from y to z. If you join these if you just write this sequence one after other you will basically get a path from x to z.

So this would imply that x and z are connected. So reflective transitive and it is also symmetric if there is a path from x to y then that would imply that y to x is also a path because you see same path in the reverse. So this relationship is a equivalence relation, so this an equivalence relation. As we know if you have an equivalence relation on a set that induces a partition on their on the underlying set, and that partition if it has exactly one equivalence class. So if the equivalence relation partitions v into a single class.

It means there is only one equivalence class, then we say that g is connected. So we define when up to vertices connected and that relation is basically an equivalence relation and that relation induces a partition and if this partition had precisely one equivalence class then we will say that the graph is a connected graph. So, Euler's theorem states that if you take any graph which is connected and if every vertex has even degree then the graph will have a Eulerian cycle. We will do a proof of this in the next section, we will stop here for today.