Discrete Mathematics
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Lecture 10
Embedding of the theories of integers and rational numbers in set theory

Welcome to NPTEL MOOC of Discrete Mathematics. This is the third lecture on set theory, right.

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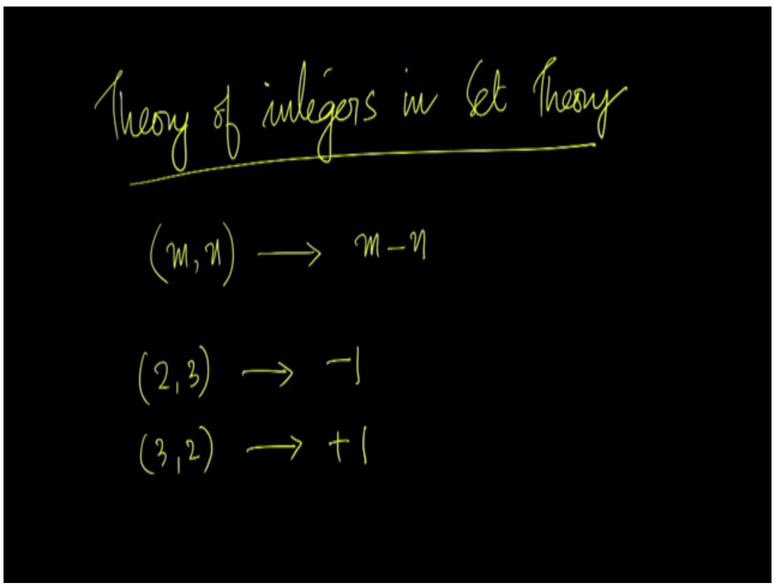

At the end of the last lecture we were seeing how the theory of integers could be embedded in the set theory. Our idea was this, an ordered pair of m n of natural numbers should stand for the integer m minus n. Then the ordered pair 2 3 would stand for minus 1; ordered pair 3 2 will stand for plus 1.

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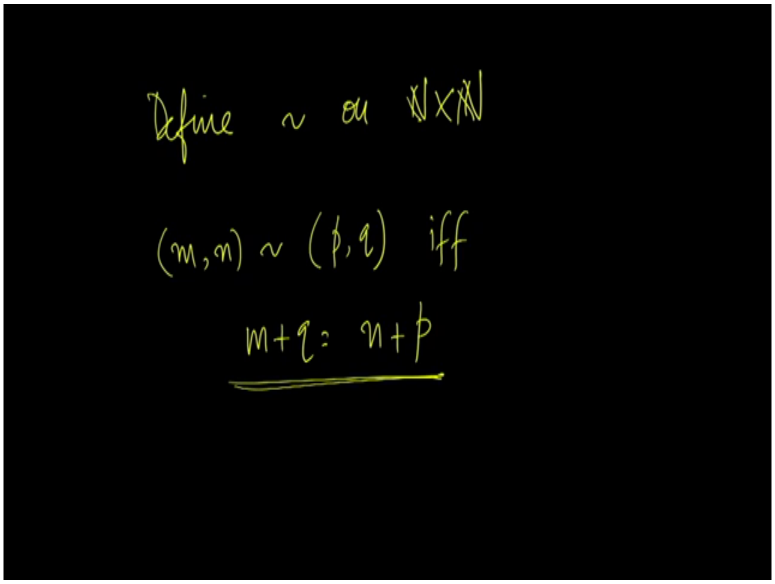

So, what we do is this. We define a relation till day on n cross n where n is set of natural numbers. The idea is that the ordered pair m n and p q would be in relation till day with each other if m minus n is equal to p minus q. But since natural numbers are not closed under subtraction, we essentially capture the same by using addition.

We write m plus q equals n plus p. We know that this would be the case precisely when m minus n is equals to p minus q and m plus q and n plus p are both well-defined natural numbers. So, we achieve the same end. We define the relationship till day in this fashion for ordered pairs m n and p q. M n is in relation till day with p q precisely when m plus q is equal to n plus p.

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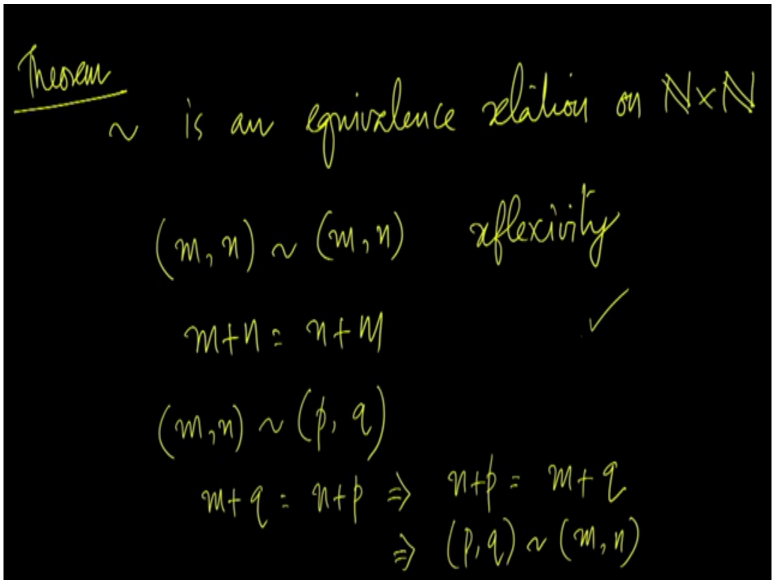

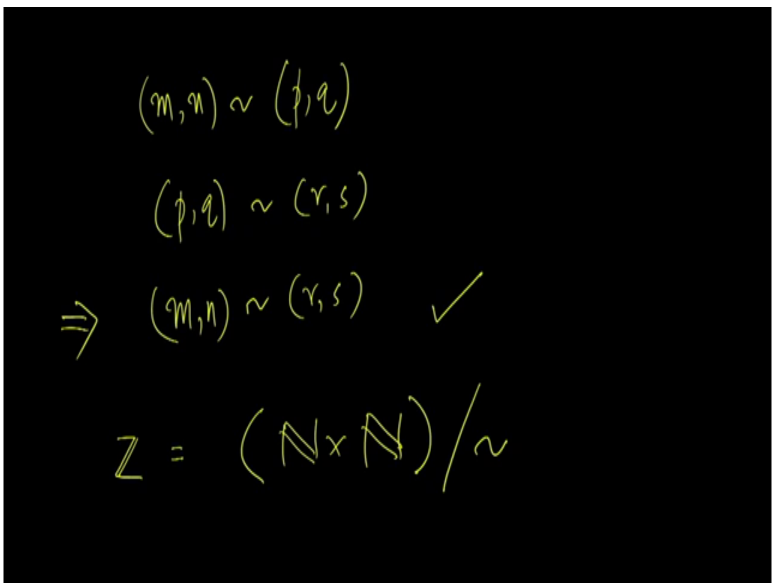

Then we can see that till day it is an equivalence relation because it is reflexive, symmetric and transitive.

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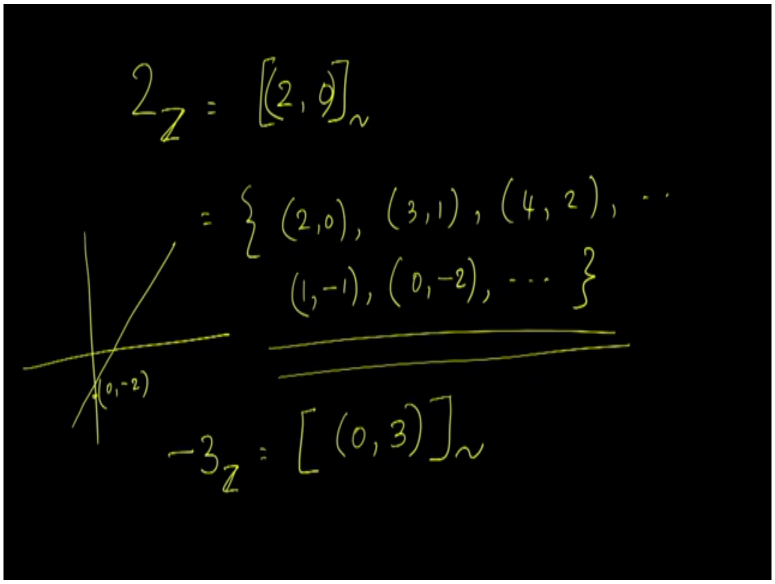

Then using this we can express integers in this fashion for example, the integer 2 which we denote 2z to distinguish it from the natural number 2 would be the equivalence class in which 2 0 belongs. So, this equivalence class contains 2 0 3 1 4 2 1 minus 1 0 minus 2 etc. All these points would lie on straight line with slope 1 and passing through 0 minus 2. Similarly, the integer minus 3 which we denote minus 3 subscript z to distinguish it from the natural number 3 would be the equivalence class containing the ordered pair 0 3.

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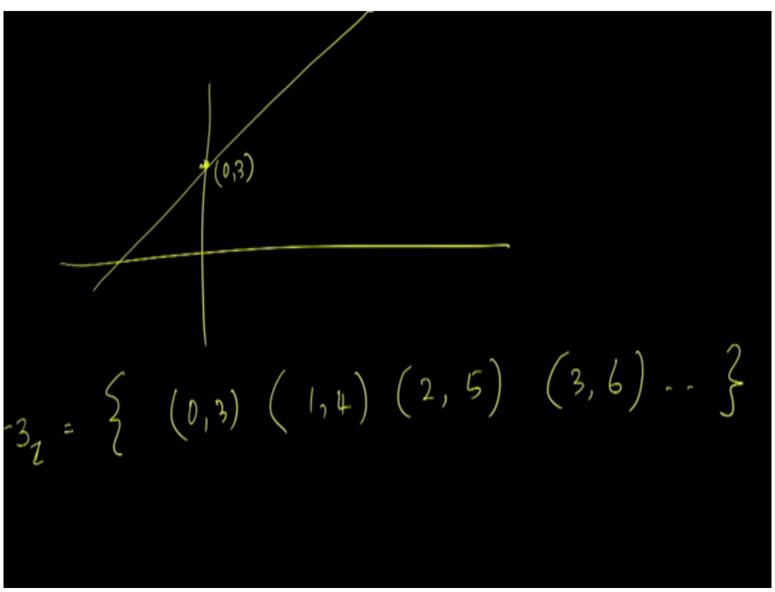

All the integral points belonging to the straight line with slope 1 and passing through 0 3 will also belong to the same equivalence class. So, all these points are equivalent under the relation t. So, that is how we define the integers. Now, comes the question. How do we define the integral operators?

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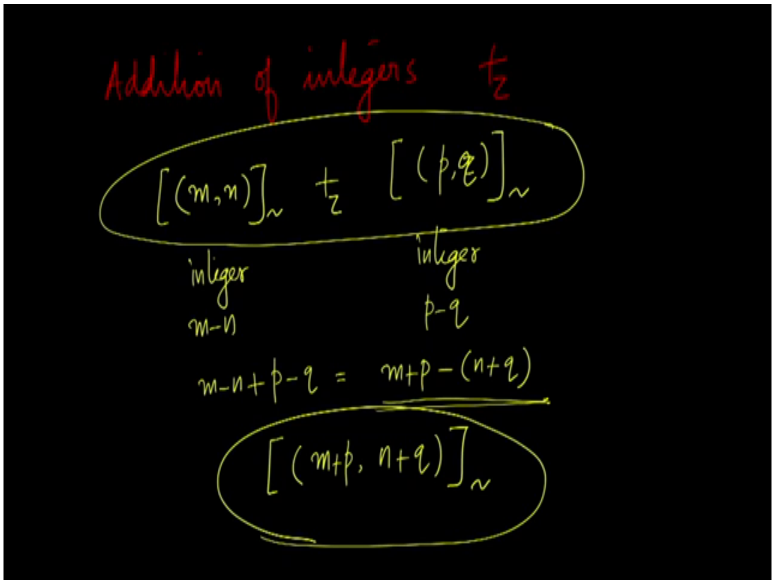

For example, addition of integers which we denote plus with subscript z to distinguish it from addition of natural numbers. Addition of integers is defined in this manner. Ordered pair the equivalence class containing ordered pair m and under the relation till day. This is an integer. This added to the integer which is defined by the equivalence class to which ordered pair p q belongs. So, these 2 are integers. This is 1 integer and this is another integer.

This integer is supposed to correspond to m minus n and this is supposed to correspond to p minus q where m n p q are natural numbers. Then, there is some or to correspond to the integer m minus n plus p minus q which could be written as m plus p minus n plus q. But, the equivalence class under till day to which ordered pairs of this sort will belong would be clearly this. Therefore, this ought to be the result of addition we have here that is when integers corresponding to the equivalence class is m n under till day and p q under till day are added.

We should get the equivalence class corresponding to the ordered pair m plus p n plus q. If this is how we define addition then it would indeed be inconsistent with the notion that we have. The notion we have is that the equivalence class corresponding to ordered pair m n stands for the integer m minus n that is indeed the case under addition defined in this manner.

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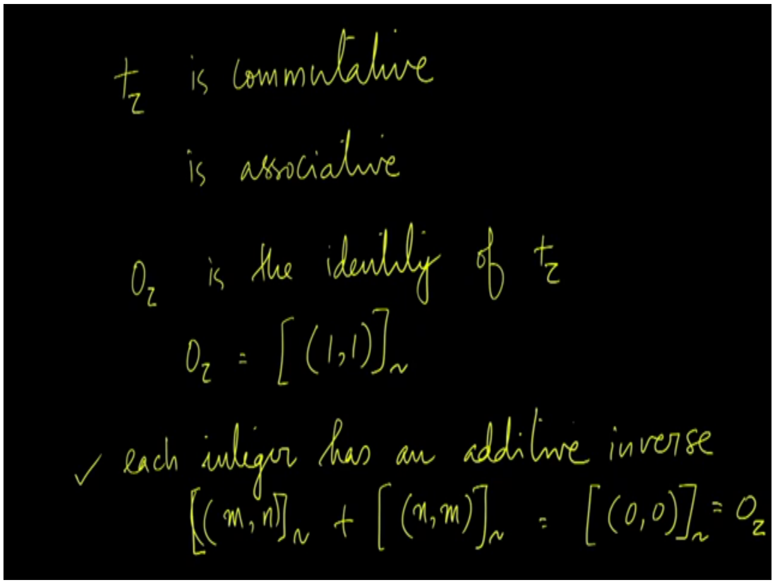

Then we can see that this addition is commutative changing the order of the order of arguments will not change the result. We will get exactly the same equivalence class. It is associative. It has an identity. This is the identity of the addition operation we have defined. Remember 0z is the equivalence class to which 1 1 belongs. Each integer you can see has an additive inverse, under addition which is defined in this manner. We can see that when this integer is added to this integer what we get is this which 0. Therefore, every integer has an additive inverse.

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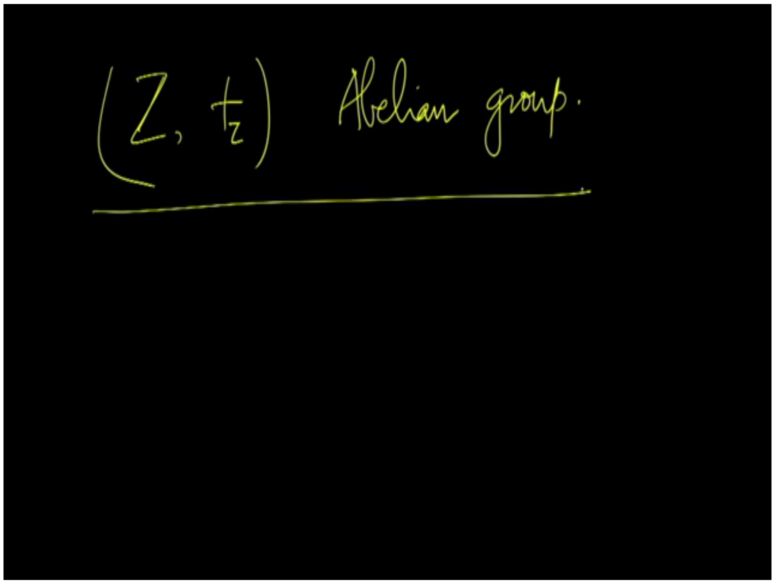

In short, the integers that we have defined along with operator plus z form an abelian group. You would recall the definition from the module on algebra. It is clear that z plus z is an abelian group.

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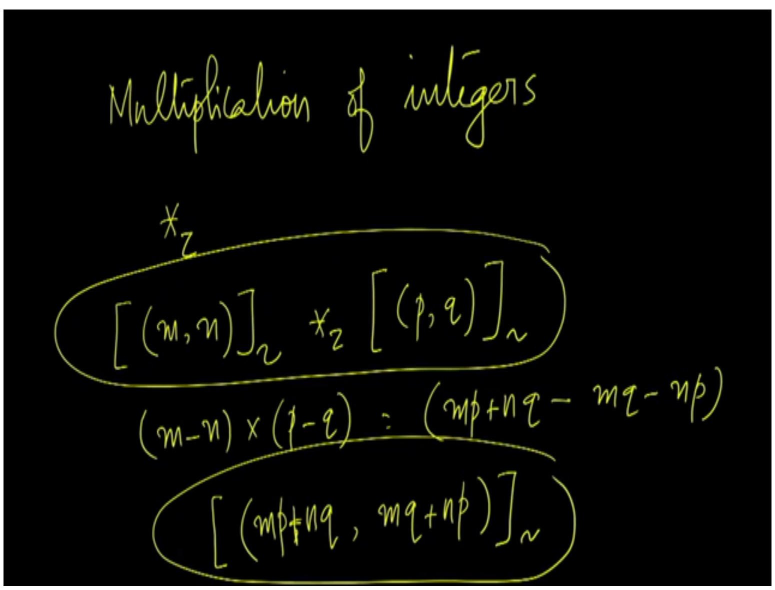

The other operator that we want to define on integers is multiplication. It is denote multiplication by star z. Multiplication would be defined in this manner. When we multiply integer m minus n and p minus q we would like the result to be mp plus nq minus mq minus np. So, with this intuitive understanding we can define the result of multiplication as the equivalence class containing the ordered pair mp plus nq. This addition is the addition over natural numbers. So, this is how we should define multiplication of integers that correspond to equivalence classes containing ordered pair m n and ordered pair p q under till day. So, if multiplication is defined in this manner it will be consistent with our intuitive notion of our integer multiplication.

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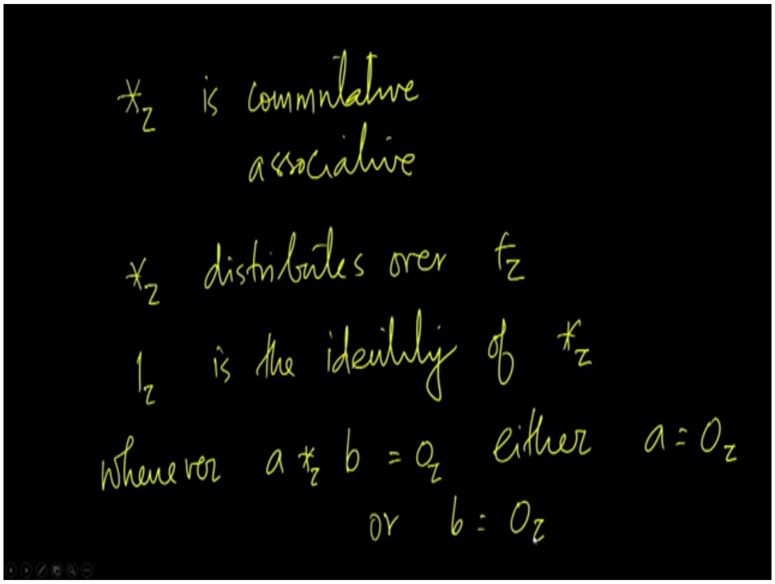

So, you can verify that this multiplication operator is commutative, associative. It distribute over plus z and 1z is the identity of this multiplication. Moreover you can also show that whenever a multiplied by b for 2 integers a and b is 0. Either a is 0 this is 0z either a is 0 or b is 0. All this is exactly consistent with our definition of integers. Therefore, what we have understood is that this definition of integers along with the definition of addition of integers along with multiplication of integers behave exactly the way the integers are supposed to behave.

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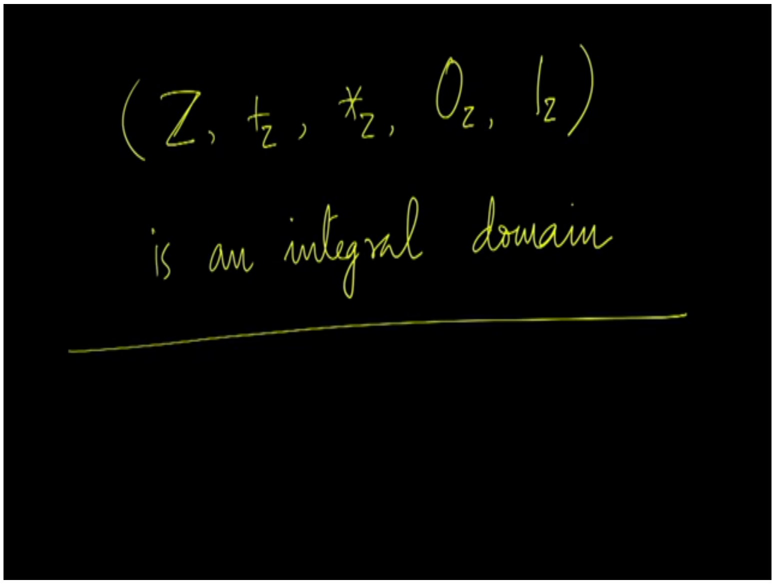

Now, in other words this definition of integers along with the addition operation, the multiplication operation the special number 0 and 1 which function as the identities of addition and multiplication respectively is an integral domain. Again recall the definition from module on algebra. Therefore, the definition of integers work exactly the way we want. So, this way of defining integer is as theory constructs achieves what we want is now the theory of integer embedded in set theory.

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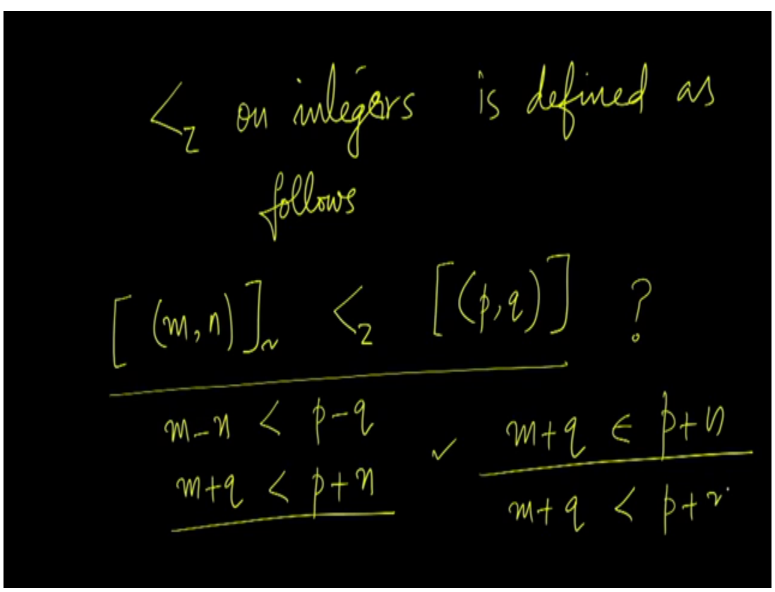

The less than relation on integers is defined as follows. We want to say that integer corresponding to the ordered pair m n is less than the integer corresponding to ordered pair p q. When do we want to say this? Intuitively we want to precisely say this when m minus n is less than p minus q that is integer m minus n is less than p minus q which would be the case when m plus q is less than p plus n. So, since we do not want to use subtraction natural numbers are not close under subtraction we try to rewrite the same condition using addition.

Therefore, this is what we achieve. So, we would be able to say this precisely when natural number m plus q belongs to p plus n. This condition would hold precisely when natural number m plus q is less than natural number p plus n. The way we have defined natural numbers using sets we know that natural number 0 belongs to natural number 1 which belongs to natural number 2 and so on.

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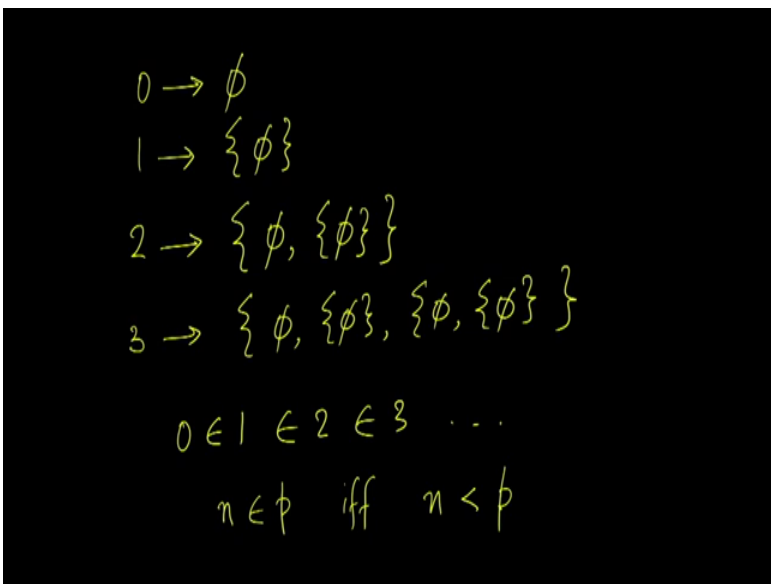

Recall that 0 corresponded to the empty set, 1 corresponded to singleton containing empty set, 2 corresponded to the 2 member set containing these, 3 corresponded to 3 member set containing these and so on. So that 0 belongs to 1 which belongs to 2 which belongs to 3 and so on. You could say that n belongs to p precisely when you want to say that n less than p.

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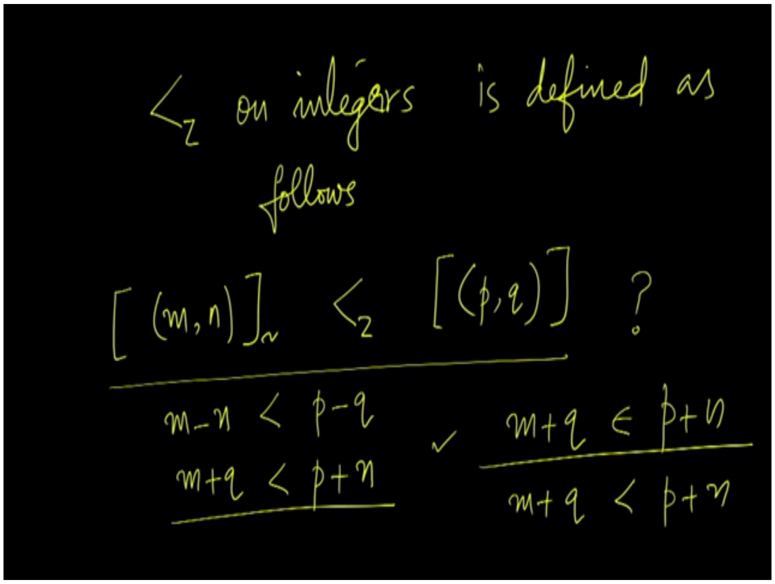

So, that is what we have done here. We say that the integer corresponding to the equivalence class containing the ordered pair m n is less than the integer corresponding to the equivalence class containing the ordered pair p q precisely when the natural number m plus q belongs to natural number p plus n.

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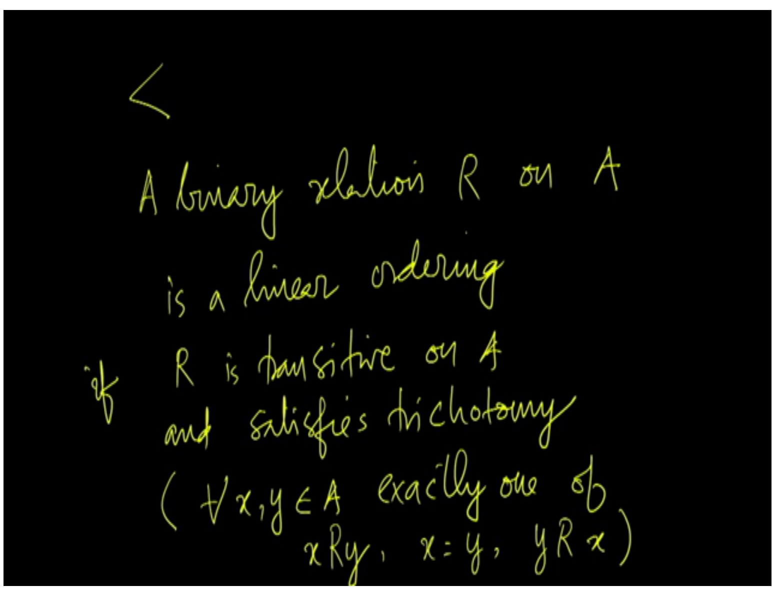

A word about the relation less than a binary relation r on a set a is a linear ordering if r is transitive on a and satisfies trichotomy. What is trichotomy? R satisfies trichotomy on a it and only if for every x and y in A exactly 1 of the following either xRy holds or x is equal to y or yRx holds. If this is the case then we say that r is a trichotomy. So, binary relation r on a is linear ordering when r is transitive and satisfies trichotomy. So, you can say that less than relation of natural numbers is a trichotomy.

For example, either x less than y or y less than x or x is equal to y and the less than relation is transitive. So, using the less than relation on natural numbers we have now defined the less than relation on integers. So, with these basic definitions we see that the theories of integers that we have evolved behaves exactly according to our intuitive understanding of integers. So, that is why we say that the theory of integers could be embedded in set theory.

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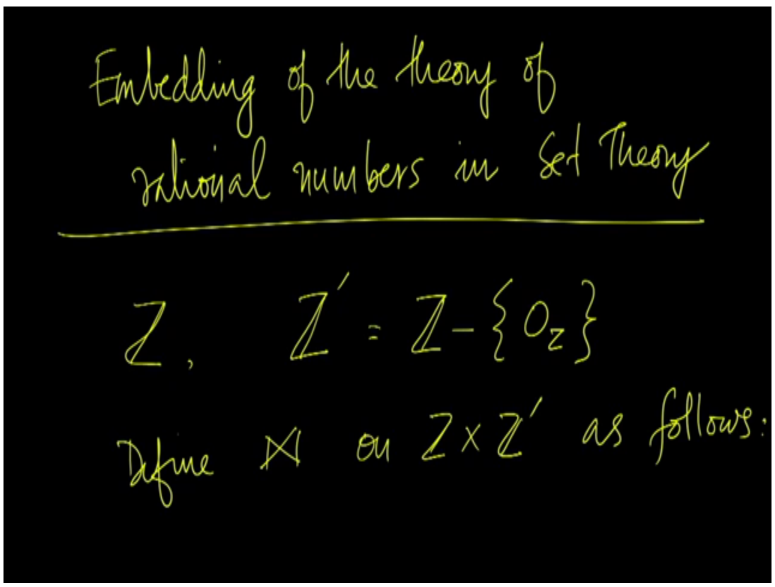

Now, let us see how the theory of rational numbers can be embedded in set theory. So, we have now defined the set of integers z. Using z, let us define z prime as z minus integer 0. Then we define a new relation which we denote like this.

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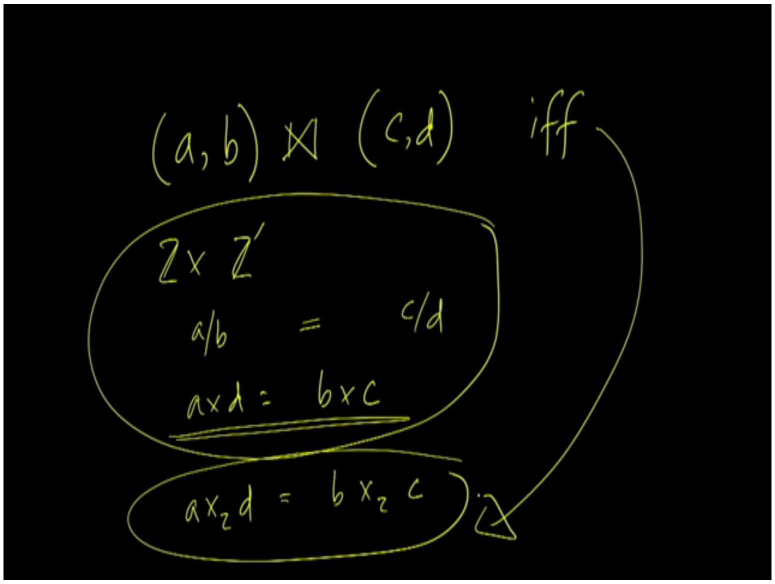

This new relation is defined as follows that ordered pair a b into c by d ordered pair c d. So, mind you ordered pairs from z cross z prime. Z prime does not contain 0z therefore, b and d cannot be 0. So, our idea is that these ordered pairs should stand for a by b and c by d respectively. But since on integers the division operation is not defined that is integers are not closed under the division operation so we do not want to mention division here.

But what we want to say is a by b is equal to c by d that is we want to say ordered pair a b and ordered pair c d in this relation precisely when the fraction a by b is equal to fraction c by d. These 2 fractions are well defined b and d are non 0. But this would be the case precisely when a into d equals to b into c. So, that is precisely what we are going to say. We would say that ordered pair a b stands in the relation to ordered pair c d precisely when a multiplied by d we use the integer multiplication is same as b multiplied by c. So, this is how we define the relationship.

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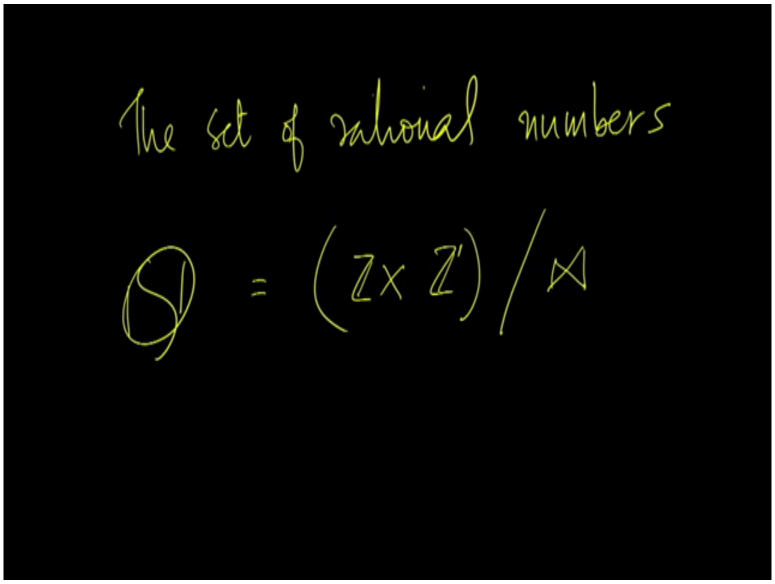

And then on this relationship we consider a set of equivalence classes. We define the set of rational numbers which we define as q as the equivalence class set of equivalence classes of z cross z prime the relation we have just defined. What does this mean?

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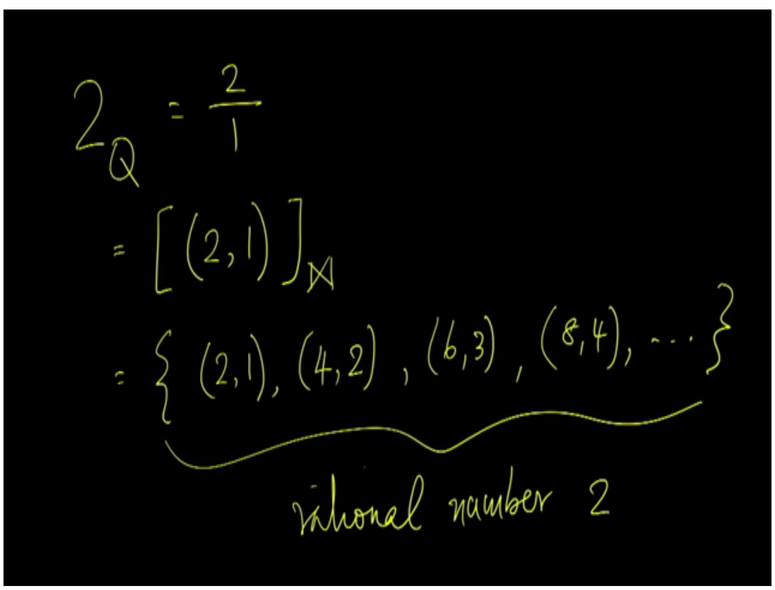

It means that the rational number 2 this is of course 2 by 1. This corresponds to the equivalence class containing the ordered pair 2 1 under this relation. This would of course contain 4 2 6 3 8 4 and so on. So, this set is defined as the rational number 2. This set will stand for the rational number 2. So, exactly as we did before we will now equate every rational number to a set.

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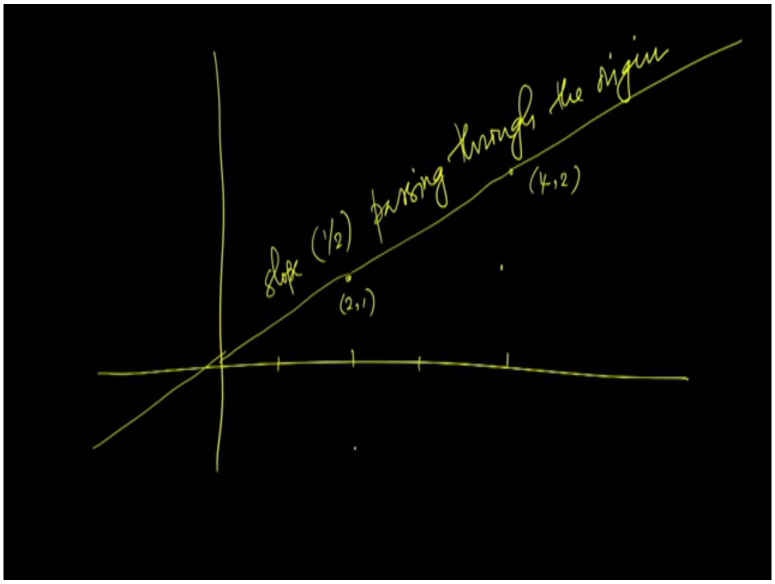

So, in particular on the xy plane, where would these points fall? We have 2 1 here 4 2 here, so this one, these points would fall on a line with slope 1 by 2 passing through the origin. So, we consider all integral grid points on the line with slope 1 by 2 and passing through the origin. These points would correspond to rational number 2. All the integral points falling on this line would form the set which is defined as rational number 2.

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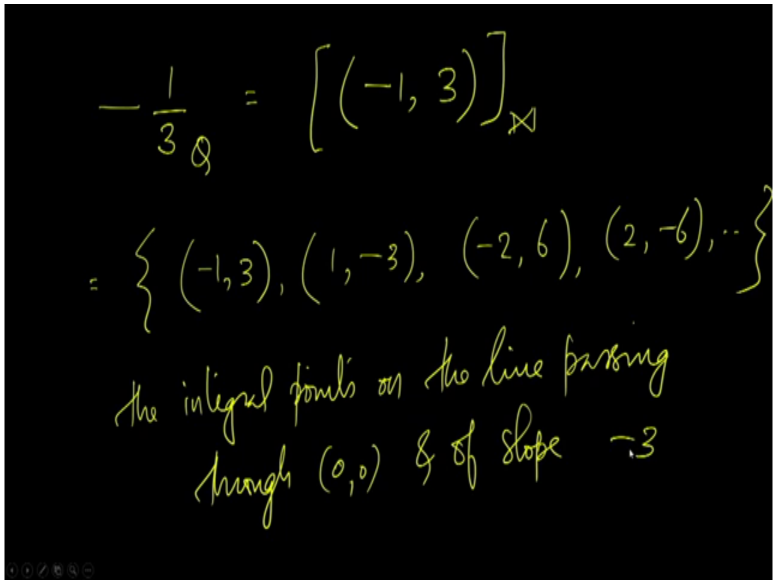

Then let us consider minus 1 by 3 rational number minus 1 by 3. This would correspond to equivalence class containing the ordered pair minus 1 by 3 under this relationship. If you enumerate some of the ordered pairs belonging to this you would have minus 1 by 3 1 minus 3 minus 2 by 6 2 minus 6 and so on. These would correspond to the integral points on the line passing through the origin and of slope minus 3.

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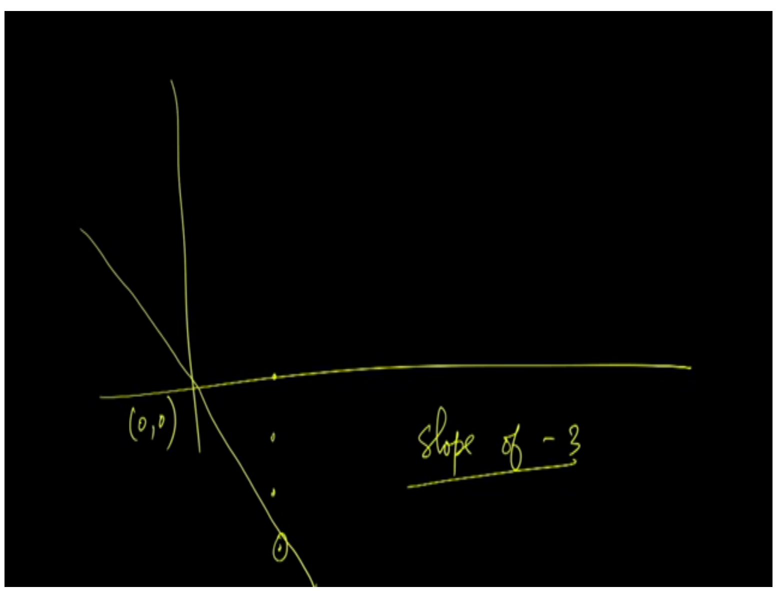

If you plot you consider 0 0 here we want minus 1 by 3 and 1 minus 3 to be on the line. So, it would pass through these 2 points. So, this has slope of minus 3 and every integral grid point falling on this line would form the set would be in the set which is defined as minus 1 by 3 the rational number minus 1 by 3.

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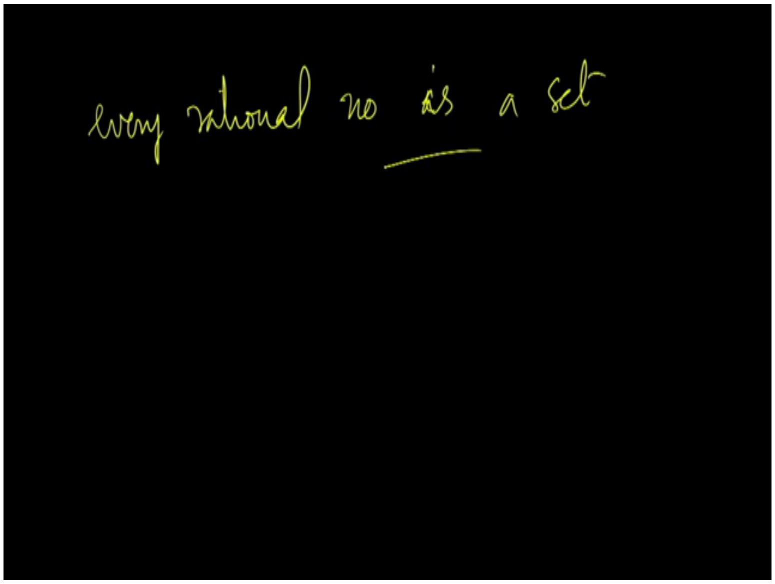

So, in this sense we define every rational number as a set. So, under this definition every rational number is defined as a set. So, this is analogous to what we did earlier. We defined every natural number as a set first of all then we define every integer as a set and now we define every rational number as a set.

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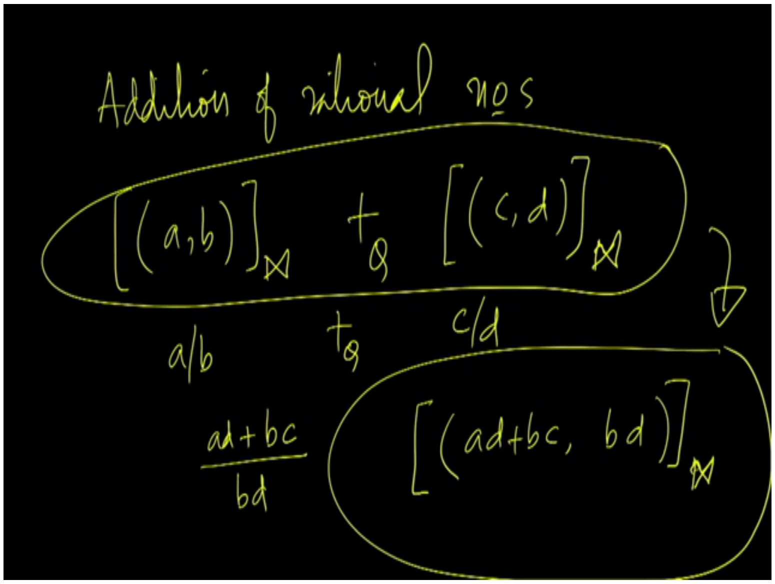

Then how would we define addition and multiplication of rational numbers. First we want to define addition of rational numbers. So, let us consider 2 rational numbers that correspond to equivalence classes containing ordered pair a b and c d respectively. These truly correspond to a by b and c by d. So, our requirement is this we want to add the fractions a by b and c by d at the rational numbers a by b and c by d.

We know that when we add them together we will get bd in the denominator and ad plus bc in the numerator. The corresponds to the ordered pair ad plus bc comma bd. So, the equivalence class containing this ordered pair under the relationship we have defined should be the result of the addition. So, if the addition of rational numbers is defined in this manner it would exactly correspond to the intuition we have.

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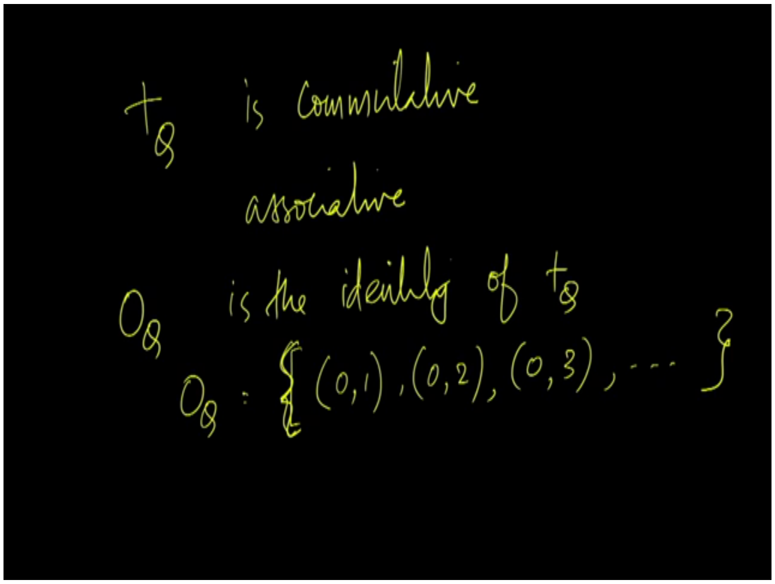

So, that is how we will define the addition of natural numbers and then we can see that according to this definition addition is commutative, associative, rational number q is the identity. Rational number 0q corresponds to this equivalence class 0 divided by a non 0 integer.

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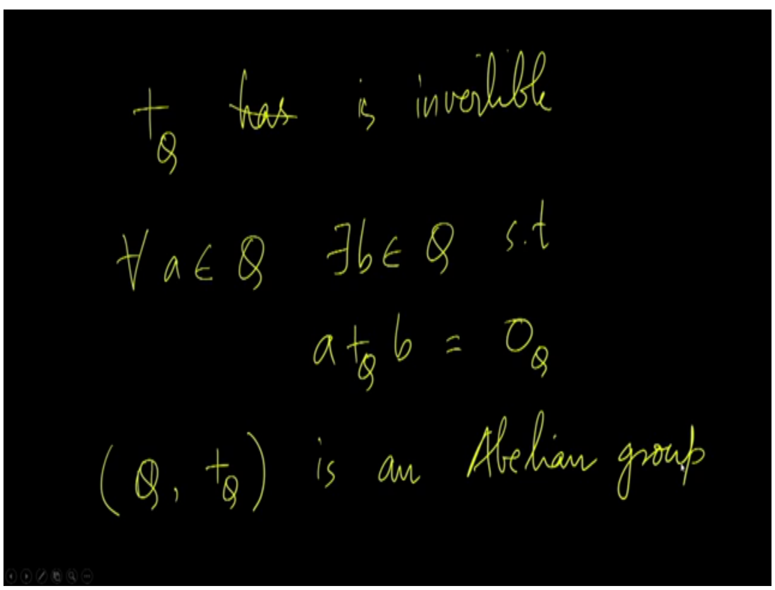

This is the identity of the addition as you have already verified. And we can also see that this addition is invertible that is for all a belonging to q the excess in b belonging to q such that a and b added using this addition operator will render 0q. So, every rational number has an additive inverse therefore we have that set of rational numbers along with that addition operation is an abelian group exactly as was in the case of integers.

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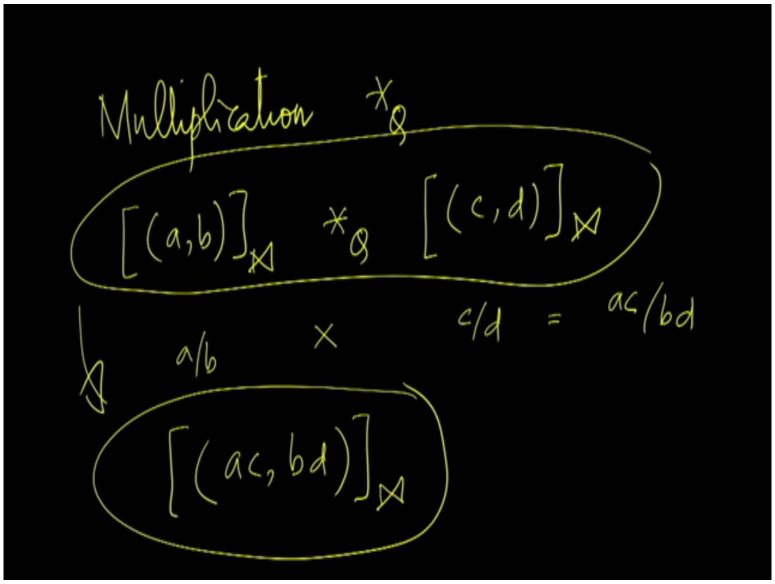

Now, coming to multiplication you can really work out how multiplication has to be defined. We consider the multiplication of rational numbers. So, let us say we want to multiply the rational numbers corresponding to the equivalence class containing ordered pair a b and the rational number corresponding to the equivalence class containing the ordered pair c d. These 2 are what we want to multiply. These equivalence classes respectively stand for a by b and c by d where b and d are non 0. These 2 are what we want to multiply.

From our understanding of rational numbers we know that the answer would be ac by bd. So, we should define the equivalence class containing the ordered pair ac bd under this relationship as the result of multiplying these 2. So, if multiplication of rational numbers is defined in this manner it is exactly analogous to intuitive understanding of rational number multiplication.

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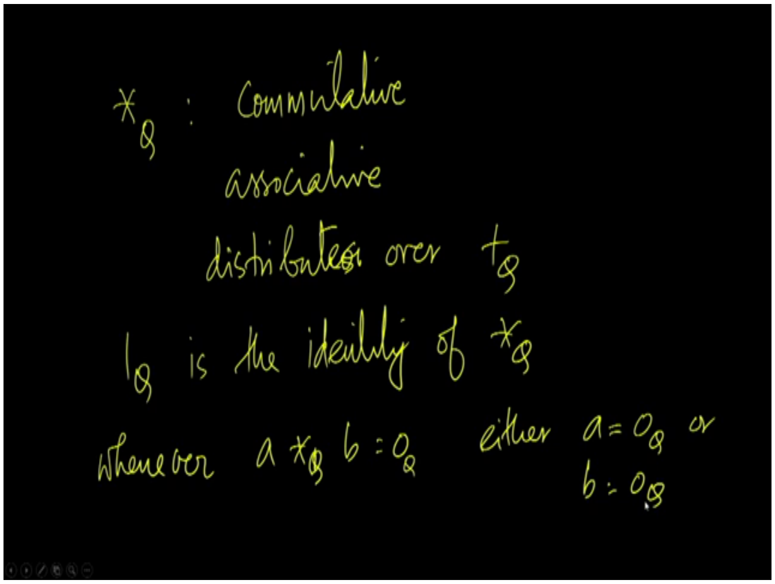

Then we can see that this operator is commutative, is associative, distributes over rational number relation. Rational number 1 is the identity of this multiplication. Whenever the product of 2 rational numbers is 0; either 1 or the other is 0 or both.

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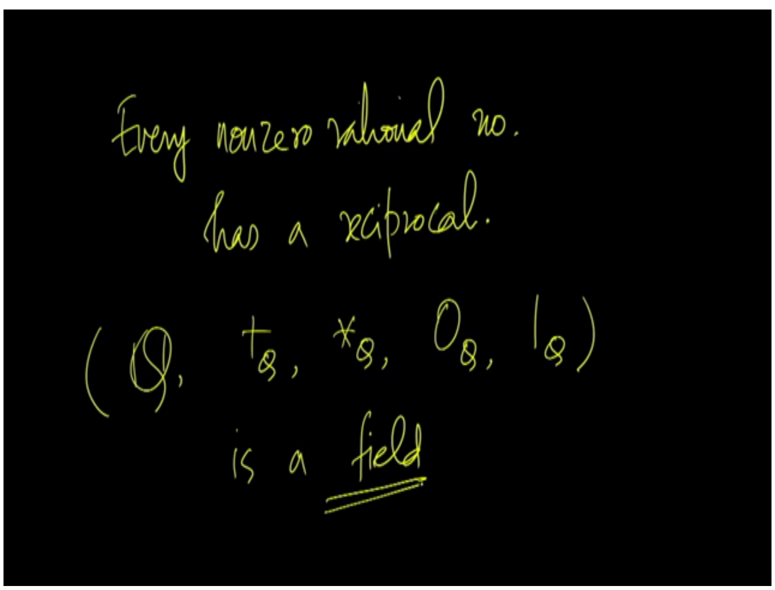

Moreover, every non 0 rational number has a reciprocal which is the multiplicative inverse. So, every rational number has a multiplicative inverse. This is in variance with the case of integers. Integers do not have multiplicative inverses that is integer multiplication is not invertible.

Therefore, this q along with the addition operation, the multiplication operation, the rational number 0 and rational number 1 is a field. Recall the definition of a field from the module on algebra.

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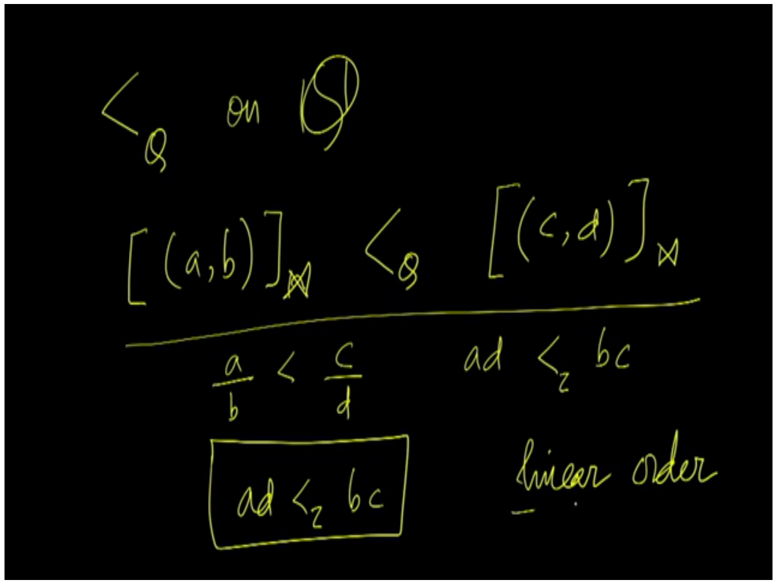

Now, let us define the less than relation on rational numbers. We want to say that the rational number corresponding to the ordered pair a b is less than the rational number corresponding to the equivalence class containing the ordered pair c d. We want to define this relationship; when you would be able to say this?

We want to say that rational number a b is less than rational number c d where ab cd are integers and particularly b and d are non 0 integers. We will be able to say this precisely when ad less than bc where this is the integer relationship. So, we will be able to say this when ad is less than bc integer ad is less than integer bc. You can readily see that this relationship is a linear order satisfies trichotomy and transitive.

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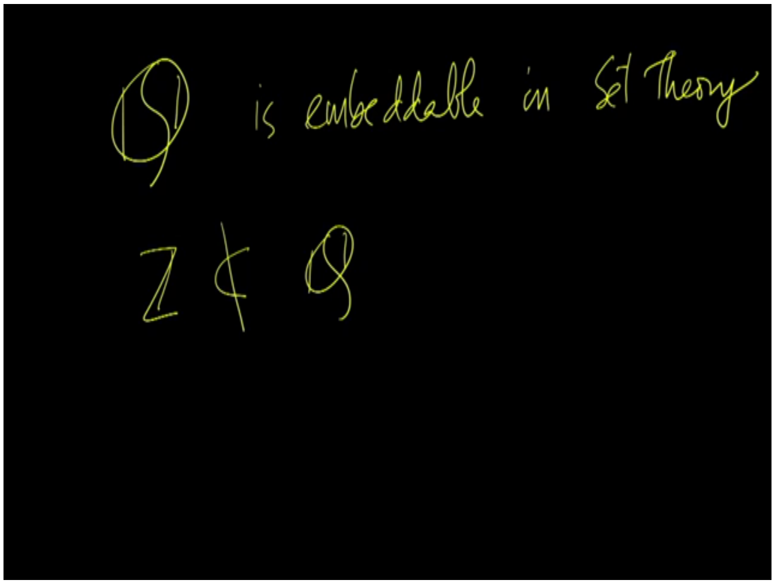

So, we have now shown that rational numbers is embeddable the theory of rational numbers is embeddable in set theory. But the way we have defined the rational numbers integers which form a subset of rational numbers is not a subset of q. The way we have defined z and the way we have defined q; z is not a subset of q. We have used different definition techniques for z and q but then is this a contradiction; not exactly.

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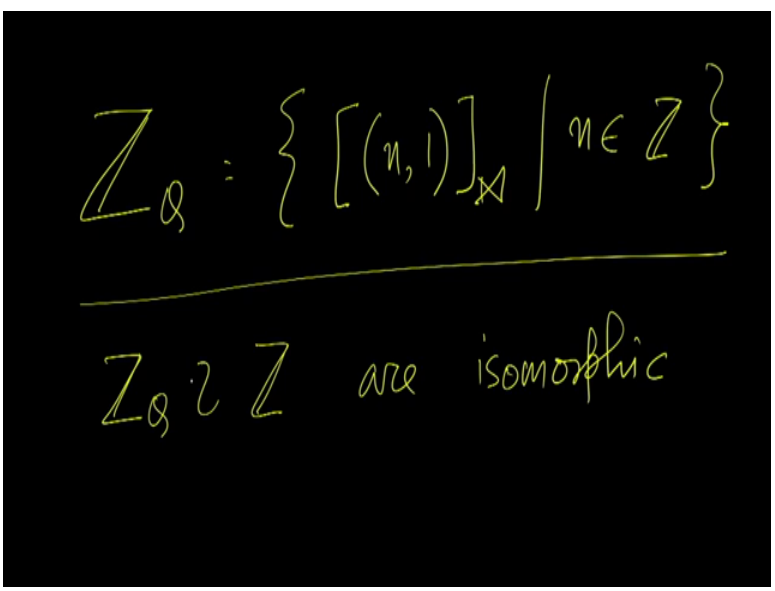

That is because we can define a set called zq as the set of all equivalence classes of this sort contains all ordered pairs in 1 and the equivalence class is defined by them. So, the set of all these equivalence classes will form zq. So, they correspond to natural numbers in which the denominator is 1 and integer is precisely a rational number in which the denominator is 1. So, we define zq as this set. So, you can readily see this zq is set of real integers convertible functions within the set of all rational numbers. So, we can see that zq and the set z we defined earlier are isomorphic. They have exactly the same mathematical behaviour.

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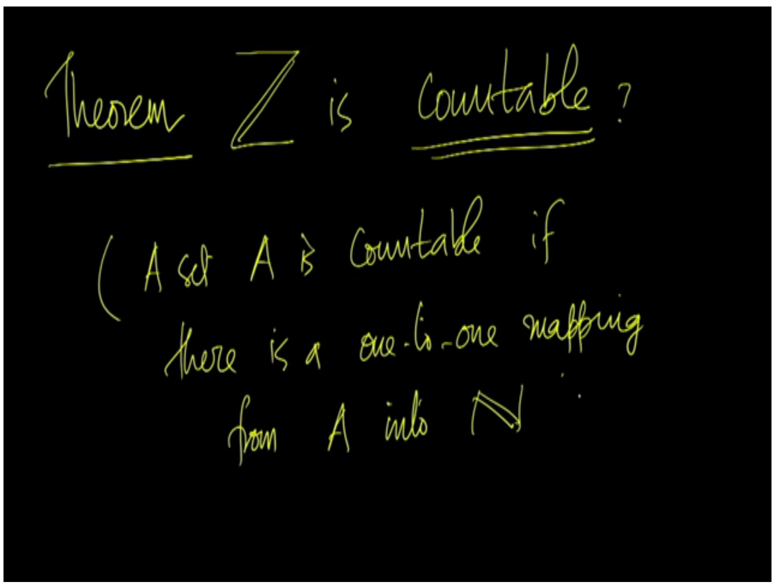

Now, we have an interesting theorem. We know that z is countable. What is a countable set? We say that a set a is countable if there is a 1 to 1 mapping from a into n. Now, how do we know that z is countable? Consider the set of all integers.

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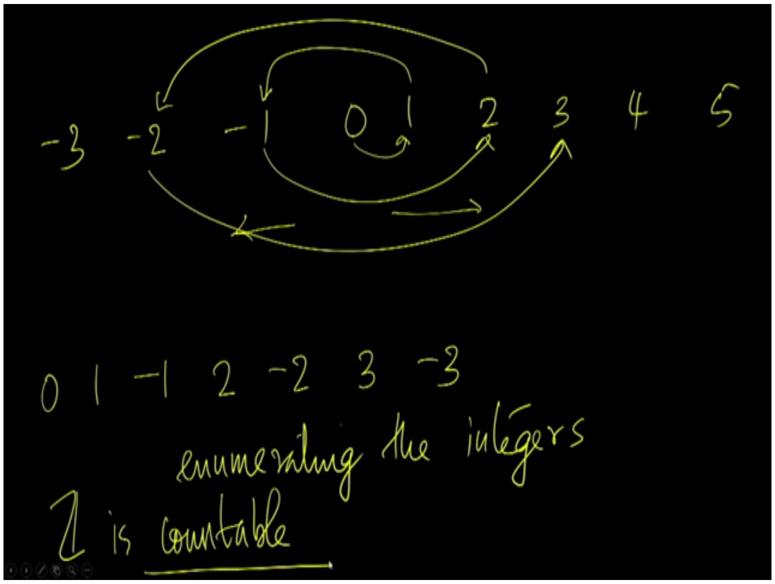

So, integers gone on both sides of 0. We have the positive side here and the negative side here. We should devise a way of counting these integers. So, you should be able to say that this is the 0^{th} integer, this is the first integer, this is the second integer and so on. You count only using natural numbers. So, if you count in this manner 0 followed by 1 followed by minus 1 then 2 then minus 2 then 3 then minus 3. If you continue in this manner then you can readily see that we are defining 1 to 1 mapping from a set of integers onto the set of natural numbers.

So, we will be threading the integers in this manner. 0 to 1, 1 to minus 1, minus 1 to 2, 2 to minus 2, minus 2 to 3 and so on. So, if you thread integers in this manner you will find that every integer will be threaded together. We are giving 1 particular ordinal position for every single integer in the enumeration that we make. So, we are enumerating the integers. In this enumeration is built in 1 to 1 mapping from the set of integers onto the natural numbers that we talk about. Therefore, the set of all integers namely z is countable. You can count them using natural numbers.

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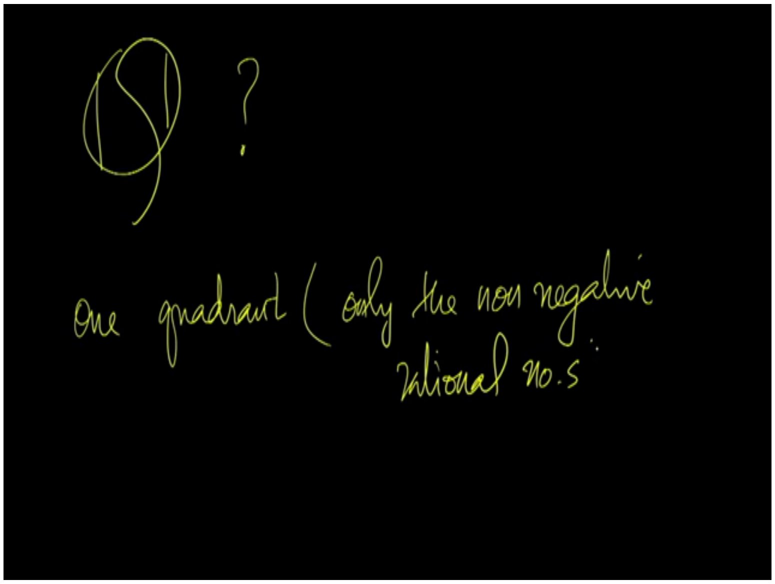

What about q? First of all let us consider 1 quadrant only the non-negative rational numbers.

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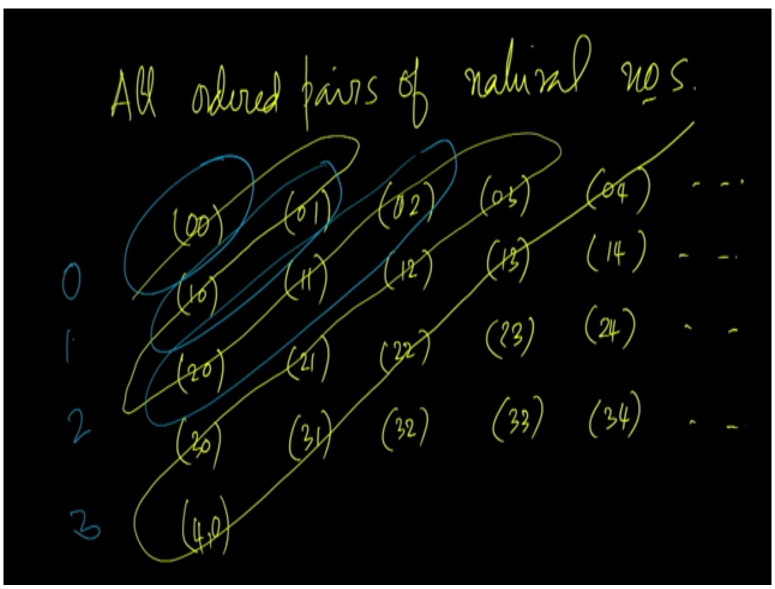

Let us first show that the set of all these are countable. But we will start off with all ordered pairs of natural numbers. So, ordered pairs of natural numbers will be 00, 01, 02, 03, 04, 10, 11, 12, 13, 14 and so on. How would we enumerate all these ordered pairs? We could thread them together in this manner. Let us start from 00 then we come to 01 then to 10 then 20, 11 and 02 then 03, 12, 21, 30. Here of course we would have 40 as well.

So, if you thread the ordered pairs in this manner you can see that every single ordered pair would be in this list that is we are giving an ordinal position for every single ordered pair when we enumerate them in this manner. So, 00 is the first ordered pair, 01 is the second ordered pair and then 10 is the third 20 is the fourth 11 is the fifth 02 is the sixth and so on. So, if you enumerate them in this manner; what exactly is the technique we are using?

We are enumerating ordered pairs in the order of the sum of the first component and the second component. For example, 00 alone stands for the class of ordered pairs in which the sum is 0. 10 and 01 correspond to the class of ordered pairs in which the sum is 1. Then we have, the set of ordered pairs in which the sum is 2; 20, 11 and 02 to have a sum of 2.

So, we have sum of 0 coming first, sum of 1 coming next, sum of 2 coming third and so on. So, this is how we will enumerate the ordered pairs. Now, you can figure out how to go from an enumeration of ordered pairs of natural numbers to an enumeration of rational numbers. So, I am leaving it to you as an exercise for now. We will discuss it again later. That is all from this lecture today. Hope to see you in the next. Thank you!