Randomized Algorithms

Prof. Benny George Kenkireth
Department of Computer Science & Engineering
Indian Institute of Technology, Guwahati

Lecture – 04
Probability Review

So, in this lecture, we will review the basic probability concepts.

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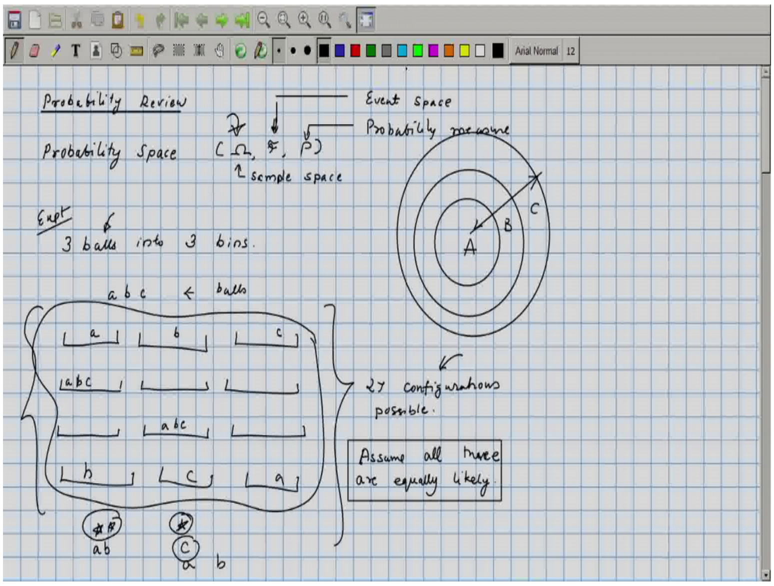

So, when we look at the axiomatic definition of probability, we start with what is called as a probability space. So, probability space consists of three objects omega, F and P, where this is called as the omega is called as the sample space and F is called as the event space and P is the probability measure. Before we do any random experiments or we look at analysing random events these three things at least at an intuitive level should be clear to us; what is the sample space, what is the event space and what is the probability measure. So, let us see this via some examples.

So, let us look at a random experiment. We will see how we can model this random experiment in terms of probability spaces. So, suppose we have three balls and we want to distribute into 3 bins. This is an abstraction of a wide variety of random experiments, ok. The balls could be the marks of students and the bins could be the grades of the students, ok. So, these marks are being distributed into certain buckets, ok. The buckets could be let us say marks of between 0 to 10 you give them one particular grade from 10 to 20 another particular grade and so on, ok.

So, the balls could represent say various it could be used to model various kind of scenarios and balls and bins as a it is an abstraction of many random experiments. So, how many possibilities are there when you distribute three balls into three bins. The questions, the pertinent questions of whether the balls are distinguishable, whether the bins are distinguishable and so on. Let us at the start think of three distinct balls and three distinct bins.

So, if we denote the balls by a, b and c ok, then there are and the bins let us just say we will just draw the bins like this, ok. The each ball could go into any one of these bins. So, a, b, c is one particular configuration and another configuration would be a b c where all goes into the same first bin. Yet another configuration could be a b c in the second bin and the others being empty.

In all there would be twenty seven configurations possible, ok. So, the twenty seven configurations basically being the first ball a can go into any of these bins there are three choices for the first ball, similarly, three for the second and similarly three for the third. So, there are twenty seven possible configurations. We could assume that all these are equally likely. Whether this is a valid assumption or not depends on the actual experiment.

If we were through if these were darts which were thrown into let us say dartboard with three concentric rings and if this was A, this was B and this was C then maybe this is not a very valid assumption because it depends upon the radius of these rings, ok. The and also the mean if you say that you randomly choose a point inside these it might not be always right to assume that all these are equally likely. So, that depends.

So, this assumption whether we can make this assumption or not is something that only the actual experimenter can tell. It depends on the situation that we are trying to model ok, but we could start off in an abstract setting where all these three means we do not know anything about the balls and bins and we could assume that all three are a priori equally likely.

Now, suppose they were distinguished they were indistinguishable then how many possibilities are there. So, here these twenty seven possible configurations we will say these are the sample spaces, these are the outcomes that we could observe when we throw three balls into three bins. The sample space now consists of twenty seven configurations. When the balls were indistinguishable does the sample space change, we look that, I mean at least what we can observe surely changes because we cannot now separate between a and b.

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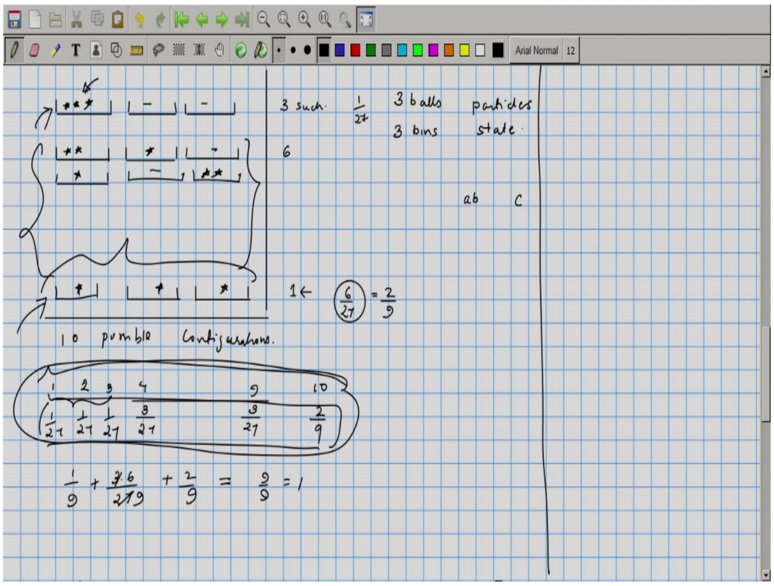

So, the kind of configurations that could arise in that would be. So, now, all that matters is the number of balls in each bin whether it was a or b or c does not really matter. So, all three in one, this is one particular configuration and these being empty and there are three such configurations. There could be another where one of them contains two, the other one contains one and the third one empty. The empty one can be chosen in the empty bin can be chosen in three ways and from amongst the other two bins you can choose the bin which contains one ball in two different ways. So, there are six configurations of this kind.

So, say something like star here this is empty and this contains two, ok. So, you can enumerate them we will see that there are six possibilities and then the remaining possibility is that each bin contains exactly one, ok. There is one such configuration. So, now, there are ten possible configurations, ok. Now, which of these mean what does the probability we asked this question. It is a bit naive to us that question we need to understand why it is considered naive to ask this question are all these configurations equally likely, ok. So, these configurations whether they are more likely or less likely depends on the mod mean on the experiment that we are actually conducting. When we are study when we look at randomized algorithms we never bother about whether these are all equally likely or on or so on. We will start with a certain assumption, it could either be that they are equally likely or it could be that they are likely with certain other probabilities and then we wonder about the consequences of those assumptions we never question whether the assumption is valid or not, ok.

So, we will see this in a little more detail. So, we could imagine that let us say these are mean these three balls are some particles some atomic particles are something like that and the three bins represent a certain state, ok. It could be whether they have a certain spin or they have a certain occupy a certain energy level. So, that is abstracted in the notion of a state. So, these balls are particles which you can measure to be present in a certain state.

So, your experiment would be you are looking at three particles and identifying the states that they are present in, ok. When we look at these particles we could say that although we are unable to distinguish between the particles, the particles are actually distinct particles; there are three distinct particles and each of them could equally choose one of these three states. If we had thought of it that way then there are twenty seven possible states and these twenty seven possible states should appear with equal probability, ok. So, a b c, b, c, a and six other configurations should all appear with 1 by 27 probability each, ok.

So, when we observe a configuration in which each bin contains one ball each this could be either a, b, c or any of the other six permutations of a, b and c. So, we could say that this particular state should appear with 6 by 27 probability or that is something like 2 by 9. We could work out a similar number for each of these states. For example, all the three in ones in the first state that could happen precisely in one way that would happen with 1 by 27, ok. There are three such. So, together they account for.

So, 1 by 27, so, if you call these ten different configurations as configuration one 1 2 3 representing these type of configurations and then 4 to 9 representing these kind of configuration and 10 representing the last kind of configuration. So, each of these appear with probability 1 by 27, 1 by 27 and 1 by 27. These appear with 2 by 9. Let us look at the second kind of configuration. So, that could come because of a, b and c or it could be because of. So, so, amongst these 27 configurations which are the configurations which will result in star star comma star, ok.

So, if a, b were in one and c was in the other that is one possibility, but instead of c this could have been a or b all those would give rise to. So, you fix whatever is the number which goes in the singleton bin, that will basically fix whatever is going in the other way. So, there are three possibilities for the singleton bin. So, that is again possible with probability. So, either this is probably will occur with probability 3 by 27. You need to check whether this adds up to 1. So, this is 3 by 9 plus 3 into 6 by 27. So, we have made a mistake somewhere yeah. So, this is not 3 by 9, this is 1 by 9 3 by 27, 1 by 9 and 3 by 27 into 6 into 6 plus 2 by 9. So, that is equal to 9 by 9 or 1. So, this is in fact, the probability if we assume that any ball could be present in any bin with equal probability.

So, if our underlying sample space was distinguishable balls into distinguishable bins with all of them being equally likely we could have twenty seven possible configurations and we could say that our inability to understand the distinctions between the ball or our inability to make measurements should not in some sense affect the ways in which the particles behave, when if we had thought of it that way this is what we would have got as the probability distribution.

But, the surprising thing in physics was the following find by Bose and Einstein for certain kind of particles and states the actual measurements resulted in all these being equally likely, ok. All these ten configurations they were in some sense equally likely. So, there are some very funny interactions going on between these particles, ok. So, the model for us is that a priori we cannot say anything about how the particles would actually behave that would be given only by the experiments, but our work would begin after we have fixed a particular model.

So, we could begin by saying that all these twenty seven possible configurations were there or with equal validity we could begin with the configurations where all these ten are equally likely, ok. So, the configurations possible is what we will call as sample space, the observables is what we call as the event space. So, we will look at the formal definitions more closely.

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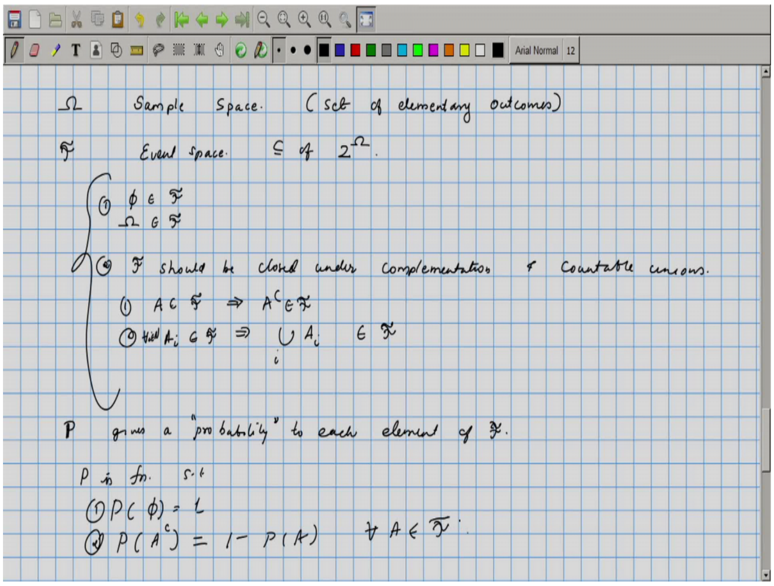

So, omega is our sample space. If were looking this is a set of elementary outcomes and F which is referred to as a sigma algebra is the event space this will be a subset of the power set of omega. But, this would have certain nice properties namely, the empty set should belong to f, omega should belong to f and then f should be closed under complementation and countable unions, ok. That means, if let us say A belongs to F this would imply that A compliment belongs to F that is closed under complementation and if A i, for i belonging to n if A i belongs to F this would imply that union over i A i also belongs to F.

So, you could take a collection of sets which are there in F and you could take that countable union they also should belong to F. The rationale for defining the event space like this would be suppose you can say that a certain event has occurred, after your measurements you can say that certain event has occurred then you should be in a position to say whether the complement also has occurred, and similarly for countable unions. So, that would basically be the event space and the function P or the probability measure basically gives a probability to each element of F, ok.

So, what does it mean to give probability to each element of F associated with each event you will associate a number between 0 and 1, a positive number. So, P is a function such that P of phi is equal to 1; second P of A plus P of A complement is equal to 1 for all just simply write this as P of A complement should be equal to 1 minus P of A, for all A belonging to F.

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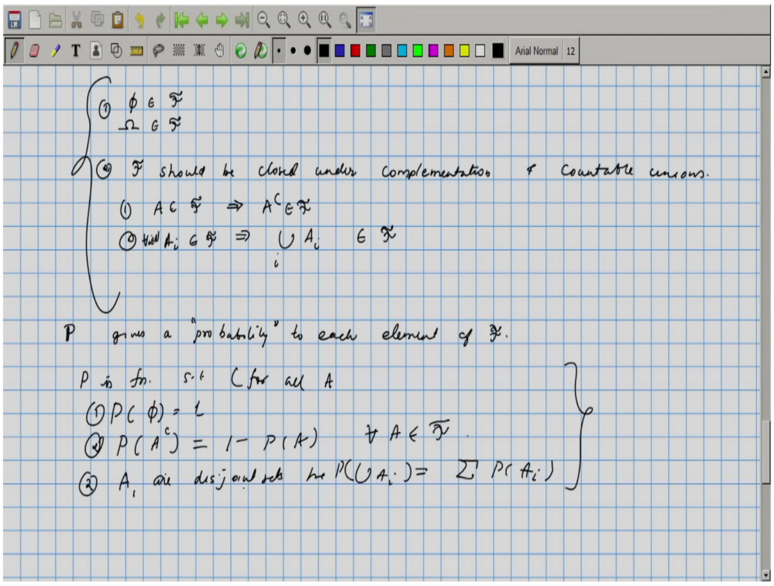

And, if so, if A i's are disjoint sets then union A i, its probability should be equal to sum of the individual probabilities. So, if these conditions are satisfied that is called as a probability measure and we want P to be a probability measure on the sigma algebra F. So, this should be true for all A, ok.

So, this is the formal framework in which we will examine all our randomized algorithm. We will first be clear about what is the sample space in our experiment I mean in are randomized algorithm and then we have to clearly articulate what is the collection of events that we are interested in and what is the probability associated with each particular event.

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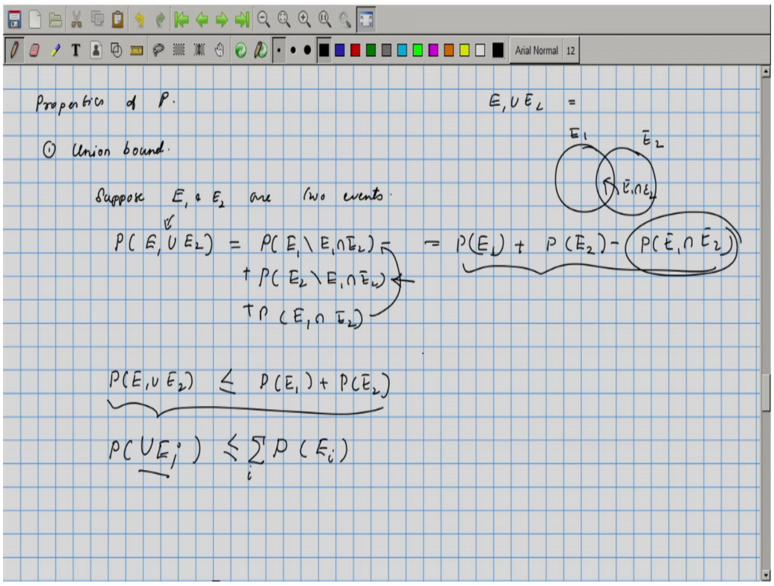

So, now, we will see some properties of P, ok. So, first is something called as union bound, ok. So, suppose E 1 and E 2 are two events what can we say about probability of E 1 union E 2. So, E 1 and E 2 are two events. So, we can write this as, so E 1 union E 2 can be written as so, this is E 1, this is E 2 and this is E 1 intersection E 2. We could write this set as the sum of the union of three disjoint sets, ok. If you write this as union of three disjoint sets then their probability of the union would be the sum of the probabilities of the individual sets.

So, we can write this as probability of E 1 minus E 1 intersection E 2 plus probability of E 2 minus E 1 intersection E 2 plus probability of E 1 intersection E 2, ok. So, this we can combine and say that this is probability of E 1 plus probability of; so, this we could write it as probability of E 2 minus probability of E 1 intersection E 2, ok. So, this is true for any (Refer Time: 22:42) so, clearly, this is true for any E 1 and E 2.

So, probability of E 1 union E 2, this is going to be less than E 1 plus probability of E 2 because the term that you are removing is a we are subtracting a positive quantity, ok. So, minus P E 1 intersection E 2 and such a statement is true even if you take a countable collection of event. So, probability of E 1 union E 2 union I mean probability to the countable union is less than summation over i probability of E i. So, this is called as the union bound.

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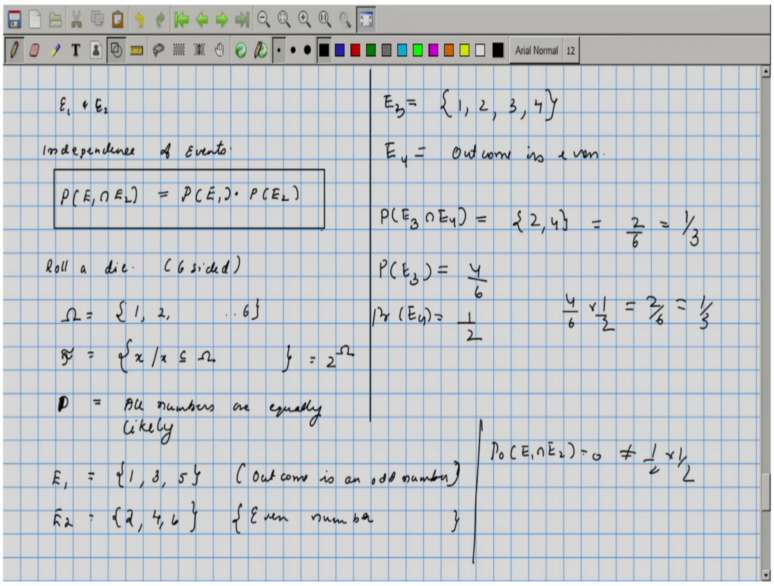

Now, when you are given two events E 1 and E 2, we will define something called as independence of events, ok. So, if you compute the probability. So, basically independence of event means when one event has occurred you have no additional information about the occurrence of the other event ok, then you say that these two events are independent that is the intuitive meaning.

For example, if you know that the event E 1 has occurred you immediately know that the event E complement has not occurred. So, there is dependence between these events, but that is a clear I mean the occurrence of E 1 completely tells you about the occurrence of E 1 complement. It need not be this certain there could be you could infer about the occurrence of the other event with some probability, with some significant probability then you would say that these events are in some sense dependent.

So, the formal definition would be probability of E 1 intersection E 2 will be equal to probability of E 1 into probability of E 2, ok. When this happens we will say that the events E 1 and the event E 2 are independent. Let us see an example. So, roll a die, ok. So, when you roll a six sided die our sample space consists of all outcomes 1 to 6. Now, sigma algebra would consist of all subsets x such that x is a subset of omega. So, basically this is equal to 2 power omega, the entire collection. So, therefore, it satisfies all the requirements and the probability measure that we will put on this P is equal to let us say the usual just I will not explicitly write down the function, but say that all numbers are equally likely. This will translate into a unique probability measure on F.

Now, let my event E 1 be equal to say 1, 3, 5 that is the outcome is an odd number, and E 2 be that the outcome is an even number ok. So, clearly if you compute the probability of E 1 intersection E 2, so, in this case probability of E 1 intersection E 2 is equal to 0 and that is not equal to half into half; half was a probability that the number is an odd number and half is a probability it is an even number. So, these are not independent.

Whereas, if you consider the event E 3 which is equal to 1, 2, 3, 4 ok. So, that is the outcome is less than or equal to 4 and sorry, E 3 is equal to this and E 4 is that outcome is even. So, when you say that the outcome is even you have no way of saying whether the number mean do we have some information as to whether the event E 3 has occurred. So, we will calculate it as so, probability of E 3 intersection E 4 is equal to 2, 4 and that happens with probability 2 by 6 or equal to 1 by 3.

Now, probability of E 3 is equal to 4 by 6 and probability of E 4 is equal to 1 by 2. So, when you multiply this you will get 4 by 6 into 1 by 2 that is equal to 2 by 6, it is equal to 1 by 3. So, we can call these events E 2 E 3 and E 4 as independent. So, two events are called independent event when the probability of E 1 and E 2 occurring is equal to the product of the probabilities of E 1 and E 2. Later on we will be defining independence of random variables. We will discuss about more about random variables and their independence and other properties of random variables in the next lecture.