Randomized Algorithms
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Lecture – 33
Perfect Matching – III

We will continue the study of approximately counting the number of Perfect Matchings in a dense graph and approach is as follows.

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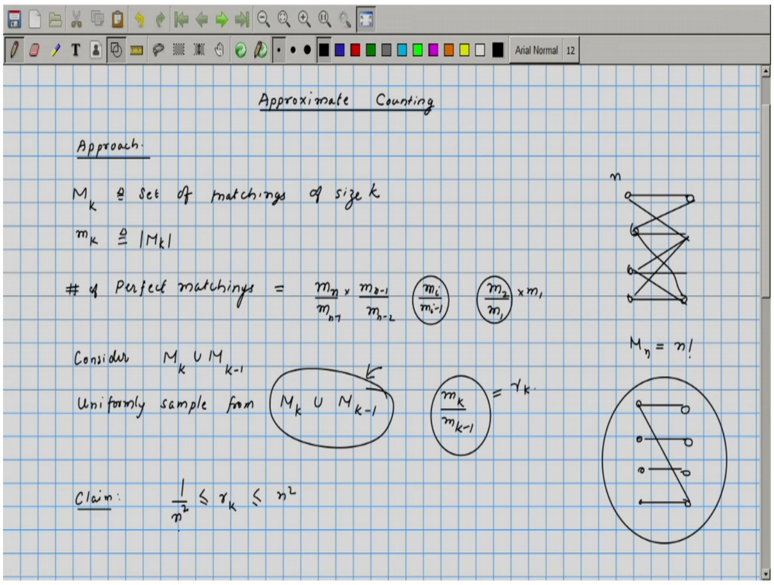

If you denote by M k the set of matchings of size k and small m k the set of matchings of the size of M k, then the number of perfect matchings was equal to m n divided by m n minus 1 into a min minus 1 divided by m n minus 2 up to m 2 divided by m 1 into m 1. So, if we could estimate each of these quantities m i by m i minus 1, then we could estimate the number of perfect matchings. So, that was the approach.

And, now the question becomes how do we estimate m i by m i minus 1. For that consider M k union M k minus 1 and uniformly sample from M k union M k minus 1. If we could uniformly sample from this set then we can compute the ratio m k divided by m k minus 1. We can approximate uniform sampling would mean that we can approximately estimate whatever is this quantity which we had denoted by r k.

For that estimate to be obtainable in polynomial time what we require is r k should have let us say values in certain range. What was our claim was r k lies between n square and 1 by n square. Note that number of perfect matchings can be very large it could be exponentially large. And, may be after removing or certain suppose you had taken this particular let us say the complete graph on let us say n vertices, m n would be let us say a when could be as large as in factorial, ok.

So, the ratio could be smaller at large, but here we are saying that it lies between n square and 1 by n square there could be matchings which does not have say which does not have more than one perfect matchings, ok. This is this one perfect matching in this particular graph has no more than one perfect matching.

Now, we had done part of this claim.

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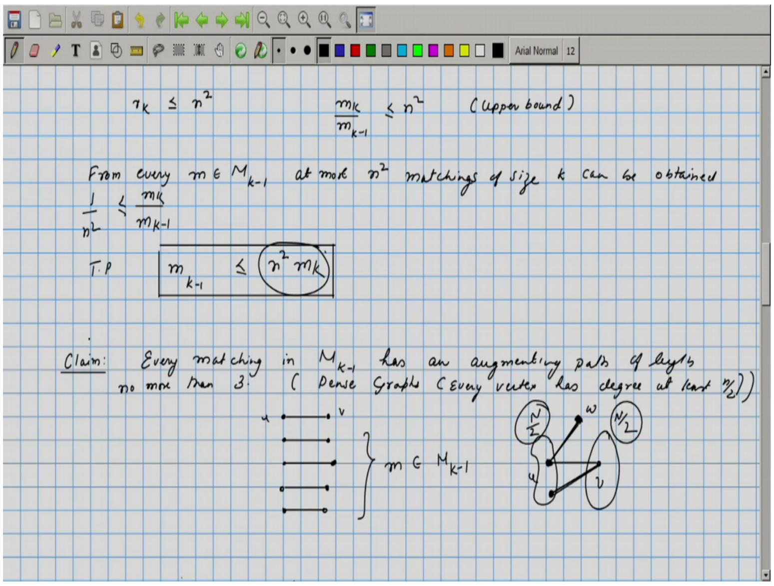

So, r k is less than n square in other words m k by m k minus 1 is less than n square, this is the upper bound. This we proved by saying that from every let us say matching belonging to M k minus 1 at most n square matchings of size k can be obtained and this basically obtains all much matchings of size k. So, by that reasoning be argued that m k is surely less than n square times the number of matchings of size k minus 1.

Now, we need to show that m k minus 1. So, m k divided m k minus 1 is greater than let us say 1 by n square or we need to prove that m k minus 1 is less than n square times m k, ok. So, what we will require is the following lemma every matching in M k minus 1 has an augmenting path of length no more than 3 at most 3. I mean if you take any matching in M k minus 1 the way to obtain a larger matching is by looking at an augmenting path and here this claim says that the augmenting paths length can be bounded by 3, ok.

So, why is this true? So, let us look at. So, this is true for dense graphs. So, dense graphs means every vertex has degree at most I mean at least n over 2, ok. So, if you take any such graph for that this claim is true. So, let us see why that is the case.

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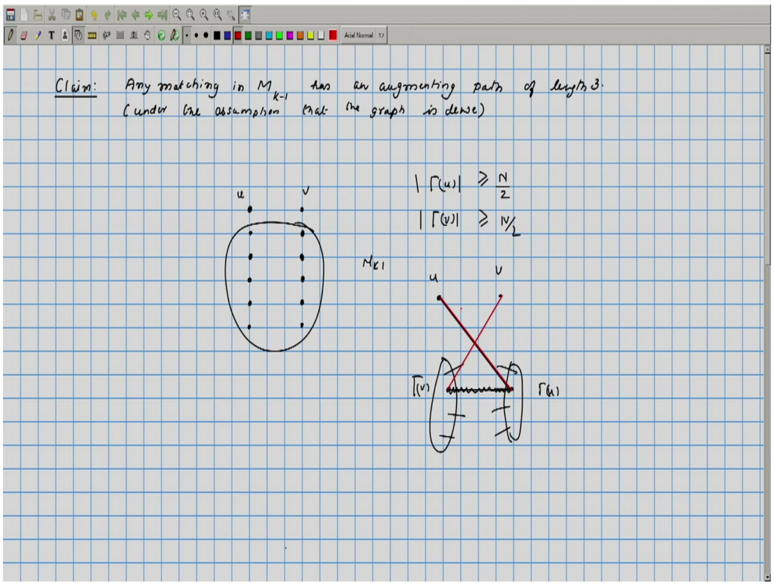

So, let us look at a particular matching in M k minus 1 and suppose this is that matching. So, these vertices are matched to each other, ok. And, if you take two vertices u and v which are unmatched; so, u is on one side and v is in the other side. And, if they are unmatched we will show we will construct an augmenting path of length at most 3, ok. The first possibility is that u and v could be there is an edge between u and v; if you v is an edge then of course, there is an augmenting path of length 1.

Suppose, there are no edge between u and v let us just look at neighbors of u, ok. So, neighbors of u the size is greater than or equal to N by 2 and the size of v size of neighbors of v is also greater than N by 2, ok. So, let us draw these neighbors. This is neighbors of v and those are the neighbors of u. Now, if you look at these neighbors they all had to be matched. If any of them was unmatched then we can of course, get an min let us say if there was an unmatched neighbor then you can connect those and get an augmenting path involving u or v, ok.

Now, so, we have to assume that all them were part of some matching edge, ok. So, these are the edges out of it, we do not know where it lands up on the other side and similarly for these. Now, since the neighbors are I mean since this is these are all sets of size at least N over 2 and if all of them were connected to each other, then you have a matching of size I mean if they were they were all different then you had a matching of size N, ok. So, since this is not a matching of size N there should be some two of them which are connected to each other, ok. So, let this be the edge which is connected to each other and this part of the matching.

Now, you can look at the edge from u to this. So, indicated in red and you can take another edge from v to the v to its neighbor and this forms a these have to be non matching at the red had to be non matching and therefore, that forms a when these three edges together forms a augmenting path of length at most 3.

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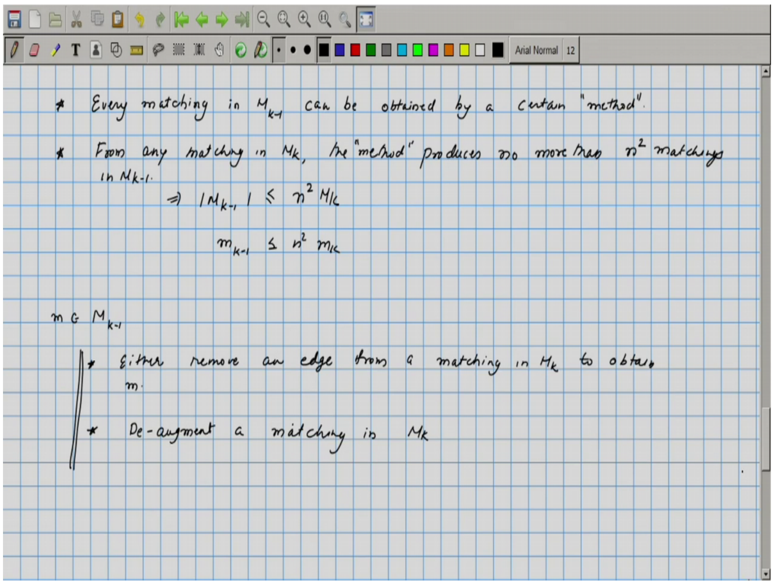

So, I will just write down these statements. What we will choose every matching in M k minus 1 can be obtained by a certain let us say method, and then we will show that from any matching in M k the method produces no more than n square matchings in M k minus 1. So, that would mean that size M k minus 1 is less than n square times M k, or in other words m k minus 1 is less than n square times m k this is what we will need to prove. So, these two statements is what we will see.

So, what is the method? If we are looking at the matching in M k minus 1 how do we obtain it from a matching in M k? So, there are two methods either remove and edge from matching in M k to obtain small m, ok. So, some matching in M k minus 1 which we need to obtain we could obtain it by just simply removing an edge from some particular matching in M k, ok. So, that is one way another way is to what we will call as D e augment a path or let me just say the D e augment the matching see what are these two procedures. By these two procedures we can obtain every matching in M k minus 1.

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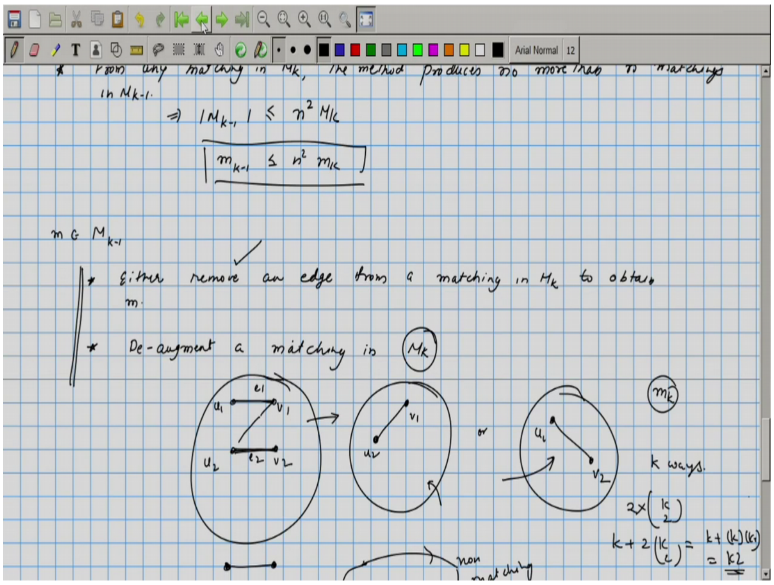

The first thing is clear just take a matching in M k and remove one edge; if you get a matching small m well and good. Another way is you take any matching in M k. Let us look at a matching in M k, let us call it as e 1 and e 2, ok. We will convert this into either. So, let us call this as u 1, v 1 and u 2, v 2 ok. We will either replace it by a matching which contains this edge which is u 2, v 1 or u 1, v 2. Once again, you take any arbitrary matching in M k. We can either remove an edge from that matching we will get a matching in M k minus 1 or you could pick two edges and replace it by u 1 v 1 or u 2 v 2, ok.

Now, this becomes a matching only if u 2 v 1 was an edge in the in the graph v 1 and similarly this becomes a matching only if u 1 v 2 an edge in the original graph. But, we can say that take any matching if these are valid edges then we do this replacement otherwise we do not, ok. Now, by this method we could get matchings and all those matchings will be in M k minus 1. What we claim is a very matching in M k minus 1 can be obtained in this particular manner, ok. Why is that so? That is where our lemma comes useful.

You take any matching m belongs to M k minus 1; there is an augmenting path of length at most 3. So, since there is an augmenting path of length at most 3, if there was an augmenting path of length 1, then we are talking about this case. If there is an augmenting path of size let us say a winning paths are of odd length, so, they are of size 3 then; that means, this edge was present in. So, e belongs to m, but these two edges they are non matching edges, ok.

So, if so there is a matching I mean since this is an augmenting path we know by replacing it with this we will get a matching of length I mean of size k, if we had taken that matching and done one of these procedures we will go into a matching which has one fewer edge. So, every edge in every matching in M k minus 1 can be obtained in this particular manner, ok.

Now, we can look at how many matchings can be obtained in this manner. So, take anything belonging to M k there are small m k of them each of them you can you can remove an edge this can be done in k ways or you could de augment. For de augmenting you have to pick an edge that can be done in k choose two ways and for each choice you can either connect the off diagonal or the diagonal. So, there are two ways. So, the total number of ways is k plus 2 times k choose 2 that is equal to k plus k into k minus 1 that is equal to k square. So, there are k square matchings that is obtainable from I mean that is the maximum number of matchings that is obtainable from any matching in M k.

Now, because of this we can conclude that m k is going to be less than n square times m k.

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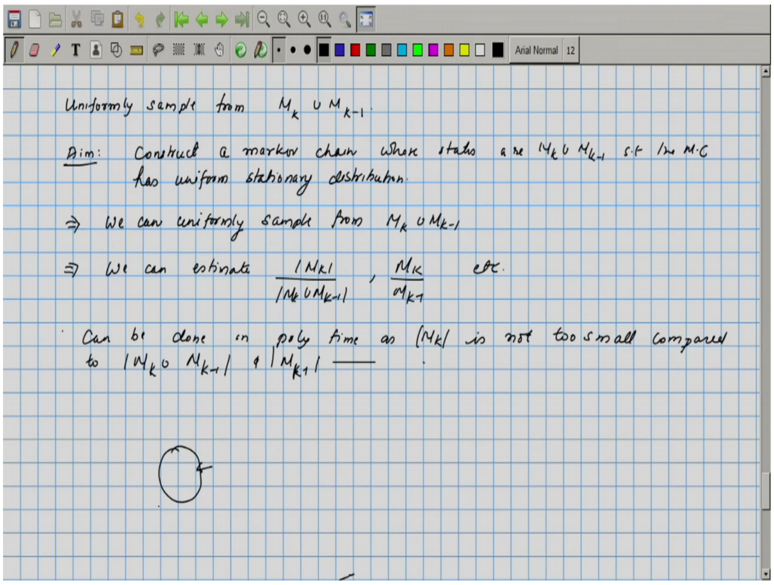

So, contingent on this claim, that every matching in M k has an augmenting path of length no more than 3, we can conclude that r k lies between n square and 1 over n square. From this what we will do is based on this claim we can try to uniformly sample from M k union M k minus 1. So, M k is the collection of all matching. So, size k and M k minus 1 is the collection of all matchings of size k minus 1. How do we uniformly sample from this? These sets could as such be very large, but we will just generate the elements as and when we require it, ok.

So, if you are at one particular matching from that matching we will go to one of the neighboring matchings in a suitable way. So, our aim is to construct a Markov chain whose states are M k union M k minus 1, such that the Markov chain has uniform stationary distribution, ok. If we manage to do this the advantage that we have is should imply that we can uniformly sample from M k union M k minus 1. We can uniformly sample that would mean that we can estimate size of M k divided by M k union M k minus 1; we can also estimate M k by M k minus 1 etcetera, ok.

And, this can be done in done in polynomial time as M k is not too small compared to M k union M k minus 1 and M k minus 1 also is not too small compared to the total size because of our claim we can do these estimations in polynomial time, now. So, now, that the reason why we are constructing the Markov chain is clear. Let us try to construct the Markov chain whose states are M k union and M k minus 1.

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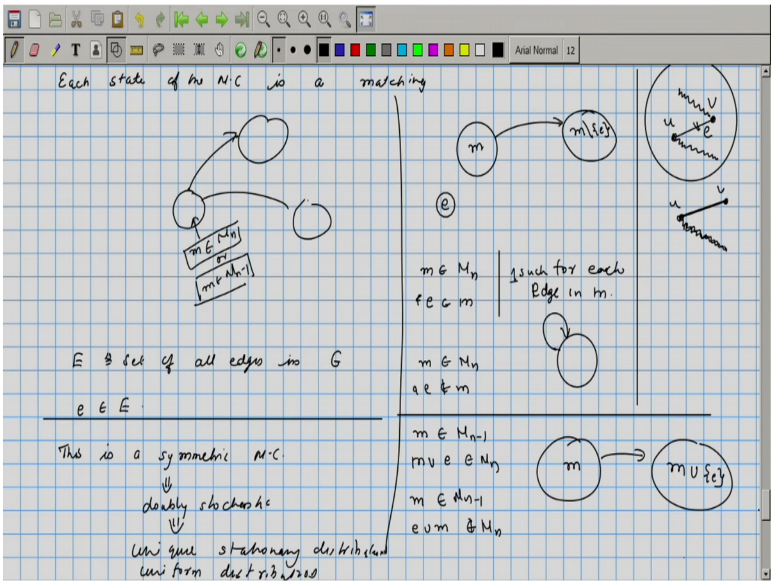

So, each state of the Markov chain of the Markov chain is a matching. So, this would be a matching. It either belongs to M n or m belongs to M n minus 1. If it belongs to M n we will move to some particular state in a certain way; if it belongs to M n minus 1 then it moves to certain other states, ok; if it is M n minus 1 it will move to let us say some other state. So, depending upon whether this matching belongs to M n or M n minus 1 we are going to the specify the transitions of this Markov chain.

So, suppose E is the set of all edges in the bipartite graph G. What we will do is we will choose an edge of E uniformly at random or for every edge of E we tell how was the transition going to be done, ok. So, let us say e is a particular edge e belongs to capital E then so, the way we do a transition is as follows. At the start with v we just pick an arbitrary matching; it might either belong to M n or belong to M n minus 1. Now, for a particular matching we randomly select an edge e ok. Based on the properties of this edge we will go to some particular state.

Now, first if your matching itself was belonging to M n and this edge also belongs to small m ok, then what you do is from this matching m you will go to a matching which is m minus e, ok. So, there are going to be lot of these transitions one for each edge of. So, one such for each edge in small m. Now, suppose m belongs to M n and e does not belong to m then you just remain at the same state that is a self loop, ok.

There is another case. Suppose, your edge m was belonging to M n minus 1, and m union e belongs to M n then you go from m to m union e. If m belongs to M n minus 1 and the edge e union m does not belong to M n; that means, you cannot add the edge e and make a larger matching out of it. Now, this happens because of certain conditions. So, let us say when you add e let us say connects u and v we were already unmatched vertices then when you add the edge G it gives you a larger matching, but when you are not able to do this; that means, u is connected via a matching edge to something and v is also connected to something by a matching edge.

Or it could be that one of v means; so, this is your edge u v, one of them is connected and the other one is not connected. If this case had appeared then what you will do is you will just remove this particular edge and add these two, ok. So, again in this there are two cases. Suppose, e cannot be added and it is because both it is endpoints are there in matching edge. I mean one of them was matched you could just flip the edge when instead of this particular matching edge you add this particular matching edge, ok. This is possible when v was not connected; if u and v were both connected then you do nothing, because if you add e that will only decrease. So, if u and v were both connected then you do nothing, ok.

So, these are the steps in your Markov chain. The property of this Markov chain is this is a symmetric Markov chain. The Markov chain is doubly stochastic. Doubly stochastic ensures that this is unique stationary distribution. So, this Markov chain since it is unique stationary distribution, the unique stationary distribution is the uniform distribution. So, doubly stochastic basically means that the uniform distribution will be the unique stationary distribution ok. So, that is our Markov chain.

We will see more of this in the next lecture.