Randomized Algorithms
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Lecture - 19
Electrical Networks

In today's lecture we will see some connection between random walks on finite graphs and related with some properties of Electrical Networks.

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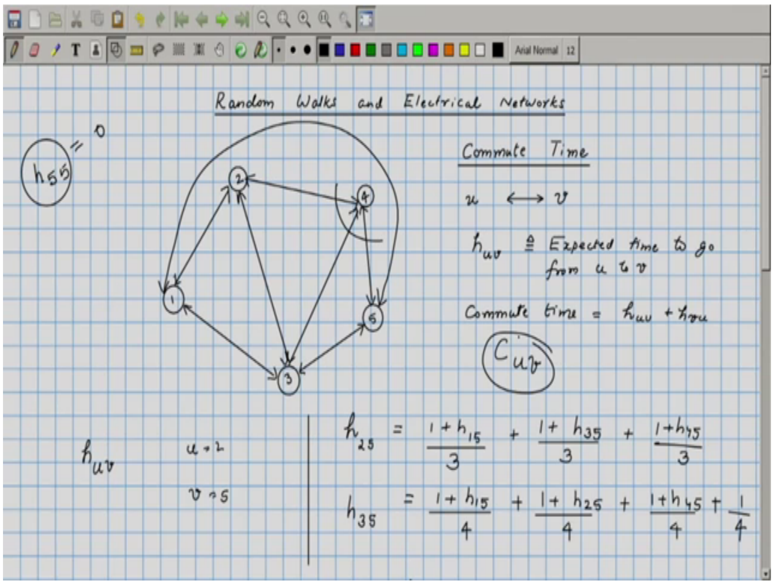

So, let us look at the following graph ok. So, consider the following graph we are thinking of an undirected graph other way, in other words the edges can be traversed in both directions. There are 5 vertices this we will name it as 1 2 3 4 and 5, when we say random walk on such a graph what we mean is. Let us imagine that you are at one of the vertices, you go to one of the neighbors by tossing a coin. So, for example, if you are at vertex 4 there are 1 2 3 neighbors of 4 and you can choose a coin with bias 1 by 3 or you can just choose one of these 3 vertices, all with probability 1 by 3 and move to that vertex.

In the next step suppose you reach at vertex 3 you go to one of the neighbors with probability 1 by 4 because 3 has 4 neighbors and so, on. You keep on doing this what you get is the Markov's chain. Now, when we do this random walk we are interested in lot of properties statistical properties of this random walk, in particular one aspect that we are interested in is called the commute time. We will relate this property of commute time to the resistance in an electrical network constructed from this particular graph.

So, let us first see what is commute time? Suppose you are at vertex 1 or let us say some graph we are at vertex u you will take some time to reach vertex from v you want to go to from u you want to go to v and then from there you want to come back. The expected time taken to do this while you are doing a random walk is called as the commute time. Once again, the expected time that you will take to go from a vertex u to v and then return back to this same vertex this is called as the commute time. We can think of this as some of the two quantities that is the time expected time take into go from u to v and then from v to u. So, if h uv is the time taken to go from u to v.

So, we define h u v as the expected time to go from u to v ok. Then commute time is h u v plus h vu, how do we compute the commute time given towards this u and v. Now, that is the question that we will be address today, let us understand this is h u v how do we if it really had to compute h uv from this particular graph how do we do it? So, let us say if u is equal to 2 and v equals 5 then what is h u v? So, since we want to compute h uv we can assume that we are at vertex. The time taken to reach vertex 5 will be, the first step you can go to either 1 3 or 4 then.

And then from 1 3 or 4 the time taken to reach 5 will give us some idea about how much h u v will be. So, we can write this equations, h 2 5 is equal to you take the first step and go to 1. So, in that case the time taken would be 1 plus h 1 5, but this happens with probability 3 I mean 1 by 3 plus 1 plus h 3 5 is if you go to step 3 and then how much hour is the time taken to go 5 from there, that divided by 3 plus 1 plus h 4 5 divided by 3.

This is an equation that has to be satisfied by h 2 5, we can write similar equations for every vertex. For example, h 3 5 would be 1 plus h 1 5 you go to vertex 1 from there you reach back to 5 divided by 4 plus 1 plus h 2 5 by 4 plus 1 plus h 4 5 by 4 plus 1 plus h 5 5, but h 5 5 is 0 because if you start at 5 I mean sorry if you are at 5th vertex you already reached there. So, it will be just 1 by 4 with probability 1 by 4 you would have reached the vertex 5 in 1 step from vertex 3, that is why you have this equation. So, we could write an equation involving all these quantities h 1 5 h, 2 5 h, 3 5 h, 4 5 h 5 5 which is I mean it should not call it as h 5 5; so, yeah you can define h 5 5 to be 0.

But in some cases when we are think of h 5 5 we think of time to start at 5 and then go through some other vertex and come at 5. So, here since there is no confusion of that kind we will just say that h 5 5 is 0, if you start at vertex you already there. But there is an associated notion of starting from a vertex and returning back to the same vertex and how much time does that, what is the expected time taken for doing that particular commute. So, you can write these equations involving different variables and you will get a linear system of equations you can solve these linear set equations that you get in this solution.

That would be a way to compute h, I mean we compute the value of h u v and compute the value of h u v, compute the value of h vu add them up you will get the commute time let us call this as C uv ok. So, that is going from u to v and coming back to u that is C uv ok. Now, let us relate this quantity commute time to the electrical properties of a certain network ok.

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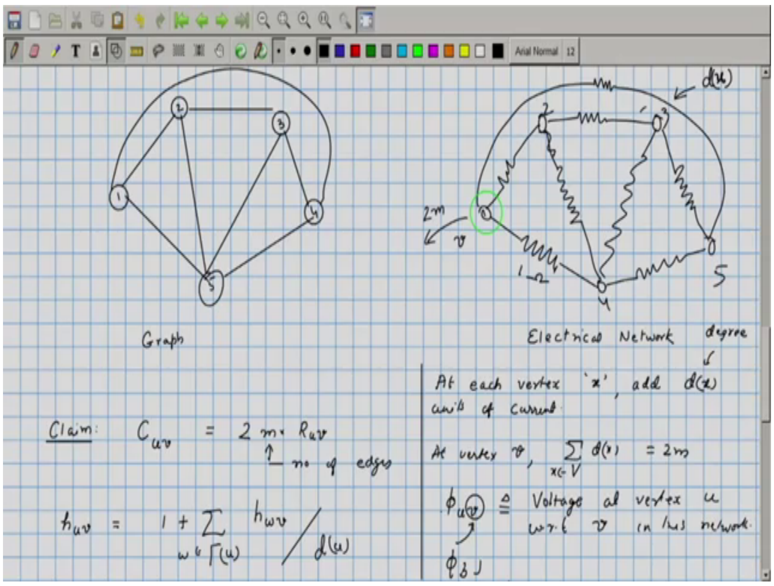

So, let us just think of a network. Now to this is the graph and the electrical network that we have in mind for this particular graph, is a resistive network all the components are essentially resistors. So, it’s a same network, but with all edges being replaced by resistor of 1 ohm ok. Now, what we will show is C uv that is commute time for starting at vertex u going to v and returning back to u is going to be equal to 2 times the total number of edges times R uv, where R uv is a effective resistance between u and v.

Now, then we look at h u v we had we can write this particular equation for h uv, the time taken to commute between u and v will be equal to 1 plus the average time taken to reach from each of the neighbors of u to so, let us just ok. So, here we not taking about the commute time, but the time taken to go from u to v that is the heating time ok. So, that will be equal to 1 plus the time taken the average time taken to go from the neighbors of u to v so, this will be sum over all neighbors of w belonging to neighbors of u, h wv the whole divided by degree of u.

Now in this electrical network we want to do the following thing, let us say an arbitrary such network we will just label the vertex u or the vertex v green to indicate that we were going to take out current from this vertex. We will take out 2 times m units of current from here and that each of the other vertices we are going to introduce d u or d x units of current ok.

So, this is a vertex of degree 3 we will introduce 3 units of current there, this is another vertex of degree 3 we will introduce three units of current there, this is a vertex with degree 4 we will introduce 4 units of current there. So, what we will do is at each vertex x add or introduced introduce d x this is the degree of so, this is the degree units of current. And then these currents that are introduced at every vertex it's been taken out through a single vertex that is v so, at vertex v.

So, how much I mean if you think of the whole system the currents are introduced at individual vertices and taken out at v; at v so, the total amount that will be the sum of the degrees of all the vertices. So, in particular at v also there would have been d v units of currents coming in, but the currents that is taken out is 2 m because sum over d x for each x belonging to the vertex vertex will be equal to 2 m.

So, 2 m units of current is taken off from v. Now, under this setting in this particular electrical network across various edges there would be some voltage that is developed voltage difference that is developed, we can compute those voltage differences using Ohm's law and Kirchhoff's current law ok. So, let us introduce some notation we will call by phi u v so, this will denote the voltage at vertex u with respect to v in this network.

So, in this network we will keep this v as fixed because v is the unique place from where current is being removed, with respect to that vertex what is the voltage at every other other node? That is phi uv, if you had numbered the vertices 1 2 3 4 and 5.

The phi 3 1 will be voltage difference between 3 and 1, phi 4 1 would be the voltage difference between 4 and 1 and so, on ok. So, this voltage difference we can use to I mean this phi u v can be combined with other phi u vs and we can use that to write Kirchhoff's current laws and Ohm's laws so, let us do that.

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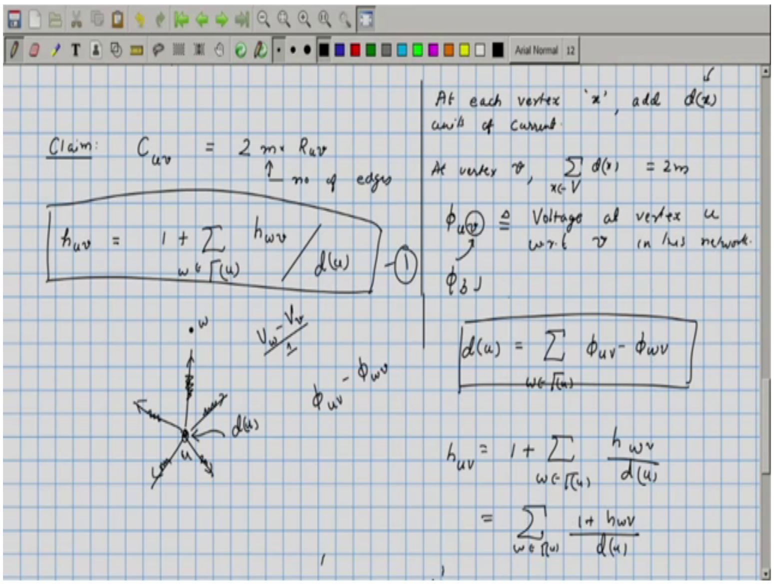

So, look at any vertex u so, phi u v. So, look at any particular vertex u the current that is coming into it is must be equal to the current that is leaving it. So, for this vertex u phi u v will be I mean sorry the; if you write down the current law; the current that is coming in we have already said that is d u ok. This is the current that was pumped into u, the current going out will be the current going out through each of the resistors connected to the node u.

So, if you look at if you think of the vertex u and there are let us say 4 5 neighbors each of them having single resistor; having a resistor of; resistor of resistance 1 ohm. We had put d u units of current so, the current flowing out should essentially be equal to d u, but this current flowing out will be the potential difference across these divided by the resistor. So, current through this particular arm would be so, if this is vertex w that will be voltage of w minus voltage of V divided by a resistance which is 1.

So, that will be the current across this, but this potential difference we can simply write it as phi u v minus phi wv so, look at. So, the potential at u with respect to v minus potential at w with respect to v divided by 1 so, this will be equal to the total current flowing out. So, sum over say neighbors w belonging to neighbors of u phi uv minus phi wv, so this is our second equation.

So, we have now two equations; one for the random walk and another for the electrical network, these equations are the essential same equation how do we see that ok. So, let us look at this equation 1 we will try and rewrite it so, h uv is equal to 1 plus summation omega belonging to gamma u neighbors of u h wv by d u, you can push this 1 inside and write it as summation w over sigma u 1 plus h wv by d u because there are d u neighbors if you take this out you would have got one out.

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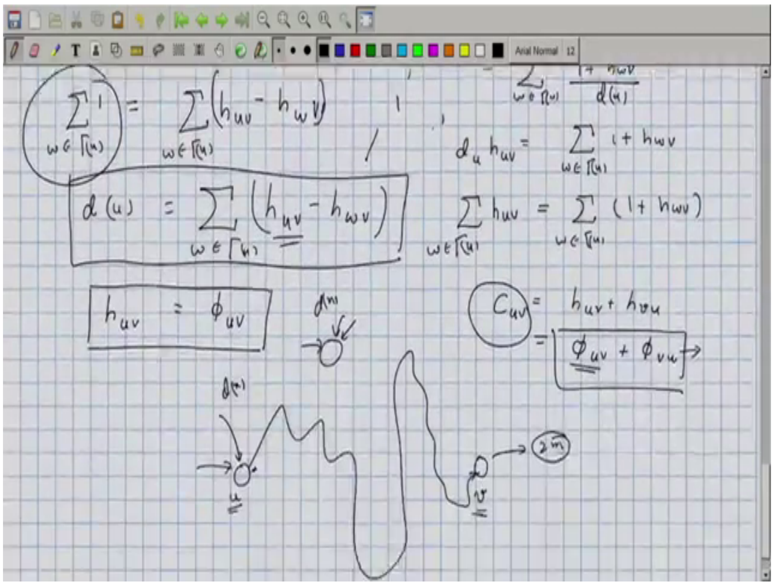

Now, d u is common so, you can write d u h uv is equal to sum over w belonging to gamma u 1 plus h wv ok. So, this d u we can simply now write it as summation over neighbor’s gamma u h uv there are d u neighbors so, that should be the same ok. So, w belonging to gamma u is equal to summation w belonging to gamma u 1 plus h wv.

We can bring the h w v to the other side and therefore, write it as so, we will write this equation here summation 1 w belonging to gamma u is equal to summation w belonging to gamma u h u v minus h w v ok. And this quantity is just d u we can write d u is equal to summation w belonging to gamma u h uv minus h wv ok. So, now, so this is the same equation as equation 1 so, we will call it as 1 prime. So, 1 prime and 2 prime now let us look at these two equations, they are essentially same ok, whatever we were calling as h u v here we calling as h uv here is replaced by phi u v. Moreover these equations the model systems which has unique solutions one is the expected value of some particular random variable, the other is the voltage differences between certain pair of nodes.

So, these are systems is physical says you can think of them as physical systems which is unique solutions. So, here you have two linear system of equations who which has unique solutions, their form is exactly the same and therefore, their solutions should be the unique solutions should be identical. In other words, we can say that h uv whatever is the value you get for h uv will be equal to phi u v so, this is key inside that we will use. So, we can say that in this random walk the expected time to go from 1 to 3 can be seen as the voltage difference between node 1 and node 3, when you have unit resistance across each edge and you introduce d x units of current at each node and take out 2 m units of currents from 3.

So, that is main idea that we that we going to use in order to prove this particular thing. So, once again if you had an arbitrary network and you had 2 nodes u and v the time taken the expected time taken to go from u to v or if you think of it is the random walk starting at u and ending at v. The length of the expected length of that random path it is going to be equal to the voltage difference between vertex u and vertex v, when you introduce d x units of current at each node x if it were u you would introduce d u.

So, you introduce d x units of current that each node at each node you introduce current proportional to the or equal to the number of neighbors and take out 2 m units of current from vertex v. When you do this you will have a potential difference between u and v that potential difference is going to be the expected time taken to go from u to v. So, the commute time we can write will be equal to so, C uv, this is equal to h u v plus h vu and that is going to be equal to phi uv plus phi vu.

Note that when you think of phi u v you cannot just think of it is the potential difference between v and u you have to think of a another network in which you are introducing current at all nodes equal to their degree and you are taking out current from vertex u, when you think of phi uv you are taking out current from v.

So, we will now introduce this second key principle that we will use in this analysis which is super positional principle,. We will take two networks their resistive components are the same and we will introduce currents in these networks which will cancel out in a certain way. And these currents will basically together they will induce potential difference equal to this across the nodes. So, let us look it carefully.

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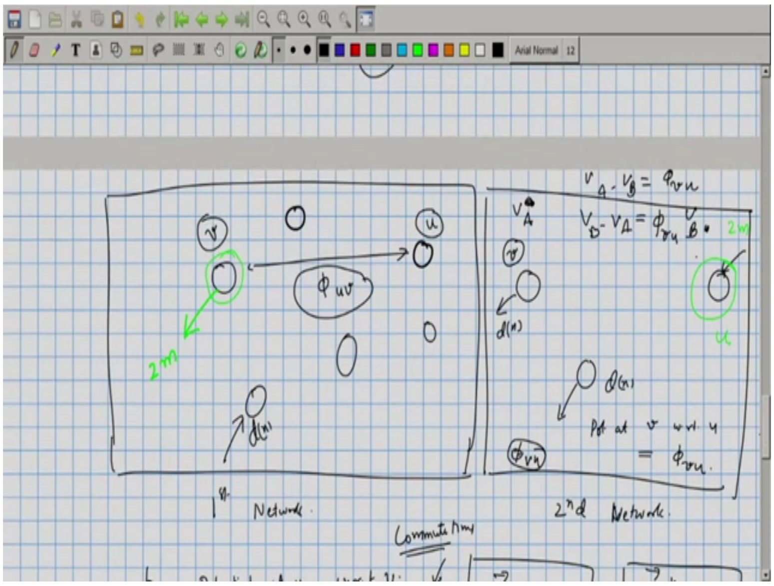

So, for elastration purpose the node at which we are introducing the current we will just draw it in green sorry the node from which the unit node from which you were taking out current we will draw it in green. So, here from here you are taking 2 m units of current out, at every other node you are basically introducing current at the other nodes you would have introduced d x corresponding to it's degree. So, this is our I am not drawing the resistive elements this is our 1st network ok. So, let us say this is a node d u this is a node v. So, the voltage difference between these two would be phi u v.

Now, let us look at another network so almost same, but what we will do is instead of taking out 2 m we will introduce you may two significant changes. In the previous network what we did is, we were introducing currents at lot of nodes and taking out current from 1 node. Here we going to change the direction ok, when you change the directions of currents the voltage is just flipped so, that is the first change. The second change is we will also change the place from which we are taking out current. So, instead of v now, we have the node u and at this node what we do is; we are introducing 2 m units of current. So, this is different from what was happening earlier; earlier we were taking out 2 m now we are introducing at vertex u.

At all other vertices what we do is; we are taking out d x units of current ok. So, you can think of this as similar to the 1st network if you are just flipping the direction of current. So, instead of wherever you are inserting current now you are taking out current and wherever you are taking out current, you are inserting current. And the second change was we are changed the node at which we are taking out current, we had a unique node at which we are taking out current now we have a unique node at which we are inserting current.

So, now what we can say about the potential difference between u and v or potential, so phi u v was in this node phi u v is equal to the potential at u with respect to v. Now, in this node what is the potential at u with respect to v? Let us say v were some other node ok, what is the potential at v with respect to at u with respect to v. That we will determine, but let before that let us make it clear why we are looking at this 2 networks ok. Now an important principle in case of a resistive circuits is that, if we take two identical resistive circuits and forget about the excitations in the circuit that is forget about the voltages and currents, if this circuits are the same.

Now add in excitations in both circuits that is you add voltages and currents which balances properly by Kirchhoff's law and Ohm's law. So, there will be some voltages across different components, there will be different currents flowing through the network. Now if you think of another network with the same resistive elements, but the excitation being some of the excitations that can be obtained simply by putting this network on top of one another ok.

So, let us say you had let us take a simple example, So you had let us say some simple circuit where some current was flowing through this let say 2 volt and I mean the some current flowing through this. And if you have another one in which say you in 10 volt and the some current flows through, it if you were giving 12 volt the current that flows through is basically the sum of these two.

And no this can be applied no matter how complex the network is, you just put 1 network on top of each other wherever there are currents if they are in the same direction you add them if they are at the opposite direction you cancel out to give the resultant current and so on.

So, this can be done for networks; which has just resistive elements so, here we have a such an network. So, if you put these two networks together the voltage difference between u and v or if you think of the voltage difference between these two vertices that is essentially going to be and whatever was the voltage difference in the 1st case plus whatever is the voltage difference between the 2nd case, but note that in this super post network at every node other than this designated nodes u and v, we have 0 current coming in ok.

Because d x units of currents were pushed into in 1 network and the other d x units were taken out. While at vertex u there are 2 m units of current coming in and there are 2 m units of current going out. So, in the super post network you can think of it as, there are 2 m units coming in at one node and going out at other node. So, the current flowing between this u and v will be so, the current flowing is 2 m. So, 2 m times the effective resistance that is R u v will be equal to the potential difference, between u and v. And we will show this potential difference is nothing, but phi u v plus phi v u ok, if you show this we have achieved what we started of that is 2 m times R the effective resistance is equal to the commute time.

So, now, we need to determine what is the potential difference between u and v in this super post network? In the 1st network it is phi u v so, if our formula has to be correct, in the 2nd network the potential difference between u and v should be phi v u so, it should be phi v u ok. So, how do we see that we show that here between u and v whatever is the potential difference that is phi u v then we are that.

From our earlier analysis what we are know is if you were taking out 2 m unit if the currents in this network were exactly reversed then the potential v with respect to u will be equal to phi vu. So, this potential so, if I call it as A and this as V B, V A minus V B will be equal to phi v u because phi v u we had said is the potential difference, when you insert d x into each and take out 2 m from u ok, when the directions are reversed we can say V B minus V A when if the current directions alone were I mean of all the currents were reversed potential also will reversed. So, V B minus V A will be equal to exactly phi u v and v b minus v a is the potential at B with respect to A ok, so, I should not written this is yeah.

So, potential difference between u and v will be potential difference in the 1st plus potential difference in the 2nd. The potential difference in the 1st is phi u v, the potential difference between u and v in the 2nd instance we have now show on the that is equal to phi v u.

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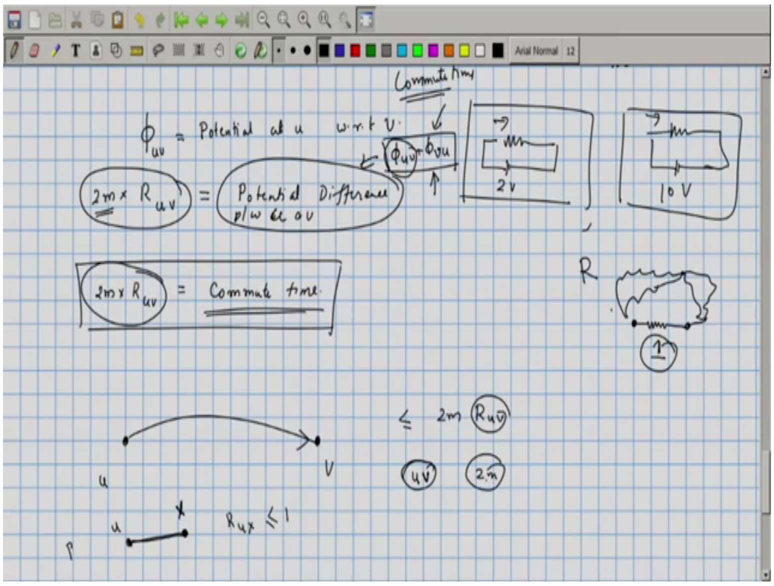

So, this means that 2 m times R uv is equal to commute time ok, that is the result we wanted to show. As a quick recap what happens is the following, we had set up the entire random walk as an electrical network in this electrical network if you have single every edge having 1 ohm resistance, we argued that the expected time to get from one place to another is equal to the potential difference between two nodes; between those two nodes.

And then by means of super position principle we showed that the potential difference between u and v for a certain super post network is equal to commute time and that super posed network, has 2 m units of current coming in and 2 m units of current going out between the nodes u and v.

Therefore by Ohm's law 2 m times u v is equal to the potential difference which is equal to the commute time, that was the main part of this lecture. Now how do we use this for algorithms ok, one thing that we know is between any two nodes u and v; the time taken to go from here to here will be less than 2 m times R uv. But in additional thing that we can say is R uv is always, I mean if you look at one particular edge say u comma v in or u comma x they are connected by an edge resistance of u x is going to be less than 1 less than or equal to 1.

Because they are connected; that means, they have a 1 ohm resistance between them and there might be other paths ok, there might be multiple other paths ok. But you can think of the other path having some resistance R and this is resistance 1, they connected in parallel their effective resistance which surely less than 1. Therefore, R ux is less than or equal to 1, therefore, we can say that if you want to take the path u comma v for any v that is connected to u that is not going to take I mean the expected time is going to be 2 times m ok.

Now, using this we can give an easy bound on the time taken to expected time taken to visit all the nodes in a random walk. So, we will stop here we will continue on random walks in the next class.