Randomized Algorithms
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Lecture – 15
Lovasz Local Lemma

In this lecture, we will learn about Lovasz Local Lemma. Lovasz local lemma is a tool that can be used while we apply probabilistic method.

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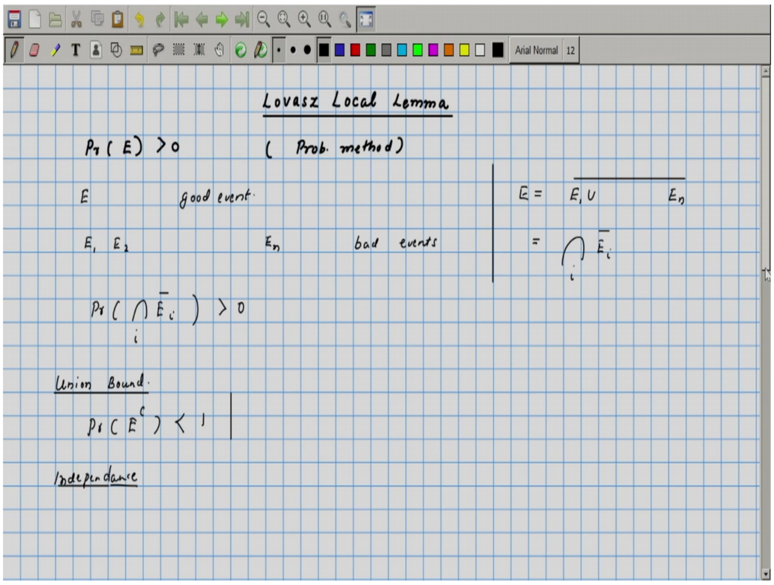

So, while using probabilistic method, we often have to show that probability of A certain good event is greater than 0. So, now, this can be done in a wide variety of ways how do we describe this event E. So, E if you think of is a good event, we could say that this good event happens when a list of bad events does not happen. So, let us say if our bad events were E 1, E 2 up to E n, and if you call this is the bad events, in order to say that the good event happens with a positive probability, we could also say that probability of any of the bad event happening is less than 1 that means, there is the probability that the good event happens.

So, we look at E as and we can simply write E as E 1 union E n, this is this means one of the bad events is happening, and the complement of it means none of the bad event happens that is equal to intersection over i, E i complement. So, instead of saying that the good event happens with positive probability we will say that the probability of this intersection this is going to be greater than 0. So, when can we say such a thing, so depends on these events.

So, let us see what are some of the common assumptions that we could make about E i one way to look at this would be if we thought that we knew nothing more about the events, just that the all belong to the same underlying probability space, we could use union bound ok.

So, probability of E complement rebar we want to say the that is less than 1, this would imply the probability of E is greater than 0. If we want to say this we could use the union bound and it can work in some cases, but in many cases when these E is are not too small and or when there are too many such bad events, if we apply union bound we might not get any useful result.

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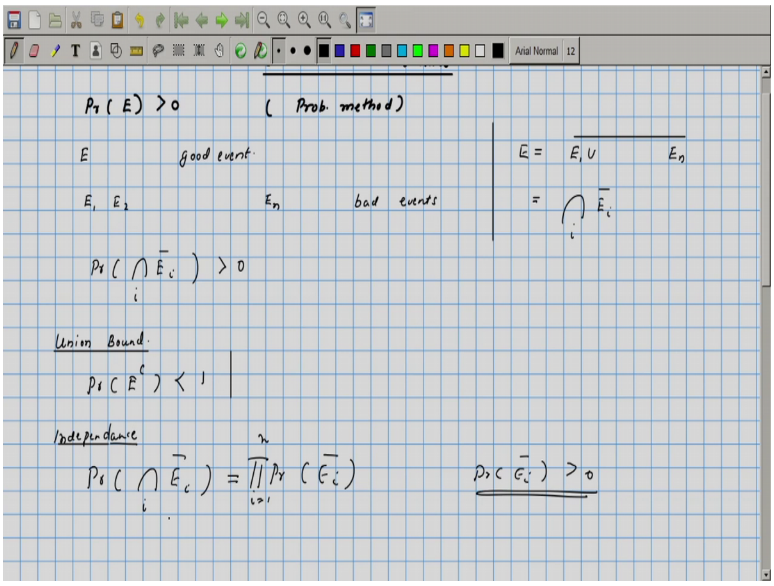

Another case is when the events are independent ok. So, when these events are independent then we can say probability of intersection of i E i complement this is equal to probability of each of the individually identity product over there i going from 1 to n.. So, if each these events if at least one of those event happen with probability less than 1, we know that the product is going to be less than (Refer Time: 03:48). If p i is greater than 0 the only assumption that we would need E i compliment is greater than 0, but independence is a very strong requirement. If the bad events are independent so are their compliments. The probability that one of the bad events occur we can bound it by using independence ok.

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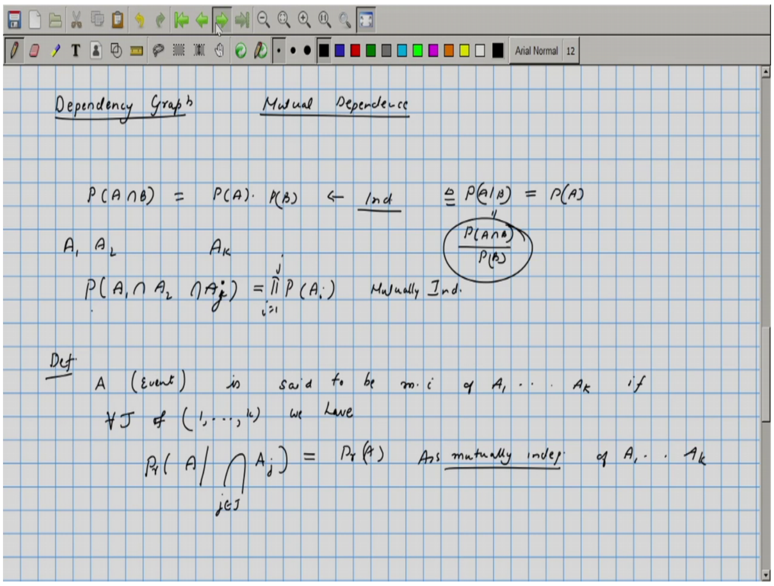

So, now, when there is some amount of limited dependence even then we can use probabilistic method and that is what Lovasz local lemma tells us. So, what does it mean to have limited dependence, we will get to it. But before that in order to specify this notion of dependency we need something called as dependency graph, we need the concept of mutual dependence. So, these two concepts we will first learn ok. We would say that events A and B are independent if probability of A intersection B is equal to probability of A times probability of B this is the definition of independence.

We will say that if you have a collection A 1, A 2 up to A k they are independent if mutually independent if you take any subset this is equal to probability or i, i going from 1 to j ok. So, you take any subset of these events if their probability if the probability that all of them happens is equal to the product of the probabilities, then we will say that these are mutually independent. We will modify this definition of mutual independence, and we will say when is one particular event mutually independent of a set of other events.

We could also cause this mutual independence in terms of conditional probabilities. So, A is independent of B if probability of A given B is equal to probability of A. Ok, this is because probability of A given B is probability of A intersection B by probability of B ok. So, they are the same thing. So, the definition of independence you can take this as the definition or even this as the definition ok. We will basically work with conditional probabilities.

In case of mutual independence the new definition that we will have is as following. An event A said to be mutually independent of A 1 to let us say A k if for all subset j of 1 to k we have the following condition probability of A given intersection over j belonging to j A j is equal to probability of A. So, if you look at an event we say that it is mutually independent of a collection of other events. If the probability of A conditioned on any subset of that collection of events is same as the probability of A. In this case we will say that A is mutually independent of A 1 to A k ok, so that is our definition.

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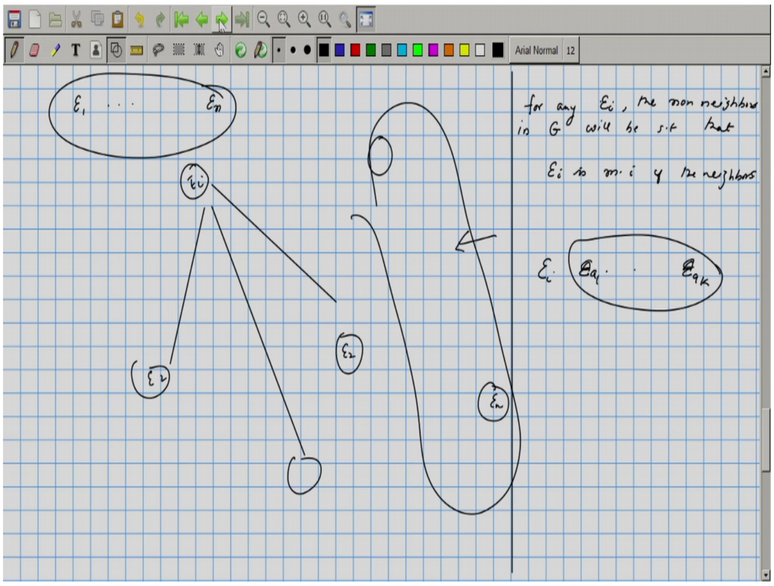

Now, we are in a position to state what is dependency graph. If we have a collection of events E 1 to E n, we could imagine a graph in which the vertices are these events ok. So, E 1, E 2, E 3 up to E n are the vertices of our graph. And we will say that this is a dependency graph for these events if so the conditional images write it down. For any event E i you look at the vertices to which it is connected those we will call as a dependent vertices and the non dependent vertices are the other ones. So, these are the non-dependent vertices.

So, for any E i the non-dependent vertices or the non-neighbours in G will be such that E i is mutually independent of the neighbours ok. So, if you look at any particular event E i and if its neighbours are let us say E A 1 to E a k, these form a collection such that E i is mutually independent of the non-neighbours. In other words, the dependency can be at most amongst the neighbours; E i can be dependent on at most the neighbours of E i and not on anything else. Events in form such a graph that is called that is called a dependency graph of E 1 to E n, of course, there could be multiple dependency graph.

For example, if you take the complete graph which is anyway dependency graph on any collection of events, because there are no non-neighbours. And therefore, there is no requirement I mean nontrivial conditions on dependency ok. So, now we know what are dependent dependency graphs, and we know what is mutual mutually independent independent.

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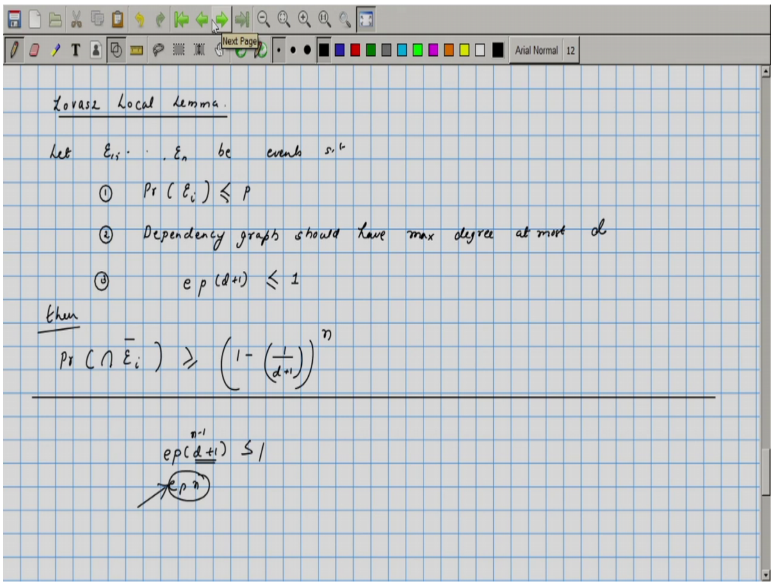

Now, we are in a position to state Lovasz local lemma. So, we have a collection of events. So, let E 1 to E n be events such that some conditions, and then under certain conditions Lovasz local lemma will say that the probability of intersection E i complement is going to be greater than a certain quantity nonzero quantity.

So, let us write down the conditions first. First requirement is probability of each E i should be less than or equal to some p. The second condition says that dependency graph should have max degree at most say d ok. So, if you construct the dependency graph E 1 to E n, every vertex should have degree less than or equal to d. And the third condition is a relationship between this p and d, E times p times d plus 1 should be less than or equal to 1.

So, under these conditions the probability of E i complement is going to be greater than or equal to 1 minus 1 by d plus 1, the whole power n ok. So, this is what Lovasz local lemma states. So, let us look at some cases suppose E i was I mean all the events were independent and of course there is not such a great bound because the condition requires that E times p times d plus 1 is less than 1, and d when they are dependent this becomes as large as n minus 1 ok. So, E times p times n should be less than 1.

But in that case we could have used union bound itself and here we get an additional factor of E union bound would have required p n is less than 1, but here we are getting an additional factor E ok. So, when they are when there is lot of dependency Lovasz local lemma may not be useful. But we will see that there are cases where the dependency is limited in that case this is the very handy tool to apply probabilistic method ok. We will first see a proof of Lovasz local lemma.

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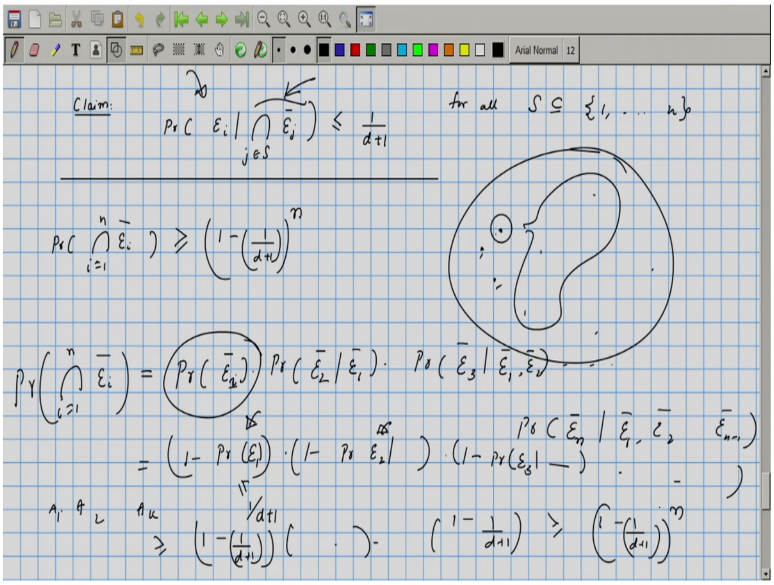

And the proof will rest on the following claim. The probability of any E i conditioned on the none of the events in a certain set S has occurred ok. So, j belonging to S. So, let us say S is some set, and what this part says is that E j bars have occurred for all j and S. That means none of the events in s has occurred. So, probability that an event i occurs. So, this is our collection of all events all bad events which we wanted to look at. And this one particular bad event occurs given that none of these other bad events have occurred ok.

The claim says that this probability is going to be less than 1 by d plus 1. Under the assumptions of our theorem the claim says that probability that E i happens given intersection j belonging to S E j bar is less than 1 by d plus 1. We will prove the Lovasz local lemma assuming this claim and later on prove the claim ok. This claim is true for all S subset of if your events are 1 to n. E 1 to E n and you take any subset of 1 to n for that this statement is true.

So, Lovasz local lemma said the probability of intersection E i bar is going to be greater than or equal to 1 minus 1 by d plus 1 the whole raised to n ok. So, this event intersection i going from 1 to n p i bar and so the probability of this can be written as a product involving conditional probability. So, if you think of the intersection is let us say A, A 1, A 2, A k, we could think of this is probability of A 1 into probability of A 2 conditioned on A 1 into probability of A 3 conditioned on A 1 and A 2 and so on.

So, this is probability of E i, E 1 bar into probability of E 2 bar given E 1 bar into probability of E 3 conditioned on E 1 bar comma E 2 bar and so on. The last would be probability of E suppose your n events E n bar conditioned on E 1 bar, E 2 bar, E n minus 1 bar. There are going to be n terms in this product, and each term looks like the complement of one of the events in the claim ok. So, this I can write it as 1 minus probability of E 1 into 1 minus probability of E 2 on the same conditioned into 1 minus probability of E 3 based on the same condition and so on.

And these events by our claim we know that they are going to be less than 1 by d plus 1.. So, this quantity p of u 1 is less than 1 by d plus 1. So, 1 minus that is going to be greater than, so this quantity is going to be greater than 1 minus 1 by d plus 1 and every quantity is going to be greater than 1 minus 1 by d plus 1. So, that gives us the lemma the whole product is going to be greater than 1 minus 1 by d plus 1 the whole raise to n. So, if we assume the claim we are done, but we need to prove the claim.

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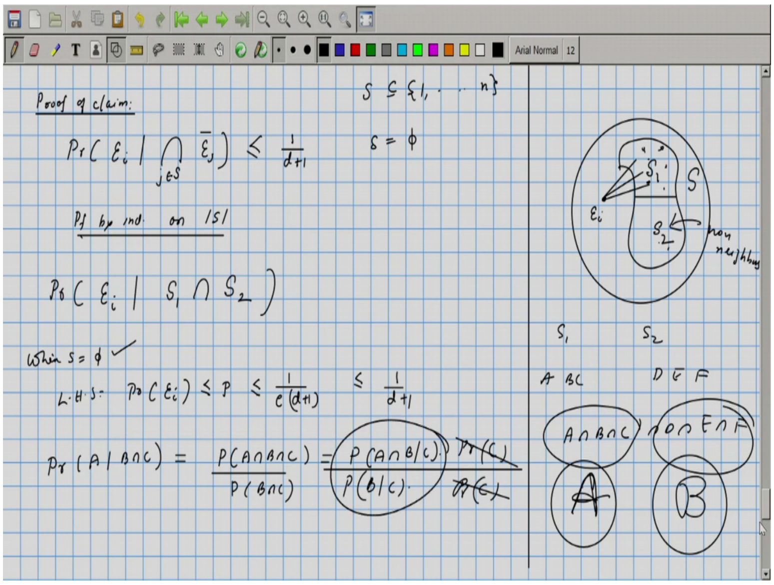

So, proof of claim we want to show the probability of any event E i conditioned on a set of other events not happening j belonging to S E j bar less than or equal to 1 by d plus 1. We will prove this on by induction on size of S. So, let us just have a small diagram before that. This is our favourite event which we are looking at E i, and this is our set of other events s ok. These are conditioning on that ok. We are saying that none of the events in S happens ok.

This S we can split into two parts let us say S 1 and S 2. S 1, where the events to which E was connect in the dependency graph. And these are the non neighbours. We can split it into two parts. So, the conditioning over intersection, we can we can write this is probability of E 1 given let us say when I will abuse a notation I will write it as S 1 the intersection of our all these events. So, if the if S 1 and S 2 where the two parts of S, what we are doing us these were A B and C, and this is D E and F. We have this A intersection B intersection C intersection D intersection E intersection F ok.

So, we could split this two parts you can think of this as let us say capital A or script A and this is script B ok. And instead of looking at these things, we just called them as S 1 and S 2. It is basically S 2 intersection S 2 ok. Here I am abusing the notation because S 1 and I mean you can think of it a an index set or you can just think of it as the collection of events. So, E i in G 1 S 1 intersection S 2, that is what we need to calculate.

Now, S could be any subset of 1 to n, if S was the empty set that would or base case of our induction. If s was the empty set, this means the intersection of an empty set is a whole universe that means, your conditioning on the entire sample space. So, the expression would be when S is equal to phi, LHS is equal to probability of E i. And probability of E i is less than some p which we know is less than 1 by E times d plus 1, because we had the condition that E p times d plus 1 is less than or equal to 1, and therefore, this quantities less than 1 by d plus 1.

So, when S is equal to phi, we are done. When S is not equal to phi, we have some set on which we are some nontrivial set on which we are conditioning that set we have split into two parts S 1 and S 2. S 1 consists of all those events on which E i could be dependent; and S 2 is all those events on which E i is not independent.

So, now probability of A given B intersection C can simply be written as probability of A intersection B intersection C by probability of B intersection C which is same as probability of A intersection B given C into probability of C divided by probability of B given C into probability of C. And probability of C, we can cancel out assuming that that is a nonzero probability event. And then we have this particular expression and that is what we will use to expand probability of E i given S 1 intersection S 2.

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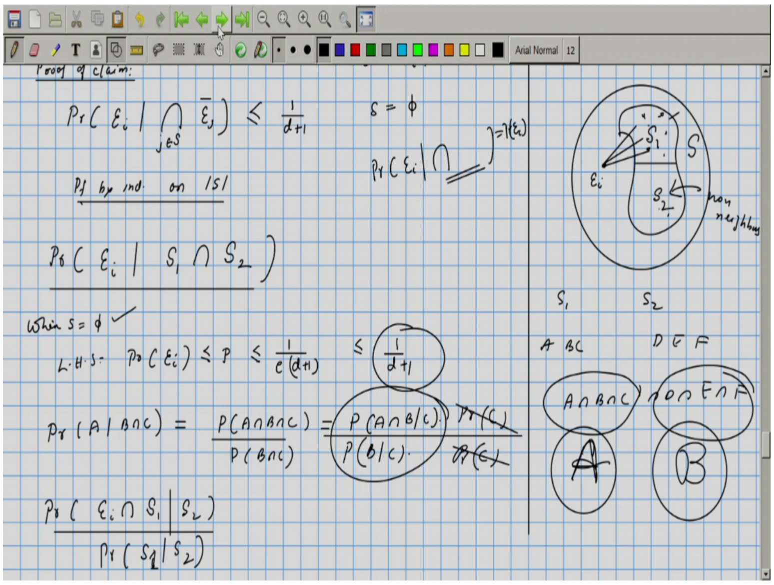

So, this will be equal to probability of E i intersection S 1 given S 2 into probability of S 1 given S 2 ok. So, this is the expression that we will need to simplify.

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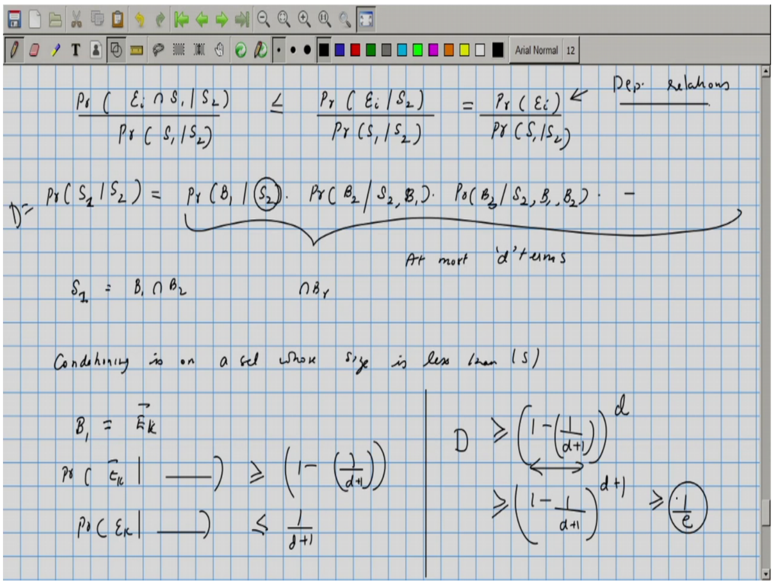

Probability of E i intersection S 1 given S 2 into probability of S 1 given S 2, now, the numerator is going to be less than or equal to probability of E i given S 2 divided by probability of S 1 given S 2. A numerator E i given S 2 since S 2 are all the events in S 2 are basically disconnected from E 1 when you drew the dependency graph or they are not connected by n h. And therefore, we know that this probability is same as probability of E i divided by the denominator S 1 given S 2 this is because of dependency relations ok. We need to estimate or bound probability of S 1 given S 2.

And one thing to note in between is that if S 1 was an empty set, it is if everything was connected to only S 2, then this probability would essentially be probability of E 1 which anyways less than 1 by d plus 1 ok.. So, if all the elements to which I mean on which we had conditioned were not connected to E i then we automatically have this. So, we can assume here that S 1 contains at least one particular element in it ok. If there were not probability of E i conditioned on certain other events and all these events are not connected by E i by n h. So, this probability is equal to probability of E i and that we know is less than 1 over d plus 1. So, from now on we can assume that there is at least single event in S 1 ok.

So, when we compute this probability S 1 given S 2, you can again apply iterated conditioning properties and write this is probability. So, S 1 we called it as a set was a set contained is B 1 intersection B 2 intersection B r where each of these B is 1 of the E 1, E

2, E n ok. So, if S 1 was this we could write this probability S probability of B 1 given S 2 times probability of B 2 given S 2 S S 2 B 1 into probability of B 3 given S 2 B 1 B 2 and so on.

Note that the number of terms here is at most d terms, because S 1 is the elements which were connected to E 1, and we are the restriction that the dependency graph had at most d edges out of any particular vertex. And therefore, there are at most d terms. And look at all of these terms the conditioning is on a set whose size is strictly less than the size of S ok. So, conditioning is on a set whose size is less than size of S ok, because here subsets of S from which at least one element has been removed.

So, we can use the induction hypothesis. And based on this we can say that each of these B 1. So, B 1 now is some event of the form E i complement, so probability of some E k complement given some other thing is going to be greater than 1 minus 1 by d plus 1 this is because what we know is probability of E k given this is less than 1 by d plus 1. Therefore, the compliment event on the same condition is going to be greater than 1 minus 1 by d plus 1.

So, product of the denominators is going to be. So, let me just called this as the denominator. So, d is going to be greater than 1 minus 1 by d plus 1 the whole raised to d ok. And since the inner term less than 1, this is of course greater than 1 minus 1 by d plus 1 the whole raise to d plus 1 in this quantity is going to be greater than 1 over E ok, because it the limit becomes 1 by E ok.

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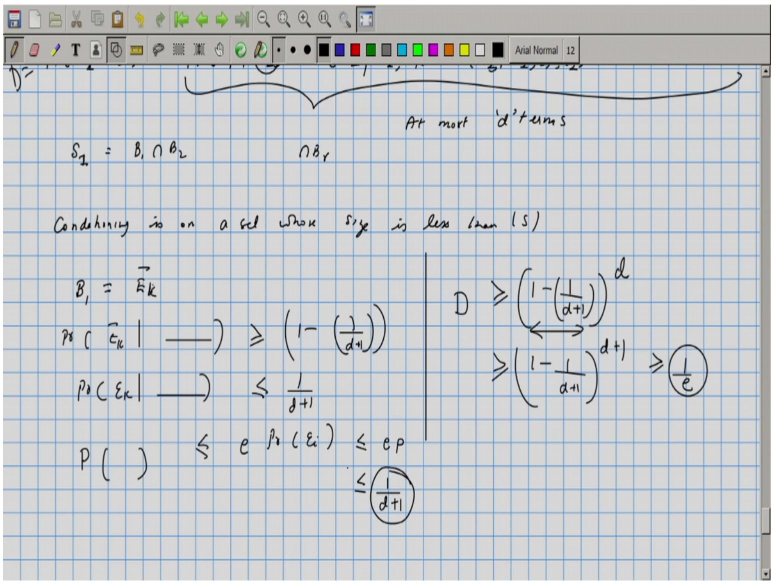

So, we can plug this into the original expression. What we will get is probability of the event that we were interested in is going to be less than or equal to 1 by E in the denominator. So, E times probability of sum E i ok. And probability of E i that is going to be less than E times p which by assumption is less than 1 by d plus 1 ok, so that concludes the proof of the lemma.

Just quickly recap the key ideas in the proof we had this lemma which says the probability of a bad event occurring conditioned on a collection of other bad events not occurring is going to be less than 1 over d plus 1. After we have that we can just apply iterated conditioning to get our lemma. In order to prove the claim what we did is we used induction the conditioning was on a set of bad events, the split the set of bad events into events which depend on E i and events which did not depend on E i.

If there is at least one event which depends on E i, then we could do this our induction case. If there was no events which depended on E i, then we can just simply say that are argument that these are mutually independent takes care of the lemma. When they where events the degree bound on the dependency graph helps us get what we want ok. So, we will move on to an application of Lovasz local lemma ok.

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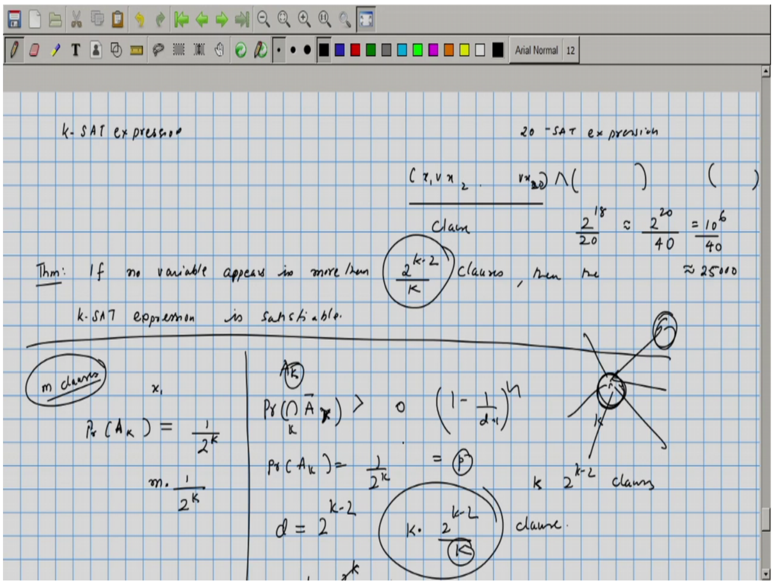

So, let us look at a K-SAT expression you can think of 3-SAT or maybe purpose of this lecture think off a 20-SAT expression and that means, expression was form x 1 or x 2 so on x k or x 20 and some similar expression and so on. So, each of this is called as a clause, there are large number of clause each clause has at most has exactly 20 literals in it K-SAT means each clauses exactly K-literals in it. We want to know that is an assignment to the variables which makes this expression evaluate to 1 ok. We will use probabilistic method to show that in many cases expression indeed evaluates to 1 ok.

So, theorem that we will show we will prove is if no variable appears in more than 2 raise to k minus 2 by k clauses ok. So, in case of 20-SAT expression that would mean 2 raise to 18 by 20. So, 2 raise to 18 by 20 that is approximately 2 raise to 20 by 42 raise to [10 is about 1000. So, this is 10 raise to 6 by 40. So, you can think of as two point 25000 and approximately 25000.

So, if we looking at a 20-SAT expressions in which no variable appears in more than 25000 clauses, so that is a good bound in the sense and if you have very long expressions I mean to imagine that there will be some variable which appears 25000 times that is the huge number. So, if that requirement is met that is no variable appears more than 25000 times, then we can guarantee that the expression is satisfiable, then the K-SAT expression is satisfiable ok.

The proof is simple. The first let us think about possible proof. So, if you think of all these clauses I mean there are let us say m clauses ok, each of these clauses had some number of variables. If you had randomly assigned true or false to the literals in to the variables in each clause, then the probability that a particular clause, this is satisfied. So, we will denoted by A k when we say that the event A k has occurred that means, clause number k is satisfied ok. So, the bad event would be that that clause is not satisfied.

So, probability of A k bar is going to be 1 by 2 power k. So, the bad event happens with probability if we just think of A k has the bad event. So, when A k occurs it means that clause is not satisfiable that happens with probability 1 by 2 raise to k because there is only one assignment which can make the formula go wrong.

So, when we use this we automatically place a bound on the number of clauses. Even if if we placed union bound, it means that the probability that one of the clauses is unsatisfied that happens I mean v i mean it is a by applying union bound we will get something like m times 2 raise to k ok. So, the number of clauses that, you could have a something like 2 raise to k, but here we have a much more relaxed requirement. We do not bother about how many clauses are there, but if no variable appears in more than these many clauses, then we know that the they surely way satisfying assignment, how do we prove that.

So, again or events are the same A k means the k-th clause is not satisfied, We need to say the probability of A k bar intersection overall k is going to be greater than 0 or 1 minus 1 by d plus the whole raise to n that is what the lemma say states. In order to apply the lemma, we need to show that the preconditions are true. So, first condition was the probability of any A k, we know it is exactly 1 by 2 2 to the power k this we will called as our p ok.

If you look at the dependency graph, look at two particular clauses being satisfied or unsatisfied now if that exactly k literals in it. So, the number of other events to which these could be connected ok, the maximum number is if you look at two clauses they are connected only if there is some kind of dependency between them, if they are variables were completely different then there cannot be any connection.

, but we have a bound on variable sharing ok, each variable can occur in at most two raise to k minus 2 clauses. And there are k variables in each clause, so each clause can be connected to at most k times 2 raise to k minus 2 divided by k clauses. So, this is a bound on the number of other events on which the event A k could be dependent ok. This k and this k are not really related, let me just call it is A r ok, so that means, d is equal to 2 raise to k minus 2 ok.

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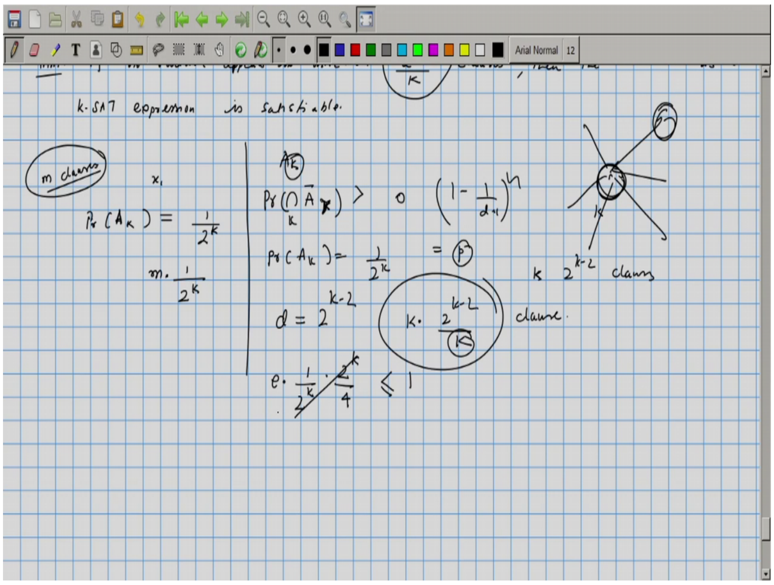

Now, the condition that we wanted was E times p, p is 1 by 2 2 to the power k times 2 raise to k minus 2 that is 2 raise to k by 4 this should be less than or equal to 1, which it is because E is less than 3. So, 3 by 4 less than 1, so we know that in this case, there will surely be an assignment which satisfies all the clauses. We will stop here. This is the final part of our lectures on probabilistic method.