Introduction to Computer Graphics
Dr. Prem Kalra
Department of Computer Science and Engineering
Indian Institute of Technology, Delhi
Lecture - 31
Fractals

Today we are going to talk about a different topic. In the case of geometrical modeling we have seen modeling of the objects which are like sphere, other quadric surfaces, parametric surfaces or it could be polyhedral objects or we can even apply CSG operations to do other engineering kind of objects. Today we are going to look at some sort of a tool which can help us doing the modeling of some natural phenomena. Now we are going to talk about fractals and then later on see how these fractals can be used for modeling natural phenomena. In fact fractal is actually from the word called fractus which basically means irregular object.

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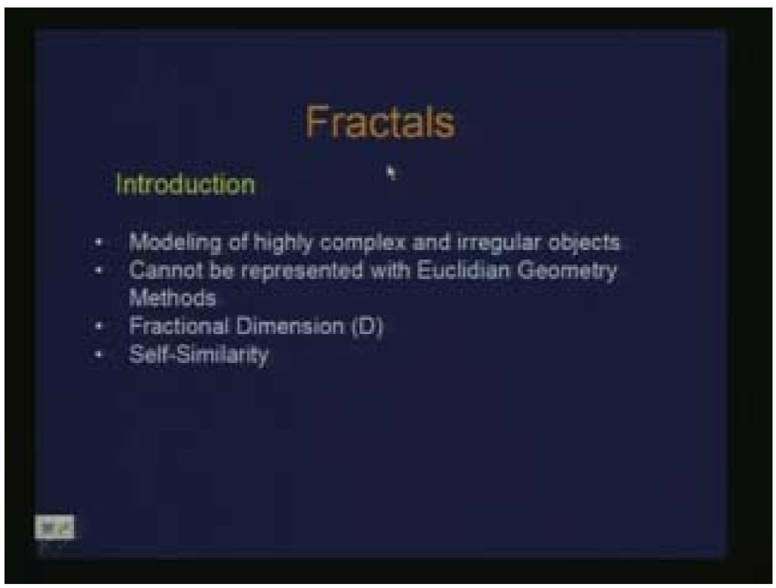

These fractals were sort of brought in the community of computer graphics and mathematics first by Mandelbrot. He even wrote a book called fractal geometry of nature. These fractals are basically meant for modeling highly complex and irregular objects which are difficult to do modeling with Euclidian geometry methods or even have the representation using Euclidian geometry. In fact as we are aware that there is notion of dimension when we are talking about the entities in Euclidian geometry there is also notion of dimension in fractal geometry so it turns out that this dimension could be a fractional dimension, unlike an integer dimension 1 2 3 it could be a fractional dimension.

We are going to see some of the fractals which have fractional dimension. They exhibit a property which is called as self similarity. So self similarity is a property where if I take a small part of the fractal which could be a curve or a surface or a pattern and if I zoom in then I see the scaled copy of the original pattern. That is what we mean by self similarity. In some of the fractals this self similarity exists in exact form and in some of the fractals it exists in some statistical form.

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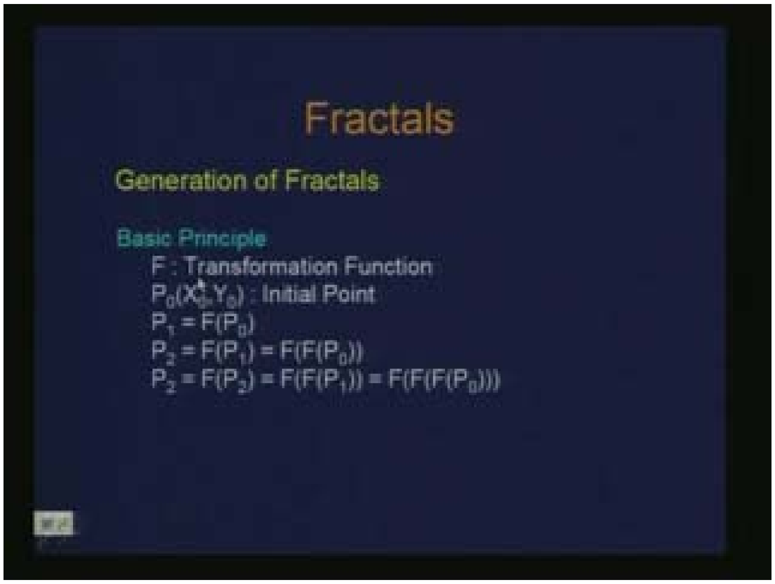

Basics of fractals: When we look at the process of generation of these fractals the basic principle of the generation involves a transformation function F. If you consider there is an initial point or initial state defined by P_0 which could be just the point X_0 Y_0.

Then the next state or the result of the application of transformation function on to P_0 is P_1 so you just apply F onto P_0. You do this process successively, repetitively so you get another state P_2 which is a further application of F on P_1. So again P_1 can be written as F P_0 and so on. So the process of generation of these fractals in fact is nothing but a successive application of some transformation function. And this could be applied forever theoretically speaking. So it is a very simple way of generating these fractals.

Here are some examples:
Here is a pattern which in fact could be obtained if I had considered a line from here to here.

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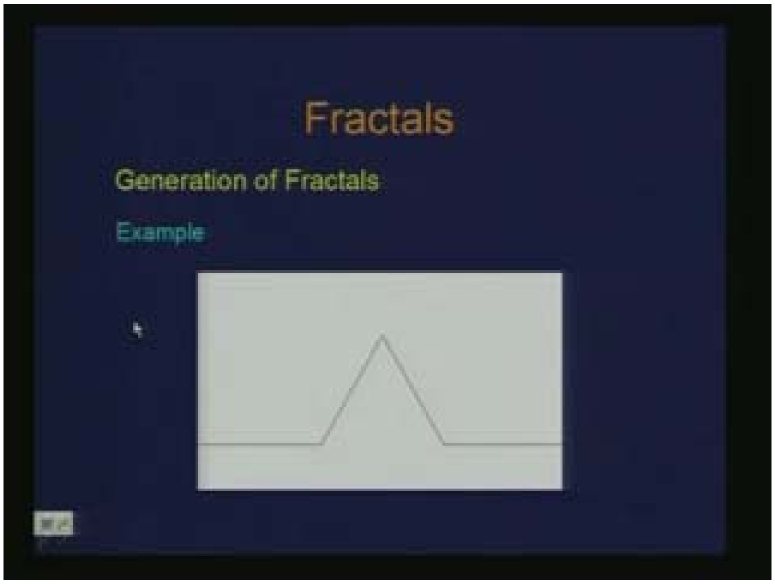

And I had applied this pattern onto this line that is from here to here which is a straight line. So this could be the first application of the transformation where the transformation is the pattern itself which I am applying on the line from here to here so this is the first apprehensive line. Now I do this successive application of the same transformation which is the pattern itself onto each of the segments I have obtained, this one, this one, this one and this one.

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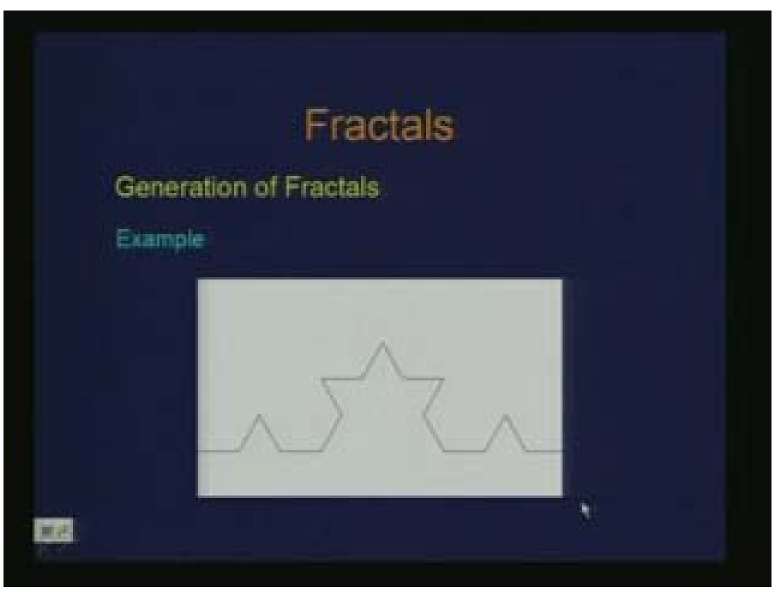

So what I get is this. It is a scaled copy of the pattern applied to the individual segments which I had obtained after the first application. Now what I would do is I would apply again a scaled copy of the same pattern to each of these segments. So all I am doing is basically applying the pattern on to a line segment so this is what I get.

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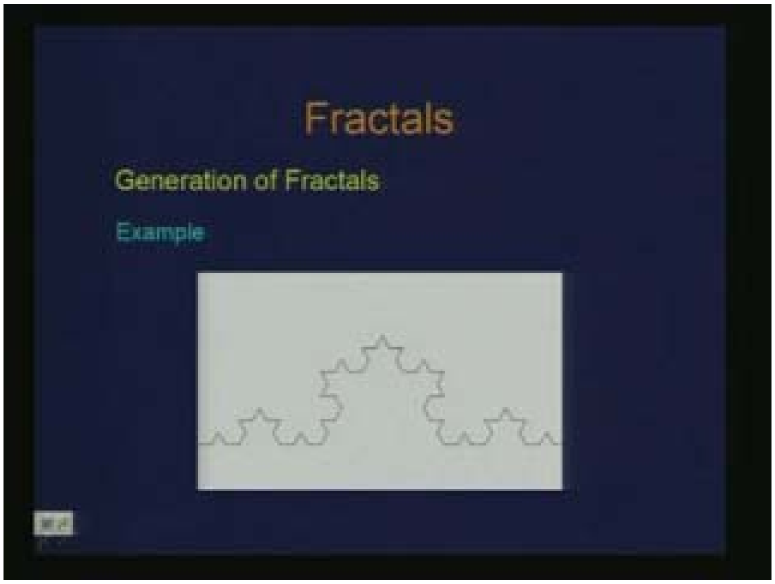

As a property of self similarity you do observe that the pattern which we have considered exist in every scale copy from here to here. If I had zoomed in this is what I would have observed as the pattern. So similarly if I further apply I will get this.

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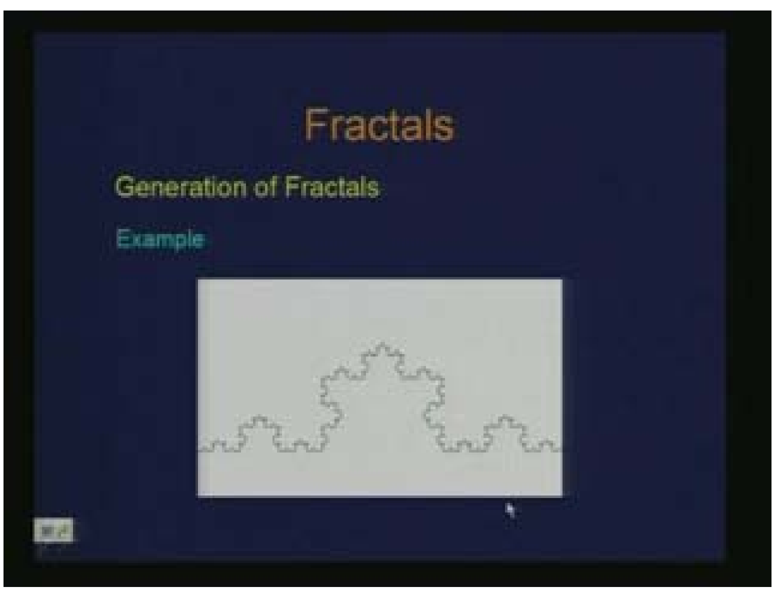

So it is a just repetitive application of a transformation and in this case it happens to be a geometrical pattern. This is the process of generation of fractals. These fractals have a dimension which could be fractional. Let us look at some of the properties which we observe in Euclidian geometry.

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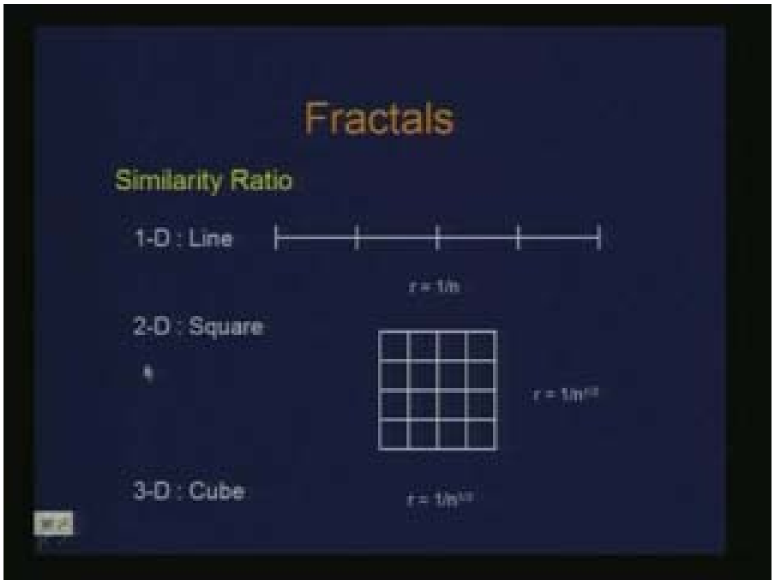

So if I have a 1-D case where there is a line and now I divide this line into equal parts. Then what I observe is that the part of this line which I have divided into has a ratio which is determined by 1 by n. So if I had considered this as 1 and n is the number of divisions I have performed then the scale or the ratio of this segment to the initial segment is given by 1 by n. This is what we call as the similarity ratio which in some sense gives you the scale ratio of the segment you have subdivided to from the initial segment.

Similarly, if I go to two-dimensions then this square which was the initial square when subdivided into the smaller squares then the ratio of this smaller square to the ratio of the whole square is given by this.

Now if you would extend it to 3D where if I consider a cube and subdivide into number of cubes then the small cube to which it is subdivided to will have the ratio to the initial cube given through this. Now I could write this as 1 by n power 1 so this similarity ratio in 1-D is this, 1 by n to the power 1 by 1. Here it is 1 by n power 1 by 2 and here it is 1 by n power 1 by 3. So you see that this is number 2, in this case 1, in this case 3 is actually from the dimension I consider. Now I can extend this idea of similarity ratio for any D dimension where I have r defined as 1 by n power 1 by D.

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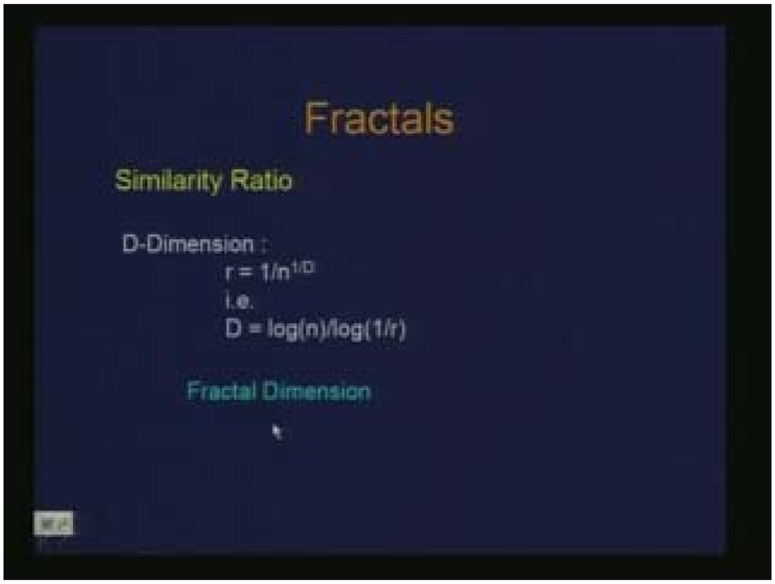

Now if you want to re write this expression to find out what D is and D is given as this. Now in fact when you look at value of D it could possibly be fractional depending on what these numbers are. This does not guarantee you that D should be always integer looking at this expression. So, that is the sort of a foundation to say that there could be a fractal dimension or fractional dimension. So the dimension which fractals exhibit or have is called as fractal dimension which could be a fractional number from here. Let us see how we can use them in some context. When we are discussing about fractals here we classify fractals of the kind which are geometric fractals from the fact that we are using these geometrical patterns as our transformation function to the generation process.

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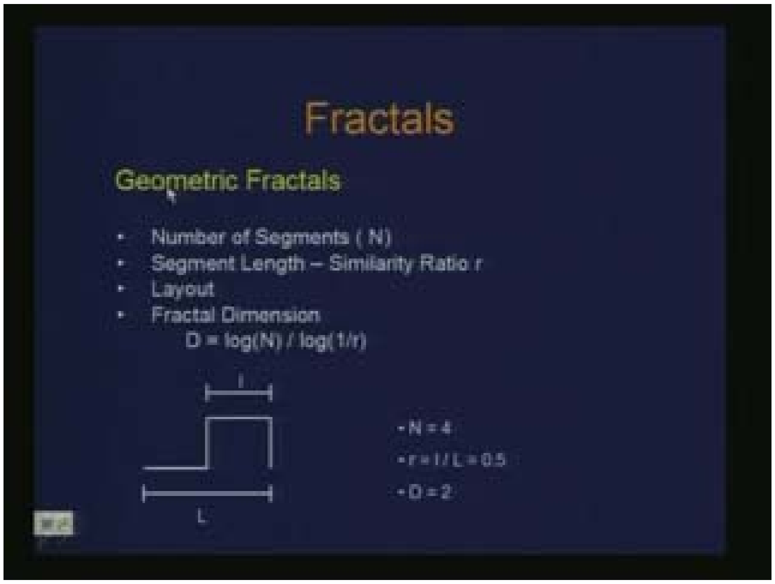

So, in order to characterize these geometric fractals the generator which is used or the pattern which is used for or as transformation function could have various attributes so there could be number of segments which define the generator. The segment length in turn defines the similarity ratio r, the layout of these segments and the fractal dimension. In fact fractal dimension could be computed from here and here and that is also an attribute towards what type of fractal is generated.

So, if you take an example here, here is again a generator which has got four segments so n is equal to 4. This length the length between two end points is given by L, the length of each segment is l. Then the similarity ratio or r is basically defined in terms of l by L which in this case is 0.5 and if you compute D using this it turns out to be 2. We will see some of the implications of this fractal dimension for the shape of the curve we get. Here is an example where again we are trying to generate a pattern which is popularly known as Herter-Heighway Dragon.

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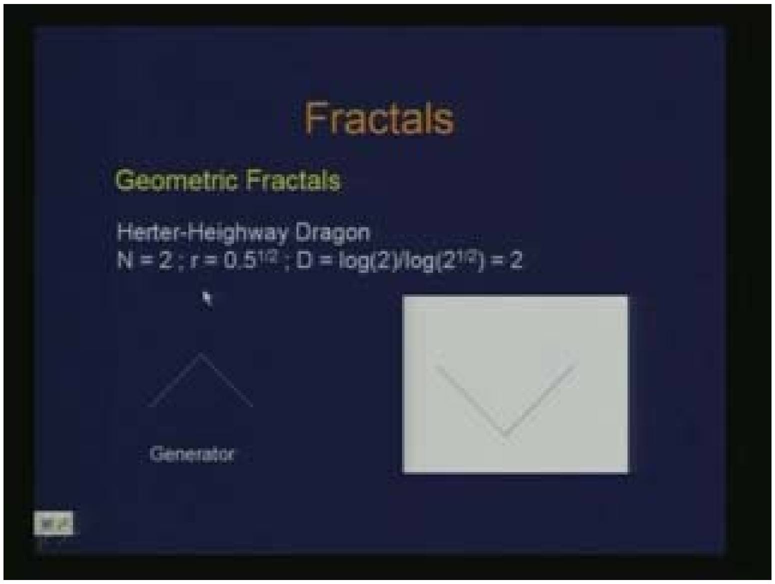

This is the generator we have and this angle is actually 90 degree. So n is equal to 2 and since this is 90 degree this is l and this distance is L. So what do you get as l by L? It is actually 1 by square root 2, root 1 by 2. This segment has got l, this distance we are referring to as L and this angle is 90 degree. So, if I am interested in finding out l by L you can see that this and this are the same, these two are the same.

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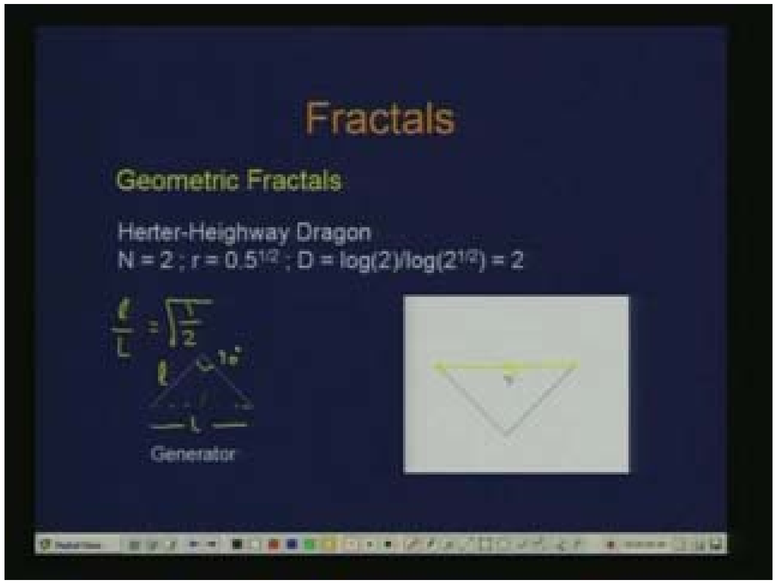

This is L by 2 so L square is nothing but this square plus this square and square root of that will be basically this. Now we look the process of generation. So what has happened is that I had basically considered these two points here and there is a line between these two points which is the line where I start applying this pattern. I have two possibilities in which I can orient this pattern. One is to go on the side of this if I go from here to here and the other possibility is on this side. What I have done here is I have basically applied to the right of this for the first time. And in the next part where I did this what I have done is for this segment I have applied to the right whereas for this segment I have applied to the left.

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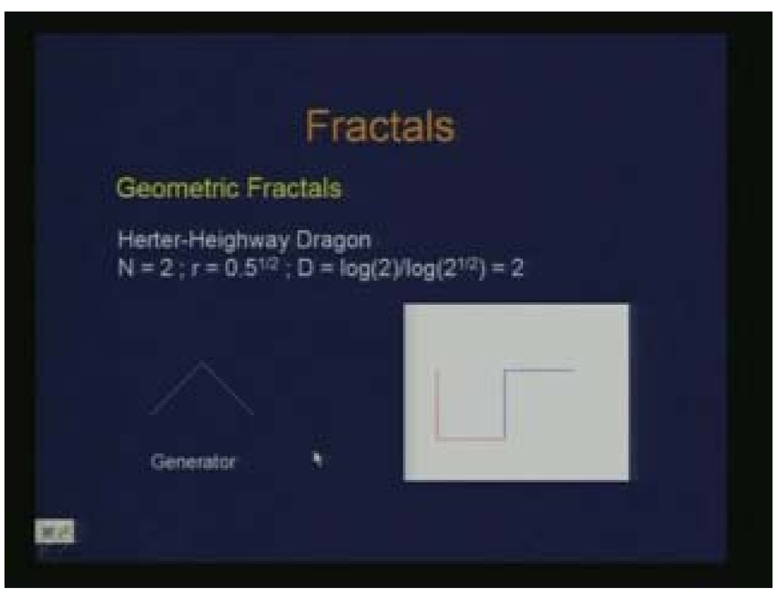

Alternate flips have been done just because it gives you an interesting shape. I can keep doing this (Refer Slide Time 20:33) again this is flipped in this, this is flipped on the other side, this is flipped on the right and this is flipped on the left.

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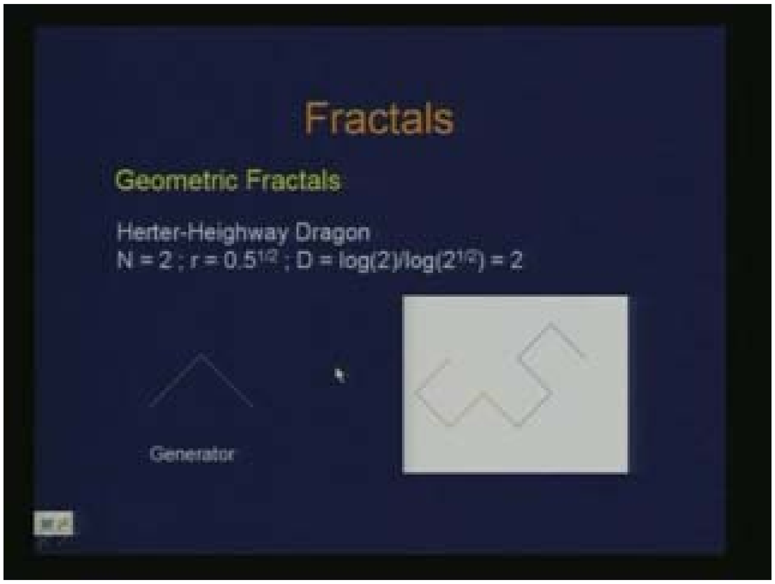

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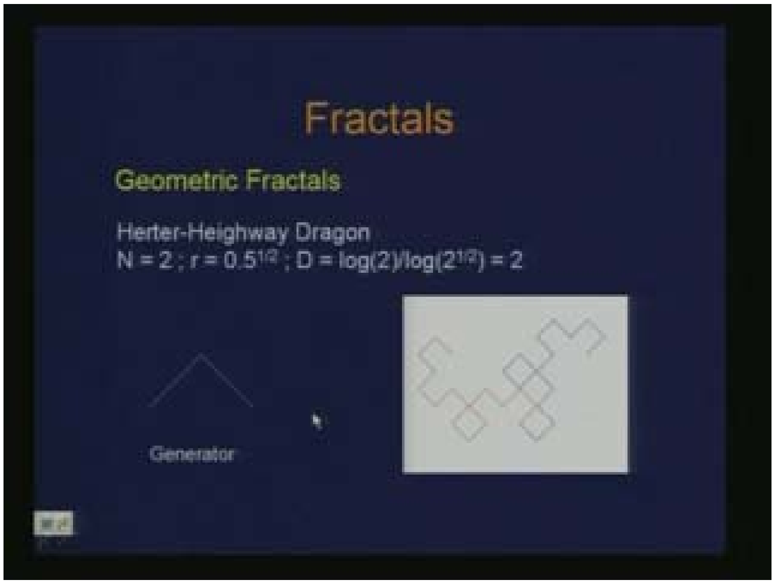

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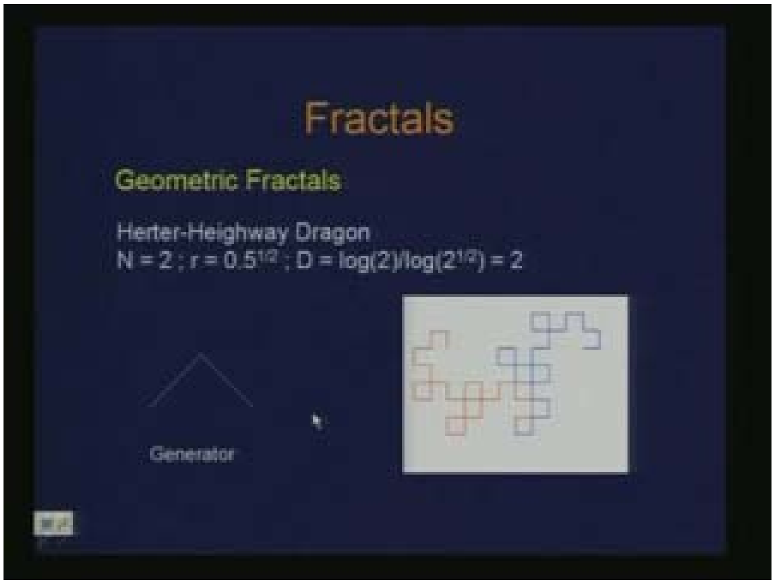

Now we are just following the basic principle of generating fractals where a repetitive application of transformation is happening. So at the end of the day you get this which is considered to be some sort of a dragon and that is why it is called a dragon curve.

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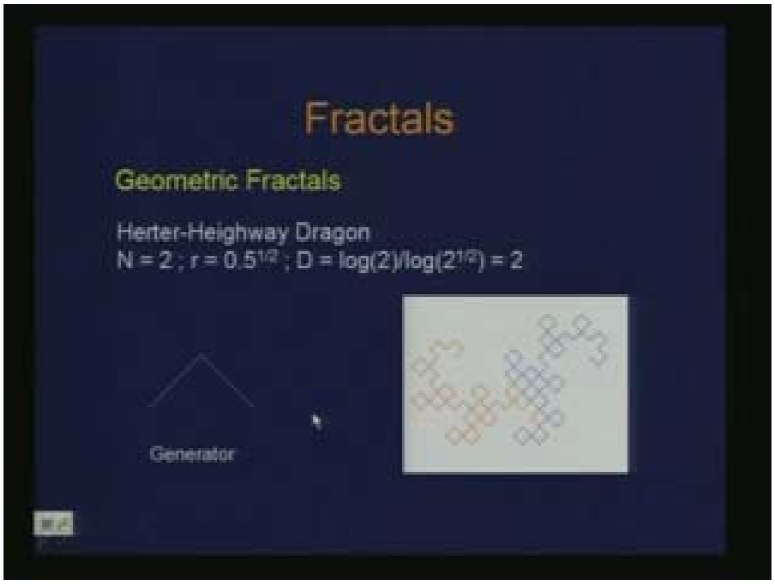

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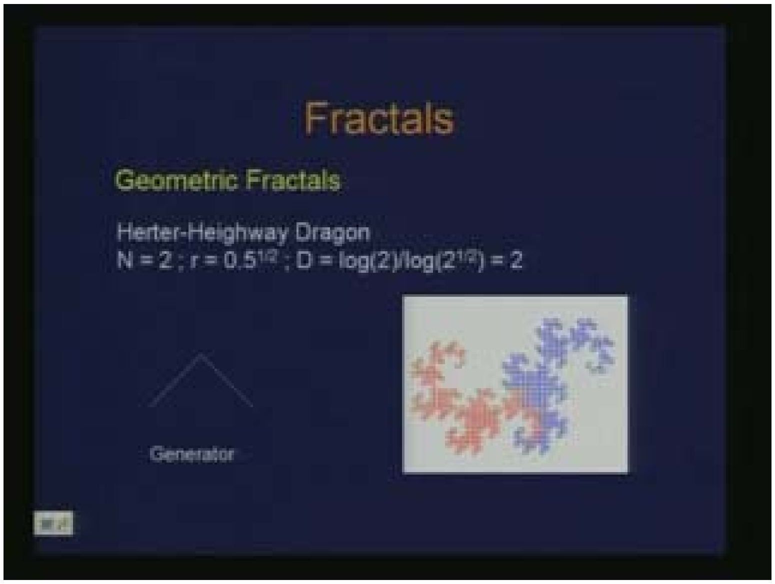

Therefore generating these is a simple process. And you do observe some sort of a self similarity in the smaller structures. When you see the whole thing and if you just take a small part here that looks similar and that is what we mean by self similarity.

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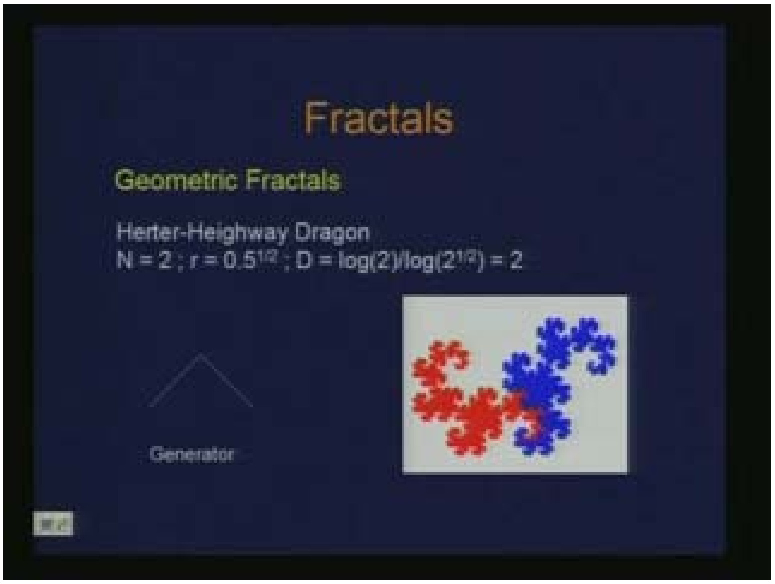

So this particular pattern exists in all scales. Here are some other examples.

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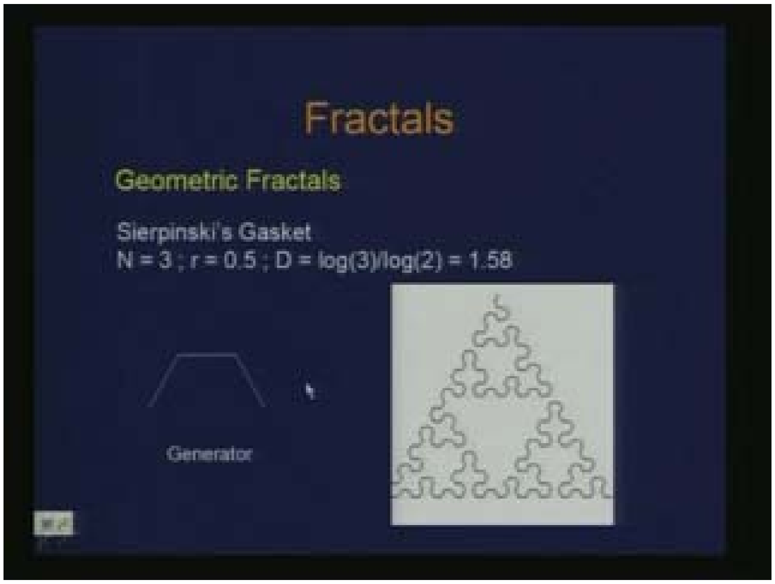

This is called as Sierpinski's Gasket. Again you have 1 2 3 n is equal to 3, r is equal to 0.5, d turns out to be 1.58 so it is a fractional number. And again there could be some sort of a flip of the two sides. If I go from here I generate on this side and from here I generate on this side and when I go from here I generate on this side. So what I am trying to say is that when I do the first iteration it will be something like this and so on.

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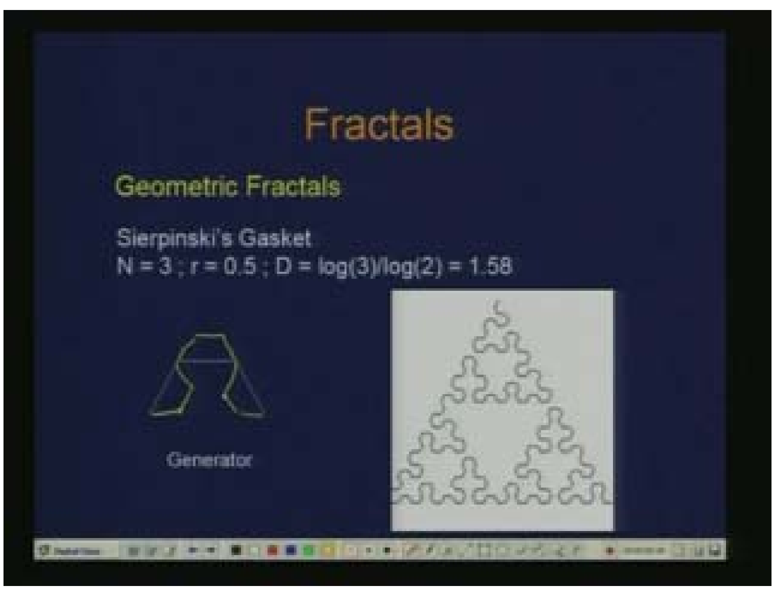

The next is this where the fractal dimension of this turns out to be 1.26.

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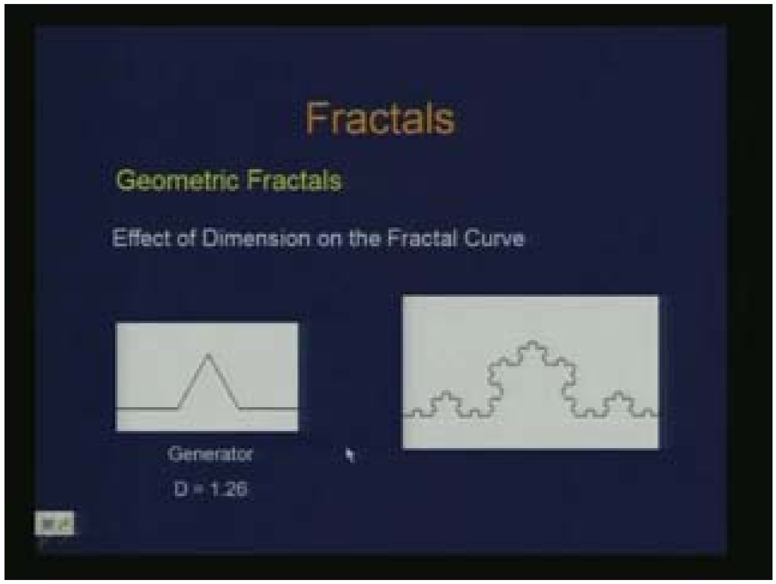

So there are four such segments and when you compute the fractal dimension it turns out to be 1.26. Now let us try to sort of observe the effect of this D the value of D. So this is the value 1.26 and this is the resulting fractal curve you get. Now I consider another generator with a different fractal dimension which turns out to be 1.89 so I consider this generator. And this is what I get as the fractal curve or fractal object.

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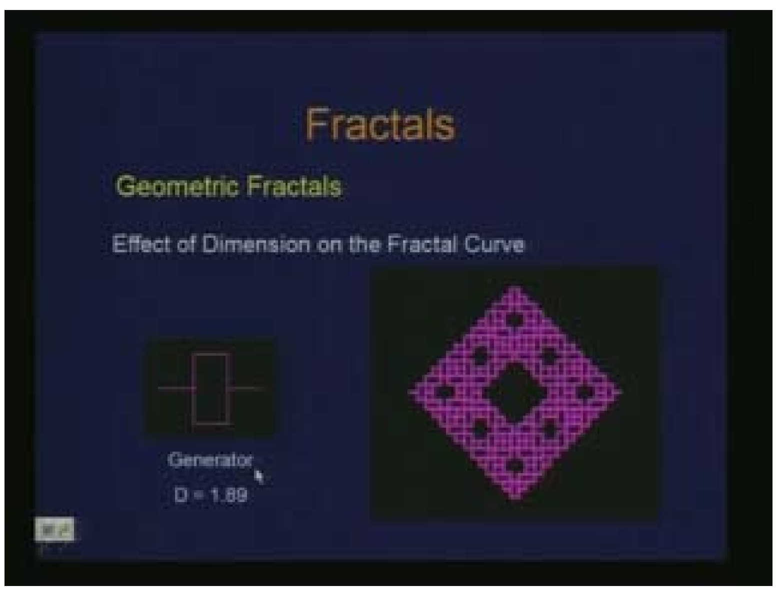

Now you see that the D value is 1.89. Previously for the curve which we had considered it was 1.26. So here it looks filled. Therefore I take another generator for which D is equal to 2. This is what you will get. And in fact if I continue for more number of iterations this may completely be filled. In fact the final shape which you observed was showing you the fillings.

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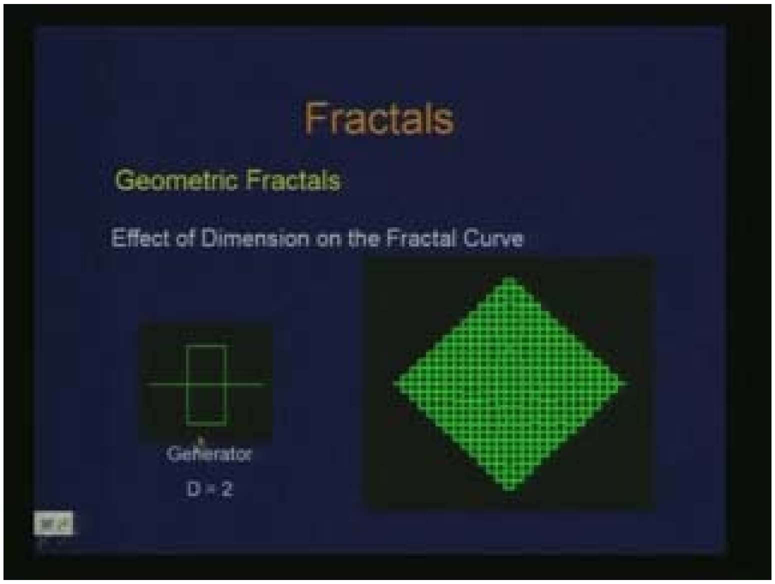

So it was having a self filling characteristic. So what we observe is that as we go from D is equal to 1 - 2 this curve shows the self filling property. And in fact intuitively what you observe is that particularly in this example when this is completely filled there is something what you would observe as a 2-D plane for which the Euclidian geometry gives you the dimension as 2. So there is some sort of an intuitive understanding of dimension. So, if I take D is equal to 1 which is a simple line and D is equal to 2 so when I migrate from 1 to 2 for various values there is this property of self filling or a pre approaching towards the plane. These are some simpler instances where you could easily compute this D and it may be so that it is not as easy as in these cases. Now this is again an illustration of how D affects the final shape.

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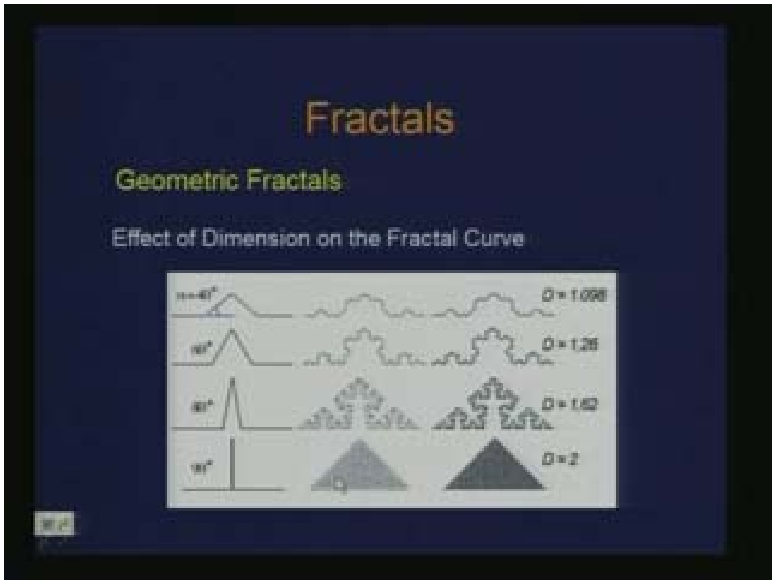

In fact we have taken a variation of the same generator which was this where the angle here was 60 degree and this is what you would have got as the shape with D is equal to 1.26. So the layout of the generator also can affect the value of D. So here if I change this angle to be 40 degree then this is what I would get as a final curve. You can also see some wiggliness to the pattern. So this is sort of less wiggliness you observe here and here there is no wiggliness so D is also capturing some amount of irregularity.

Now, if I change the angle further D goes to 1.62 and you see there is more sort of filling here, more filling, more wiggliness to this shape. And if I consider this, this is what you will get. If they are not then the computation of D is a little tricky because what we are doing is we are doing l by L where l is considered to be this length of the segment. So this actually holds the way we have looked at the computation of the fractal dimension and the length of segment to be the same for all segments. There are situations where the computation of D could be more involving. So you observe how this D basically affects the final fractal curve. Here are some of the applications of these geometrical fractals. One of the motivations was to be able to apply these fractals for modeling natural phenomena.

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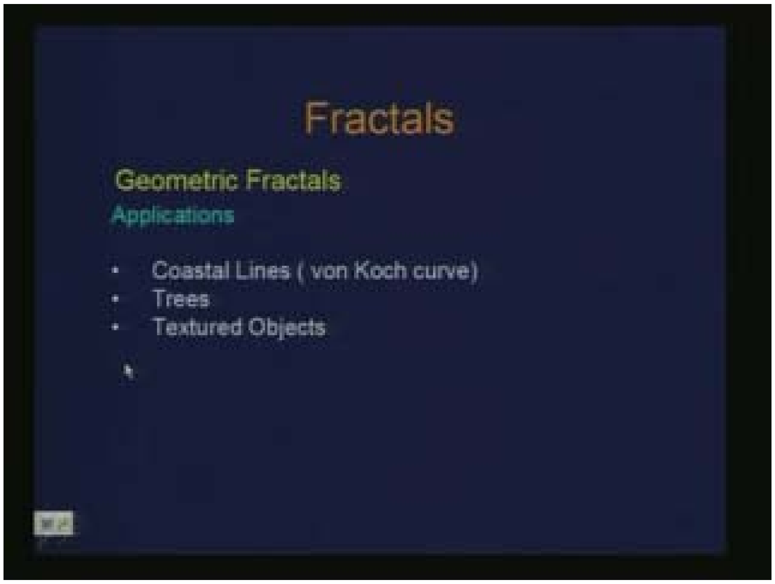

So the coastal line is one of the natural phenomena you want to model. And there is also this property which these fractals exhibit is the self similarity and this is what we also observe in many of the natural phenomena.

For instance, if you consider a coastal line, a plot of a coastal line given on your map and you start zooming in so you keep getting the details of the coastal line as you keep zooming in and the pattern of the coastal line are some what similar is to what was the entire coastal lines. So one can actually use for modeling the coastal lines. Similarly structures like tree can also be built using these fractals. And there is another kind of application one can think of is to generate some textured objects. First of all let us see how we can model the coastal lines. In fact a very simplistic model of the coastal line was this Von Koch Curve.

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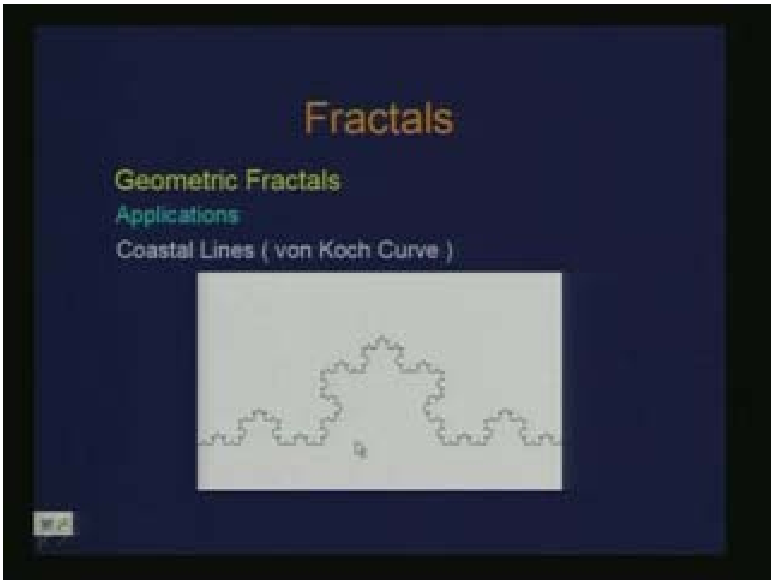

This is considered as a model of coastal line though when we observe this we immediately say that this does not really look like a coastal line. All it is doing is some sort of a wiggle in the structure. However, if I just modify the generator, in fact I sometimes consider what is referred as generalized Von Koch Curve where I break this symmetry of the generator which was like this.

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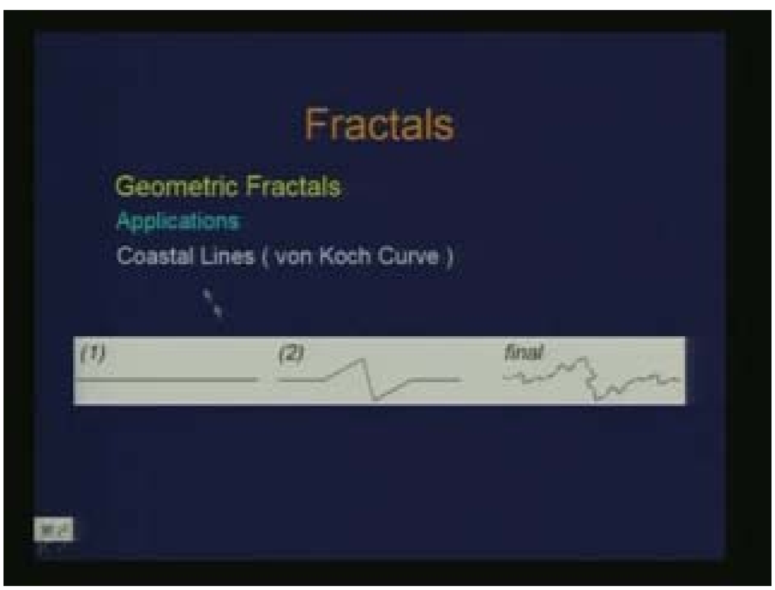

So, instead of having that generator I have a generator of this type. Then when you do the successive application of this you will get something like this which is not too bad.

It is better than the other Von Koch Curve we have looked at. Clearly there is one observation one can make out. When we are using a repeated application of a geometrical pattern what we get here is some irregularities but these irregularities are regular. But in natural phenomena there is some randomness to these irregularities. Even these irregularities need to be stochastic or random in nature and that is where we study these random factors. But even here what we obtain as the final curve is not too bad. In some sense it is capturing a characteristic of coastal lines. There are other applications like trees. So what we can look at is that tree also has some notion of generator. It may not be the same exact definition of generator the way we have looked at for geometric fractals.

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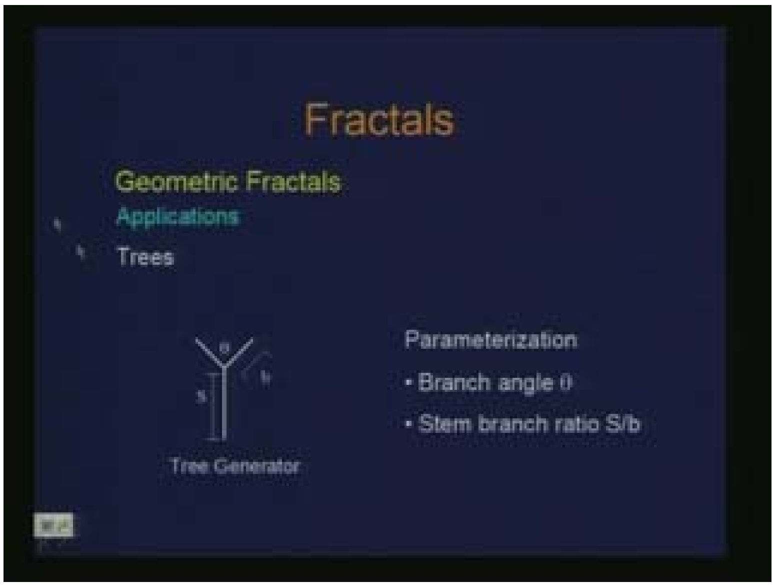

But here, there is a notion of something like a stem of the generator, there is also a branch of the generator and there is this angle between the branches. These are some attributes to the generator or in some sense we are doing a parameterization of this generator through the branch angle and stem branch ratio. So varying these things or even the number of branches which come out we can have variety of trees.

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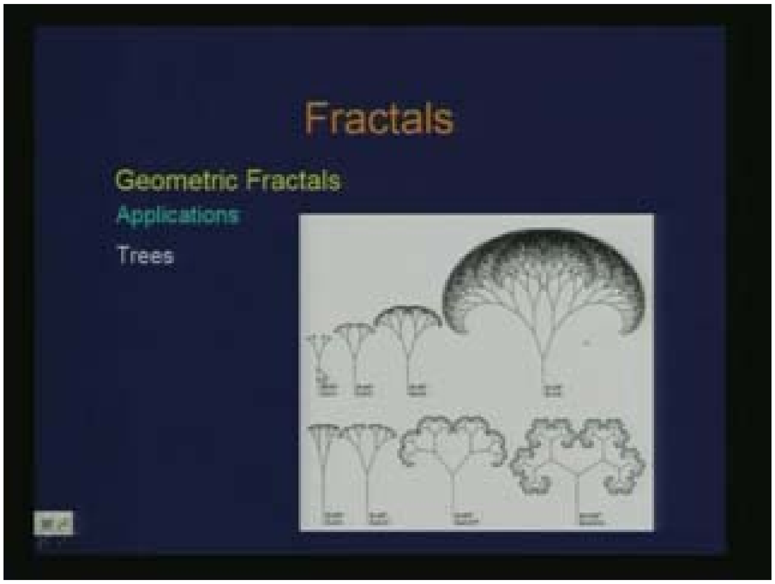

Here is an example; this is what we had considered as the generator. Next time this would have acted as stem. These two branches would come similarly here on this side and the next time this would act as the stem and these branches would come and so on. So, at the end the whole process would be something like this and here what is being changed is the angle between the branches. So here you have a smaller angle, here you have a larger angle, here you have an even larger angle and here you have much larger angle.

Therefore various kinds of shapes can be obtained for these trees. Similarly, if I consider three branches instead of two branches 1 2 3 and I have the branch to stem ratio as this so these two branches have the ratio of 0.6 and this branch is shorter which has got 0.5 so this is the kind of tree you would obtain. Now if I further change it to something like this as 0.4, this as 0.4 and this as 0.8. So what you observe is that the tree is going on one side which could be modeled.

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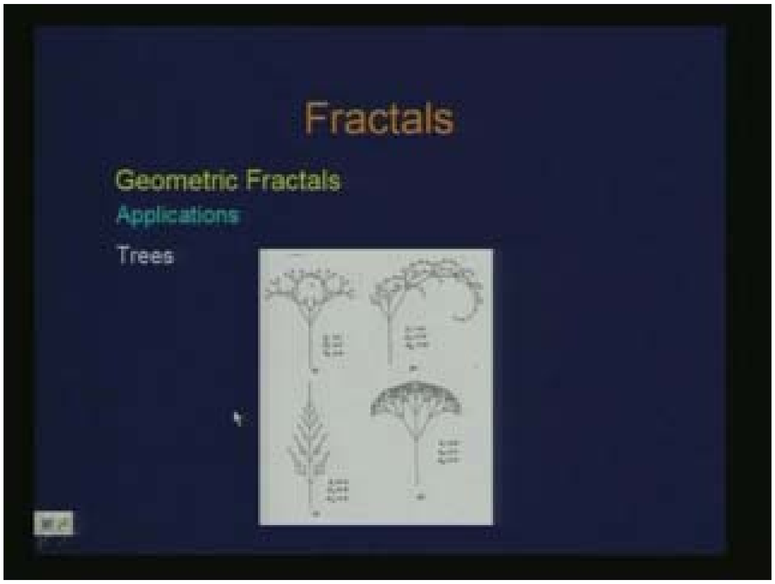

So, even changing these ratios may help you modeling some sort of a wind effect, the whole thing is bending or the kind of trees as there are kinds of trees which sort of bend. So one needs to study about the various types of trees and then incorporate those parameters into the modeling. Similarly you have a tree like this corn fest kind of a tree where you have this as the ratio 0.4, 0.8, 0.4 and here you have all these ratios to be the same. Hence, there are number of parameters which are associated to the tree generator which you can change and obtain a variety of trees. So this illustration which has been given to you is in 2D.

But one can also think of constructing these skeletal kind of trees in 3D and instead of using these line segments you can use some solid shapes or surfaces. And even you can attach the terminations here not as branches but as something else for instance flowers or fruits etc. So you can do lots of things using this simple process of generation. The next application is something I was referring to as the textured objects. So what is basically meant here is how you can use these fractals which basically offer to you some irregularity of patterns of geometry or structures.

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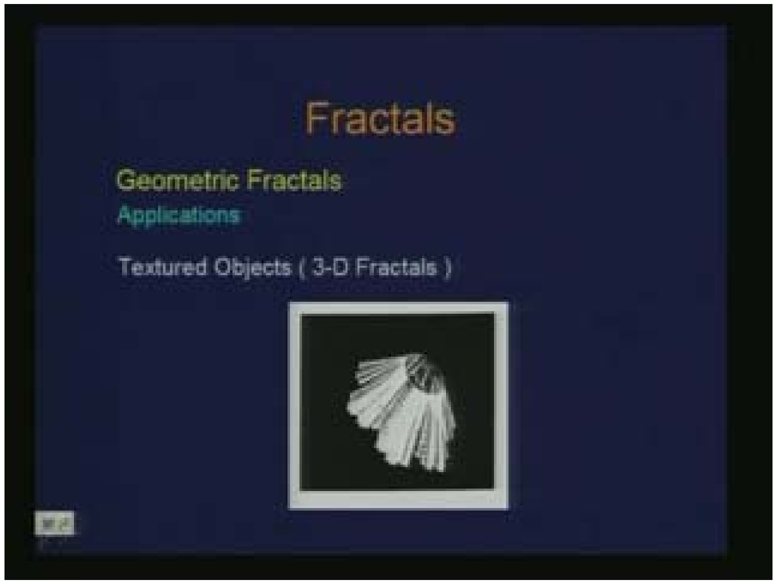

Here is an example where there is this stock contour of the object which is a circular contour. And then there is some sort of a bottom contour which I term as fractalised where you have basically taken a simple shape of square or something and then applied one of the fractal generators and obtain irregular shape of that contour and then what you have done is joined this contour and this contour which is at the bottom. So it has given you an object which is irregular or has got textured in the geometry. Similarly, you can apply this process of fractalizing which is some geometric entity and obtain an object which is in some sense irregular or textured. So, for instance you have these vase kind of objects which are nothing but some sort of surface of revolution. What I basically mean is that, this is the generating curve I have for the surface of revolution which is rotated about this axis so what you would get is a shape like this.

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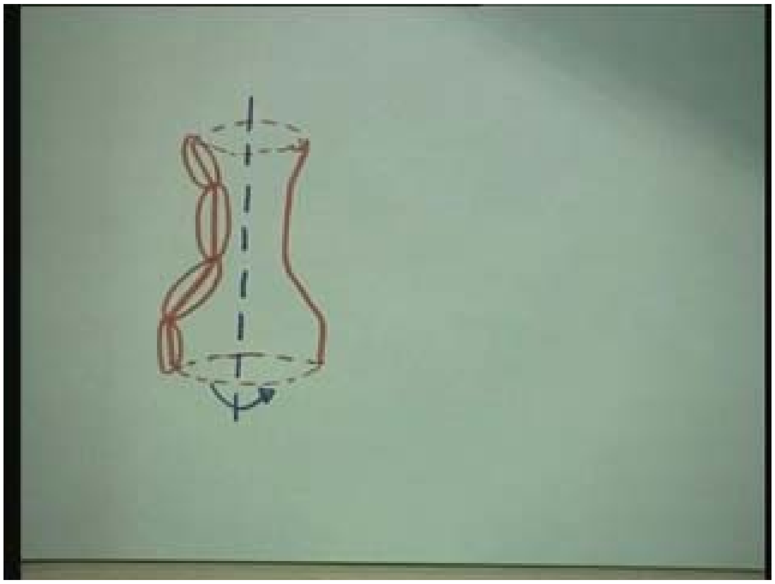

Now in order to make it like a fractalized object you can consider these segments and apply one of your generators and do this surface of revolution using this. So what you would get is a vase now which has some irregularity in the geometry or some roughness in the geometry. So you can sort of augment or enhance your simple objects by adding these patterns on the top. This is what we call as textured objects. So clearly here we have looked at the fractals which exhibit some sort of an exact some self similarity, their irregularities are regular. Random fractals are something where randomness would be incorporated in the process of generation. Later on we will study about those with illustrations.

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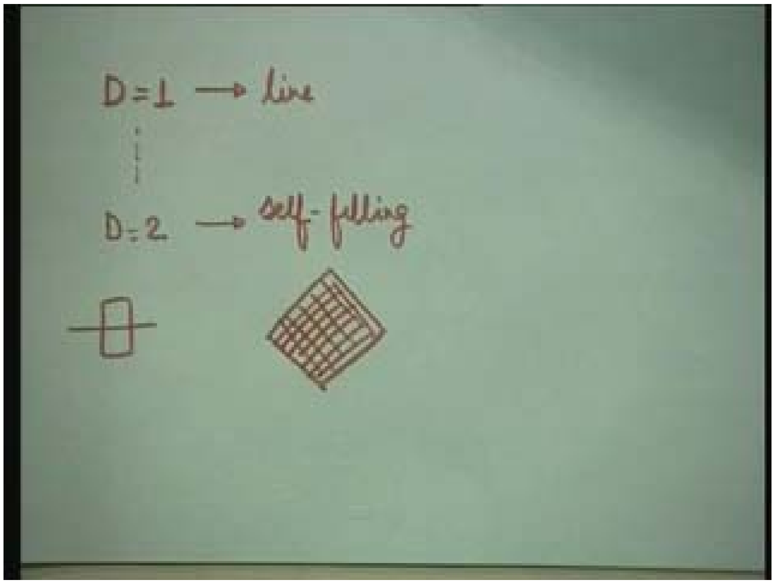

Basically if I have D which is something like 1 then it is close to line and when I have D is equal to 2 there is this property of self filling. For instance, when we looked at one of the generators of this kind it gave you a shape like this in which if you do it many times it basically fills the whole thing. Therefore, in a Euclidian space you would call it as a plane. So intuitively it matches to the Euclidian notion of the dimension.

I could have considered entities in 2D and do the similar process for generating 3D. For instance, if I consider an entity which is a tetrahedral kind of an object where I do not consider a completely filled tetrahedral so I am already saying that this particular shape does not have a dimension 3 but has got a dimension less than 3. So consider a shape as something like this and it has a hole like this and similarly it has a hole on this side and on the other face also and every face. So there is this hole which is there in the generator.

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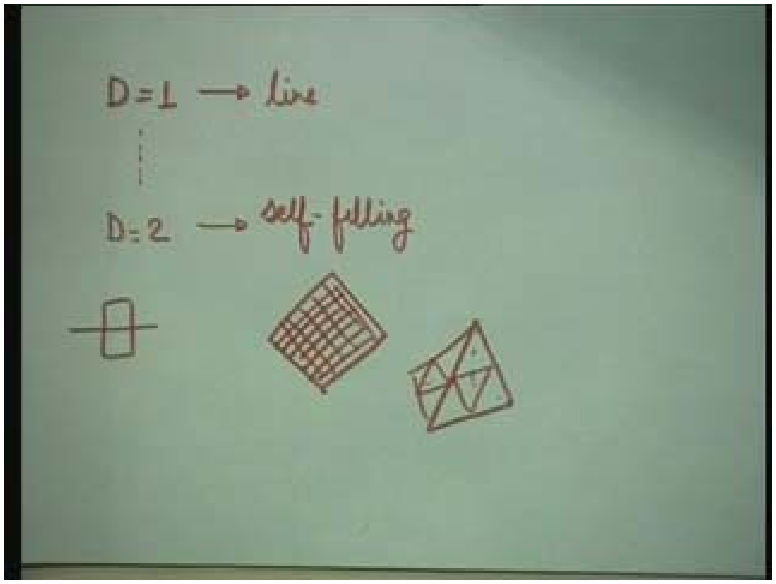

So I would compute the D using similar process not may be the exact process which would give me D less than 3 greater than 2. And when I do this repetitive application of this pattern I would get a fractal shape which may not be self filling in 3D as far as the volume of the object is concerned. So there could be a fractal dimension between 2 and 3. For instance, when we look at the case of dragon there was no line there and the generator of this was this.

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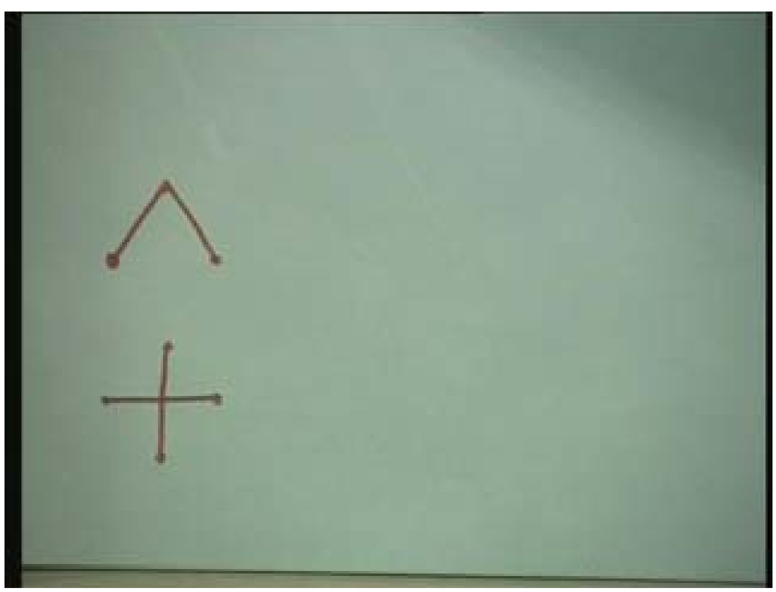

And in fact if you take another generator this would also give you a self filled curve. The problem with this is that there is no single rule which will always hold good. Here if I do not do the flip the kind of curve I get will be sort of weird. So the dragon curve which is obtained is a self filling curve also requires you to have this alternate flipping.

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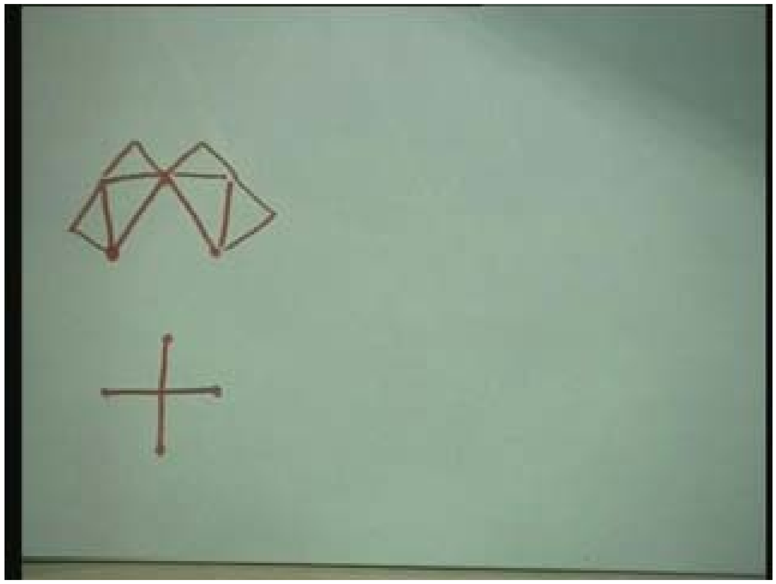

It will self fill in a very scattered manner unlike we see in this diagram and the curves shape will not be interesting because next time it will go to this and the next time it will go to this. Now this is growing in the boundary.