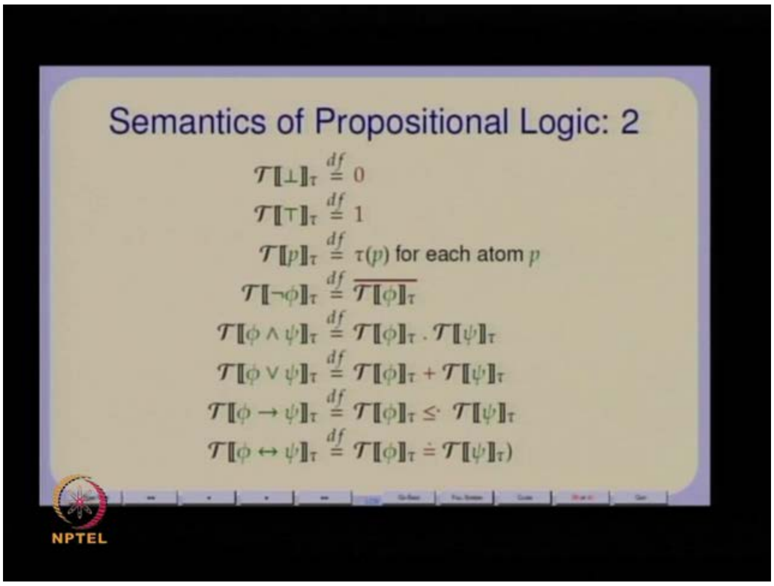
Logic For CS

Prof. Dr. S. Arun Kumar
Department of Computer Science
Indian Institute of Technology, Delhi

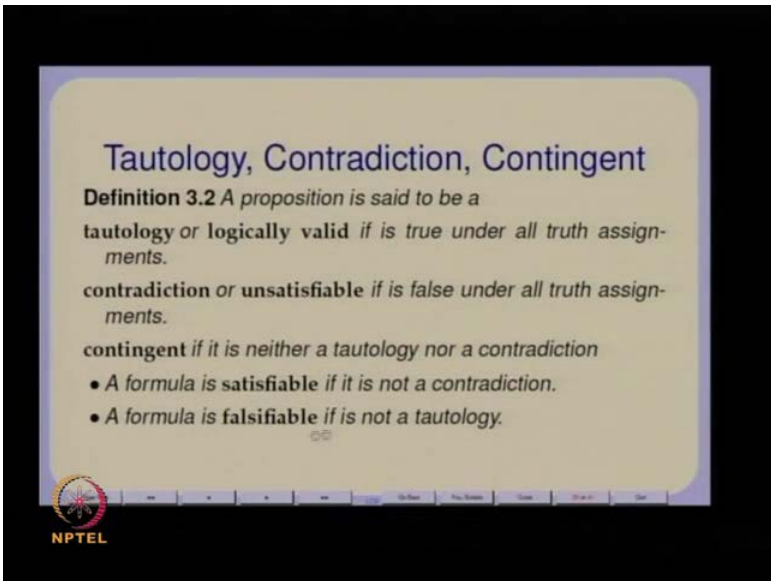
Lecture - 4
Logical and Algebric Concepts

(Refer Slide Time: 00:25)



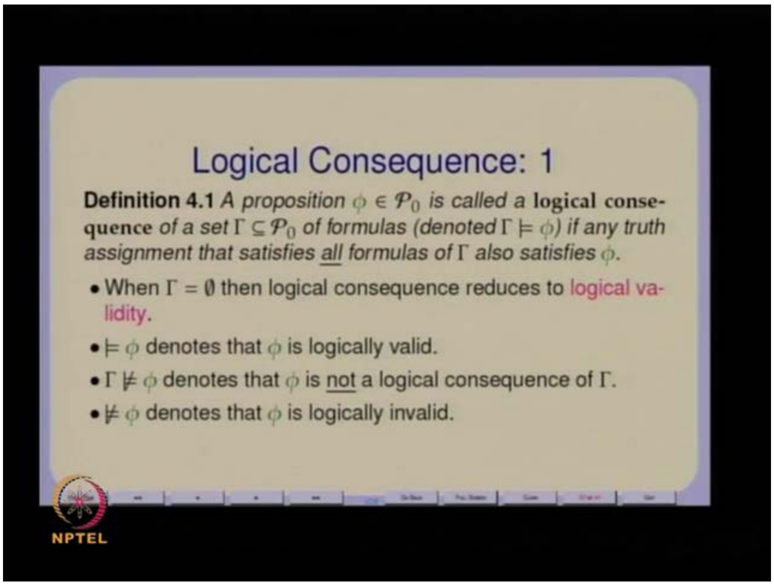
So, we did the semantics of propositional logic and so this is what so basically the whole idea is to look at proposition. So, one way of looking at propositional logic is as in algebraic system and in fact one of the reasons for adding this bottom and top, is to sort of bring in a one to one correspondence between the signatures of propositional logic and the Boolean algebra right. So, you have a correspondence between the operators this order have the same signature or similar signature right. So, that was also the reason for defining this operator less than or equal to dot and equal to dot.

(Refer Slide Time: 01:16)



So, then we look at this basic concepts is concept of a tautology or a logically valid formula, when we say formula. Of course, I mean a well formed formula according to the syntax abstract syntax tree and so on and so forth. And contradiction and there is a notion of satisfiability and falsifiability also. So, we would call in terms of logical concepts the most important logical concept that we will look at is a notion of logical consequences.

(Refer Slide Time: 01:52)



So we would say that so supposing there is a set of formulas gamma and another formula phi, we say that phi is a consequence of gamma and denoted by this symbol. Notice that the symbol is black in color right so it's a metalogical concept it is not part of the logical language. Of course, I can make mistakes with color, but normally if I have not made a mistake then this black means, that it is not part of the language the object language under discussion.

And essentially we say that phi is a logical consequence of gamma, where gamma is a collection of formulas, this collection p naught. Of course, is an infinite collection of formulas right we started with an infinite collection of variables and built up, the language. And that language would also be infinite will have an infinite number of well formed formulas right. And if the set of variables of the set of propositional atoms is countable is accountably infinite then p naught could also be accountably infinite. I guess everybody is clear about those things so p naught even though it is constructed from an infinite set and it's not uncountable.

So, it is so this gamma could potentially be an infinite set could be accountably an infinite subset of p naught, and you would say that phi is a logical consequence of gamma, if any truth assignment that simultaneously satisfies all the formulas of gamma. So, every under that so you take a truth assignment tau in which every formula in gamma is true, then phi is also true so that truth assignment also satisfies phi. Then we would say that phi is a logical consequence of gamma, remember that we are talking about propositional logic which is essentially meant deal with reasoning.

And in general remember this in the background of the fact that I said all of logic is essentially symbolic in nature. So, which means that and the notion of meaning that we have given is restricted to essentially declarative statements, and it is restricted to the truth value of declarative statements under certain truth assignments.

So, in any logical argument, it is not necessary that the individual sentences of gamma or a phi, need to be somehow meaning related in natural language because meaning for us is restricted to. So, you can take completely unrelated sentences form a collection and take a completely unrelated sort of sentence form phi. And if it so happens that for all truth assignments in which every formula in gamma is true phi is also guaranteed to be true then phi is said to be a logical consequence.

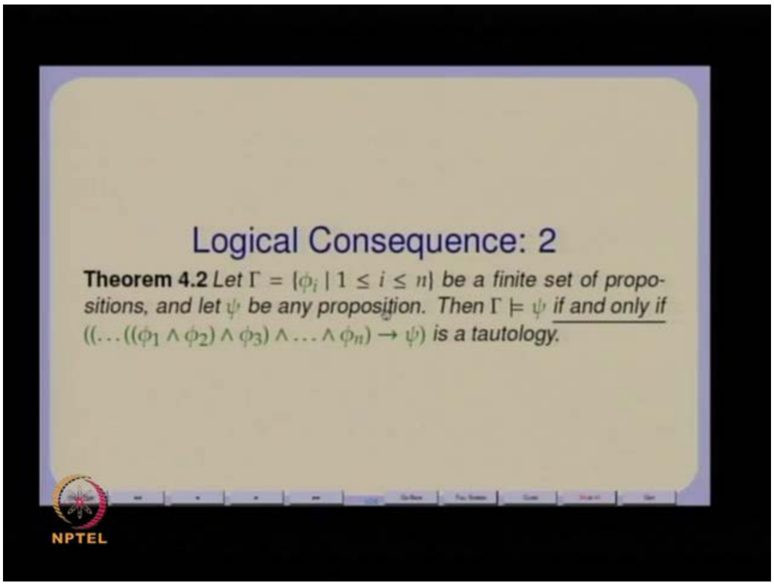
So, I am emphasizing logical consequence because its not semantic consequence, if it was semantic consequence then we would actually assign a meaning which is much finer than just whether it's true or false right. In which case semantic consequence may not hold, but logical consequence would hold correct. So, we are talking about logical consequence only so in general thus the sentences of gamma need not be related to each other they can be completely arbitrary sentences like the sky is green and I am present in room number three zero one today and so on and so forth and some other sentence might form phi and if it so happens that for all truth values truth assignments, all these sentences are true. Then we would say that one is the logical consequence of the other two.

When gamma is empty then logical consequence reduces to logical validity right so essentially to a notion of to something like absolute truth, by the way so that that top symbol that we used for absolute truth is a trivial example of a tautology or a logically valid formula and that is absolute truth right. And of course, as far as notation is concerned, so logical validity there are certain things the notion of tautology in contradiction is only used in propositional logic, it is not used in other logics. But validity is the concept that is used in all logics right, logical consequences is used in all logic right.

We will see that we will see the difference when we come to first order logic, but till then as far as propositional logic is concerned validity and tautology are the same yeah. Something being logically valid and something being tautology at this are really the same. And of course, if you strike the symbol logical consequence symbol that is like negating it and it is a meta negation right.

So, it's not the same as the propositional operator naught it is in the meta language that we are saying we are essentially saying that phi is not a logical consequence of gamma and that holds if and only if it is possible to find a truth assignment in which every formula of gamma is true, but phi is false right that's the extent of in which we will talk about something not being a logical consequence. Or when gamma is empty of course, we are saying that something is not valid not logically valid or its logically invalid, which essentially means that there is a truth assignment which will falsify this phi. So, if phi is falsifiable then that means there is a truth assignment in which phi can be false. And therefore, phi is not logically valid and it's not a tautology yeah.

(Refer Slide Time: 08:56)



So, this is the first theorem such theorems are usually not put up in any logic book therefore, what we will do is I will prove, I will give a proof of this theorem which shows exactly how it has to be done. And all other following theorems which are of a semantical nature I will leave it as an exercise for you to do at home right. So, let's look at this something that is actually fairly obvious and the whole purpose of proving this theorem is to show is to establish how from the semantics one can derive certain meta truths about let's say the meta concepts that we are defining right about some other concepts.

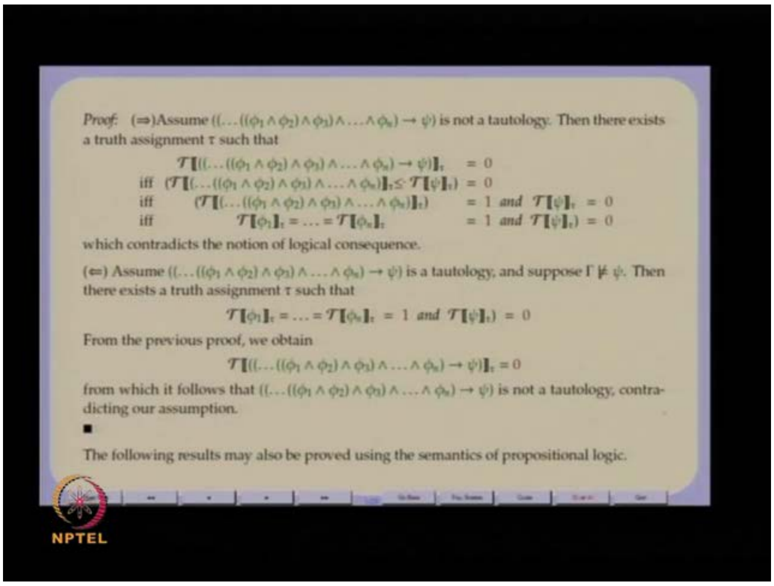
So, let gamma be a finite set yeah if gamma is a finite set of propositions and let psi be any proposition, then psi is a logical consequence of gamma if and only if this formula, which essentially is obtained by taking a conjunction of all the formulas in gamma, arrow psi is a tautology it's important to look at every aspect of this theorem.

Firstly this theorem really holds and is well is formulated only if gamma is finite, as opposed to the normal notion of logical consequence where gamma could be infinite. This theorem actually is a restriction of the assumption, basically it the restriction comes from the fact that you have to form this formula. All formulas are finite string objects so if gamma were an infinite set then I would not be able to form a well formed formula in the language of propositional logic is that clear.

So, this theorem because of this restriction because I want to form a single formula out of the entire set gamma and the formula psi, I the single formula has to be of only finite length of finite depth. And therefore gamma cannot be infinite right so this is in some sense a restriction on, so there is a question of so what it means is supposing is a logical consequence of only an infinite set there is no finite set of which its a logical consequence then what happens that’s a question that you have to ask, for which we will not give any easy answers now we will postpone the answer later, but for a moment to establish our notation and so on and so forth lets prove this theorem yeah right.

So, the thing is that this theorem is something that is obvious you have grown up with these things, but there is a there is a formal way in which one has to do these things, in order to ensure that there is no circularity of reasoning. In order to ensure that you are not mixing up the language and a meta language to make clear where your reasoning is your reasoning in the is at the meta language level, or is at the language level to make all those things clear let’s look at this proof.

(Refer Slide Time: 12:50)



I hope you are all able to see it, but if you are not able to see it so what I am going to do is I am going to use a standard proof by contradiction. So, firstly this is this is an if and only if characterization which means that there are proofs there are two parts to which the higher path is logically implies the conclusion. And the conclusion logically implies the hypothesis. So, there are there are actually two aspects to it so those two aspects are delimited by these arrows, which are defined these are so, there is one part of the proof in which I take the hypothesis, I assume the negation of the conclusion and I prove that there is a contradiction.

The second part of the proof I assume the conclusion and the negation of the hypothesis and prove that there is a contradiction right. So, I am doing both this is this is to establish how one would expect proofs dealing with the semantics of propositional logic, to look like right. Firstly and of course, as I said part of this is the fact that we are using standard mathematical reasoning also in our proofs. So, assume that this formula is not a tautology then that means they exist at truth assignment tau such that I take this entire formula and that gets the value 0, in the Boolean algebra. And this is possible by essential properties of Boolean algebra and from the semantics of the meaning function T of tau that this is possible only if this holds this yeah.

So, notice there are semantics for structural induction so the natural thing to do is to go down the formula. The root operator of the formula is this arrow is the conditional arrow so replace that by and that conditional arrow exactly represents l e q dot less than or equal to dot and that’s what this means yeah. So, this Boolean object which should actually be which is brown in color T of in some green thing in under tau is something that comes in brown in color right yeah. And similarly, this is some brown object that should be equal to 0, which is possible only if this conjunction is 1 and this is 0. Otherwise l e q dot will not be z0 right we know that from the properties of l e q dot.

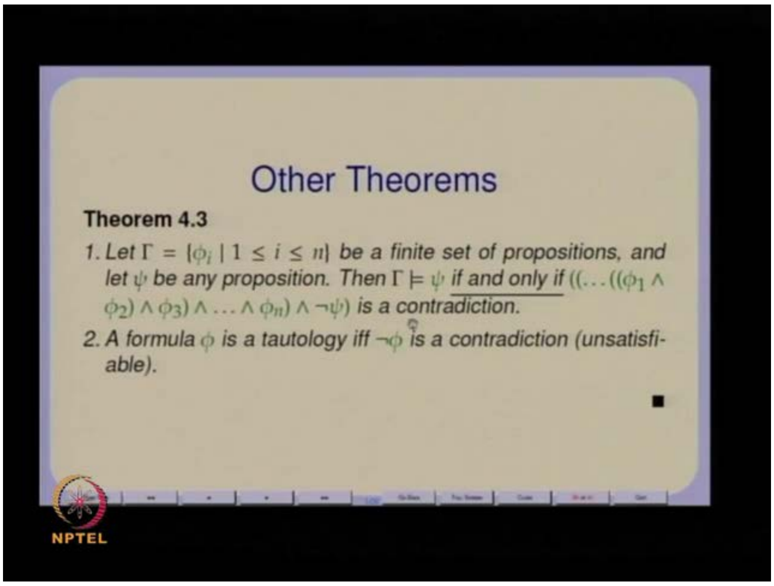
So, we are looking at all the properties of the underlined Boolean algebra the properties of the less than or equal to dot yeah. So, that’s why this is an if and only if all this is if and only if remember that so this works both ways. So, I can go from the top down or from the bottom up. So, which means that this is possible only under the semantics of and which is a dot and if that is equal to one if this conjunction is equal to one, then every individual in this conjunction should also be equal to one right. And this I am sorry this is right bracket should not be there, this tau of psi tau should be equal to 0 right.

Now, this of course, this last statement contradicts the notion of logical consequence because what we are saying is therefore, that the individuals are all true, but psi is false. And therefore, gamma cannot be a logical consequence of thin, but I did not I should have actually initially said assume gamma is part of this, hypothesis is that gamma is that psi is a logical consequence of gamma and that gets contradicted here right. Now, as far as this the second part is concerned, so you assume that the conclusion is a tautology and assume that the hypothesis is false. Which means that psi is not a logical consequence of gamma and you just apply this semantics.

So, this implies that there must be a truth assignment tau such that this holds where of course, this red bracket should not be there. And this is the same as this and I can work this backwards which leads to the fact that this whole thing should be equal to 0, but we assume that this is a tautology so under all truth assignments this the truth of this would be equal to 1.

And so there is a contradiction and therefore, this so we have therefore, we have shown both ways right, we have shown the if and only if. Now, this is a tedious proof right it is meant essentially to state that we are using existing mathematical principles ourselves to reason meta logically about the concepts of logic. So, we are essentially treating logic also as a branch of mathematics and using the same principles of reasoning right.

(Refer Slide Time: 18:54)

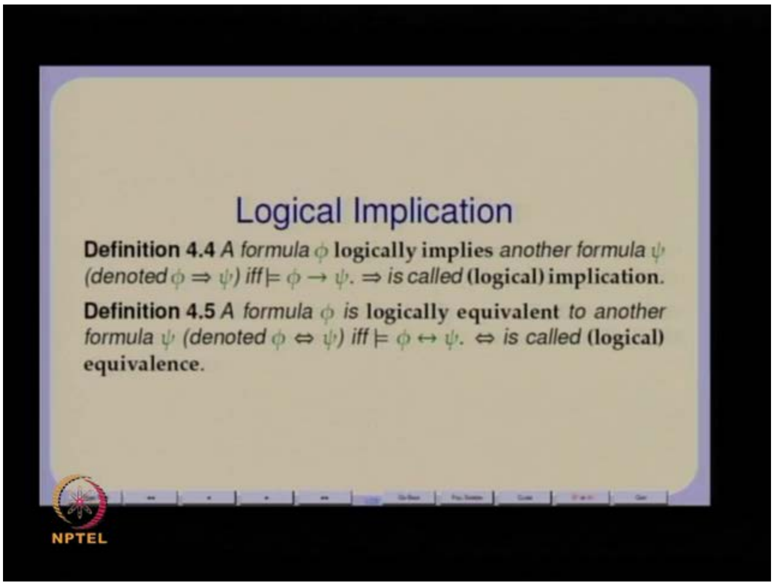


So, we might actually prove several other result so one is for example, that which for which I am not going to give proof these proofs are too tedious, we will do more interesting proofs later. So, let gamma be a finite set of propositions and let psi be any proposition by psi should be green in color, then gamma then psi is a logical consequence of gamma if and only if, this whole thing is a contradiction where I take all the assumptions.

So, gamma is essentially like the set of assumptions and psi is like a conclusion from those assumptions and essentially, we are saying that if you take all the assumptions and the negation of the conclusion then what you should and form a single formula that has to be a contradiction. If that is a contradiction and this is the reason for this theorem is that many of our implementation mechanisms will actually rely on this right its trivial, but tedious to prove this theorem, but it Can be proven in the same way by using the semantics of the individual form yeah.

So, I will not go to prove this theorem right so a formula phi is a tautology if and only if naught phi is a contradiction or naught phi is unsatisfiable is not it. So, these are things we will use later in our algorithms for proving for proofs basically, but the notion of proof itself is something that I will have to formally define right.

(Refer Slide Time: 20:47)

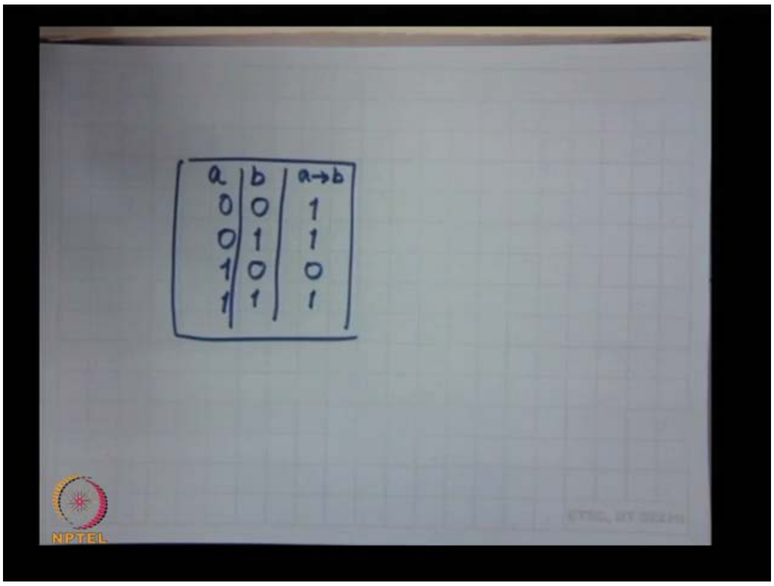


More importantly lets come to some so there is a notion of logical implication and here is that it is important to distinguish between what I write and what most books use. So, there is a notion of logical implication which essentially means that and that is denoted phi implies psi. So, this black implication sign is exactly the same not that we used here right, so I will keep using this for all characterization. This black implication remember is not part of the language of logic it’s not part of that’s why it is black in color right all that we are saying is that this logical implication is a meta logical concept. And logically it refers to the validity of a conditional statement in propositional logic.

So, you take the conditional statement phi arrow psi if that is logically valid or if that is a tautology in the case of propositional logic, then and only then can you say that phi logically implies psi logical implication is so are notion of logical implication. Of course, so this conditional arrow what is it its that less than or equal to dot right what is less than or equal to dot just less than or equal to dot, essentially says phi logically implies psi if and only if phi is no more true than psi. The truth of phi is always less than or equal to the truth of psi the less than or equal to dot also says that false is less than true, a false statement is less true than a true statement basically.

So, logical implication essentially says, therefore that the truth of the left hand side of the implication is in some degree less than or equal to is no more than the truth of the right hand side of the implication. So, you so you have to look at so one of the major problem, a lot of people have in dealing with sentential logic is this notion of material implication. The standard problem has always been that you the material implication has always been defined as something.

(Refer Slide Time: 23:40)



With the truth table 0 0 1 0 1 1 1 0 0 and 1 1 1 right use you take any standard book a arrow b is defined like this and its always a matter of confusion to the beginner. What exactly is meant by this kind of a truth, when everything in the when everything is false how can you still have true, that's one thing right that is that is one problem.

So, the way to understand material implication which way I am calling logical implication, but it is called material implication is to understand it as some kind of measure of the content of truth. And you are just saying that the if phi logic if a statement phi logically implies psi then the truth content in phi is no more than the truth content in psi under all circumstances.

And all are mathematical theorems are stated essentially in that fashion, so in a certain sense if you look at the hypothesis of a theorem and a conclusion. Basically all that you are saying is that the truth content of the hypothesis is no more than the truth content of the conclusion. And in the case of if and only if characterization theorems here essentially saying that their truth content is exactly the same. So, truth statements phi and psi are logically equivalent, logically equivalent not semantically equivalent necessarily yeah. Logically equivalent means where the semantics is restricted to the notion of truth or falsehood, they are logically equivalent if and only if the bi conditional is valid.

Most books on logic do not distinguish between the conditional and logical implication, most books also do not distinguish between the bi conditional and logical equivalence. But the bi conditional is actually an operator within the language and logical equivalence is a concept outside the language right.

And in our mathematics also it is very common to replace one by the other, but then you have to understand in what context it is done. And also any formalization may have may have to distinguish between the two and that you have to keep. Unfortunately our logicians are also careless with their notation so most books do not distinguish between these two.

The fact that logical implication is a validity of a conditional, where the conditional belongs to the object language, but validity itself is a logical concept. And similarly, with the bi conditional and logical equivalence that distinction is usually not drawn in most books on logic yeah, but I would like to draw that because that's how we like to clearly keep the object language different from the meta language yeah. Now, what you are saying by this logical equivalence is that the truth content if phi is logically equivalent to psi, then the truth content in phi is exactly the same as the truth content in psi.

So, what are you saying in particular from the meaning of truth what you are essentially saying is that you take any truth assignment to the atoms of propositional logic. Phi and psi will have will be true at exactly the same points and will be falls at exactly the same points yeah right. So, we call this logical equivalence and not just equivalence though often it is just called equivalence.

Student: ((Refer Time: 28:21)) Single arrow beautiful language.

Yeah this.

Student: For a given truth assignment that it may or may not be true for the given truth assignment but, the double arrow is for is a tautology for all.

Yeah its for all truth assignments, but what we are saying is this you talking of this logical implication.

Student: The single arrow within that green.

This single arrow is the conditional, conditional less than or equal to dot right operator which essentially says that truth of phi is less than or equal to the truth of psi.

Student: This depends on what particular domicile.

Whether this whole statement is true or not depends on the particular truth assignment, but this validity means that for all truth assignments, the truth content of phi is less than or equal to the truth content of psi. If it is so for all possible truth assignments then phi logically implies psi right. Remember that the number of truth assignments, that you can make is uncountable is unaccountably infinite there are an unaccountably infinite truth assignments.

Inspite of that so which means that any kind of formalization will have to make it comment to some finitely domain and keep it atleast at most accountable accountably infinite. So, this is what this is part of the object language this arrow single arrow is part of the object language this. This double arrow is part of the meta language, it is a logical concept independent of truth assignments right.

So, we are in all these logical concepts we are abstracting away from truth assignments from individual truth assignments, that is what most of the logical concepts will do.

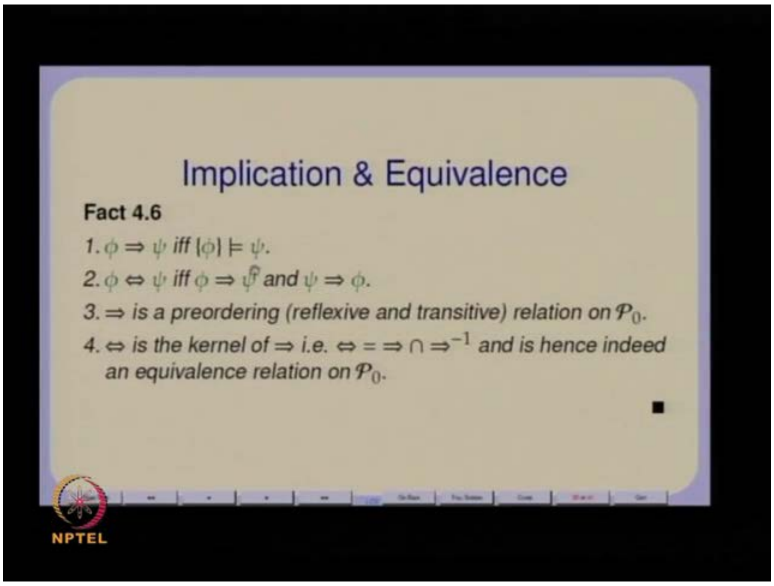
Because you have to abstract away from the individual truth assignments because of the fact that there are unaccountably large number of truth assignments, and we still want some symbolic way for dealing with objects of the language right. So, initially so you so you understand what the enterprise of mathematics and logic, for the last two thousand years as always been characterized by how to give a finite characterization of some infinite object.

So, linguistically it reduces to saying is there a finite like sentence that I can use to characterize some infinitary property. So, it starts with it starts with basic sets enumeration of sets right, you take an enumeration of a set it if its a finite set of course, it can be enumerated in finite time, but if its an infinite set you can't enumerate it in finite time.

So, what you do you look for a property of the elements of the set, which can express it by abstraction right this in standard school mathematics is called I think there is this roster form. And the set builder form that set builder form is a definition by abstraction where you are trying to use a finite pro a finitely expressible property to express properties that to express the collection of all objects, which might be an infinite set right this has always been one of the problems in one of the things. You always try to do an abstraction so as to characterize some infinite property, in some finitely fashion. And this is one this is a this is a first in this course.

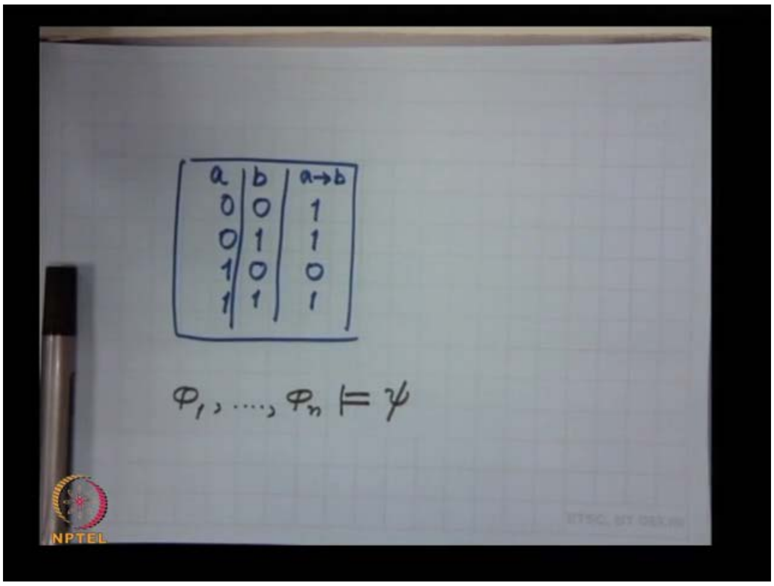
But if you if you just go through a flashback of all your mathematics starting from lets say fifth standard or sixth standard, where you know when where algebra came up for the first time not arithmetic algebra the use of that variable x is a finitary is a is a finitary expressible thing for possibly an infinite set of values. So, various tools in mathematics essentially try to express the infinite in some finitary fashion. So, that is there throughout the history of mathematics actually and so this is what logical equivalence is about.

(Refer Slide Time: 33:37)



And now we come to essentially an back to an algebra because now this any programming language or any formal language is actually an algebra right. So, we can also think about these concepts as algebraic concepts, so you take now a phi logically implies psi if and only if essentially this holds right.

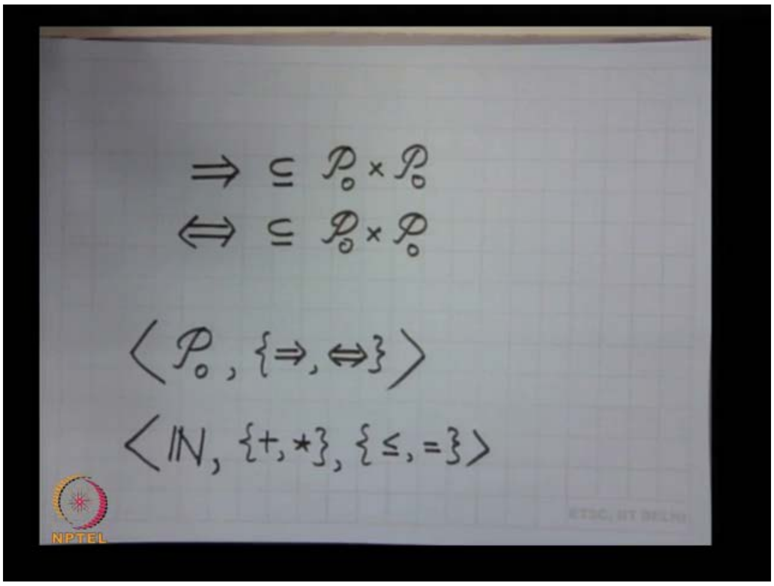
(Refer Slide Time: 34:54)



Remember from our from the previous theorems you are saying that phi arrow psi should hold, and then from that first theorem of logical consequence phi is like a singleton finite set gamma. And psi is a logical consequence of phi and so this is essentially what it does by the way this set braces is something that we will give up, if there is going to be only one. Even if there are going to be a finite number we will simply write it as something like this you know phi one phi n psi right, this is to indicate that its a finite set.

Psi is a logical consequence of finite set so these set braces will all be given up as part of the its no longer going to be part of the notation, but at the for the moment since I said that gamma is a set i had i put it yeah phi is logical equivalence of psi if and only is if phi logically implies psi and psi logically implies phi. Remember this and is black its a meta logical and right it is not a logical. Logical implication is a pre ordering relation on the set of propositions right so essentially I can think of this.

(Refer Slide Time: 35:45)

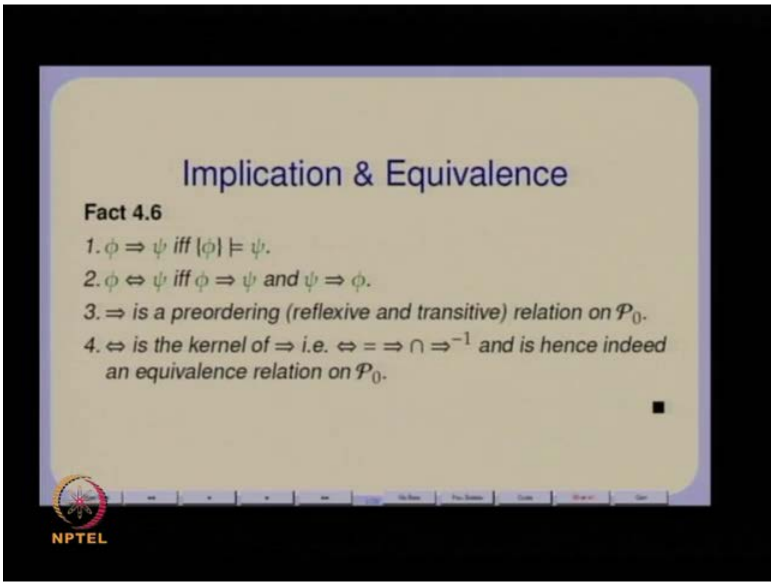


So, all I am saying is that I can think of this logical implication and logical equivalence as essential subsets of p naught cross p naught. So, I can take a purely algebraic view and think of this as a relation on so I can think of an algebraic system in which I actually have a carrier set p naught. And I have these two relations this is an algebraic system, it has no operators, but it has two bi relations it's like you take an algebraic system like the natural numbers under less than or equal to right. So, it's really an it's really an algebraic system, natural number under let's say addition and multiplication and less than or equal to. And so, its really the set n which has a which is a carrier there are two operators let's say plus and addition and multiplication.

And then I might have these two relations so this is an algebraic system right so I can take propositional logic p naught under logical implication and equivalence as an algebraic system with no other operators. So, if I consider this system, the other thing of course, and this is also easy to prove so a relation also has its inverse right. So, infact this usually the inverse is represented by implied by so one of the things we say is when phi is equivalent to psi we say that phi implies psi, and is implied by psi right. So, this of course, is the inverse of this relation so you take the cornel of this relation.

So, this is a pre ordering relation on this on this set p naught so logical implication is a pre ordering relation, which means its reflexive and transitive. And if I take the cornel of this pre ordering relation, which means I take the intersection of this pre ordering relation with its inverse. Whatever common elements I get that forms an equivalence relation that's an equivalence relation in the in the algebraic sense. So, all I am saying here is that logical equivalence is also an equivalence relation on p naught.

(Refer Slide Time: 38:57)

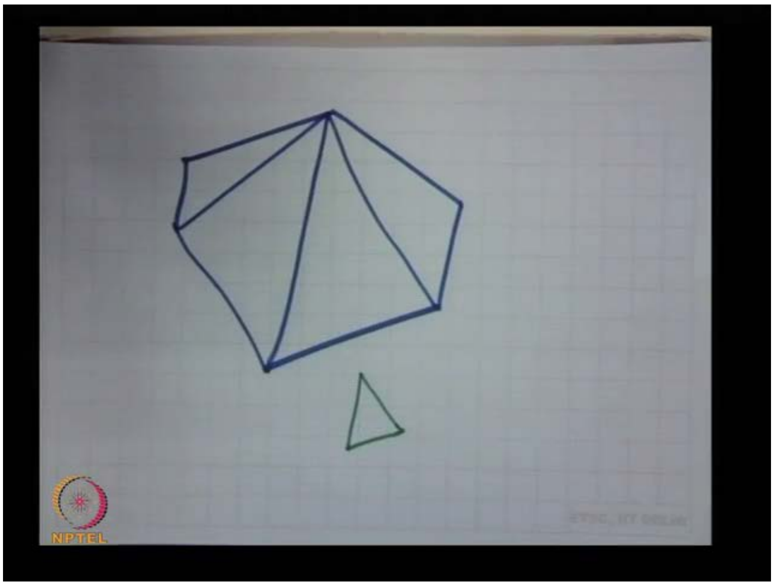


So, its an equivalence relation so its reflexive, transitive and symmetric right. So, essentially if phi logically implies psi then psi logically if phi is logically equivalent to psi, then psi is also logically equivalent to phi. So, its an equivalence relation in the standard way in which we talk about relations on sets, and the set here is the set of propositional sentences, defined over a collection an infinite collection of atoms right.

So, you have an algebraic system, but there is more to it it's not just an equivalence relation, it is a congruence is everyone familiar with the notion of a congruence relation.

If you want some analogy take uh go back to Euclidian geometry triangle congruence is a congruence relation triangles on the plane. Similarity triangle similarity is an equivalence relation, what they are both equivalence relations both the congruence of triangles and the similarity of triangles are both the equivalence relations, but there is a difference. Triangle congruence is actually a congruence relation on planar triangles. Whereas, similarity is not a congruence relation does anybody can anybody tell me what the difference is does anybody know, what an algebraic congruence is it has to be preserved under all context.

(Refer Slide Time: 41:02)

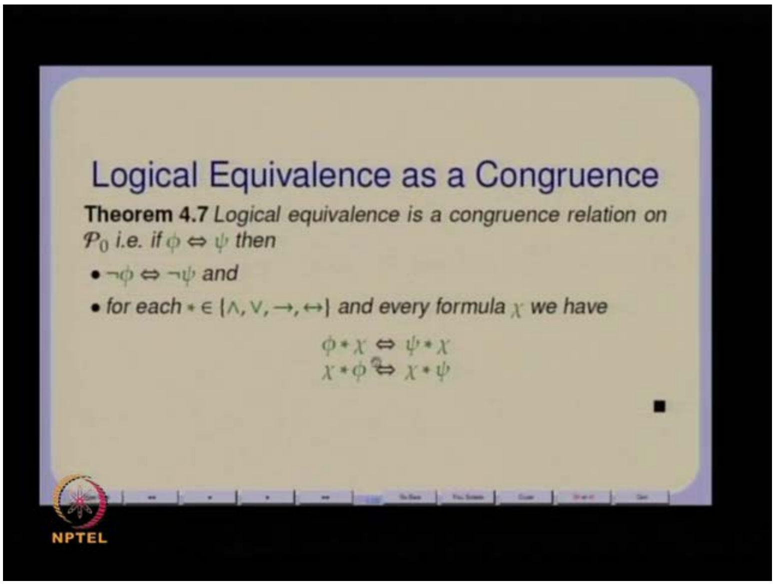


See if you were to take so if I were to take a take some polygon and divide it up into triangles. Now, this is like an algebraic this is like a geometric figure, now what we are saying is if, have you come across this tan grams this game called tan grams? Tan grams you have asset of plastic pieces on planar they are all basically various kinds of triangles, and you put them together in order to form some other, some other geometrical figure. Convex or non convex it does not matter right, so this is like five tan gram pieces put together to form this pentagon, let's say right. Now, it's possible now if I take a triangle similar to this middle one, but not congruent to it and if I remove this triangle, and try to fit this in I will not get the same figure.

However if I had another triangular piece congruent to this triangle, then I can replace one by the other. So, that means there is a context or a framework in which, if you replace congruent objects by congruent objects then the then the framework does not change. Whereas, if you replace congruent if you replace an object by only another object, which is not congruent to it, but is even though it might be similar their reserved of the replacement does not give you the same structure again right.

So, that structure preservation or context preservation is what distinguishes a congruence from an equivalence. And the interesting thing about logical equivalence is that its a congruence relation on this set p naught. So, the standard algebraic method for proving the congruence of any relation is to actually look at it under various contexts right.

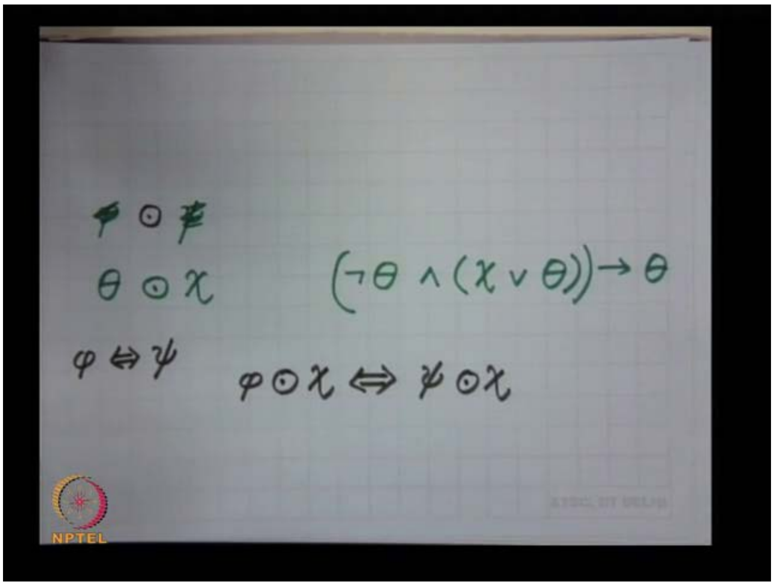
(Refer Slide Time: 43:32)



So, while all that we are saying is so here the framework is provided by some arbitrary formula kai lets say. So, you take the as you said propositional logic is also an algebraic system with operators like and the green, and the green or and naught double arrow and the bi conditional. So, you take any context made up of these logically equivalent propositions can be replaced by logically equivalent propositions and up to logical equivalence, the truth value will not change. And that is what this is usually so what it means is that if phi is logically equivalent to psi, then naught phi is logically equivalent to naught psi and for every formula kai. And each of these binary operators of the language I can replace phi by psi in each context.

So, this indicates you take any of these any or all of these if you if it its so it will happen that phi scar star kai or kai star phi is logically equivalent to psi star kai or kai star psi respectively right. So, this is how contexts are created. The interesting thing about any algebraic congruence is that we prove it only for these but, it holds for any of complicated framework, you might form out of these operators. So, if I define a new operator for example, if I define a new operator called zero dot…

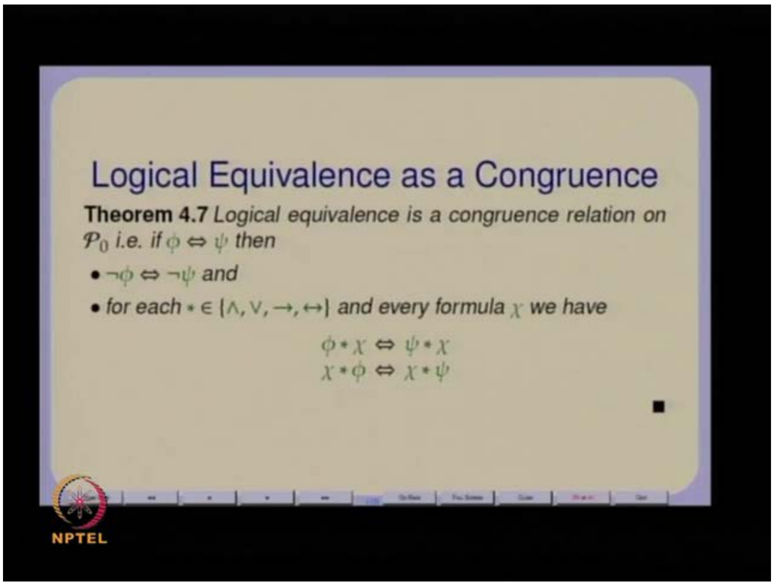
(Refer Slide Time: 45:28)



Which I say shouldn’t use phi and psi maybe I should use some I should use some theta and kai lets say if I define any operator theta and kai lets say bi as naught of theta. And or of kai or theta for example, I can define some new complicated operator I can even go further than this. What we are saying therefore is that you take any of these operators then if phi is equivalent to if phi is logically equivalent to psi, then I can replace wherever phi occurs.

So, for example in this context phi o dot kai is going to be logically equivalent to psi o dot kai right. And for that by structural induction, it suffices to prove just for each operator right this is true not just of this system, but of any algebraic system. So, if you have done any kind of elementary abstract algebra congruence would have been defined like this.

(Refer Slide Time: 47:05)



But what they actually mean is that for any complicated operator that you might define, with the existing operators, in that context the replacement of equals by equals does not change the value. So, it is sufficient the number of different such operators like this o dot that I have done here given here, the number of such different operators is actually at least accountably infinite.

But all you are saying is that by structural induction it is possible to prove that for any such, if for any operator that you might define this will also this factor will also hold if you have proved this theorem for the individual operators in the language. It is actually a so congruence is defined in most mathematics textbook simply as being the relation being preserved under each operator, but the effect of that is that it is preserved under any of the any operator that you might define using the individual operators. So, it is sufficient to prove this case analysis.