

Logic for CS
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Lecture - 35
First-Order Theories


So, what we proved fairly successfully I think is that first order predicate calculus actually the Hilbert style system is complete. We, have proved the resolution method is also complete. We have proved the tabular method is also complete. And, since essentially the natural deduction system can also be derived from the Hilbert style system with that extra existential elimination rule and so on so forth. So, the natural deduction system is also complete and so that so we do not even bothers about proving the completeness of that natural deduction system. So, and we and especially since we showed that even though the existential elimination is not a derived rule of the Hilbert style system every proof which involves that can also be converted into one which does not use existential elimination. So, the adding that existential elimination as a proof rule does not make your system inconsistent that is a first thing the second is that it still complete.

So, the natural deduction so genesis natural deduction system is also completes so and the tabular methods are also complete. So, now what it means therefore is that any logical theory any mathematical theory you build on top of a studer predicate calculus is essentially a first order theory. And, will see that a meaning of that so there is a first of all there is a limitation of expressibility in first orderwhich, I will not expressiveness properties which I will not go into. So, as for as the first order theory is concerned its incompleteness therefore depends entirely on its aximatization. And, does not depend upon any incompleteness of reasoning within first order predicate calculus. So, what we are essentially saying is the reasoning mechanisms of first order predicate calculus are complete and they are consistent. So, any incompleteness that you see in a theory in a first order theory is essentially used to the axioms that you what are called the non logical axioms that you introduce as part of the theory. So, let us just look at some first order theories. So, when I go through these examples you will also be you should be able to connect it up to whatever you have already done in these theories and see the differences also. So, for example so let us start with this actually the simplest first order theory is the theory of the directed graphs.

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Directed Graphs

$$\Sigma = \{e : s^2\}$$
$$\phi_{e\text{-irreflexivity}} \stackrel{df}{=} \forall x[\neg e(x, x)]$$
$$\Phi_{DG} \stackrel{df}{=} \{\phi_{e\text{-irreflexivity}}\}$$


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Let us say what does it have the theory of Directed Graphs in its most fundamental form has absolutely in its signature it has just one edge relation that is it a directed edge relation there is nothing else to it. So, that is what I am calling e here, it does not have any operators and your carrier set is usually a set of nodes that is like the model. I mean that is like this structure that is good to be brown in color if you like. But, otherwise the only relation you have is directed edge relation and in general all the only axiom you have is that the edge relation is irreflexive. I mean if, you are talking about simple directed graphs normally when we are talking about directed graphs in graph theory they mean simple directed graphs. So, therefore there are no self looks on the nodes.

If, there are no self looks on the nodes then essentially your edge relation is irreflexive there could be cycles of course. But, then those cycles have a length of at least two and, so there is just this irreflexivity just says there are no self looks on nodes there are no length one cycles. And, so essentially what you can derive from here is almost nothing. But, then you go back to what you studied about directed graphs in let us say a discrete math codes. You, find that all those theorems that you had about simple directed graphs they involved counting. Basically, this simple axiom with all the with whatever first order logic axioms first order predicate calculus axioms and inference rules. You can derive very little you will not be able to get any interesting theory at all. The interesting aspects start with for example counting. One saying one which says

that the number of n degrees in a directed graph should be equal to the number of out degrees for example. The, degree of a node this is for the in degree of a node, the out degree of a node where automatically they bring in integers or at least the naturals.

So, in fact if you see so directed the first order theory of directed graphs viewed in isolation as simply in edge relation gives you almost nothing it gives you just essentially whatever, you can derive from irreflexivity and the axioms of predicate calculus the proof will of predicate calculus. So, nothing else and that is not very interesting the interesting things really come from certain combinatorial aspects. So, in that sense this is too trivial and it is a looking at this first order theory is not very interesting.

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Undirected Graphs

Σ	$\stackrel{df}{=} \{; e : s^2\}$
$\phi_{e\text{-irreflexivity}}$	$\stackrel{df}{=} \forall x [\neg e(x, x)]$
$\phi_{e\text{-symmetry}}$	$\stackrel{df}{=} \forall x, y [e(x, y) \rightarrow e(y, x)]$
Φ_{UG}	$\stackrel{df}{=} \{\phi_{e\text{-irreflexivity}}, \phi_{e\text{-symmetry}}\}$

Equivalently $\phi'_{e\text{-symmetry}} \stackrel{df}{=} \forall x, y [e(x, y) \leftrightarrow e(y, x)]$ (see exercise 20.1.12) may be used in place of $\phi_{e\text{-symmetry}}$.

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From, directed graphs of course we can go to the next Undirected Graphs. So, essentially what you have is the signature and again this signature has only an edge relation nothing else. By, the way you could have a inequality relation. So, very often I am not going to mention the equality relation if there is some strange proof in which especially you want to say that. Two different nodes which for purposes you called a some nodes with different names for purposes of argument. And, then prove that they are the same nodes that would constitute an equality relation. So, I am not mentioning equality here but in general what we are asking for is actually a first order predicate calculus with equality always that. Without, that the theory will become

even less interesting than it would be. So, in the case of undirected graphs for example you also have an addition to you have the edge relation which is irreflexive of course again we are talking about simple undirected graphs. So, it is irreflexive however there is symmetry that is what you want about the edge relation. But, normally in any discrete math both for undirected graphs the edge relation is not really thought of as a relation.

But, it is thought of as a set of two elements set in order to in fact as the undirectedness. But that is I mean, within that will take us to axiomatic set theory and so on so forth those are complications which we do not require at the moment. So, the first order theory of undirected graphs essentially says that you have an edge relation which is connected. And, that is sufficient and in fact most of the times when we draw and directed graphs so will direct we essentially think of it as essentially as symmetric relation here. So, the interesting thing here of course is that you can take this symmetry axiom to be either directed implication like this or, at by condition like this. And, there is an exercise which shows the above the equivalent as for as first order predicate logic is concerned. But, remember what that means.

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(b) $\models \exists x[\phi] \leftrightarrow \neg \forall x[\neg \phi]$

9. Let $\exists x[\psi] \in SF(\phi)$ and $y \notin FV(\psi)$. Let ϕ' be obtained from ϕ by replacing $\exists x[\psi]$ by $\exists y[(y/x)\psi]$. Then ϕ and ϕ' are said to be α -equivalent and denoted by $\phi \equiv_{\alpha} \phi'$. Prove that two α -equivalent formulae are equivalent.

10. If $x \notin FV(\psi)$ then $\exists x[\psi]$ is equivalent to ψ .

11. Give examples of interpretations (A, v) to show that the following formulae are not valid.

(a) $\exists x[\psi] \rightarrow \forall x[\psi]$
 (b) $\forall x \exists y[\psi(x, y)] \rightarrow \exists y \forall x[\psi(x, y)]$

12. Prove that for any binary predicate ψ ,

$$\forall x, y[\psi(x, y) \rightarrow \psi(y, x)] \Leftrightarrow \forall x, y[\psi(x, y) \leftrightarrow \psi(y, x)]$$

13. Prove that the following formulae are valid.

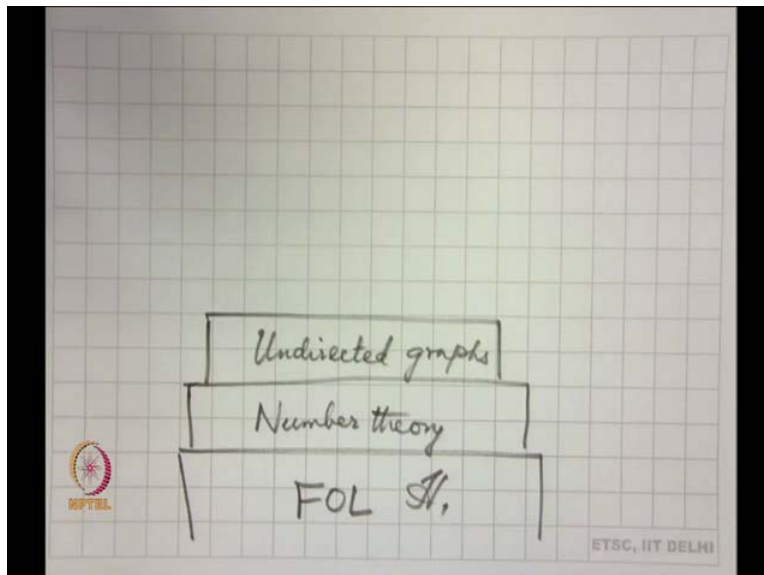
(a) $\exists x[\psi(x) \rightarrow \forall x[\psi(x)]]$.
 (b) $\exists x \forall y[\psi(x, y)] \rightarrow \forall y \exists x[\psi(x, y)]$

Prove that $\phi \Leftrightarrow \psi$ iff $\phi \models \psi$ and $\psi \models \phi$.

It means that where is this problem 12. Here, for any binary predicate side for all x, y psi of x, y arrow psi of y, x is logically equivalent to for all x, y psi of x, y is by conditional psi of y, x. Note, that this logical equivalence does not mean that the corresponding by conditional is a

tautology. It is not a tautology. We have defined what are tautologies because this equivalence cannot be derived only from the propositional axioms and modest pronence. It requires exist universal elimination instantiation of the variables x and y again generalization. So, it does with it is not a tautology but it is logically valid. So, corresponding by conditional is logically valid but it is not tautology that is a so that is what you get in your theory of directed graphs undirected graphs so this is all. So, basically from a purely logical point of view there again in the theory of undirected graphs most of the interesting theorems has to do with counting comminatory x and so on and so forthwhich, means that what you are actually doing is you are actually it means essentially you are imposing a structure.

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The stratification I spoke about is essentially that you have your first order logic axioms here. Let us say H1 and then you have essentially Number theory. And, many of the interesting results actually come from combinatory x which is essentially number theory. And, then you actually have your theory of undirected graphs directed graphs on top of this. So, basically one of the things that we implicitly assume our facts from number theory in our theory so whatever you learn let us say in discreet math on directed or undirected graphs essentially has a stratification of this form. So, otherwise the theory would be quite on interesting and there is very little you can actually prove. So, that is what the theory of undirected graphs is.

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The slide is titled "Irreflexive Partial Orderings" and contains the following definitions:

$$\Sigma \quad \quad \quad = \{; <\}$$
$$\phi_{<-irreflexivity} \stackrel{df}{=} \forall x[\neg(x < x)]$$
$$\phi_{<-transitivity} \stackrel{df}{=} \forall x, y, z[\{(x < y) \wedge (y < z)\} \rightarrow (x < z)]$$
$$\Phi_{IPO} \quad \quad \quad \stackrel{df}{=} \{\phi_{<-irreflexivity}, \phi_{<-transitivity}\}$$

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Now, since we were on relations there is a theory of Irreflexive Partial Orders I used times most of the other books confused have confused me for many years with their terms. So, I am actually using some explicit terms which may not be found in books. So, when I say when the notion of they use notions like strictness or total and word total is the very ambiguous word for various reasons, totality of a relation is different necessary form totality of an order and so on and so forth. So, I am using what I consider to be precise an unambiguous terminology for these things. So, we usual less than relation on any structure is an irreflexive relation and it could be a partial ordering or total ordering. Though, just as the word total is ambiguous the word partial is also ambiguous. So, we have to be very carefully how to use these words.

So, let us think of this so I am just so there is a single relation it is irreflexive and it is transitive in that sense it is different from the edge relation of in directed graphs. However, so but however supposing you take the transitive closure of the edge relation in directed graphs. So, then essentially that the transitive if you take that as your signature rather than the edge relation itself. Then, the theory you get is the theory of irreflexive partial and so that is this kind. What actually logic allows you to do? Is to do this clear stratification and clearly compartment less things what goes where. And, it allows you to formalize these things therefore so essentially what we are saying is you whatever is there in the theory of irreflexive partial orderings is just the theory of directed graphs that is it theory of parts in directed graphs. So, if you replace the edge relation by

the path relation that is it what you have is an irreflexive partial order. Then, of course there is a question of being things being a cyclic. So, if there are cycles in the directed graphs then this goes beyond that. So, we will come to that at some point but some in direct fashion. I am not going to be directly interested in that.

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Irreflexive Linear Orderings

Σ	$=$	$\{<\}$
$\phi_{<-irreflexivity}$	$\stackrel{df}{\equiv}$	$\forall x[\neg(x < x)]$
$\phi_{<-transitivity}$	$\stackrel{df}{\equiv}$	$\forall x, y, z[(x < y) \wedge (y < z) \rightarrow (x < z)]$
$\phi_{<-trichotomy}$	$\stackrel{df}{\equiv}$	$\forall x, y[(x < y) \vee (x = y) \vee (y < x)]$
Φ_{ILO}	$\stackrel{df}{\equiv}$	$\{\phi_{<-irreflexivity}, \phi_{<-transitivity}, \phi_{<-trichotomy}\}$

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Then, you have Irreflexive Linear Orderings. So, this is so very of many books total orderings and then very often they do not specify whether the total ordering is strict or not strict. So, there is a difference between the less than and less than or equal to. The, less than is an irreflexive relation the less than or equal to is a reflexive relation and, in between irreflexivity and reflexivity you have a lot of stuff. You have things which are neither irreflexive nor reflexive you have relations which are which could be which essentially there is at least one element x . Such that x is related to itself and there is at least one element x such that x is not related to itself. So, all those kinds of orderings that you can get lie between irreflexivity and reflexivity and that is a usually of matter of confusion very often in if you when you do not formalize the theory in an precise fashion. So, the notion of irreflexive orderings is essentially this. So, all you have is a less than relation which is irreflexive. So, for there does not exist any x such that x is less than x that is the you are and this is that is logically equivalent of this irreflexivity. There, is transitivity as we had in the case of partial orderings.

So, if essentially a is less than b , and b is less than c and a is less than c that is what the transitivity is says. And, how does irreflexive partial orderings differ from irreflexive linear orderings? There, is a trichotomy law in the case of a linear ordering which, is essentially to say I mean I am calling this trichotomy here again books can vary in various ways. But, given array to elements x and y there are exactly three possible relations between them. So, one possibility is that x is the same as y and for purposes of argument I have considered arbitrary x and y . But, they could be the same element so if they are not the same. Then, either x is less than y or y is less than x . So, that is so this trichotomy is what makes a difference between irreflexive partial orderings and irreflexive linear orderings are irreflexive total orderings are so you call them. The notion of well ordering essentially is that of an irreflexive linear ordering which is bounded below well orderings in fact. If, you think of it carefully all the proofs that you did using structural induction or using the principle of mathematical induction they can all be generalized to essentially proofs saying that.

There, is a well order structure there is a an irreflexive linear ordering which preserves the property. You, just go back to all those proofs essentially they bounded below in the case of structural induction the bound below is the basis in the case of any induction the bound below is the basis that you use. And, it might be unbounded in the other direction so and in the case of structural induction actually what you are using is an irreflexive partial ordering. But, what you also use is a particular path a linear path in that partial ordering which, any ways fine is bounded below which has a least element. So, what you are actual well orderings can be generalized to irreflexive partial orderings. Because, I can consider each individual path of that partial ordering to be a well ordering to be a linear well ordering in itself. So, that is the theory of irreflexivity linear orderings.

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(Reflexive) Preorders

$$\Sigma = \{; \leq : s^2\}$$
$$\phi_{\leq\text{-reflexivity}} \stackrel{df}{=} \forall x[x \leq x]$$
$$\phi_{\leq\text{-transitivity}} \stackrel{df}{=} \forall x, y, z[(x \leq y) \wedge (y \leq z) \rightarrow (x \leq z)]$$
$$\Phi_{Pre} \stackrel{df}{=} \{\phi_{\leq\text{-reflexivity}}, \phi_{\leq\text{-transitivity}}\}$$

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And, then of course you have the notion of Reflexive. So, this is the notion of the Preorders and I will implicitly assume that a preorder is a reflexive these in many books are also called Quasi orders. Some books distinguish between Preorders and Quasi orders. Some, of them say one of them is linear one of them is irreflexive. And, the conclusion is total but the most precise way of looking at it is through these axioms you have a reflexive relation. And, you also have a transitive relation and that is a Preorder. So, all you are usual less than or equal to relations are essentially those of reflexive preorders.

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(Reflexive) Partial Orderings

$$\Sigma = \{; \leq : s^2\}$$
$$\phi_{\leq\text{-reflexivity}} \stackrel{df}{=} \forall x[x \leq x]$$
$$\phi_{\leq\text{-transitivity}} \stackrel{df}{=} \forall x, y, z[((x \leq y) \wedge (y \leq z)) \rightarrow (x \leq z)]$$
$$\phi_{\leq\text{-antisymmetry}} \stackrel{df}{=} \forall x, y[((x \leq y) \wedge (y \leq x)) \rightarrow x = y]$$
$$\Phi_{PO} \stackrel{df}{=} \{\phi_{\leq\text{-reflexivity}}, \phi_{\leq\text{-transitivity}}, \phi_{\leq\text{-antisymmetry}}\}$$

But, what you study more often in discrete math course is that of Reflexive Partial Orderings. Where, there is an extra property of anti symmetry. This anti symmetry source of much mental trauma to students is really this question of identity of objects which have to different aliases. So, this we considering first order predicate calculus with equality however, that equality is such that syntactically distinct terms are considered different. And, when you say that x is less than or equal, to y and y is less than or equal to x, implies x is equal to y. What you are actually saying is? That x and y are the same object I mean the edges given them the aliases given that same object two different aliases that is it. And, so this is anti symmetric property which essentially identifies objects which might have more than one name is what this what gives you a partial order here.

So, partial ordering of course so this is and I will assume that by partial ordering we mean a reflexive partial ordering. So, there is a reflexivity axiom there and there is transitivity and there is anti symmetry and, of course so the theory of. So, this phi PO in each case this capital PHI with a subscript is the set of all axioms which just define the theory. So, all the logical consequences of this along, with the axioms of first order predicate calculus constitute the first order theory all the logical consequences constitute the first order theory of this structure.

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(Reflexive) Linear Orderings

$$\Sigma = \{; \leq : s^2\}$$
$$\phi_{\leq\text{-reflexivity}} \stackrel{df}{=} \forall x[x \leq x]$$
$$\phi_{\leq\text{-transitivity}} \stackrel{df}{=} \forall x, y, z[(x \leq y) \wedge (y \leq z) \rightarrow (x \leq z)]$$
$$\phi_{\leq\text{-dichotomy}} \stackrel{df}{=} \forall x, y[(x \leq y) \vee (y \leq x)]$$
$$\Phi_{LO} \stackrel{df}{=} \{\phi_{\leq\text{-reflexivity}}, \phi_{\leq\text{-transitivity}}, \phi_{\leq\text{-dichotomy}}\}$$

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Then, you have reflexive linear orderings or remember here that if it is reflexive. Then, you actually have a dichotomy law and not a tracheotomy law. Because, I mean that is sufficient that is what I am trying to say I mean I do not need a tracheotomy law. So, I have a dichotomy law for less than or equal to so that is my notion of a reflexive linear ordering.

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Equivalence Relations

$$\Sigma = \{; \sim : s^2\}$$
$$\phi_{\sim\text{-reflexivity}} \stackrel{df}{=} \forall x[x \sim x]$$
$$\phi_{\sim\text{-symmetry}} \stackrel{df}{=} \forall x, y[(x \sim y) \rightarrow (y \sim x)]$$
$$\phi_{\sim\text{-transitivity}} \stackrel{df}{=} \forall x, y, z[(x \sim y) \wedge (y \sim z) \rightarrow (x \sim z)]$$
$$\Phi_{EQUIV} \stackrel{df}{=} \{\phi_{\sim\text{-reflexivity}}, \phi_{\sim\text{-symmetry}}, \phi_{\sim\text{-transitivity}}\}$$

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Now, well then you have Equivalence Relations which, are reflexive symmetric and transitive and here I mean that is what I am saying. So, here is notice that this equivalence is something that might be define external to the set of objects. So, if x and z are equivalent that does not mean that they are the same object. So, if should not confuse with the anti symmetry in the partial ordering case. So, that is why I use different symbols for this but, what does happen normally in algebra is if, you quotient out the structure on an equivalence relation. Then, what you are actually dealing with the objects that you are dealing with are equivalence classes. So, if you when you talk about an object A you are essentially talking about the set of all objects which are equivalent to A by your equivalence relation. And, therefore if you are first so then what you are dealing with on the quotient it structure the first order theory of the quotient its structure actually has the anti symmetry property.

If, a and b are equivalent then their equivalence classes are exactly the same classes with the same object. So, the theory of equivalence relations is automatically takes you also back to allow you to map between, the theory of using just allows you to map between just a first order predicate calculus with equality on the quotient its structures. And, separately as the first order theory of equivalences. Now, these are some examples of first order structures and what we normally do is we just whenever we are talking about a theory in computer science logic mathematics or anything. If, you just gives these defining axioms that is it. We, just say that a relation is an equivalence relation if it satisfies these axioms and most often we do not even use we do not even put the any universal quantifier. Because, we know that validity is preserved only under universal closure. So, the universal closure is implicit and this is how we work with it.

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Peano's Postulates

P1. 0 is a natural number.

P2. If x is a natural number then x^{+1} (called the *successor* of x) is a natural number.

P3. $0 \neq x^{+1}$ for any natural number x .

P4. $x^{+1} = y^{+1}$ implies $x = y$

P5. Let P be a property that may or may not hold of every natural number. If

Basis. 0 has the property P and

Induction Step. whenever a natural number x has the property P , x^{+1} also has the property P

then all natural numbers have the property P .

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So, let us in order to do something concrete I thought we will start with Peano's Postulates. And, the theory of numbers we just do some basics of formal now what is known as formal number theory sometimes it is also known as formal Peano arithmetic. But, there are subscript of it also which are interesting so from decidability point of view. But, I will not get into that but let us just look at this these are the Peano's Postulates which, essentially so if you have studied Peano's Postulates somewhere in school or whatever you see that it has basically five postulates. Where, the first two postulates 0 is a natural number and if x is a natural number I am using this class one now, the successive function usually s is used. But, I since I am using s for let defining sorts and so on and so forth as a symbol for sorts. I am using plus 1 which I think intuitively is first. So, it is called the successor of x .

So, this P1 and P2 essentially define the signature they essentially states that there is a distinguish constant element called 0. And, there is a unary function called successor and they basically these postulates do not do anything rather than define the signature. And, in programming terms it is like P1 and P2 are actually define the data type which generates all possible terms of this data type of naturals. So, the third postulate essentially says that 0 should be distinct from any successor that 0 is not the same as any successor. So, these are completely syntactic forms. So, what we are saying is that. Whatever, terms you can generate from P1 and P2. You, cannot identify any of the terms generated from P2 with something in P1. If, you were

to take the theory of Booleans for example. One explicit axiom you would require in any first order theory of Boolean though we take it for granted. One explicit axiom you would require is that 0 is not the same as one. If, you do not have the axiom then there are models for all your predicates on Boolean algebra. Where, there is a singleton single carrier set and truth is the same as the false. So, unless you have these axiom the 0 is not equal to 1. You, have not actually defined a boolean algebra and in its proper form you have not defined a two element.

So, that essentially says that your carrier set must have at least two elements it could have more of course. But, that it is important to specify the 0 is not same as the 1 after all 0 and 1 could be different names to the same object. So, that possibility has to be excluded. The, fourth axiom essentially says that your construction is deterministic. And, it also says that this successor function this successor can be thought of either as a function or it can be thought of as a constructor. When at the time when Peano defined it the notion of a constructor was not very well understood since, there was no programming and so on and so forth no computers and so on and so forth. But, what the only thing that you could relate to is this syntax of terms. So, all that this says is that this successor regarded as a function is an injective function.

So, which means that if there are two objects not x and y not necessarily distinct from each other and if I apply the successor function to each of them. And, I look at the resulting objects and they are just same object then the original object must have been the same that is. So, this says that the successor function is essentially injective. And, lastly Peano had this I am its important here to think of it this way. So, let P be a property the word property here already is ambiguous it is not clear exactly what is meant by that. And, Peano left it open he himself was not very clear about it. But, he knew that it is important it is necessary to have such a thing if you have a property P that is true for 0. And, the property P is preserved under the successor function that is actually what your induction step it has. It says that the successor function does not affect the property and the property preserved is preserved under successor which, is also equivalent in algebraic terms to say saying that you can push the successor inside the property itself.

Then, all the numbers that you have generated all the terms that you have generated from P_1 and P_2 also satisfies that property. So, this is what this principle of induction was which, translated in first order let us I will talk about this a little later. So, this notion of a property he left it ambiguous but basically A to the any property. And, when you look at the first order theory of

numbers then this property essentially becomes a unary predicate expressible in first order logic. And therefore, it does not include properties of numbers which might be properties of finite subsets of naturals for example, those would be second order properties. So, whereas Peano's original formulation left it ambiguous with the idea that it can be any order actually. It need not be first order but the first order induction principle would have essentially a property P specified in first order logic on the theory of individuals. Where, the individuals are drawn from the data type generating natural numbers through the 0 and the successor operation.

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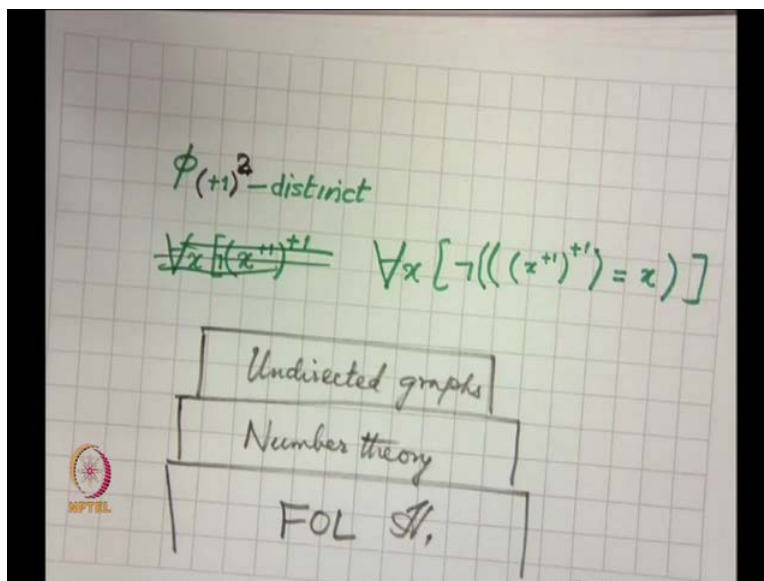
The Theory of Naturals

Σ_S	$= \{0 : \rightarrow s, +1; = : s^2\}$
$\phi_{0\text{-notsuccessor}}$	$\stackrel{df}{=} \forall x[\neg(x^{+1} = 0)]$
$\phi_{+1\text{-injective}}$	$\stackrel{df}{=} \forall x\forall y[x^{+1} = y^{+1} \rightarrow x = y]$
$\phi_{\neq 0 \rightarrow \text{successor}}$	$\stackrel{df}{=} \forall y[\neg(y = 0) \rightarrow \exists x[y = x^{+1}]]$
$\Phi_{(+1)^n\text{-distinct}}$	$\stackrel{df}{=} \{\phi_{(+1)^n\text{-distinct}} \mid n > 0\}$
$\phi_{(+1)^n\text{-distinct}}$	$\stackrel{df}{=} \forall x[\neg(x^{(+1)^n} = x)]$
Φ_S	$\stackrel{df}{=} \{\phi_{0\text{-notsuccessor}}, \phi_{+1\text{-injective}}, \phi_{\neq 0 \rightarrow \text{successor}}\} \cup \Phi_{(+1)^n\text{-distinct}}$

So, let us look at the Theory of Naturals. Will start with so essentially we start inspired by Peano. And, you have sort of reformulated into first order predicate calculus so the theory of so here I have I do not have induction. I am induction is something I am going to talk about later. So, the first thing is that so the first two of Peano's postulates essentially give you the signature. And, there is an equality also so we will assume first order predicate calculus with equality so there is an equality relation. And, there is a 0 and successor function plus 1. By, the way I am not written the sort of this I should have written it as s arrow s. That will have to be corrected. The, first of Peano's postulates essentially says that 0 is a natural number. There, is something else we need P3 that 0 is not the successor of any natural number is what this axiom says. So, for all x this it is not true that the successor of the x is equal to 0.

The other axiom that it is injective is this trivial and then, we need an axiom which says essentially that if a sort of a natural number is in successor form it does have a predecessor. And, that is what this one says if something is not equal to 0 then it does have a predecessor. This, in conjunction with this injectivity property actually ensures that this predecessor is unique but that will have to be proven. So, that is one thing that is a let us say that is a first thing you have to prove that the predecessor whenever it exist is unique. Then, there is what we want from our theory of naturals is that the entire set of even though the entire set of naturals that we get a countable model. We, should not have only finite models because, the set of natural numbers as we know it is infinite is count ably infinite. And, so in order to ensure that there is a countable model you have to essentially say that no successor element these are all distinct. This says that for any x so here this is an abbreviation are you able to distinguish the colors. So, this I have put plus 1 enclosed in black parenthesis and there is to a black n. The, black indicates the metal language it is an abbreviation, for an n fold application of plus 1 for each n greater than c. So, essentially what we are saying is.

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So, I have phi plus 1 black n and green distinct. Now, this one essentially if I take n equals let us say 2 then, this essentially is stands for all x I need purple of course but well just leave with one. Plus 1 by the way there should be a naught here naught for all x naught x plus 1, plus 1 equal to x. So, this for each value of n its essentially an abbreviation for this so this n distinct. So, what

we are saying is that for any x any number of applications of successor to x cannot give you back x . And, this is an infinite collection of axioms there has to be an infinite collection of axioms. Otherwise, what you can it opens up the possibility of having finite models. In particular some of the finite models is a very popular in number theory are this modulo n for some appropriately chosen value of n .

So, therefore this theory of naturals which guarantees an infinite which guarantees that there are only infinite models consists of all these axioms. So, basically that 0 is not the successor of any element the successor operation is injective. And, if something is not equal 0 then it must be the successor of some element. And, for each n successor applied n times to x is not equal to x for each x . For each x and for each n successor the n th successor of x cannot be equal to x that guarantees essentially infinite tree models. But, there is something else one of the things I did let us go back it some point it must be here did we do the isomorphism lemma here. We spoke about the distinguish ability of structures. We spoke about isomorphic structures.

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Isomorphic Structures

Example 21.3 Let $\Sigma = \{0 : \rightarrow s, +1 : s \rightarrow s, = : s^2, < : s^2\}$. Now consider the structures

$$\mathbb{N} \models \langle \mathbb{N}; 0, +1; =, < \rangle$$

$$2\mathbb{N} = \langle 2\mathbb{N}; 0, +2; =, < \rangle$$

where $2\mathbb{N}$ is the set of even natural numbers, $+1, +2$ denote the respective "successor" functions, $<$ is the usual "less-than" relation on both structures. The two structures are clearly isomorphic and there is an isomorphism which $\pi : \mathbb{N} \rightarrow 2\mathbb{N}$ such that $\pi(n) = 2n$ which along with the inverse map $\pi^{-1}(2n) = n$ maps one structure exactly onto the other.

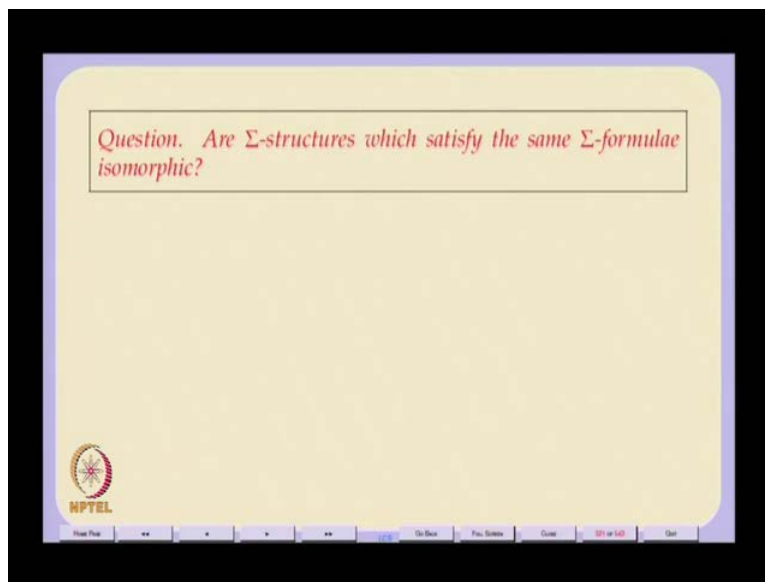
Isomorphic Σ -structures cannot be distinguished by $\mathcal{P}_1(\Sigma)$.

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So, we said that the Isomorphic sigma structures cannot be distinguished by \mathcal{P}_1 sigma. So, if you look at our theory of naturals there are at least two different models. The theory of even number is one model and the theory of odd numbers is another model. And, the theory of naturals actual the actual set of naturals is another model they are all infinite tree models which, satisfies the

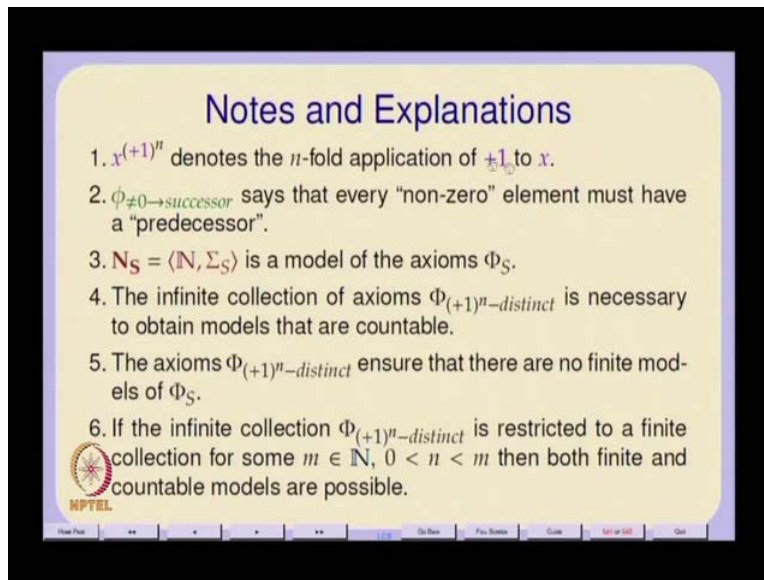
axioms we satisfies Peano's postulates for example. And, you can actually go further I mean you can just talk about 2^n for example. For each natural number n takes the set of all powers of 2 where, your successor is a mixed higher power of 2. Because, though so all these are isomorphic structures and any statement that you make about the naturals is also true about these structures translated appropriately that was one thing. So, this distinguish ability you cannot so basically first order logic cannot distinguish isomorphic structures. Basically, isomorphic structures are different only in their names. And, that is an isomorphism which is a name isomorphism which makes things and in interpretation which makes them essentially the same. So, we had this isomorphism lemma which is essentially said that first order logic is not powerful enough to distinguish isomorphic structures.

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Then, we ask this question which we never answered. Are there sigma structures which satisfies the same sigma formulae but are not isomorphic? And, now let us get back to our theory of naturals.

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Notes and Explanations

1. $x^{(+1)^n}$ denotes the n -fold application of $+1$ to x .
2. $\phi_{\neq 0 \rightarrow \text{successor}}$ says that every "non-zero" element must have a "predecessor".
3. $\mathbf{N}_S = \langle \mathbb{N}, \Sigma_S \rangle$ is a model of the axioms Φ_S .
4. The infinite collection of axioms $\Phi_{(+1)^n\text{-distinct}}$ is necessary to obtain models that are countable.
5. The axioms $\Phi_{(+1)^n\text{-distinct}}$ ensure that there are no finite models of Φ_S .
6. If the infinite collection $\Phi_{(+1)^n\text{-distinct}}$ is restricted to a finite collection for some $m \in \mathbb{N}$, $0 < n < m$ then both finite and countable models are possible.

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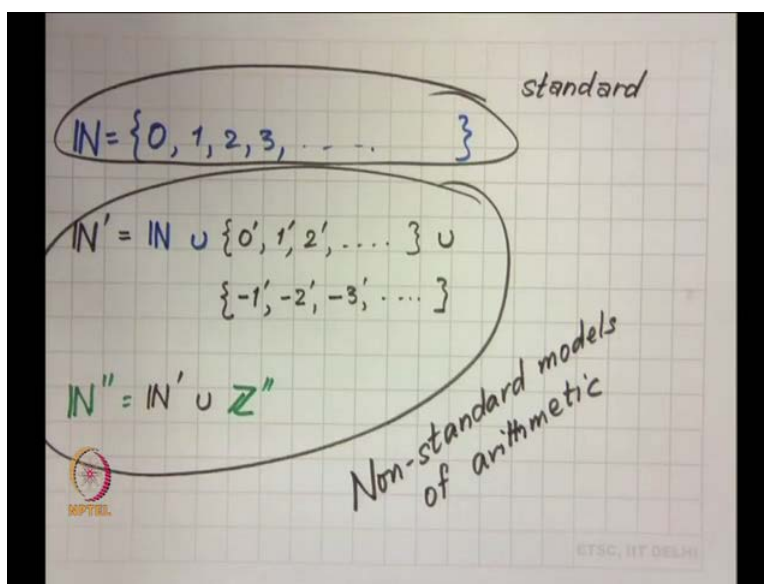
So, let us just quickly go through this so x raised to plus 1 n denotes n fold application of plus 1 to x . Says the every non 0 element must have predecessor and this set of natural numbers under sigma S is the model of the axioms ϕ of s . The infinite collection of axioms is necessary to obtain models that are accountably infinite. The, axioms 5 plus 1 ensure that there are no finite models. What you can do is? If, you supposing if you restrict this set of axioms ϕ of plus 1 n to supposing you restrict this to some instead of n greater than 0. You have 0 less than n less than m for some given m . Then, what you essentially get you get both finite and infinite models. You get finite models like modulo m models and modulo m is not the only model you can even take a modulo two m and so on and so forth. You, get all those finite models plus you do get infinite models if you want only finite models which, actually what happens if you done a course on number theory in some systematic fashion what you will notice is that at some point after Euclid in algorithm it is much more convenient to deal with just modulo and structures.

And, usually that modulo n is that n is some prime that is usually you do you try to reduce all your properties for large numbers from infinite sets to finite sets by doing a division model of ϕ modulo n modulo p for some prime p . And, this is very useful and the whole of number theory becomes useful precisely because you can do that there are large number of models which are large number of useful theorems especially RSA for example which, allow you to move between

modulo p structures and the whole of the naturals if you want a whole of a integers if you but there is something else coming back to our question.

So, what is now supposing you want a only finite models. Then, you will have to add an extra axiom which says for some m let us say that m successor of x for all x is equal to x . So, then you will get all 0 to m minus 1 . So, you will get modulo m structures and then you will not get any infinite models you will get only finite models. But, if you do not add these axioms but you still restrict n to less than m when you have both finite and infinite models. So, coming back to this question coming back to this. What we can do even with the current thing is?

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So, we have a essentially constructed 0 and I do not want to keep writing plus 1 and so on so forth. So, essentially we have constructed $0, 1, 2, 3$ etcetera the entire set N . And, now what we can do is I take this. So, this is my set N generated entirely by this by Peano's postulates one and 0 , and one or one and two. There, is nothing in those postulates and nothing in this aximitiazation that we have donewhich, this is 321 no go back to back to $541, 540$. So, between this there is nothing which prevents me from adding a new fictitious element. I mean this is a model that satisfies this all these axioms. There, is nothing that prevents me from considering an N primewhich, consist of the whole of N of course. And, then I am going to add a new element and I am going to call this element, I am going to call it 0 prime.

Then, what do I do I will look at all these and 0 prime this thing allows me to generate successor of 0 prime, successor of 1 prime and so on and so forth. But, that is not all that is only from the process of generation from this successor function it has to be total so it should be possible to if you add an element 0 prime you should be able to generate successor of successors of 0 prime. And, if you generated successor of 0 prime and it has to be close from s. So, it is closed but the problem is here, if 0 prime is distinct from 0. Then, this one says that it must be successor of an element that means it must have a predecessor. So, what it means is that now this axiom essentially says that I cannot stop with this. I have to start adding new elements. Let, me call it minus 1 prime. And, by the same token I would be generating minus 2 prime, minus 3 prime and so on and so forth. Now, consider \mathbb{N} prime. \mathbb{N} prime is a countable model of this axioms.

Absolutely, no reason why I should stop here I create an \mathbb{N} double. And, which is essentially going to \mathbb{N} prime union a new 0 double prime. And, this is \mathbb{N} prime essentially union u actually I am generating all the integer because, of that so let me call that \mathbb{Z} double prime. The, point is that the theory of naturals does not says you should not generate the integers, does not say that you cannot generate two different copies of integers. The, important thing going back to this question and what are we saying now? \mathbb{N} prime, \mathbb{N} double prime, \mathbb{N} triple prime whatever you might generate they all have copies distinct copies of the integers within them. And, if I have an infinite number of different copies of the integers also I still get only a countable model.

Now, none of these models is either not fit to each other there is no isomorphism between them. Because, it clearly want \mathbb{N} is a sub set of \mathbb{N} prime, \mathbb{N} prime is sub set of \mathbb{N} double prime that is all there is no isomorphism that you can define. So, the notion of first order in distinguishability goes beyond the isomorphism lemma there are also non isomorphic structures which, cannot be distinguish by first order form. So, these and in fact what you can do is you take any expansion of the theory of numbers all these models are called the Non standard models of arithmetic. So, first order indistinguishability is limited I mean first order distinguishability is limited very limited. What we showed was that it cannot distinguish isomorphic structures. But, what this shows is that there are also non standard models that you can create which, I am not isomorphic to each other what might be called as a Standard Model this is called a Standard Model.

So, Peano's original assumption that he can generate all the naturals that exact to the naturals by these axioms is not true I mean you can generate things which, are not isomorphic to the standard model too. And, your first order logic will not be able to distinguish that every property that you prove for this theory. Every property that you prove of the naturals would also be true in this in these Non standard models. Actually, the problem goes further in the sense that there are people who I mean this was the existence of the non standard models came probably in 1920s-30s. But, after that there are people who actually create a non standard analysis. So, the entire theory of real analysis or complex analysis you can create non standard models for all those axioms. And, so that is by the way even if you are mathematics text books do not explicitly say so most of the time all your mathematics text books are using some form, of using first order reasoning applied to higher order logic. Because, most of those axioms of first order logic are also applicable in higher order logic their differences will come in terms of decidability and so on and so forth. And, finiteness and infiniteness but otherwise whatever you so all your presentations of things and analysis like Dedekind cuts all the construction of reals the construction of rationals so on and so forth.

What, you can do is you can start with number so basically you can start with the naturals and create fractions create integers and then create reals through Dedekind cuts. But, now I can take this trans standard models. And, do the whole thing I mean it is very reminiscent of that thing of there is ancient Indian methodological story of the sage Vishwamitra constructing a new universe he started constructing a new universe that is like a constructing a non standard model of the existing axiom of the Brahman. So, you can actually create non standard analysis and there are very many different kinds of non standard analysis possible. And, those have been created so notions of continuity which the whole purpose of analysis was to formalize Newton's assumptions about continuity in some logical framework. And, that those acquire a new meanings when you create non standard analysis, non standard Dedekind cuts, non standard real numbers from using higher order logics from the naturals.

And, of course my purpose was my purpose is basically to go through just standard model but to show that they do exist non standard models. To show that first order logic is not expressive enough for many properties. But, most of our reasoning's still is first order what we do is we use the same reasoning mechanism of first order. When we go into higher orders because, the

reasoning mechanism of first order are fairly general in any higher order logic your propositional connectives have the same kinds of meaning. And, your existential and universal quantifiers also have the same kind of meaning you have to instantiate them you to initiate instantiate them and then you have to generalize them. So, the axioms of higher order logic will also be the same but we keep transcending this borders. So, I stop here we will do more about Presbever arithmetic and Peano arithmetic next time.