Logic for CS
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Lecture - 34
Completeness of the Hilbert System

So, today will study the Completeness of the Hilbert system for first tautological may be what one should briefly look at are these rules.

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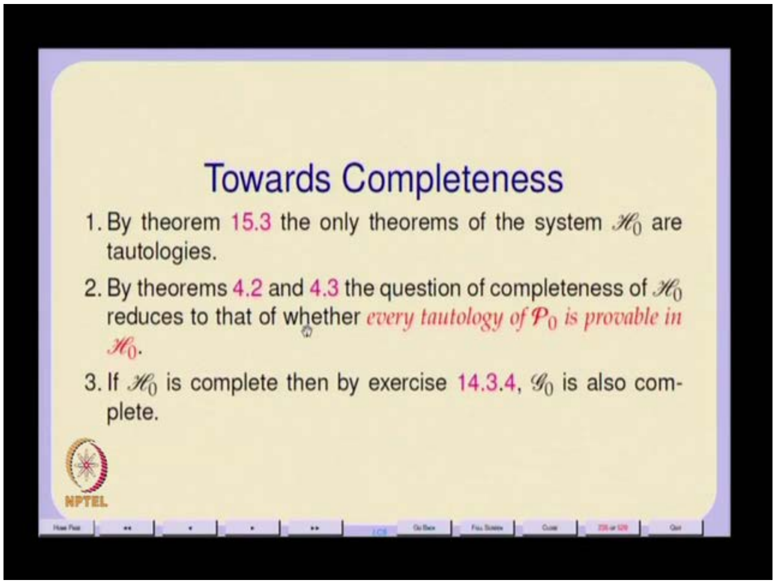

So, as well as the Hilbert system is concern first thing of course, they for propositional logic we had this interesting concept of being able to simulate the truth table.

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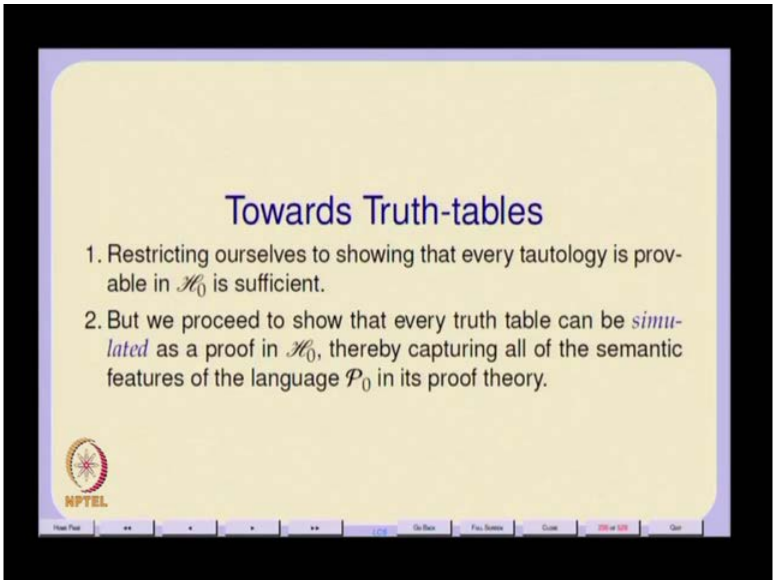

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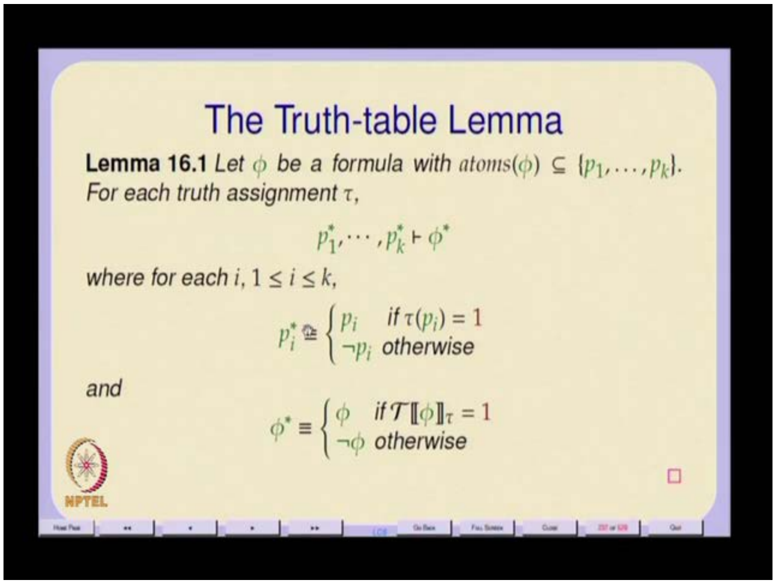

So, we actually simulated Truth table we showed that every row of the truth table can be simulated by a proof in the propositional Hilbert System. And, then after that it was that was the hardest part simulating each row and having simulating each row.

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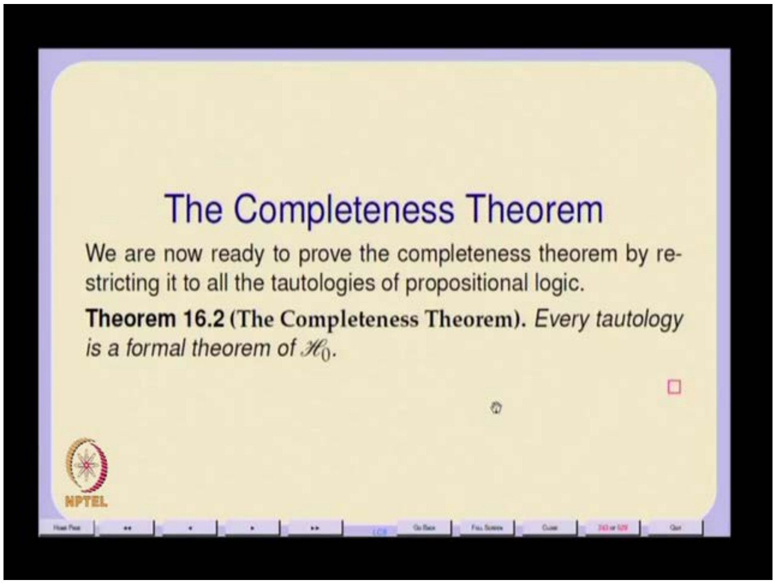

Then, we could just prove that every tautology is of formal theorem of H naught. In the context of first tautology, we have to make certain things clear I mean so in the context of first tautology there are free variables that is one extra complication and terms. And, essentially Herbrand's terms and if there is at least one constant and at least one function symbol then, the number of Herbrand's terms ground terms is infinite automatically. So, as a result so which means that truth tables are not possible I mean so this whole concept of trying to simulate truth tables should go. The second thing is that so what we essentially showed in the proposition Hilbert style system is that you can prove every tautology as a formal theorem. And, your and the soundness essentially shows that anything that is provable is a tautology. So, that is how the soundness in completeness went. And, the deduction theorem allowed you to move formulae to the left of the terns talent and therefore give you a complete system for logical consequence also. So, it was sufficiently just to look at tautologies and whether every tautology can is provable and for soundness it was sufficient to say whether anything that is provable is interial tautology.

Anything that is provable from no assumption basically and that is if that is a tautology then you are essentially done. So, here of course and the technique involved using a trying to simulate truth tables. But, of course within the first tautological because of the presence of free variable and ground terms of the Herbrand's universe truth tables are never finite anyway so therefore there is no question of any kind of finite proof simulating each row of the truth table and trying to take a conjunction of all those rows I mean, it makes absolutely no sense. On the other hand so this technique that was used for propositional logic is clearly not suitable for first tautological even with just a single constant and the signature and even with the single operator function. The other thing that is interesting is a notion of a tautology I mean this is something I did not provably clarify before the difference between validity and tautology.

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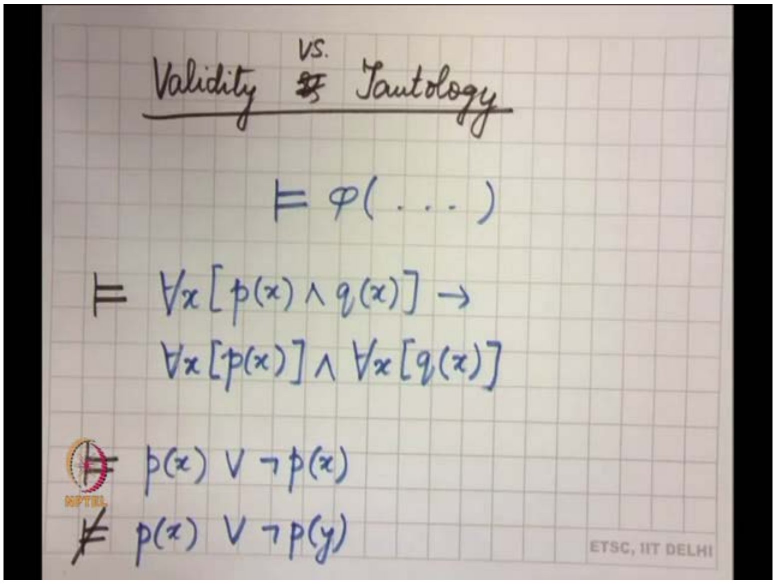

So, what will do is I will take the view let me call this Verses Tautology. So, we have used two different terms but then in first tautology in the two term are actually different. So from so there different in certain sense and the sense is that you would we would think of. Let, say a predicate phi which might have various free variables. We defined this as validity and basically we said that it is valid for all models for all sigma models it is indeed true. So, there are formulas like for all x p of x and q of x implies for all x p of x and for all x q of x. So, if you are to take a formula like this. This, formula is clearly valid so this is a valid formula but it is not a tautology because, a notion of the tautology is essentially comes from proposition logic. And, loosely speaking what I have said before about a tautology was that it should have one of those tautological forms.

So, for example this even if it has free variables I do not care but take this if you take this. This has a tautological form let us say let us stick since am using any way the entire language does not matter. So, this is also valid for all models it is true that with a unilery relation p in it this is always going to be true. So, this is valid because it is a tautological form I mean it has the same shape that tree looks like a typical propositional tautology with the difference that instead of the leaf being a propositional atom it is actually a predicate with free variables. But, then the free variables match so for example this is not a tautology. Say, p x or naught p y mean this is not tautology form and it is therefore also not a tautology. And, it is clear that this is not valid because I can always find a model in which by giving x and y certain different values and by interpreting p in a certain way and make this false.

For example, so this is not a tautology where as this is a tautology and it also a valid this formula is not a tautological form but it is valid. So, in general if you want of so this notion of a tautological form is something like that I did not actually formulized. But, now in the light of the Hilbert style proves system we can actually formulize it. So, will just say a tautology in first set of logic is any formula that can be proved using only the axioms k s n and models points that means you do not use any of the quantifier rules of any of the quantifier axioms. So, even if the formula is open in the sense that it has free variable.

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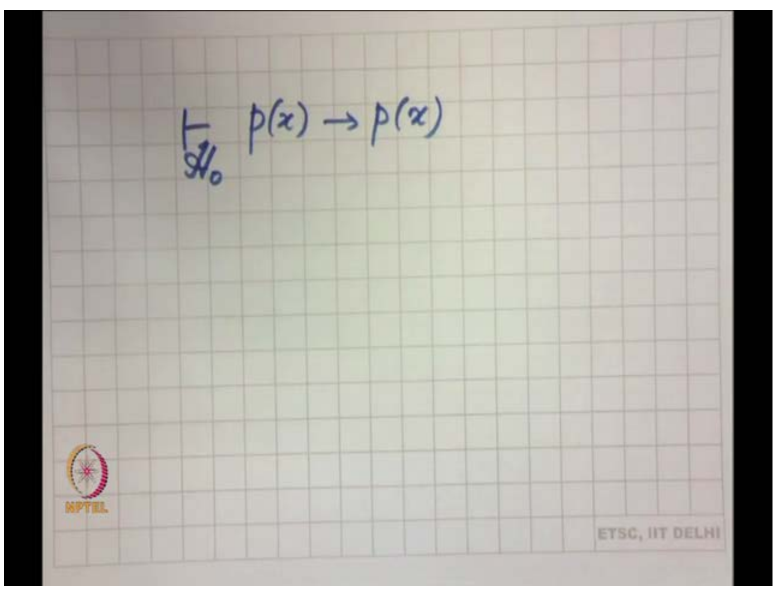

If, the proof so for all these things like so therefore p x arrow p x we have already seen a proof of this in proposition. So, a proof of this in will involve entirely just H naught I mean just the axioms and proof rules of the propositional fragment of the Hilbert stress rule system. Whereas, the validity of this formula cannot be proven in just H naught because, of that fact that you are distributing quantifiers you have to so you will have to use you will have to first eliminate the universal quantifier by an instantiation. And, then you will have to generalize the resulting formulae again by universal quantifier introduction. So, you will be using some of the proof rules of H1 which are not there in H naught. And, so even though this is valid this is not a tautology. So, the corresponding analog for first tautology is really just that we want to show that every valid formula can be proven. The soundness theorem that we prove already showed that any formula that you can prove is that actually valid. I mean that is one of thing that we where do we show that some over here, this is resolution did we prove the alphia conversion theorem.

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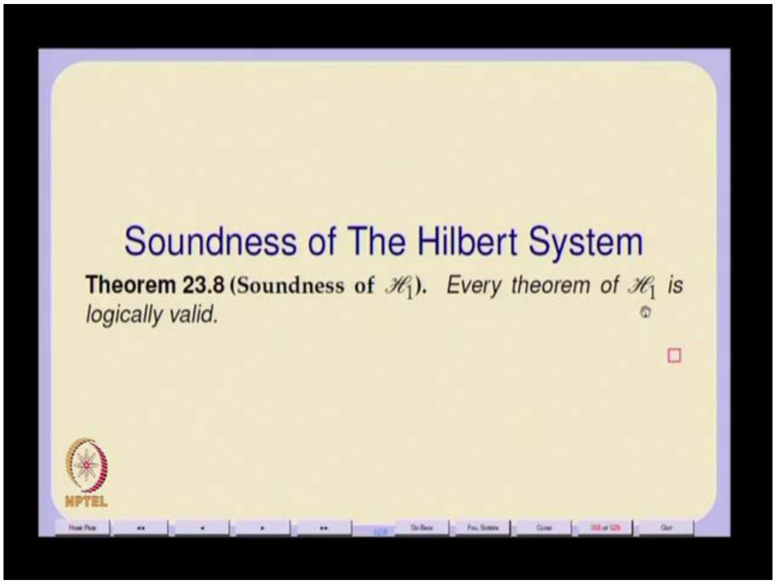

Every theorem of H1 is logically valid so we have shown that. Now, what we need to show is a converse that every theorem that you can prove in H1 is actually every valid formula can be proven basically. So, for that what will use is so now that automatically brings in the fact that there might be infantry set and so on and so forth. So, what will take now is there was we had in our propositional logic we had the notion of maximal constant sets and if you remember the notion of maximum constant sets we actually did.

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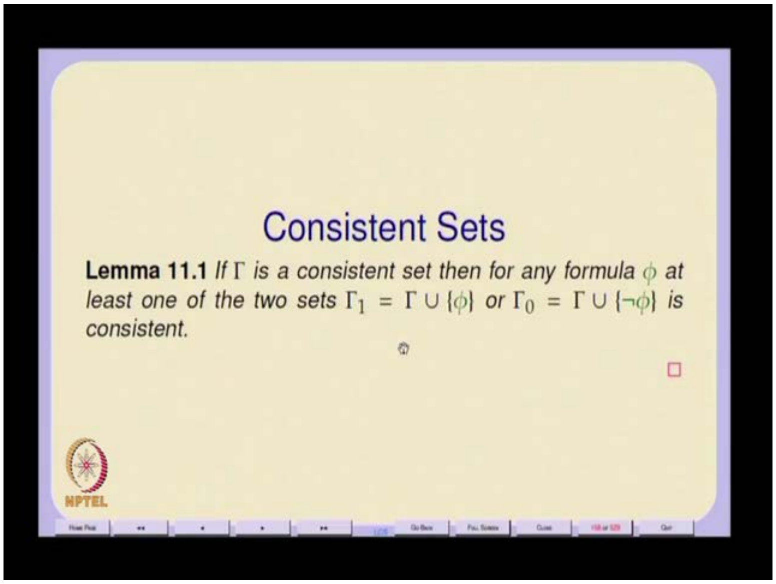

Here, if you remember this was all done model theoretically in the sense that we used the notion of consistency based on truth assignments. So, we where basically looking on models mean remember I mean my color coding is such that they no anything that is brown in color is a model and anything that is green and or black so on and so forth was a proof theoretically I mean.

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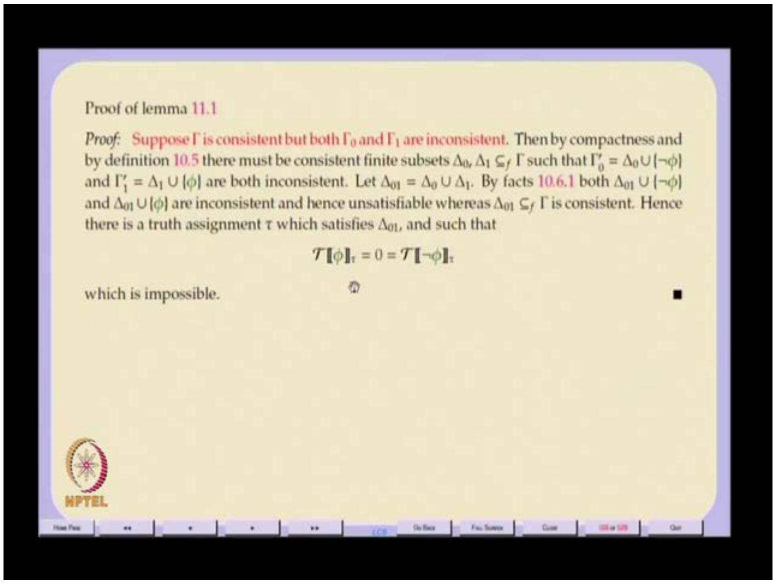

So, in the case of maximal consitant sets and so on. We actually used the semantics therefore we used models basically and we defined notions of properties of a finite character and we showed that you can extant a set consistent set to a maximum consistent set we showed there many that hintica sets are in some sense maximal reconsistent or any maximal equivalent set is also a hintica sets and so on.

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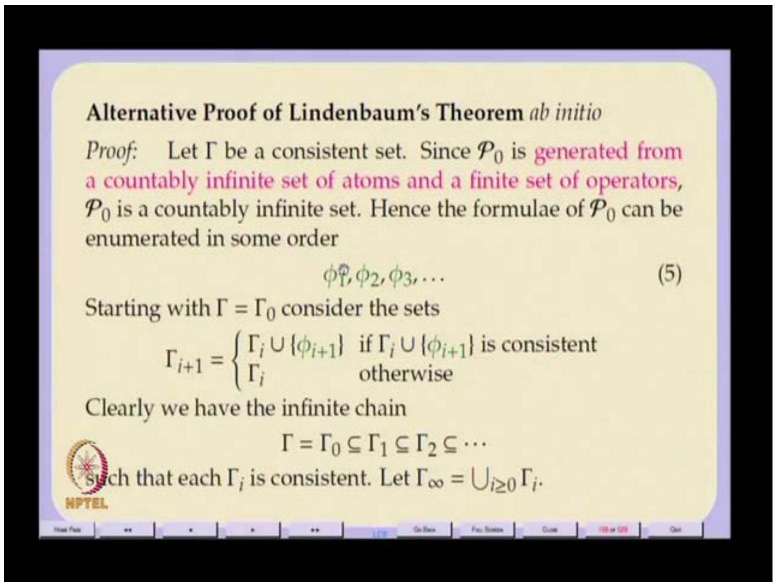

And then, we had Lindenbaum's Theorem and so on all this was done with by referring to the notion of truth and therefore the model the underline truth valuations model that you had for proposition logic. And, it was possible to do these things in propositional logic because our truth valuation model was just a two element truth valuation model. And, therefore anything that was find if you started of this something finite continue to remain finite always. Now, that luxury is again not available directly in first tautology so the only finiteness that is you have in first order set of logic is a notion of proof. So, the exact analog of these terms can also be defined proof theoretically in terms of what am saying is this is the analyses. The notion of the tautology was defined model theoretically through the notion of truth the notion of validity was also defined through the notion of truth and so on and so forth. But, the notion of a tautology in first order of logic and defined proof theoretically by just saying that. If, there is a proof that uses only the proposition axioms and rules of inference then it is a tautology.

And, in fact that is a preferly sound ways so there is so they you have if you have soundness completeness then you have proof theoretic analog of also for model theoretic concepts. So, the notions of validity and the notions of tautologies and so on. The notion of maximal consistent the notion of compact that is another thing. So, we define compactness entirely through the tablo proves system and we actually used this notion of unsatisfiability and so on and so forth.

What, we use lot of model theoretic concepts. But, it is possible to also transform that notion of compactness to what is known as reductive compactness.

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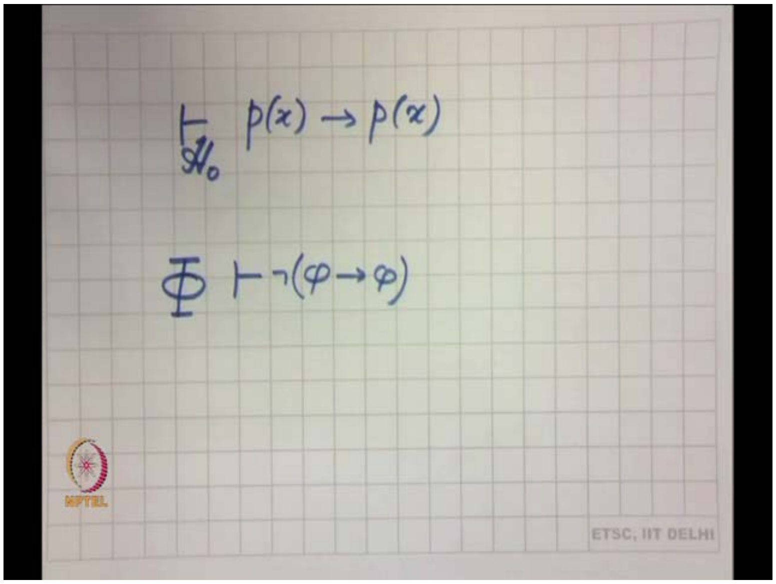

So, what are you saying now by deductive compactness I am saying that a certain formula phi is a logical consequence is mean given a set of a formula capital PHI. This is, unsatisfiable if and only if, it is possible to prove a contradiction such as naught of phi arrow phi for some formula it is consistent if and only if you cannot prove any contradiction. So, note the use of these symbols are very important. I mean, these are model theoretic symbol logical validity a prove ability is a proof theoretic symbol. So, you have to distinguish in between the two. So, what we will do now as a technique for proving to get us technique for proving completeness of the Hilbert style proof system. Is essentially, that of transforming all those model theoretic concepts into corresponding proof theoretic concepts. And, so that is what will that is what actually happens here.

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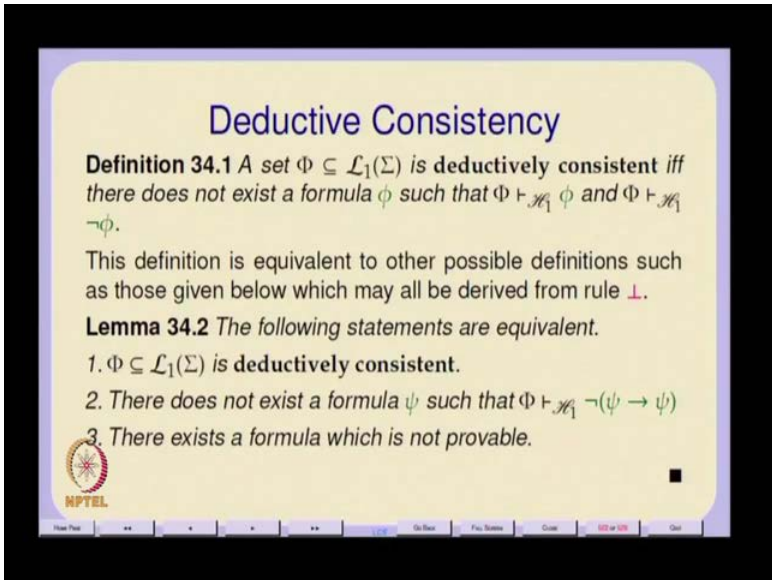

So, will so with that introduction let me start this so firstly the notion of consistent sets was defined in terms of whether it was possible to find a model for it, whether it was possible to give a truth assignment which will make all the formulas in set true. Now, will talk about Deductive Consistency and so by the way this is my term. I want to clearly separate our model theoretic concepts and the corresponding proof theoretic concepts. So, you may not find this term express this deductive consistency it might depending upon the view of the author whether, you taking a proof theoretic view or a model theoretic view he might just call it as consistency. So, given a set phi I and of course we are resisting ourselves to the language consisting of just the universal quantifier not and arrow that is one of sigma. So, this set phi is deductively consistent so am making it completely deduction is proof mean you reduce something to give a proof.

So, it is deductively consistent as oppose to just being consistent in a model theoretic fashion is deductively consistent if and only if there does not exist a formula such that. Both formula phi such that both phi and this negation can be proven from the Hilbert style system notice again this is completely proof theoretic. So, we are not looking at interpretation and we are not looking at models now we are looking at consistency just from the point of view of the consistency of a formal theory if you like. So, if you cannot proof both formula and its negation from the given set of formulae phi then you some all say that your this set phi is consistent. And, in fact a consistent formal theory will be one in which not all formulae are theorems.

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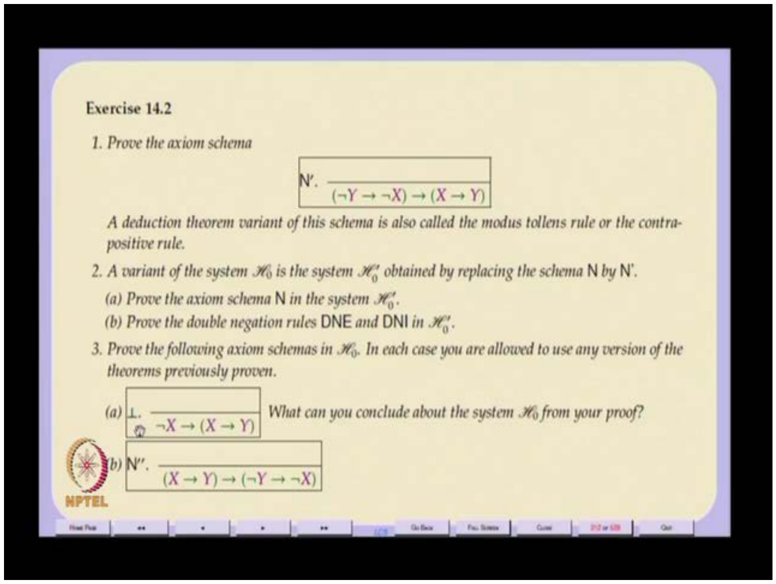

And, that follows from this fact remember this axioms of bottom axioms and then if you use the deduction theorem essentially. If, you can proof a so for some formula phi you can proof both naught phi and phi. Then, you can conclude any formula of phi. So, which means an in consistent theory is one in which every formula is provable. And, model theoretic what does that work out to it works out to to just saying that a set phi is inconsistent if and only if every formula is a logical consequence of that sense. Which, it will always be is phi itself conjunction of all formula and phi is always false then, that tells any way imply in any other formula and that is for all formula will be implied by that. So, the two notions go in sort of parallaxes so however so this is perfectly valid notion of consistence. And, it is a it just says that you should not able to prove a quantradiction in anywhere using just. So, if you think of your phi as set of axioms of our formal theory then you should able prove a contradictions.

So, they should so the other possible so here are equivalent statements. So, you should not be able to proof both the formula and its negation as valid proof theoretic consequences of set phi we should not be able to prove a single quantradiction like naught of phi of phi for any phi. Or simple you should not be able to prove every formula they should be at least one formula which you cannot prove. Then, your set phi consistency so all these are equivalent definitions of consistency and any or all of the may be found in books in various forms. But, let us live with this actually we can live with all these three definitions and there all equally valid. And, sometimes we can it is convenient to you once rather than the other.

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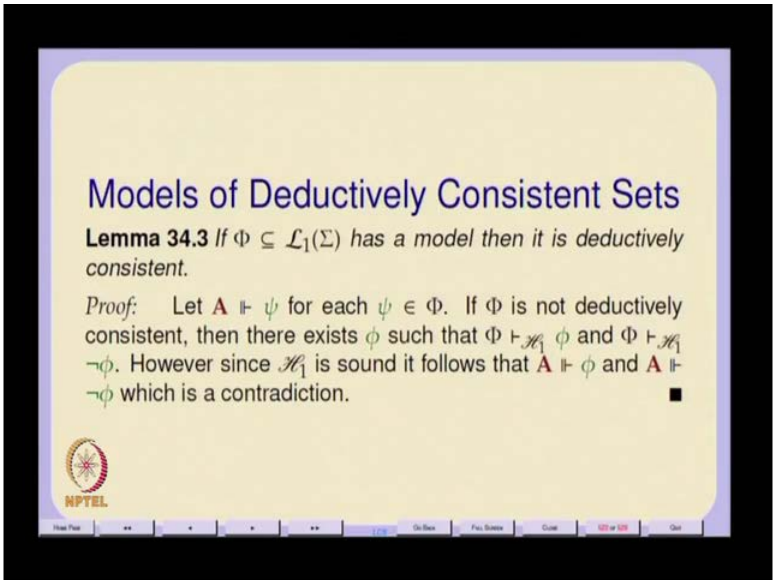

Now, going back to the model theory. So, what we are saying? We have to have this plane assurance that if a set of formulas has a model. Then, it must be deductively consistent if it is not deductively consistence then something seriously wrong with our entire frame work of logic. And, that is why this Lemma is important. It is important for the overall point of view of the whole of logic. This is how for the last two thousand years we being worry we have been looking at that mathematical theories. So, this lemma has a very (Refer Time: 23:08) proof however, that is that supposing you take this set phi which has a model A. That means under in A naught is a brown color every formula in phi is true in A. If, phi is not deductively consistent then there exits some formula phi such that both phi and its negation are provable in the Hilbert system. But, Hilbert system is sound that is something we have been seen which, means that if it is sound then it follows that whatever you prove it also valid. So, this model should satisfy both phi and its negations and that is not possible and so therefore our notion of deductive consistency is fine correct.

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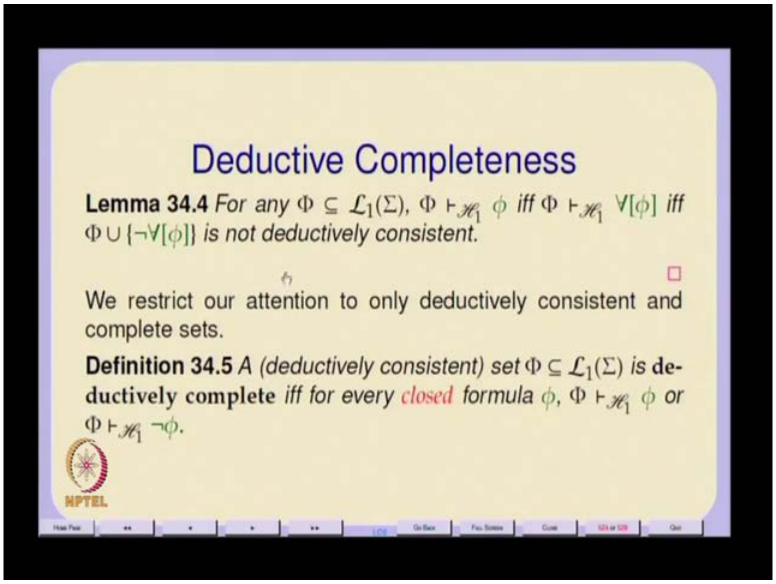

Then, just like we did maximal consistent sets I will talk about deductively complete sets. So, this it is really like a notion of maximal consistency. What was the notion of maximum constancy? That for any formula phi so capital PHI is maximally consistent if for every formula small phi either phi or naught phi is in the sets. And, this is however this Deductive Completeness for us comes as certain price and that is that. We cannot with the presence of pre variables we cannot just take phi and naught phi we have to take this universal collusion. So, completeness and from this notion of deductive completeness the completeness of the Hilbert strile system will follow and that is only for essentially for universally quantified formulae for closed formulae. So, for any set phi a formulae firstly you we know that even a phi has free variables, phi can be proven from capital PHI. If and only if, it is universally closure can be proven from capital PHI.

Otherwise what happens is you have all those problems about free variables and admissibility of substitutions and so on and so forth. So, if you can prove phi without worrying about all that then it means what you are essentially saying is. You do not care how the free variables are substituted it is still going to be true which is equivalent to say that if universally quantified it will always be true. So, for any set capital PHI small phi is I will is provable from capital PHI if and only if the universal closure of small phi is provable from capital PHI. If, and only if phi union not of the universally closure of phi is not deductively consistent this is so you just take this. So, this portion is something I have repeated several times before. But, this one this notion of we always went between logical consequence and unsatisfiability or inconsistency.

We took the negation on the right hand side and push it into the to find the given a set of formula and we said that is unsatisfiable or inconsistent. But, now what we are saying is that just taking this negation of phi itself and putting it into adding it to capital PHI may not make this deductively inconsistent. But, you take the universally closure of phi negate it and then put it in then you have some chance of being getting something that is unsatisfiable.

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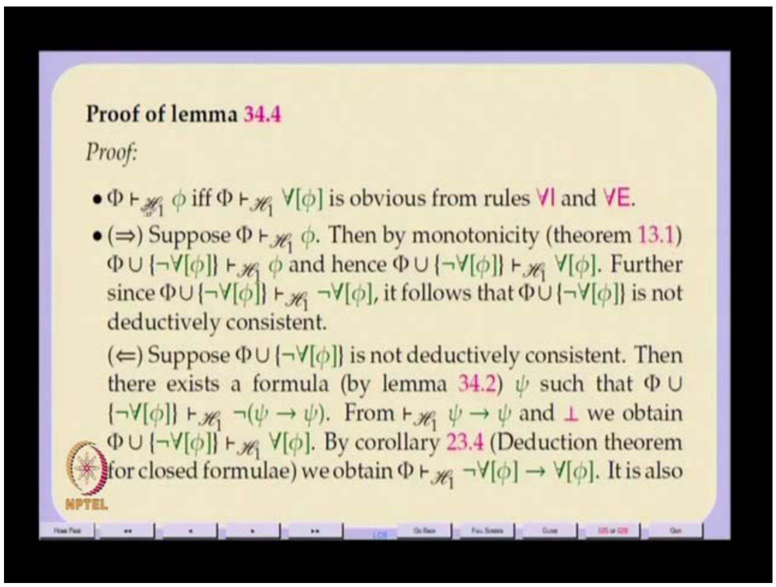

So, this so let us just look at the proof of this lemma. One thing is so the fact that this first part small phi is provable from capital PHI if and only if it is universally closure it is provable from capital PHI follows from these rules.

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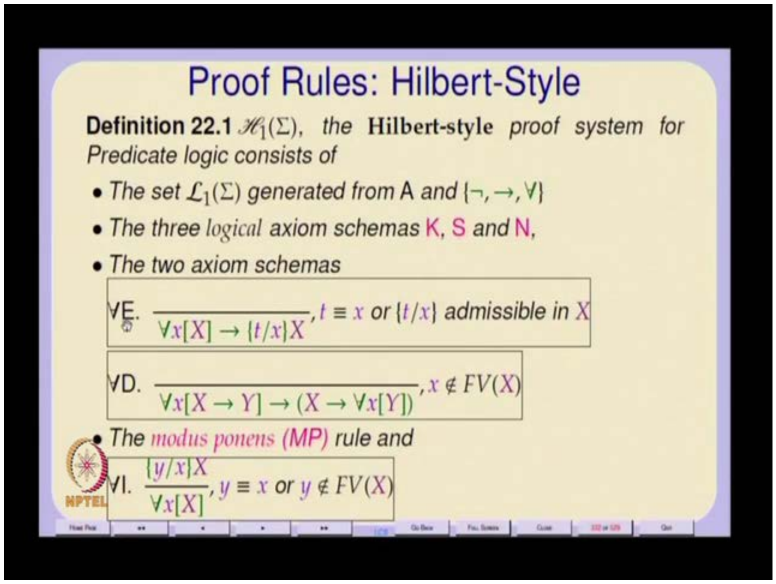

For all elimination and for all introduction. It follows from them and I leave it as obvious. So, which means so now supposing, you know what we have to show is that if phi is what we are going to show is that. If, phi union negation of the universally closure of phi is not deductively consistent if and only if phi is provable from capital PHI that is what you going to prove. So, let us assume that you can prove phi from capital PHI. Then, one thing is by monotonicity by adding extra assumptions you do not destroy that proof even if you added contradictory assumptions it does not matter that is.

So, let us add this naught of for all forms and I can still prove phi from the same proof holds basically this is by monotonicity. So what this so now, from this first part it follows that if I can prove phi from this set of assumptions. Then, I can also prove the universally closure of phi from this set of assumptions. In other words and further any assumption itself is provable from this set of assumptions so I have proven both universal closure of phi and then negation of the universal closure of phi. And therefore, it follows that this set phi union the negation of the universal closure it is not deductively consistent.

The other way suppose phi union naught of universal closure is not deductively consistent. If, it is not deductively consistent then there must there exists some formula. Which, we there exists some formula phi such that phi and its negation can be proven. So, there exists some formula psi such that phi union this proves naught of psi arrow psi. So, I may essentially use this characterization. If, so can I prove naught of psi arrow psi from this assumptions and then I can use this psi arrow psi is a theorem of both H naught and H1 it does not matter what it is, it is a tautology. So, it is it always holds and then we had this we have this bottom rule somewhere. If, you can prove a formula and it is negation then you can basically in for anything you like. So, by using this we obtain that I can prove the universal closure of phi.

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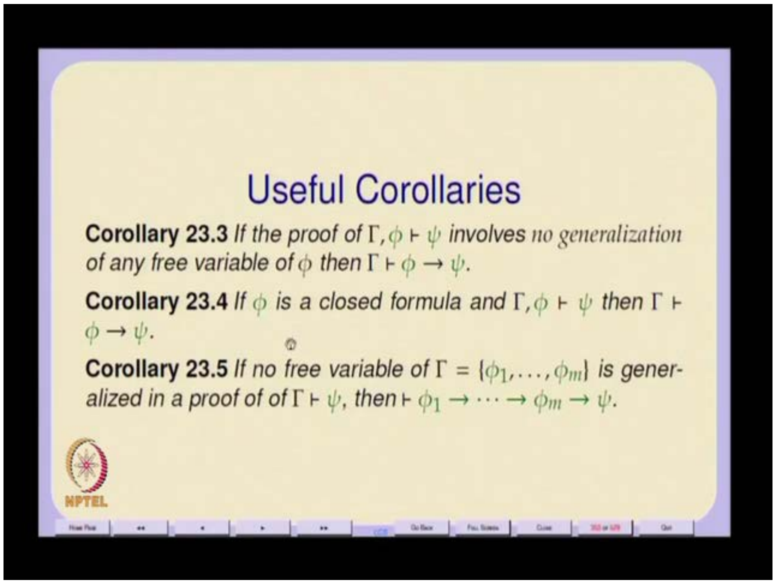

Then, if you remember when we did the deduction theorem we said that. If, phi is a closed formula then I can actually move phi on either side of the term style appropriately. So, I can use this corollary to show that and since we are dealing only with closed formulae there are no free variables. So, all those side conditions are no longer are not applicable the side conditions for all elimination and for all introduction are not applicable the deduction theorem is also applicable since we are dealing with only closed formulae. So, which means from phi I can prove naught of the universal closure of phi arrow universal closure of phi.

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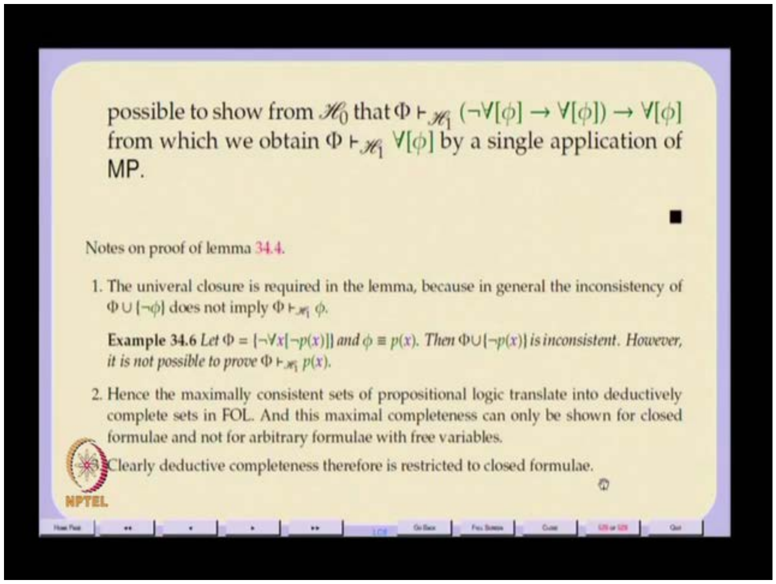

So, you then it is also possible to show by this I should have probably made this an exercise. But, I did not it is possible to show that actually since it is possible to show that if I can if naught phi arrow let us say naught psi arrow I mean that is the theorem that you can easily show using one of these derived rules that we have done. So, if you can prove that then by a single application of modespongence we have proved that the universal closure of phi can be proven from phi. So, the importance of that universal closure in first order logic as opposed to whatever we did in propositional logic is absolutely required. I mean that without supposing you did not have that universal closure then the general inconsistency you add naught phi it will not imply that. So, the if phi arrow naught phi is inconsistent then that does not imply that you can prove phi from phi. Because, you take this set capital PHI to be naught of for all x naught p of x and take small phi to be p of x.

Then, phi union naught of p of x is inconsistent because for whenever this formula is true this can under no valuation can this naught p of x be so phi union naught p of x is inconsistent. But, it is not possible to prove that p of x follows from naught of for all x naught p of x is it clear. When there is a lot negation going on but you guys are so this is essentially equivalent to saying that there exists an x such that p of x is true. And, saying there exists an x such that p of x is true and naught p of x that is inconsistent because, I can find some model which under any under any valuation that will be actually inconsistent.

However, I cannot move this thing around because of this free variable precisely for the constraints that were there in the deduction theorem for the same kinds of reasons of the use of free variables and so on I cannot prove this that p of x follows from phi. Because, anyway this is inconsistent so what happens therefore is that maximum what was the notion of maximally consistent sets in propositional logic translates into deductively complete sets in first order logic. But, deductively complete sets are deductively complete sets require universal closure and when you do universal closure there are no free variables and that is essentially like having a proposition they are all propositions.

So, they directly translate into essentially close sets of closed formulae and that is what and this is the maximal so essentially what we are doing therefore we are restricting ourselves to basically just closed formulae. What can we proven using closed formulae? And, our notion of completeness is also going to be restricted to that which is not a severe restriction because in any mathematical theory. What are you trying to do? When you prove a theorem you are trying to prove essentially closed formulae there are so you are essentially trying to prove closed formulae and if there are particular cases to be considered.

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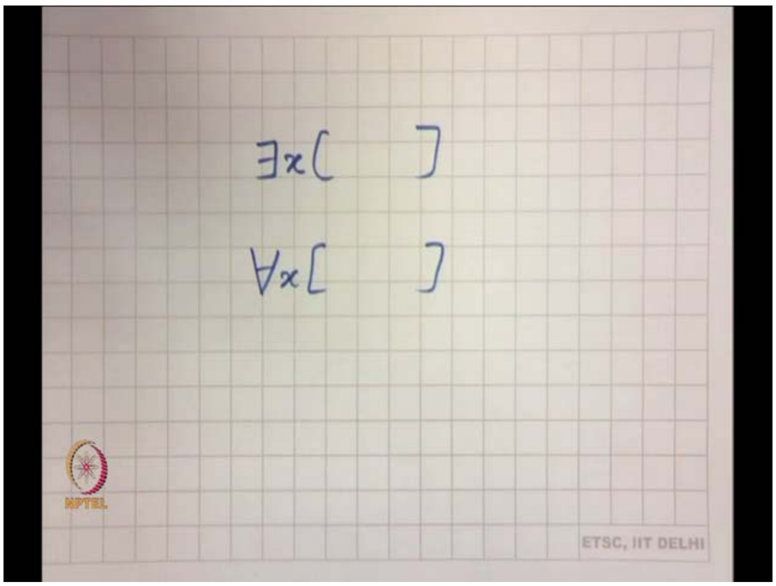

Then, you are essentially proving closed formulae of the form there exists something it satisfies some problem. Otherwise, you are trying to prove essentially universal form in either case you are essentially trying to prove closed from the point of view of the consistency of the theory and the completeness of the theory this is not a serious restriction.

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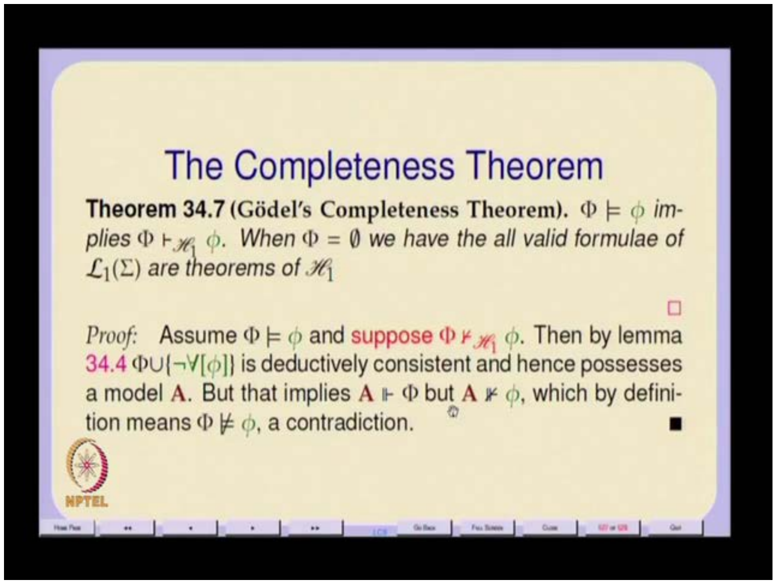

And, so which brings us actually to Gödel's Completeness Theorem. So, this just says that if phi is a logical consequence of capital PHI. Then, phi is provable from capital PHI. Which, is equivalent to now of course since, we cannot move things since the deduction since phi might have some free variables. And so, on you cannot moving it on either side of the turn style because of the restrictions posed by the deduction theorem the notion of use of free variables and so on so forth.

However, when this set phi is empty you what you are essentially saying is that every valid formula would be provable in H1. Which, is what you want the corresponding notion from tautologies is that of valid formulae and what you are essentially saying is that every valid formula can be proved in the Hilbert strile system. So, an actually after all that preparation this proof is actually truing. So, you assume that it is a proof contradiction. Assume, that small phi is the logical consequence of capital PHI notice that capital PHI can be an infinite set we are not putting any restrictions of finiteness.

However, for any proof you will be using only a finite number of assumptions. Because, your proof has to be a finite tree and if it is a finite tree then you are using only a finite number of the assumptions as leaf notes. So, supposing phi is not provable from capital PHI under H1. Then, what it means is phi union the negation of the universal closure of phi is deductively consistent. And, hence possesses a model A. But, that implies that every formula in capital PHI is true in model in this model A but, this formula phi itself is not true in model. Which, by definition means that this phi that means A is a model which shows that even the every formula in capital PHI is true the small phi is not true so it shows that small phi is not a logical consequence of capital PHI.

So, and that is a contradiction and it therefore follows that this assumption that phi is not provable from capital PHI. So, this once you have done this once you are translated all your notions from propositional logic and from model theory into first order of logic. It this the proof of this theorem becomes quite simple. So, now what we should look at actually is so. What is this completeness mean? This, completeness just means we have proven that there is a structure to mathematical theories. Supposing, you take the theory of partial orders for example then what we are saying is.

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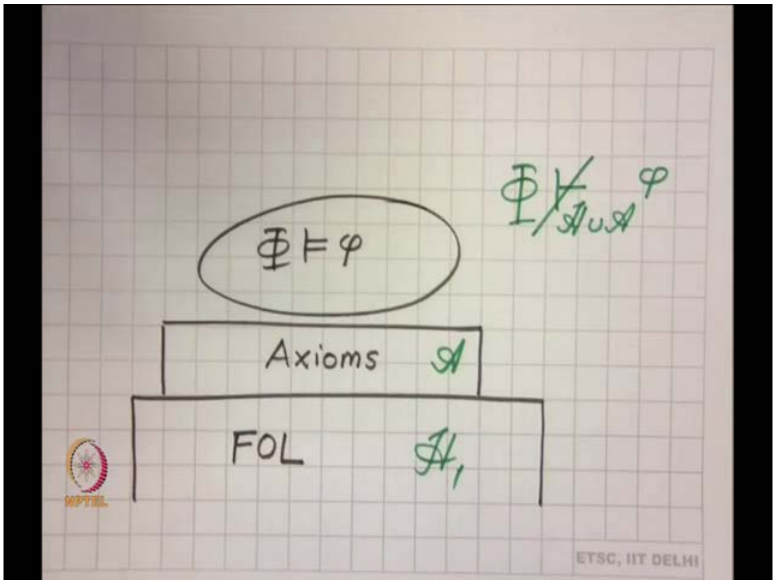

You take some you supposing you have created a new mathematical theory with some axioms. Then, what you are saying is let us look at the first order theory of that structure of that system of axioms. Then, you have a hierarchy of using reasoning from first order logic on top of which you have you will also add your axioms of your theory. So let us so if it is group's theory then you going to add the group's axioms. Then, it is a theory of monoids then you going to add the monoid axioms. If, it is Euclid in geometry then you are going to add the including geometry axioms. But, those Euclid in geometry axioms will reside on top on the axioms for real numbers there is what I am trying to say is that there is a modular structure in mathematics such that you actually build on one on top of the other.

So, you take any theorem supposing there is some valid formula here expressed supposing there is some valid form. If, this formula in this theory is valid but supposing you are not able to prove it supposing it is provable with these axioms. So, what are you saying? You going to take this is your Hilbert style axiom system. You, are going to take this set of axioms and now you are going to essentially try to prove theorems. So, you try to prove using H1 union A you essentially try to prove this phi and you find that it is not provable. Then, what does that mean? It means that your axioms are not sufficient the fact that you cannot prove a logical consequence. Which, holds in that theory in first order logic is not due to the insufficiency of the logical axioms but it is due to the insufficiency of the axioms you have used for that theory. That, is what this completeness says. What this completeness says? If, I am going to mean cannot put the blame for incompleteness in a mathematical theory in first order mathematical theory on anything to do with just the axioms of first order logic. I have to put the blame for incompleteness roundly on my axioms systems.

So, I have to look for may be extra axioms to complete it so you that is what this completeness means. Remember that we are showing that this completeness independent of any particular theory we have taken it just parameterized on sigma that is it. So, what does that mean? Your theory is also for have a certain modular structure in your mathematics and it is necessary to be convinced. And, so the importance of this theorem lies in the fact that what Gödel's essentially has said is that. If, you are finding that there is something logically valid formula that you cannot prove in your theory do not put the blame on the on first order logic. Put the blame on your axiom system you probably do not have enough axioms. Or, you do not have sufficiently powerful axioms which will capture all the logical consequences that you want. So, it was it is interesting that so, that is what completeness is that as for as first order theory is themselves are concerned the Hilbert style system is complete for particular theory is the incompleteness comes from the axioms of that theory and not from first order logic. But, what does happen in the larger picture is that. If, I look at the first order theory of topologies or neighborhood systems or whatever then very often properties that I am expressing are not first order expressible.

I may not be able to express it as a finite formula in first order logic that is problem exists in which case I may not be able to prove then do not know whether I am not able to prove because, I do not have sufficiently powerful axioms. Or because, I am not able express my properties in first order logic so the larger structure of mathematics or mathematical reasoning is that it just it does follow this hierarchy of reasoning system. So, what we are saying is that your basic reasoning system is basic first order reasoning system is complete. And, valid all the incompleteness that comes is either because of inexpressiveness or because you are the axiom system for the theory that you are building on top of this is not powerful enough. So, there is this was the first step Gödel has to take before he could prove of that. You know first order number theory is incomplete you know that is first order number theory is incomplete. Then, it was natural for Gödel to look for the reasons in first order number theory itself. Once he has proved this theorem he did not have to look at reasoning systems within first order logic. He had to look for it in number theory that is a clear stratification that stratification absolutely important otherwise nobody had that idea including Hilbert.

That there is a stratified way of looking at theories. Even though, Hilbert had given a complete axiom system for Euclidian geometry he did not fully realize that he had already a complete first order reasoning system on top of that he had he may not have had a complete theory of the real's. But, he had enough axioms in fact just the field axioms of the real's which basically delt with addition multiplication. What, you require for Euclidian geometry? You do not require more that addition and multiplication the field axioms in the properties for rehearse using just addition and multiplication were enough. And, you could build Euclidian geometry on top of that and prove that your axiom system for Euclidian geometry was complete.

He did not use any because you require addition because you want to add angles you have to look at interiors you want to look at line segments divide them up to look at between. And, well in the case of the similarities you looking at ratios in the case of the Pythagorean theorem looking at squares. But, both of them correspond to essentially multiplication so you as long as I had all the axioms related to addition and multiplication I was doing fine. I Just had to take care of ratios which for example the denominator should not be 0. So, in every theorem where there was problem of similarity you have to make sure that for example if one side of a triangle is got 0 length. Then, you do not actually have a triangle and you have to make that clear if one angle of a triangle is 0. So, all those things that the angles have to be greater than 0 this slide segments have to have legs greater than 0 which has put in all those conditions and then to ensure that you get proper triangles.

And, of course in the case triangles you should ensure that you are interior angles do not exceed around is less than 180 you should put that also always you do not have a really a triangle. And, once you add all those conditions rigorously you can and with the theory of rehearse need not be complete. It is just that the field axioms for rehearse should be there because those are the only operations you are using your first order theory of Euclidian geometry is going to use only those addition multiplication that is it. Squaring is just multiplication ratios are also essentially expressed as there exists some things are that when you multiply something you get something and so all those things can expressible and you could prove.

So, Hilbert could prove the completeness of Euclidian geometry but he did not realize that there is a stratified way of looking at it which Gödel realized. So, if you just look at the first order theory of numbers as just being built on top of first order logic. Then, Gödel Gardy's incompleteness which basically shows that just the finite set of axioms that you put on numbers is not sufficient to guarantee all that you can prove valid properties. I will not going to get his incompleteness theorem which is saying that will take us too far away. And, we do not have enough time for that but what I will do next time onwards is that I will just look at some simple basic first order theories. And, basically we will this kind of stratification so we will assume that our first order logic is complete and you just add this sort of axioms and we just say whatever can be proven this is the first order theory of that structure.