Logic for CS

Prof. Dr. S. Arun Kumar
Department of Computer Science
Indian Institute of Technology, Delhi

Lecture - 26
Skalemization

So, last time we looked at normal forms and we also looked at actually the notion of Herbrand algebra. So, this specifically goes through this so we have this usual quantifier movement and you can move quantifiers in such a way that. You, can construct what are known as prenex normal forms.

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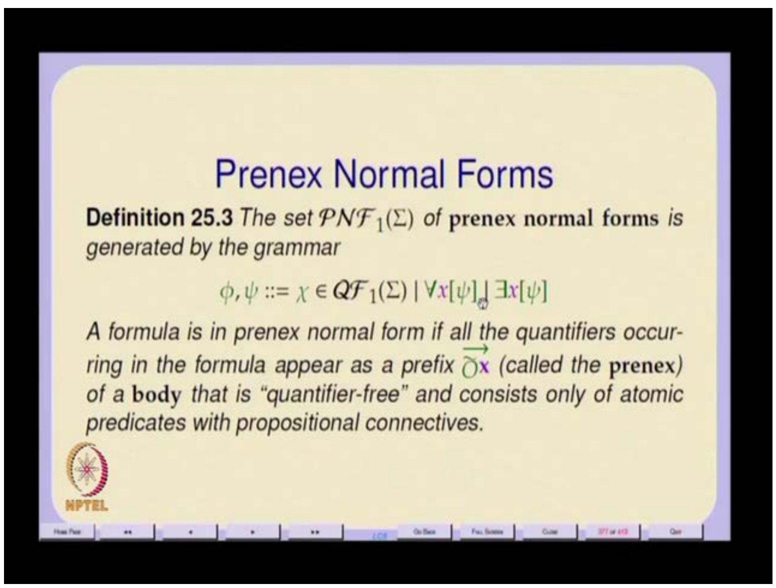

Which, are essentially mean they are up to logically equivalence you can transform any first-order logic of formula into a form. In, which there is a sequence of quantifiers followed by a body which is quantifier free so this is important.

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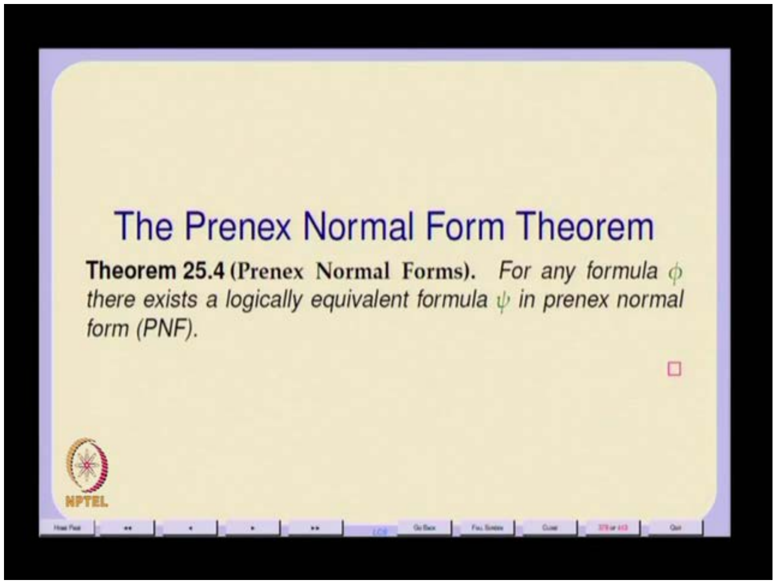

And, then we have the Prenex Normal Form theorem. Which, essentially states that you can convert any logical any formula phi into a logically equivalent formula psi and Prenex Normal Form.

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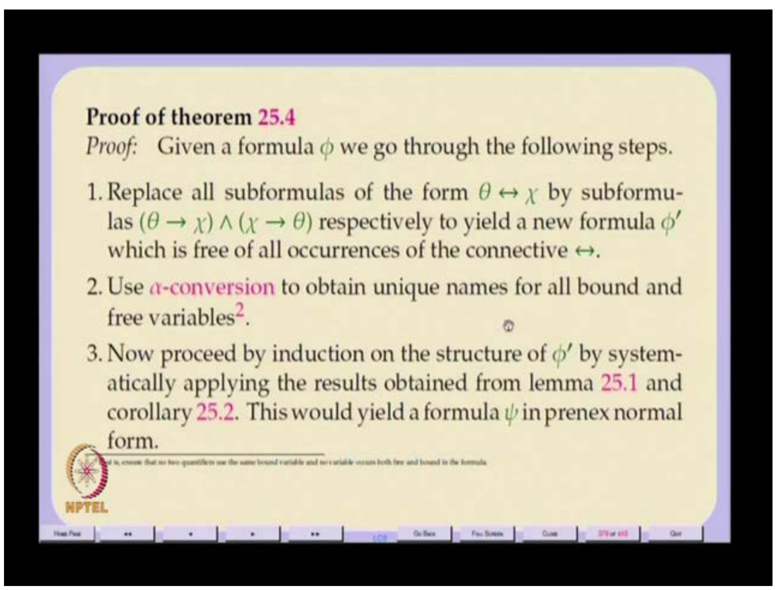

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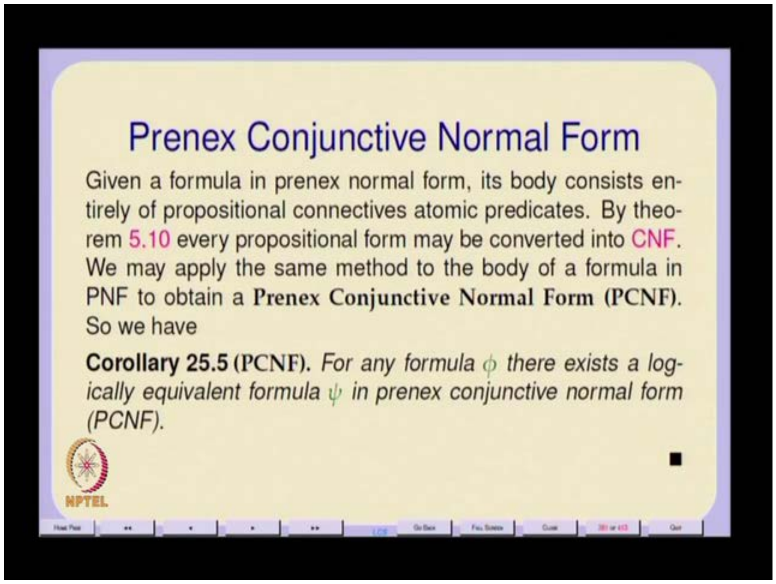

And, of course since the body is quantifier free it can be transformed into a Conjunctive Normal form essentially the way which propositions can be converted transformed into conjunctive normal forms.

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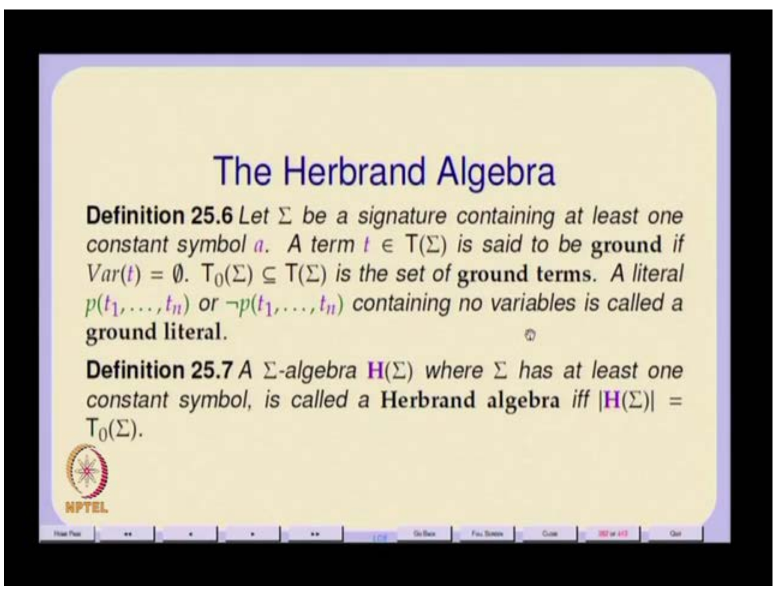

Then, we looked at the notion of a, Herbrand Algebra where I said that basically we are looking for models within the language of terms itself.

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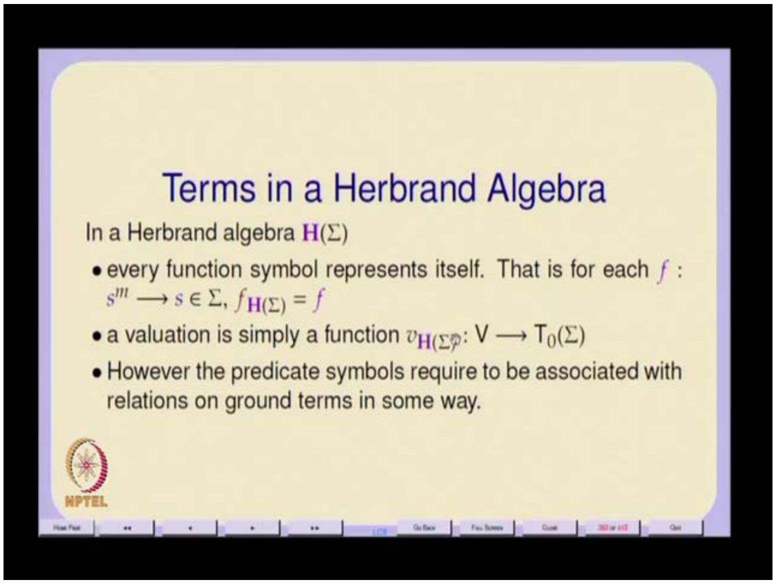

So, that is why so essentially the Term Algebra itself is your carrier set or your domain of this course. And, the functions automatically give you new terms from this set according to their arity and so on. And, the only thing that is left on specified is really the meaning of the predicate symbols the atomic predicate symbols.

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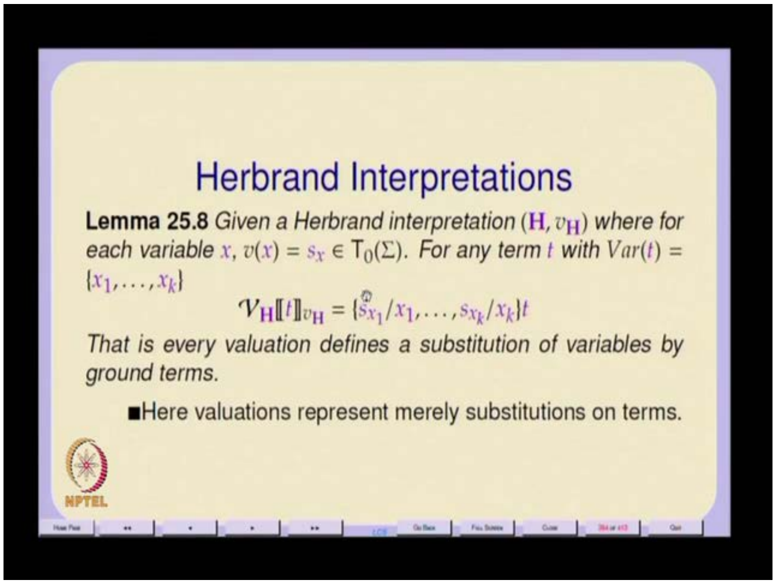

And, so the notion of an interpretation therefore essentially depends upon what kinds of. So, firstly so the notion of an interpretation in this case then works out to just a substation of the appropriate terms by terms for the variables.

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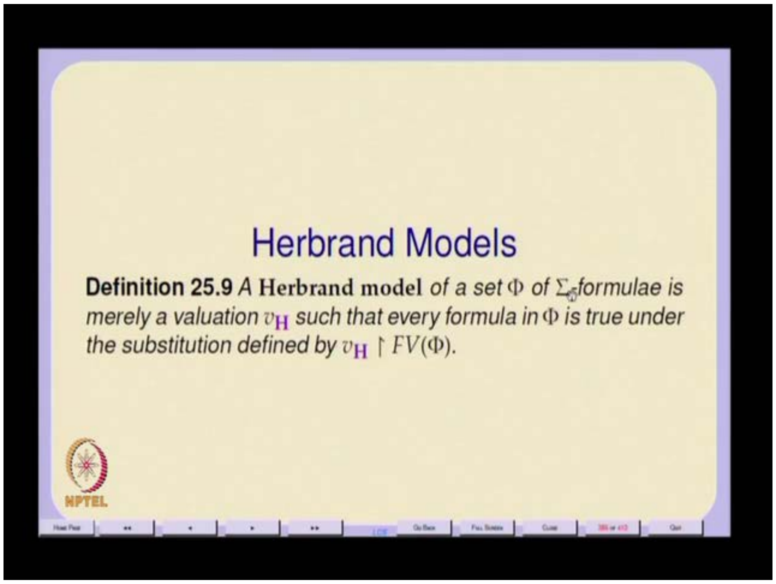

And, the notion of model essentially reduces to lat of finding an appropriate valuation or an appropriate substitution actually that is enough. Since, we have got only the finite number of atomic predicate symbols in any formula or any finite set of formulae. Though of course if the set of formulaes infinite you might have an infinites collection of atomic formulae also.

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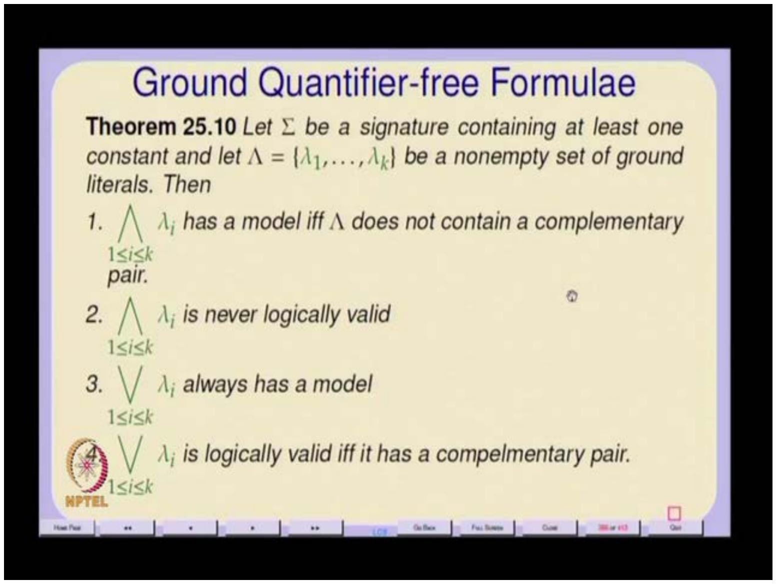

So, we had the notion of ground term so this requires that there should be at least one constant in signature. And, if is not a constant in the signature what we do is we just expand the signature to include the constant symbol. And, what we essentially showed was that for literals so ground literals. So, that means these lambda i’s are essentially atomic formulae or negations of atomic formulae in which there are no variables. And, the parameters of the formulae are ground terms of the Herbrand algebra.

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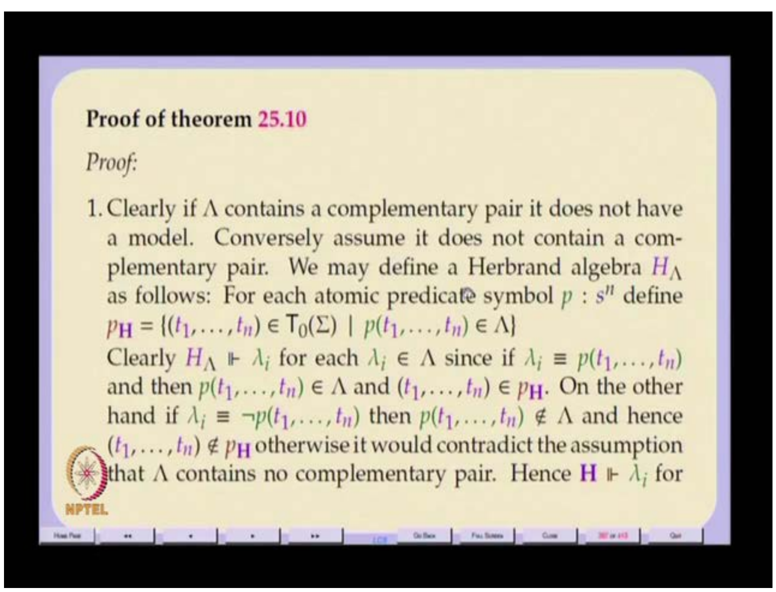

So, there are no variables at all they completely variable free so this first of all these are. So, we had this theorem on ground quantifiers. Which, clearly gives you a gives us a syntactic characterization that is important thing it gives us a syntactic characterization of logical validity of and of it will it gives us a syntactic characterization of unsatisfiabilty. Essentially here, in this for and of finite set of form literals. And, here it gives you a syntactic characterization of validity for an or of a finite set of literals. So, next today will do what is known as Skolemization which essentially comes from the name of Foresscolem who came up with this whole thing. And, it will see its relationship to a Herbrand algebras. So, the first thing we have here is a Skolem normal form theorem say supposing phi some in some prenex formula. Which, is of this form that means there is a sequence of universal quantifiers. Now, we are not a quantifiers are not any general quantified q. So, there is a specific form in which we are looking at these formulae.

So, there is a sequence of universal quantifiers so this is like for all x1 for all x2 dot for all x n followed by a the say single existential quantifiers. So, there exists y and this psi is of course quantifier free and it can contain these variables x 1 to x n and y as free variables. And, of course we will assume that these are all distinct variables. So, and will assume that psi is quantifier free. So, this is essentially a prenex form in which there is a sequence of universal quantifiers followed by an existential quantifier. Then, what will do is we will expand this signature with a new function symbol g. Which, has which is of arity n where this n is the same as this n x 1 to x n. So, you can expand this signature to sigma to sigma g such that then. And, what you can do now is instead of you can replace all free occurrences of y in the quantifier free formula psi by g of x 1 to x n. So, take this formula for all x for all x 1 for all x 2 for all x n this formula phi prime. For, any model of this formula is also a model of phi. Because, if phi any way is a sigma formula therefore it is automatically sigma g formula also so it is. So, that way there is the fact that you expanded the signature keeps phi also in sigma g so, there is no problem. So, and every model of phi can be expanded to a model of phi prime. Of course, phi prime is not in the phi prime is the sigma g formula it is not a sigma formula. Because, of the addition of the extra function symbol g.

So, what the Skolem normal form theorem essentially says that is that you can always so every model of this phi prime. Where, the existentially quantified variable y is replaced by a function of all the universally quantified variables that precede that existential quantifier. So, that is the connection so this so, you can be replaced by a g of x 1 to x n. Where, x 1 to x n are all the universally quantified variables preceding this existential quantifier of y. So, you can right so this is what so this is essentially like what we are saying is the analogy is readily there for example in the theory of groups right I mean. So, if you take a group axiom like for every x in the group there is right universe. Here what, we are saying is that you are formulizing that there is a right universe by a function. Which, is essentially the inverse operation. But, you have to remember one thing in sigma g these two formulae phi prime and phi are not logically equivalent.

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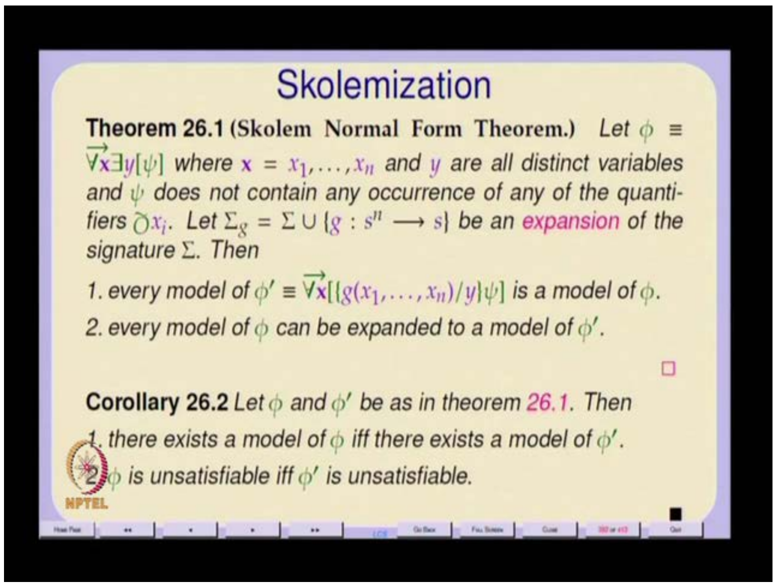

So, what we are doing now is we are getting away from logical equivalence. And, we are getting into the domain of pure model theory. So, the existence of a model so, all that we are saying is that if there is a model for phi prime. Then, the same model can also be used for phi satisfy phi but phi prime is not logically equivalent to phi simply. Because, it is quite possible let us go back to example of that group and group inverse it is possible perhaps for me to construct something very much like a group. Except that it does not have unique inverse right universe in this system or an element might have more than one inverse let us say. So, in which case that inverse operation is no longer function but, all that you are saying is that every group is a model of that axiom and not necessarily the other way. And, that but we are saying something more important you are saying that I mean my search for models might go anywhere. But I just have the guarantee therefore that if this does posseses a model then that also posseses a model and, in fact same model can be used. And, you take any model of phi you can expand it to a model of phi prime basically that is. So, I can always find a system with unique universes which will be a model of which will be a model of phi prime.

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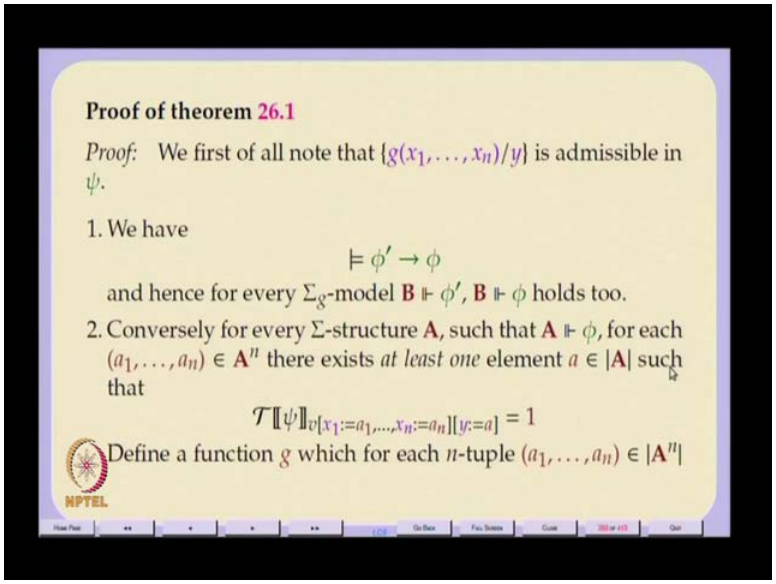

So, this theorem is not actually I mean it is not very hard to prove. But, first of all we should note that g of x1 to xn for y is admissible in psi. And, the other thing you have to note is that this phi prime logically implies phi as I said they are not equivalent but, it logically implies phi. So, which means that for every sigma g model of phi prime phi also holds. Now, if you were to take this take the converse the second part every model of phi can be expanded to a model of phi prime. Then, essentially you consider some model of phi and there exist one at least one element. So, what we are essentially saying is you take this formula phi there exist at least one element a for every anti opel a1 to an. Such that, phi in which in which a1 to an replace x1 to xn. And, a replaces y would be would be true.

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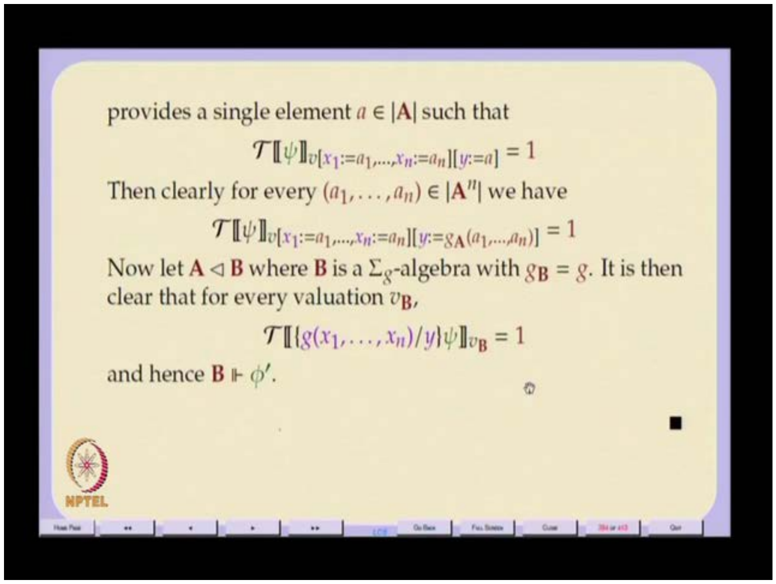

So, then you define this you just chose an arbitrary element. So, in a certain sense you using the axiom of choice in axiom if you were to think of it as part of set theory. And, you just define a function. So, you chose an arbitrary element a that satisfies this predicate psi for each a1 to an. And, you chose one arbitrary element for each ant opel and define this function g. And, then it is clear that you expanded the signature by this and it is clear that truth value also holds. So, essentially therefore you take any model of phi prime that is the model of phi. So, now that the fact that there exist at least one and is clear from the function g. Remember that in our signature all are functions are totals. So, there is no partial definedness in anywhere so the g actually g is a total function so for every anti opel a1 to an there is an element a. Which, will satisfy psi for example and also wee are shown that basically we can expand the signature from of and create a model of for phi prime using this function g.

But, what this means and the corollaries are actually more important is that there exists a model of phi if and only if there exist a model of phi prime. And, conversely phi is unsatisfiable if and only if phi prime is unsatisfiable. So, that is so what it means is that now is there we are going to work with this phi primes. So, remember the phi prime is not logically equivalent to phi. So, we come to the notion of the Skolem Normal Form in which we do not preserve logical equivalence. But, we preserve satisfiability so the some books. Which, use the term equi- satisfiable. So, that is like saying that that is essentially this part two of this corollary two. We, say is that phi is satisfibale if and only if phi prime is satisfiable.

So, here is an equivalence relation which is well finer than logical equivalence. So, if they were logically equivalent they would satisfy exactly the same models. Whereas, what we are saying now is that they are not logically equivalent. But, the quest for models is such that you can replace the quest for models for one formula by the quest for models for other formula.

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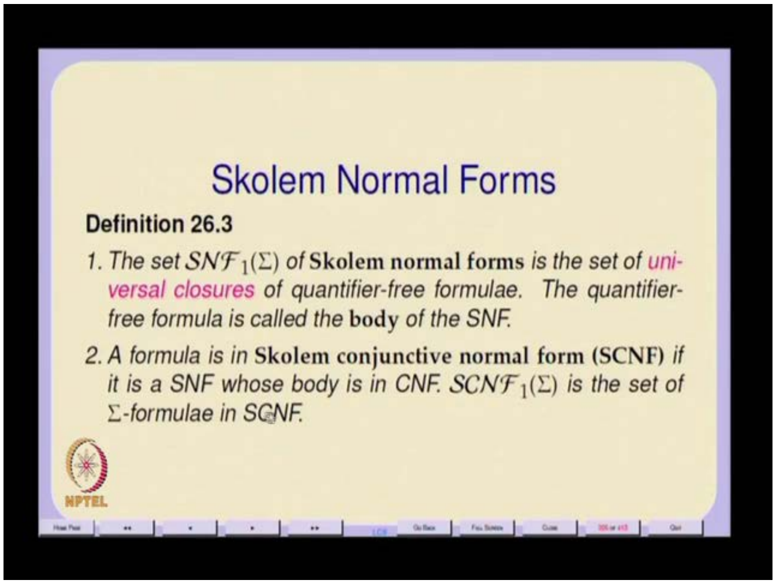

So, this so what is known as a, Skolem Normal Form. So, what this, what this also means is that you can get rid of these existential quantifiers thorough a skolem normal form. If, you get rid of this existential quantifier then you can define equi-saisfiable formulae. And, so in a skolem normal form we essentially consider only universally closed formulae. So, the set of skolem normal forms is the set of universal closures of quantifier free formulae. So, you take any quantifier free formula essentially using proposition connectives and just close it on all the variables universally close it on all the variables and that is a skolem normal form. So, and of course the quantifier free from body can be written in conjunctive normal forms so you have a skolem conjunctive normal form right.

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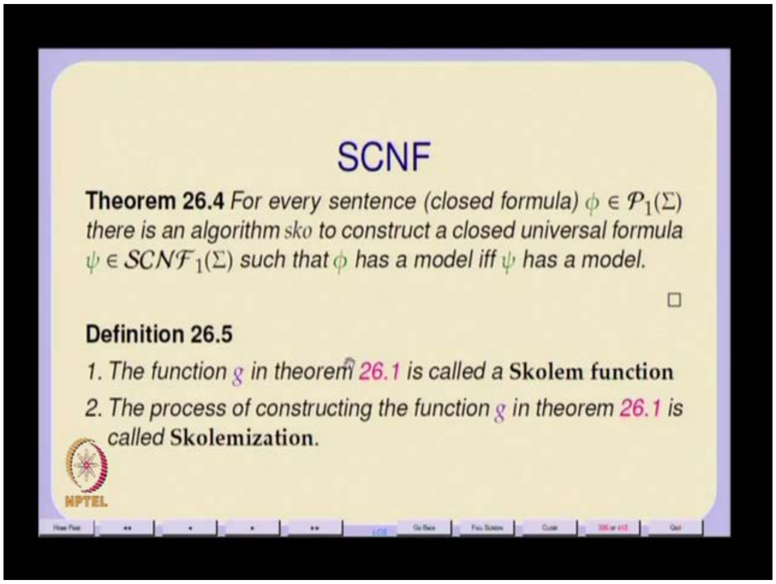

So, here is a simple theorem for every sentence of every sentence of first-order logic there is an algorithm. Which I am going to call a function sko to construct a closed universal formula psi. In skolem conjunctive normal form such that this formula phi has a model if and only if psi has a model. And, the algorithm is actually trivial you just take the formula phi transform it to prenex conjunctive normal form and what you do you create what are known as skolem functions.

So, that that function g that we created in terms of all the universal quantifiers preceding it is called the skolem function. So, what you do you take a formula in prenex conjunctive normal form reading from left to right. Read it from left to right take the first existential quantifier and skolemize it. When, you skolemize it what you are doing is you are adding an extra new function symbol. And, of arity equal to that arity of the preceding sequence of universal quantifiers. And when you do that and when you replace all occurrences of that existential existential quantified variable in the body by the corresponding function. Then what you get is you have a longer sequence of universal quantifiers. And, then you probably have another existential quantifier in which case you skolemize it in the same way. So, if you started if the first existential quantifier occurred in.

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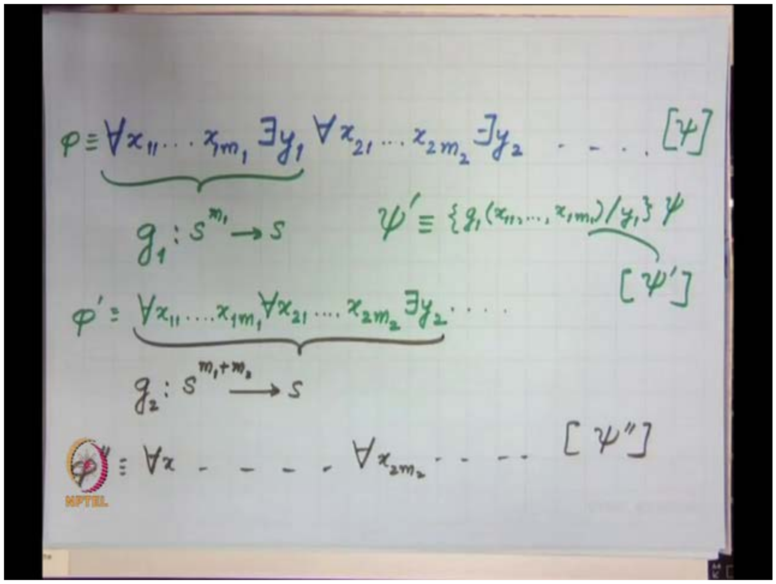

So, essentially what you are saying is so, if I have for all x1 to xm1 and there exist y1 then I have for all well i should probably 1m1 x21 dot x2m2 there exist y2. And, so on and so forth my corresponding skolem functions will essentially be firstly a skolem function g1 of sort Sm1 arrow S. And, this formula phi by this replacement so there is some body here somebody psi here, it becomes a phi prime which looks like for all x11 dot x1m1 for all x21 dot x2m2 there exist y2 whatever. And, the psi is replaced by a psi prime where psi prime is just equal to psi in which g1 of x11 to x1m1 replaces y1.Once, you done this you get this phi prime and essentially what you are going to do now is you are going to take you are going to skolemize this y2 by a function g2. Which, has a, sort Sm1 plus m2 arrow S. A1nd, you are going to get a formula phi double prime which is essentially have this up to for all x2m2. And, then psi double prime where that psi double prime is essentially a psi prime in which all free occurrences of y2 are replaced by g2 applied to these m1 plus m2 arguments. So, you can sequentially from left to right skolemize each existential variable one by one. And, you can actually create the final formula let us say psi. Which, will consistent only of universal quantifiers and then a body which is in conjunctive normal form. So, this process of constructing these functions is called skolemization.

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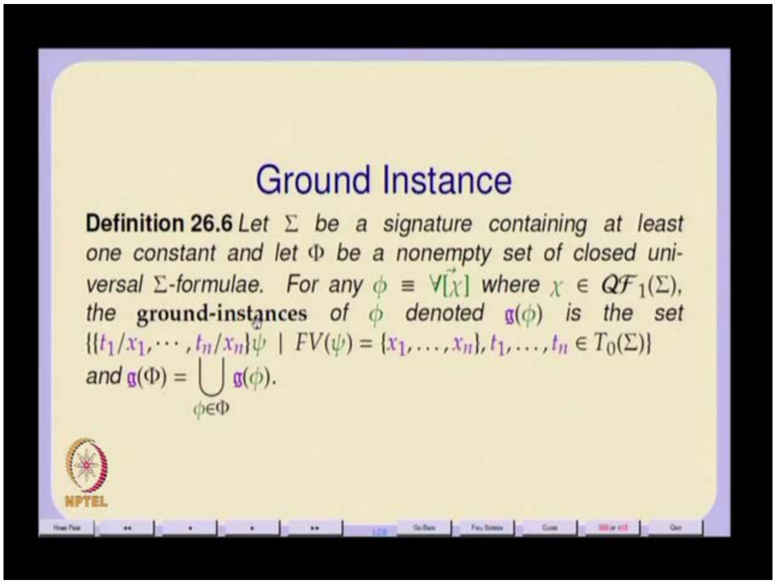

Now, what happens we look at our Herbrand algebras and we look at the notion of skolemization. So, what this means is that up to the existence so, the problem of satisfiabilty. Therefore, can be restricted to just universally closed predications in skolem normal form. And, if you need it skolem conjunctive normal form. So, now what will do is let us look at an interesting theorem by Herbrand. So, you take any signature and you take any formula phi I can talk about the Ground Instances of phi. Where, the ground instances of phi which I am going to call it g of phi. Where, this g is a gothik's is a gothik g. So, essentially what we are saying is you take all the free variables in psi, phi is of course of the form universal closure of this in this is an this is a mistake in I should not have used psi here this should be chi here, and this should be free variables of chi. So, essentially what we are saying is replace all free occurrences of the free variables in psi. Let us say x1 to xn by terms t1 to tn respectively such that t1 to tn do not contain any variables.

So, they are they are what are known as ground terms. So, what you essentially getting by ground instances is a collection of formulae that essentially look like propositions. Basically, they are like propositions our predicates are essentially parameterized propositions. You, replaced all the parameters they appropriate terms which do not contain any variables. So, they become essentially like propositions of indexed by indexed by the actual parameters. So, now so for each universally closed formula which is existential quantifier free for each universally closed formula which is existential quantifier free you find the ground instances for all over all the terms of the algebra. And, for any set of formulae of this kind while you just take the union of all the ground instances. So, this is so will call this g of phi that is ground instances of phi.

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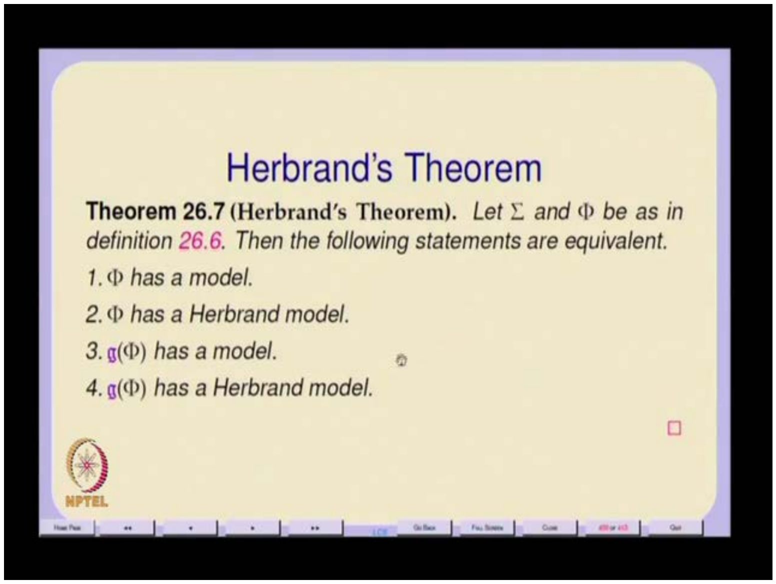

So, now we have an important theorem which says that lets sigma be any signature. And, actually you want sigma to contain a constant at least because otherwise there are no ground terms. If, there is no I mean there are only terms with variables in them then there are no ground terms so sigma should have a constant. But, if necessary we just expand the signature sigma arbitrarily to include a constant include at least one constant. Let, phi be some set of formulae which do not have existential quantifiers anywhere in them.

They, were universally closed on a each formula in capital phi is a universally closed formula over a quantifier free body. Then, the following statements are equivalents phi has a model phi has a Herbrand model. The ground instances of phi has a model and the ground instances of phi has a Herbrand model. So, the significance of the significance of consisting considering only skolem normal forms is that normally in any first order in any theory that you are talking about. the axioms of the theory are essentially universally closed formulae in a skolem normal form.

So, you look at your group axioms you look at axioms for number theory whatever all of them will be axioms of the form. Some, universally quantified universal closure over essentially all the free variables of the of a body which is quantifier free. And, now there is given essentially a set of axioms you want to know whether this set of axioms has a model. And, for that what Herbrand's theorem essentially says is two, fold. Firstly given a set of axiom you do not need to look at the space of all possible mathematical theories to find it. If, there is a model anywhere in that space of mathematical theories. Then, there is a Herbrand model also. Secondly what it says is that I do not need to look at satisfaction from a Hilbert system proof theoretic point of view. And, prove everything all I need to look at other ground instances. And, in fact the ground instances in the Herbrand model that is it. So, I can restrict my search to essentially just ground instances of the formulae within a by just looking at the terms. So, it is a completely syntactic it just completely syntactic quest. Which, does not have to look at any abstract algebra's at all so this is being this is an important theorems.

So, let us try it and what we are essentially saying is that all these statements are equivalent. Which, means essentially your quest for models comes down to just checking whether g of phi has a Herbrand model. And, so we have to proof all these so there are four statements which are meant to be equivalent. But, what happens is you can see that, if phi has a Herbrand model then it definitely has a model right so that is. So, similarly if g of phi has a Herbrand model then g of phi has a model so two implications are out of these are obvious. If, phi has a model and every formula in phi is in Skolem normal form whether universal closure over all the free variables of the body of the formula. Then, you phi has a model then clearly g of phi has a model all the ground instances of phi have a model because it is a universally closed formula. So, for all possible substitutions of terms for the variables of the body it should be true. And, so therefore one implies 3 and that is also true here. By, the same sort of argument to implies 4.

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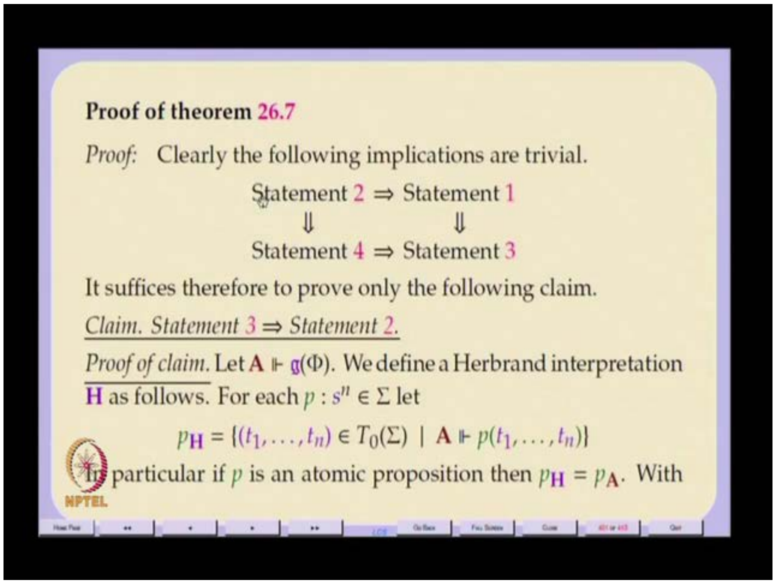

So, finally what it means is that you have got this Statement 2 implies Statement 1 Statement 2 implies Statement 4 Statement 1 implies Statement 3 Statement 4 implies Statement 3. So, in order to prove that all 4 Statements are equivalent we just need to prove that, Statement 3 implies Statement 2 then, you will get two cyclic triangles. So, that you can traverse so that your implications and transverse all the way. So, that greatly simplifies our problem so, he has a, claim that Statement 3 does imply statement 2. So what we are going to do now Statement 3 is essentially says that the ground instances of phi has a model. And, we are going to prove that therefore phi has a Herbrand model. So, given a model of the ground instance of phi. We, have to construct essentially a Herbrand interpretation to show which will be a model of phi.

So, assume that the ground instances of phi have a model. Let us say some algebra A. Notice, that it is an its brown in color right I mean I cannot assume anything arbitrary. I mean I have to assume something totally arbitrary this A may not be Herbrand model it could be a Herbrand model also. But, it may not be a Herbrand model because we do not know for a fact that a, Herbrand model exist. All, we know is that a Herbrand model exist. So, this A is some structure with a same signature and the signature of course contains at least one constant. So, we define a Herbrand interpretation as just follows. So, for every so of course in the case of a Herbrand algebra the terms anyway they are interpretations are already redefined. It is only the predicates the atomic predicates was interpretations need to be defined as relations on the terms.

So, for every ground tuple t1 to tn by the way t1 to tn has to be ground down. Which, I have not as specified its ground every. So, for every atomic predicate symbol p that occurs anywhere in phi in the formulae phi. I, look at what I mean if a is a model for all formulae of phi then it its true for all these tuples also it is true for some tuples. So, I take all those n tuples and I include them in my interpretation of ph, p h is the corresponding relation on the Herbrand algebra. So, in particular if p is an atomic proposition of course then by atomic proposition I mean that p is a 0 relation. So, which means that it only has a truth value true or false then, I assign p the same truth value as in case of assigned in a.

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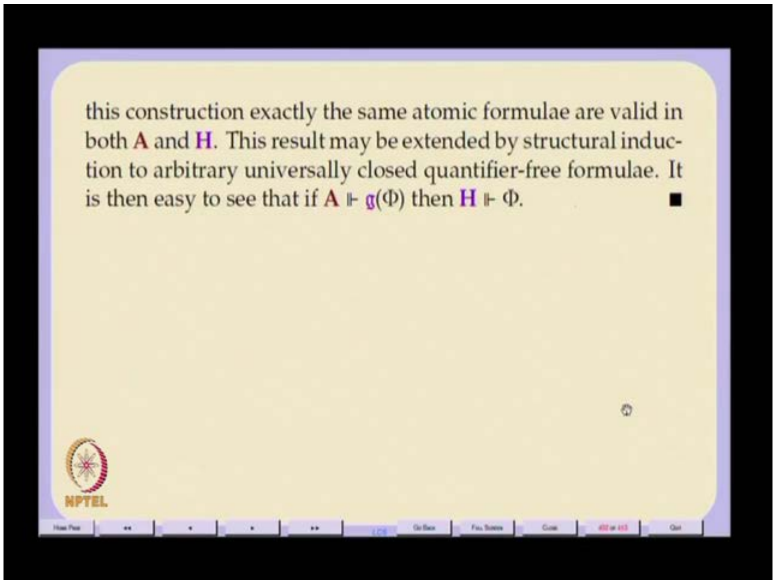

And, with this construction exactly the same atomic formulae are valid in A and H. And, therefore you can just use structural induction to show that H under this interpretation of atomic formulae is a model of g of phi and therefore also a model of this Statement 2

So, phi has a model so, this is an important theorem and in fact this is the theorem which actually makes say things like logic programming possible it also restricts all our attention to I mean it focuses our attention from arbitrary extract mathematical structures. So, that we can look at more concrete terms just terms constructed in the Herbrand algebra. So, we need to focus exclusively on just the formation of terms in this language of a, first-order logic.

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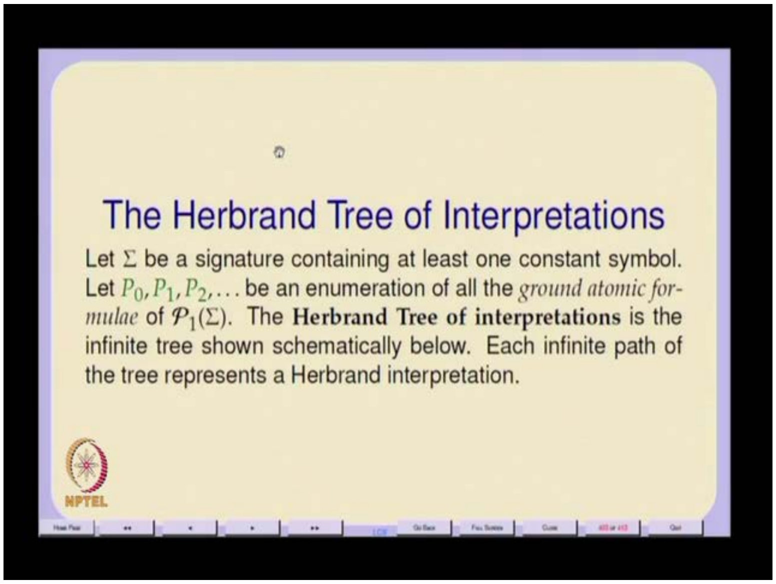

So, let us look at so what happens now so we are going so we need to restrict ourselves only to Herbrand algebra's. And, we need to restrict ourselves only two terms and in fact that is what mathematicians have been doing for centuries. Now, they constructed some algebraic system there some mathematical system and they looked at all the terms of mathematical systems that is it. And, in that sense they were focusing only on the Herbrand algebra. And, they were coming up with essentially the theorems based on the patterns of these. So, an algebraic theory would essentially based on the patterns of the terms. So, if you look at arbitrary structures like so if you done if you done some number theory. For, example it is, clear that if you are to take say the natural numbers modulo some prime number p. So, you essentially take n quotiented by p by where you have congruence modulo p in a certain sense. So, what a standard number theoretic theorem would say that by the way this you take the integers, the integers under addition form only a monoid. But, you take the integers modulo p where p is a prime. Then that forms a group under addition if you do not believe me go home and, work it out. And, in fact what you can do is it is possible to get an isomorphic structure of zp, z p is integers modulo p it is possible to get an isomorphic structure for zp. By, just choosing a group by just choosing a single symbol A. And, putting in the operations of let u s say concatenation. And, essentially you are able to generate all the elements from essentially from 0 to p minus 1 as this putative multiplication operation. Which, is just concatenation of this element with itself. The 0th concatenation of this will give you the essentially an empty string if you like. And, that empty string is identity element of the group. And, of course it will circle in multiples of p right it circle through but till you come to p you have generated p elements. So, essentially A raise to i where A is just some arbitrary symbol A raise to i denotes the equivalence class i. Where, i is lies in the range 0 to p minus 1.

So, any so, the group zp can be completely simulated by just using a, symbol A and concatenating to itself. So, you have so that this is a transformation of group operations from zp to this to this essentially string of symbols. Where, the moment you so A raise to p plus 1 in this concatenation would give you A again I mean that is that is what will happen. So, that is an equation of the group which you will. And, essentially for every i Ap minus i is a inverse Ap minus i of course is equal to A raise to minus i also I mean in this in the same thing. So, you so what it means is to play around with zp I do not need to consider abstract number theoretical notions I just need to consider a single symbol. And, that and its concatenation with itself up to p so that it cyclic so cyclically generates the group. That, is very much like just looking at Herbrand interpretation . So, you just you have a more concrete thing if you look at numbers as being abstract you have a more concrete thing in terms of strings. Which, will give you exactly the same kinds of properties. So, all the properties of zp will be captured by this string concatenation by this group of on a single symbol A. So, then let us look at what is the notion of a Herbrand Interpretation.

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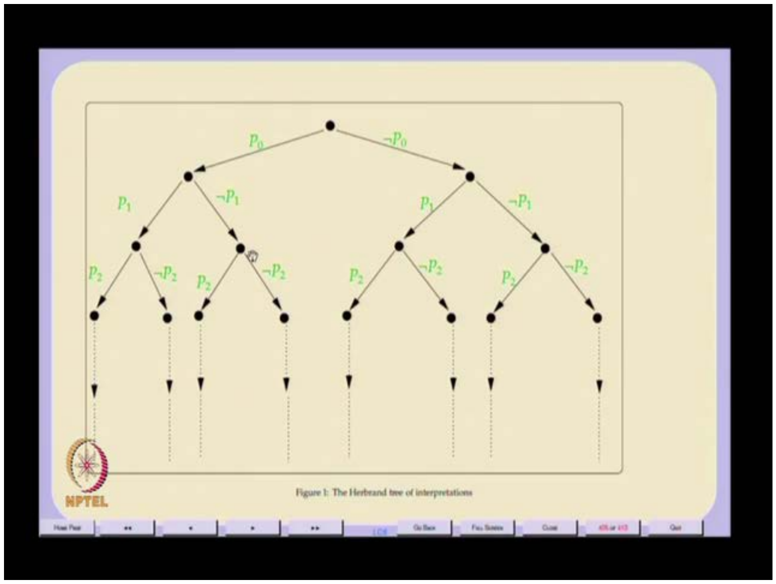

So, what we are saying is so for every formula which has a model there is a Herbrand model. And, of course in the case of a Herbrand model basically you have to find out a Herbrand interpretation. A Herbrand interpretation means that you just basically have to assign to all the predicate symbols appropriate relations in the set of ground terms. So, a Herbrand interpretation is essentially this tree so assume you got you you got your atomic predicates symbols P naught. And, you got your ground terms you assume any enumeration of the ground terms. And, you got your atomic predicate symbols small p's they are all small p's. Now, for each small p and of an appropriate clarity and collection of ground terms I still have your t naught sigma is countable infinite. You, have in your sigma a finite collection of atomic predicates pq etcetera.

And, I can take all the ground atomic predicates which are essentially like atomic propositions with index by the tuple of ground terms. That, set of all ground atomic formulae is still countable infinite. And, therefore capable of enumeration so I will call that enumeration capital P naught. So, whether this is this capital P naught means that it is some small p with a tuple of appropriate tuple of ground terms in violet. So, I have this if I construct this collection this collection is going to be countable infinite. And, therefore it has some enumeration so I take this enumeration. And, if I construct this tree such that they will I for each of this P naught the left edges labeled P naught. And, right edge is labeled P naught. And, I and I go through this enumeration then I get essentially an infinite tree. Such that, each infinite path in this tree corresponds to a certain assignment of relations to the atomic predicates in my signature sigma.

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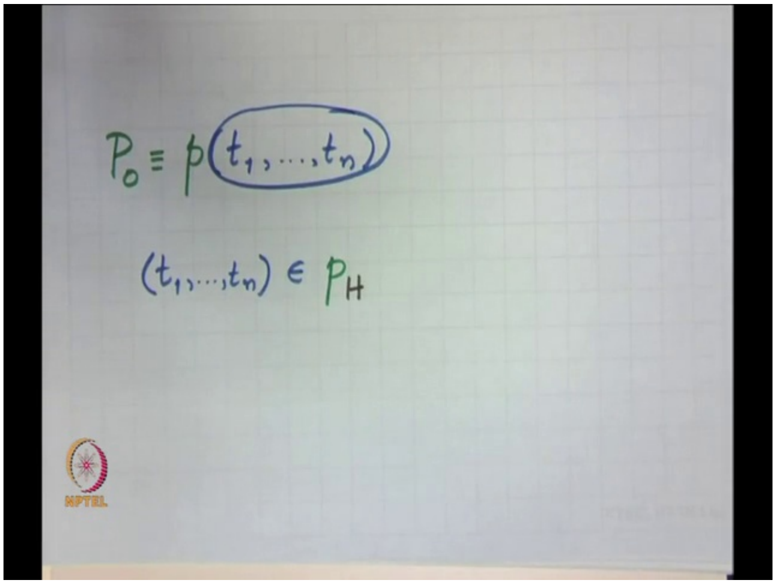

So, all I am saying is supposing your supposing you are the first element in enumeration is essentially some atomic predicate small p. Which, is with some n tuple of ground terms supposing this is capital P naught. So, what you are essentially saying in your interpretation is that, this tuple t1 to tn belongs to the Herbrand interpretation of this small p. If, you take this left edge and it does not belong to PH if you take this left right edge. You got all terms all n tuples have terms. So, essentially and this and you got enumeration of all these big P naught P1. Essentially got enumeration of all possible ground terms applied to all possible atomic propositions in your signature. So, any path in this tree you isolate a, single path in this tree that single path corresponds to essentially of full interpretation of all the atomic predicate symbols. This, is what you want right so this is I am going to call this the Herbrand tree of interpretation it is an all standard name I need some way of call referring to this tree. So, I have given in that name. So, this tree emphasis all possible interpretations of the atomic predicate symbols P. Why do we require this tree? I mean one thing of course is it so the tree is a binary tree if but its infinite. So, one thing is that is important is that its finitely branching. That is at any level there will only be a finite number of nodes.

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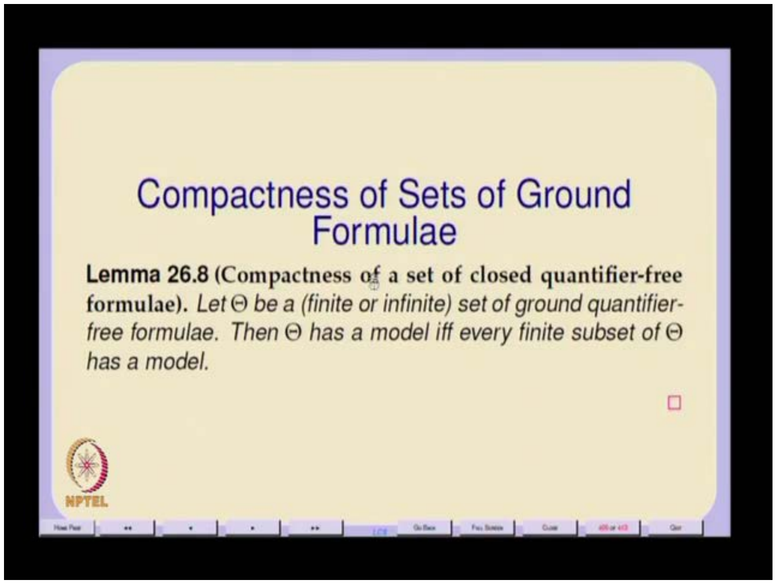

So, with it actually brings us to Compactness right so you take. So, what I am so here is the lemma which we are going to prove compact. Now, we have we have shown by other means compactness of propositional logic ground terms apply to atomic predicate symbols is closest to coming to propositional logic. So, in certain sense they are like propositions however we need to prove a compactness theorem essentially to show that there is a, compactness also holds for quantified formulae. So, this lemma is a first step towards that. And, this lemma is going to use that Herbrand tree interpretations. So, let theta be a finite or infinite set of ground quantifier free formulae. Remember that when we are talking about proof theory we are talking about a, gamma being finite. Whereas, now I am saying theta could be infinite 2 this not matter. So, essentially the compactness theorem for closed quantifier free formulae closed quantifier free formulae all basically, ground quantifier free formulae. Then, this set of finite or infinite set ground quantifier free formulae has a model if and only if every finite subset of theta has a model.

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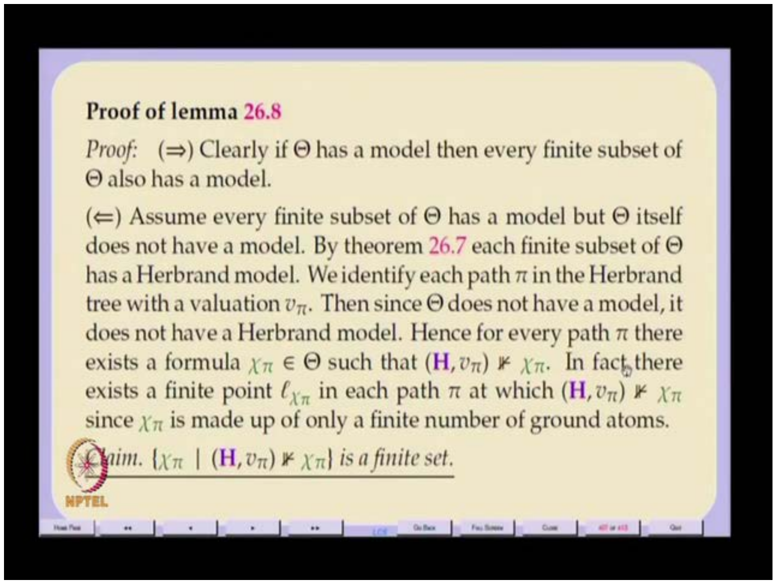

So, we are going to prove this and we are going to use a, Herbrand tree interpretations one is that if and only if. But, if theta has a model then every finite subset of theta also a, has model. And, in particular we need by a Herbrand's theorem we need to concentrate only on the Herbrand models. So, it is enough to consider only the Herbrand tree interpretations. So, what will do is will prove this, will assume that every finite subset of theta has a model has a Herbrand model. But, theta itself may not have a model suppose and then we prove that there is a contradiction. So, what do we do so now we it is enough to concentrate on Herbrand models. So, this is equivalent to saying assume every finite subset of theta has a Herbrand model. But, theta itself does not have a Herbrand model. So, if theta does not have a Herbrand model that means none of the paths none of the infinite paths in this tree is a model of theta, take any arbitrary path. If it is not a model of theta then, there is at least one formula in theta. Which, is contradicted at some finite point in that path. So, you take any what you are saying is if theta does not have a model anywhere here. Then no path is a model of theta so each path corresponds to a Herbrand interpretation. And, therefore you are saying there are no paths which will completely whose which basically are assignments of this kind. Which, not satisfy which make at least one formula in theta is false.

So, every path in this tree makes at least one formula of theta false. And, it makes at least one formula of theta false means there is a specific point at which that formula is known to be false.

That, specific point basically corresponds to your point when the assignments to all the relevant assignments to the when you encountered all the ground terms that appear in that formula. And, when you have encountered all the atomic predicate symbols attached to those ground terms also have appeared in that formula at some point you know that it is false. There, has to be a, if it is falsify then there has to be a finite point at which it falsify in each path.

So, let me call at finite point essentially as a level. So, you take so here is the here is the crucial sentence hence for every path pi there exist a formula chi pi belonging to theta. Such, that the Herbrand algebra with this valuation v pi each path is corresponds to a valuation with this valuation v pi does not satisfy chi pi which, means chi pi becomes false. And, this so this point let me call this level there is there is a level in the tree starting with the root node become at 0 and so on so forth. So, let me call this point l chi pi the point where it is first known the chi pi becomes false along that path. So, now we have this now what of course the tree is infinite and therefore has an infinite number of paths. Now, I choose all, these chi pi's and I construct the set of these formulae my claim is that this set is finite. At, the moment all I know is that in every path there is a point at which some formula becomes false some formula theta becomes false. I identify the first occurrence of that point and call that level l at the formula that gets becomes false i call it chi pi. And, basically for the valuation v pi and the level at which it occurs is a l chi pi. So, having identified these there are infinite number of paths and every path has at least one such point. So I collect all, these l chi pi's into a set it could be an infinite set.

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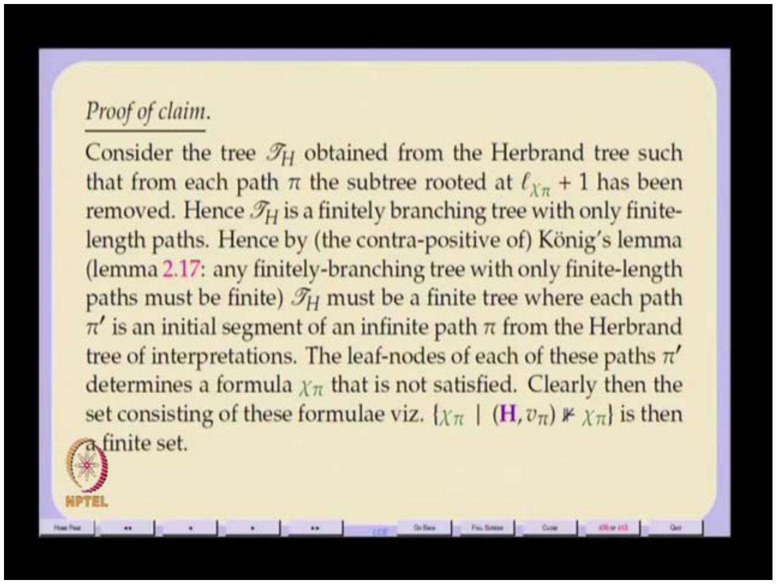

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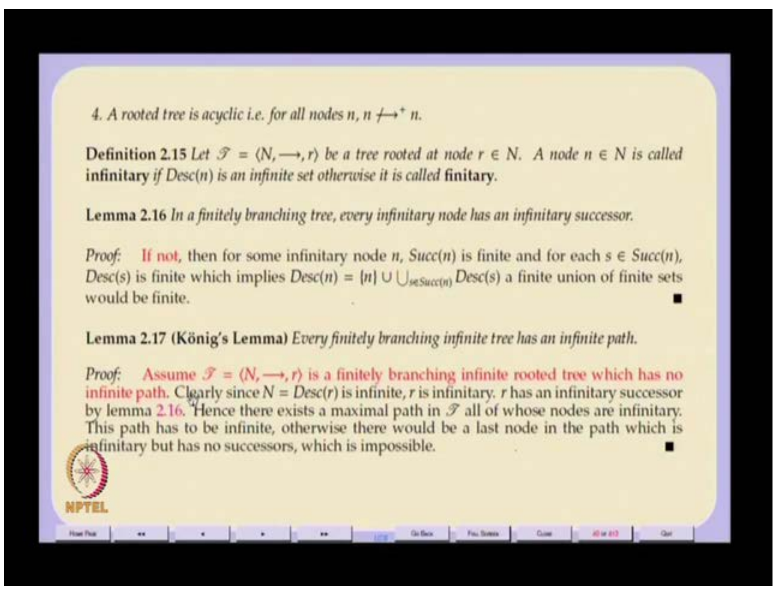

But, the set is going to be finite and why is it finite it follows from konig's lemma actually it follows from the from a contrapositive of the konig's lemma. What, is here konig's lemma is that every finitely branching infinite tree has an infinite path. If, you take every finitely branching finitely branching as an assumption. Then, every finitary branching tree which has no infinite path should be finite. So, what I am saying now how may I going to construct this finite tree I have got this level l chi l chi pi for each formula. Now, I take this Herbrand tree of an interpretation and chop it at that point. So, that that level l chi pi is a leaf node of that path. So, the entire sub-tree that becomes below it and chop it, off. So, in every path there is one such point and I have chopped off everything below it. And, I have got this finitely it is still a finitely branching tree but now by konig's contrapositive of konig's lemma that tree cannot be infinite because, there are no infinite paths. So, that entire tree must be finite which means it has only a finite number of leaf nodes which means that set of formulae chi pi and that I am collecting from the leaf nodes that set must be finite. One of the most engineers proofs have come across.

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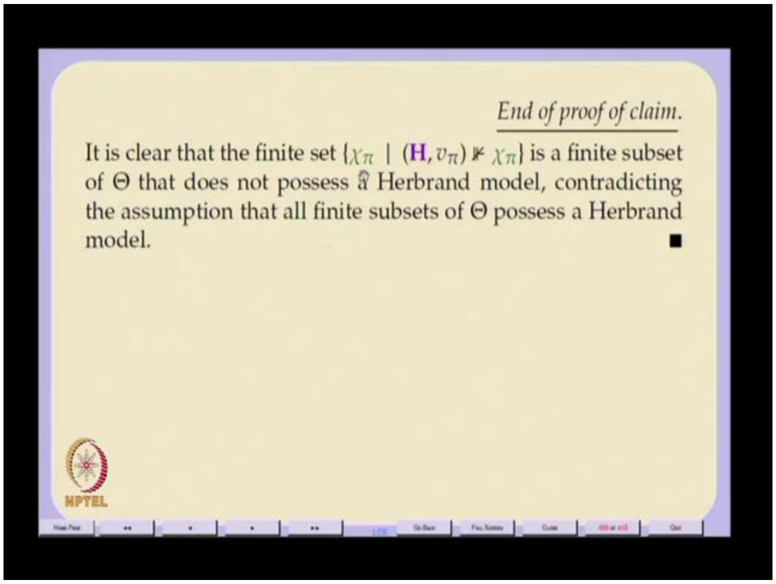

So, now here is a finite set of formulae which does not have a model which contradicts are assumptions that every finite subset of theta has a model. And, therefore are the original assumptions that theta has no model is false. So, theta must have a model in the Herbrand tree of interpretation.

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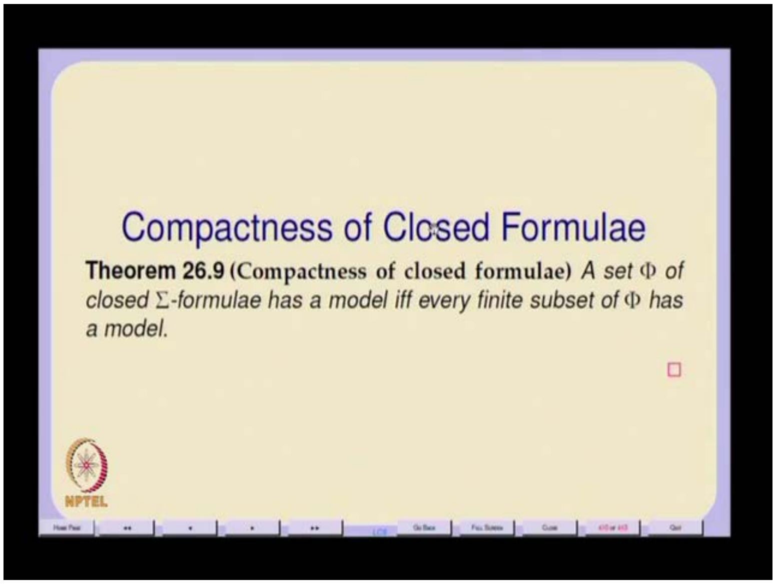

But, once you have proved this then it is very simple Compactness of a first-order logic is just that you take a set of Closed Formulae sigma formulae this set phi by the way this set phi can be finite or infinite closed. Now, they are quantified formulae has a model if and only if every finite subset of phi has a model.

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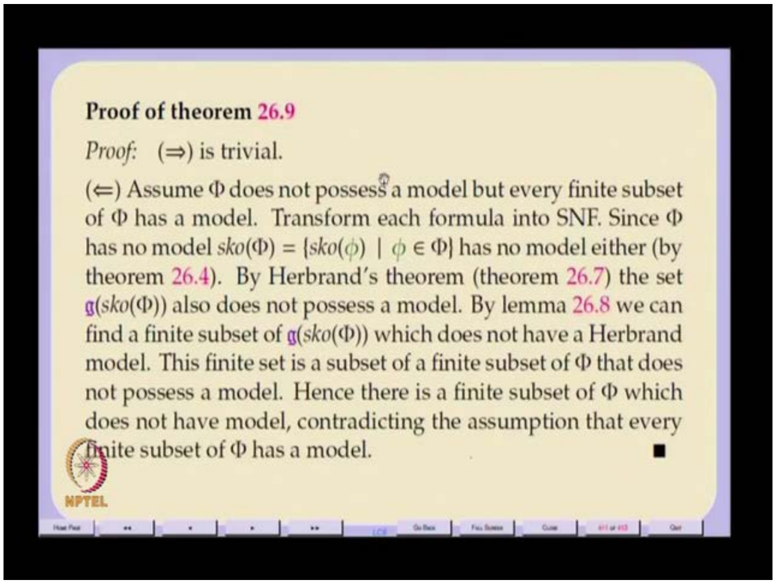

And, what do I do I just take each formula phi again I go through the same kind of argument I take each formula in phi and transform it into a skolem normal form. But, now I get sko phi which is in no sense in no formula in sko phi is equivalent to any formula in phi. But, what I do know from Herbrand's theorem is that phi itself will have a model if and only if sko phi has a model. And what I also know is that phi has a Herbrand model if and only if sko phi has a Herbrand model. So, now what do I do I concentrate on finding the Herbrand model of sko phi. And, now sko phi basically I can it just consist of universal quantifiers and a and a body that universal quantifier just means that I instantial instantiate the body with all possible terms.

And I apply this same reasoning that I applied here. So, it follows from the previous lemma that you'll come to the same contradiction. So, now your closed formulae therefore will so, what it means is that you will be able to show that there will be able to show that there is a finite subset of phi which does not have a model which contradicts their assumption. And, therefore phi must have a model if every finite subset has a model. so we have done compactness now as its fairy engineers proof . And, after looking at the class I think I stop here is become a heavy quite of you. So, what we do have to do more on first-order logic before we get into actually formal theory is completeness of your predicate logic and decidability of pure predicate logic. Once you have all these results and you would not have unreasonably expectations about doing theorem proving in formal theories.

So, you will have only reasonable expectations so and then this all this also lead into dovetail into essentially do domain of logic programming. So, this is a fundamental theory so this Herbrand theorem is a fundamental theorem of a logic programming. Basically what it says is it says me the trouble of trying to look for satisfaction of form of a set of formulae in arbitrary domains. And, just look for satisfaction within the domain of terms itself. So, that logic programming therefore becomes possible within the notion of a term of terms and constructors. And, you do not need to actually look at arbitrary theories. So, these two are important so we have proved the compactness.