Logic for CS

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Lecture - 22
First-Order Theories

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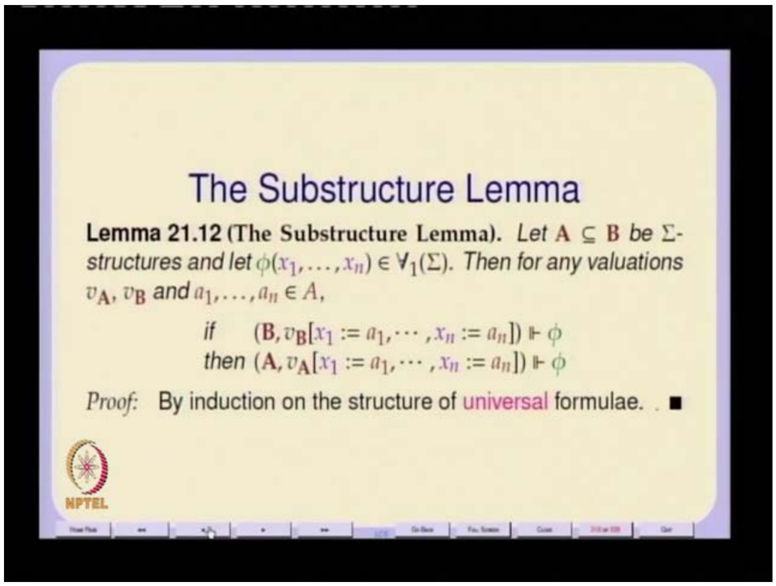

So, as we can see from our previous lectures actually the whole emphasis on most of the specification of first-order logic is actually of a very algebraic nature. And, that historically speaking of course what happen was that the inspiration for all this really came from Euclidean geometry. And, what Hilbert did which will be it will go well beyond this course but at least it is a good idea to know that it was something that he could do is that he actually based,

so, even though it was inspired by geometry and essentially Euclidean geometry. Hilbert based his entire axiomatization of Euclidean geometry on the real numbers, so on the real field. So, in that sense he had already algebraist Euclidean geometry. Because, the main things in geometry actually were of the form of measurements. So, measurements of line segments congruence's measurements of angles and they were all essentially. So, he required the real field on which to base all his axiomatization. So, what Hilbert actually did was that he looked closely at Euclid's geometry. And, found that there were a lot of there were quite of few loopholes logical loopholes in many of Euclid's statements and proofs. For, example one of the things was that Euclid's proof of the congruence of triangles would typically goes as place this triangle on top of the other triangle. And, just as and it was sort of hand writing proof which said that this vertex coincides with that vertex and this vertex coincide with that and before the two triangles are equal congruent.

And, which is a it is a completely hand writing proof. Whereas, the whole idea about Euclidean geometry was that it was suppose to be essentially. So, the what Hilbert got from that kind of proof was that he is essentially saying that Euclidean geometry is invariant under translation rotation and possibly even reflection. But, those invariant properties should not be part of the formulization and which you can prove invariant properties. So, the model for Euclidean geometry which actually comes from coordinate geometry is not really invariant under translation rotation. And, so on because notion of a point in that is a pair of xy coordinates. So, when you translate a triangle you get a different triangle with different vertex points. And that is it might be congruent but, it is not translation invariant for example or rotation invariant. So, the emphasis at that point late in the 19 century and 12^{th} century was actually for looking at invariants through homomorphism and so on so forth.

So, what you what Hilbert actually found was that it is possible to define in addition to Euclid's axioms and postulates is possible to assume the field of real numbers. And, actually he add to define extra some extra axiom and postulates. So, the final list in Hilbert's axiomatization of Euclid's geometry was something like 18 axioms 18 postulates. And, many of them were derived from real's because they had to deal with congruence's and measurements and so on so forth. And, besides that of course these were non non-algebraic features like parallelism and so on and so forth. And, but essentially he could reconstruct the whole of Euclidean geometry a regressly within a first-order logic framework with about 18 axioms. And, he could actually prove all this theorems. So, in particular one major change that this formulization required was that you had this 3 congruence after angles rules SASSSS and ASA Hilbert found that you had to assume one of them as an axiom as a postulate. So, if you if you do not want to go along with Euclid's notion of move this triangle on top of the other triangle and so on and so forth. Then, one of these 3 congruence things actually what Hilbert did was he took the SAS as a postulate and he proved the others essentially through one to one correspondences and so on so forth. So, for example Hilbert's proof of the base angles of an isostral triangle are equal I mean.

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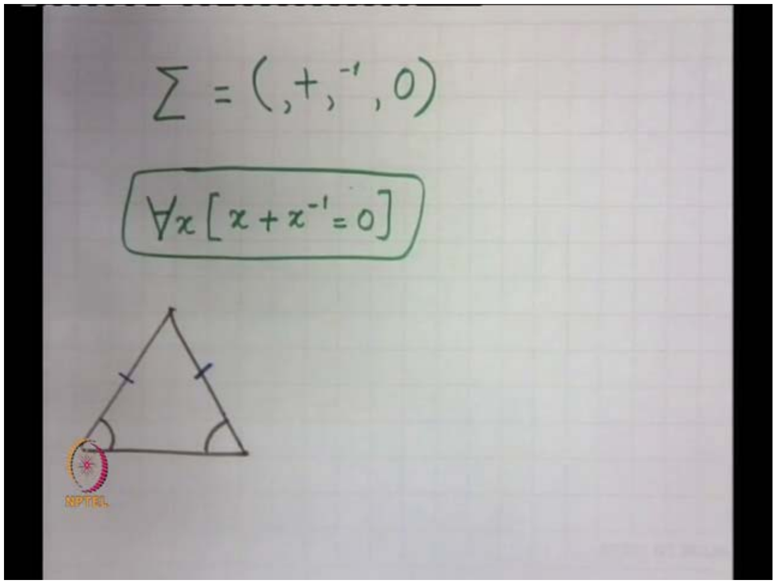

So, that actually involved this, looks too much like an equilateral triangle but let us assume it is just as an isostral triangle. So, that there are there were are lots of settle things one of this standard things that happened was that so you are given that you are. For, example given that these two sides are equal and you have to prove that these two angles are equal one of the standard things was that you drop some perpendicular or some such thing or you drop a bisector. And, then prove that those two angles are equal but there is a there is a certain logical flaw, How do you know that if I drop I mean. So, the notion of if I drop a perpendicular for example then there is no guarantee that it lies between these two points

So, logically speaking how do i know that without drawing that diagrams, How do How can I guarantee things like the perpendicular drop between these two points? So, which means the notion of between’s the notion of one point line between another other two these were definitions which you had to be made in terms of measurements and so on and so forth. And, then you had to given additive axiom also to save that well if there is a point between two different points on a line. Then, the some of the measures of the two line segments equals the measure of the larger line segment and all those things.

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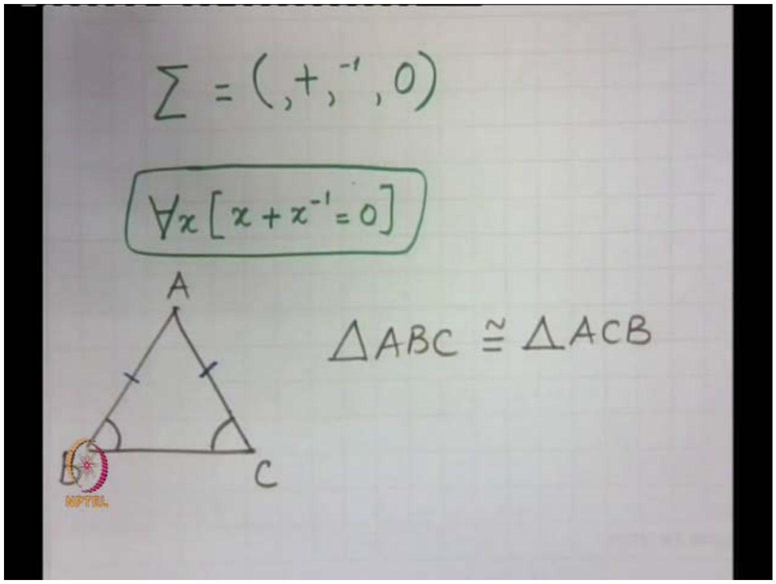

So, what Hilbert did was he instead used something like one to one correspondences. So, he actually defined triangle congruence. For, example as one to one correspondence we should preserve measures. So, there are 6 measures 3 segments and 3 angles. And, if all 6 measures are preserved then, the two triangles are congruent. So, in the case of this without getting into between's and so on and so forth what he did was he proved. That, this triangle ABC was congruent to the triangle ACB without dropping perpendiculars are even bisectors I mean it is possible that if you are defined betweens and defined the notion of bisection you would have got between's. And, then there is a guarantee that bisectors would lie there that is another way of proving. But, this is an innovative proof using just one to one correspondence. So, it basically proved that this triangle was congruent to itself with a different correspondence. And, from which it followed from the definition of congruence that these two angles would be equal.

So, there are lot of innovations in what Hilbert did in the formulating geometry. And, the other interesting things that, he did was after having reformulated he showed that his version of geometry was consistent and it was complete. So, all the theorems could be proven I mean the its its an amazingly great result for an for an essentially an elementary thing. But, we will that it is takes a too far away from logic for computer science. So, we will not get into it but if you are interested you should actually look at some of those books of the 1920s which actually detail Hilbert's axiomatization of geometry.

So, these are look at other first order theories from more modern view point. So, essentially Euclidean geometry is axiomatizable in first-order logic all of Hilbert's axioms can be proven to be independent to each of each other no axiom can be derived from the others. And, the axiom system is sound and complete you know that is it is an amazingly complete piece of what that Hilbert did.

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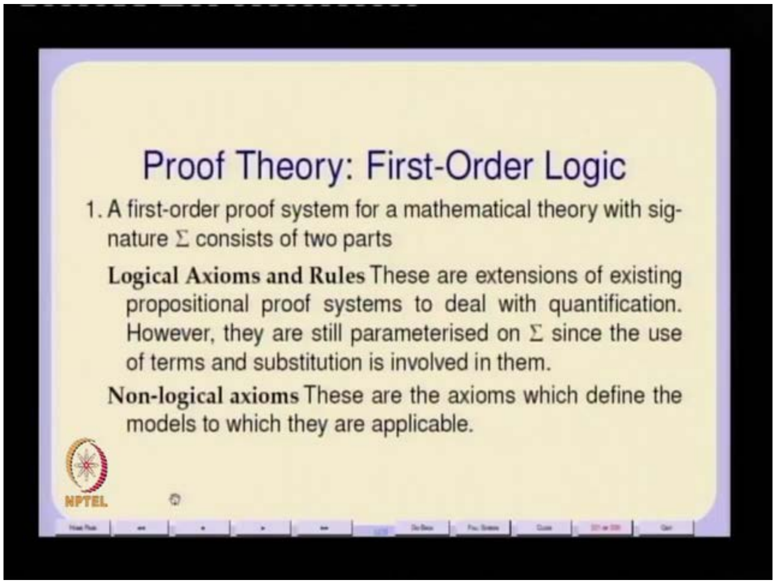

So let us get into Proof Theory and again will follow Hilbert's think that we want our essentially a proof theory of a First-Order logic what distinguishes. Now, first-order logic from propositional logic is existence of signatures and terms. And, that something we have to worry about and quantification. So, if you look at first-order logic as a descriptive mechanism formal descriptive syntactical descriptive mechanism of a mathematical theory. Then, firstly we require two kinds of axioms and proofs. One is the fact that we have got quantifiers means that we are required axioms and proofs for quantifiers. So, those are the logical axioms and logical proofs so, in that sense you got a two tier language. So, you got a two tier proof system then, there are the non-logical axioms. These non-logical axioms are what are used now a, days to briefly described any kind of structure. So, you just give the three group axioms and you have defined essentially all groups. So, what you are so those three group axioms which by the way have to be what we have done what we gave as a three group axiom as three group axioms is not exactly the same. Because, you have to include inverse the existence of unique inverses and you have to give universally quantified axiom for that unique universe. So, something like you assume that you assume here, we have it we you have this and you have to give this axiom in addition to the associativity. And, you have to replace that right inverse axiom by this universal axiom the unique this is like the unique right inverse axiom it is clear that you can prove that unique that left inverses are also unique. And, that your right inverse is that left inverse. So, is possible to prove those things but essentially the group axioms will be those 3. So, what we will look at so you can think of therefore first-order logic as since it is a two tier structure we actually make it a three tier structure I will give the reason for it. So, we first of all require first-order logical axiom and rules.

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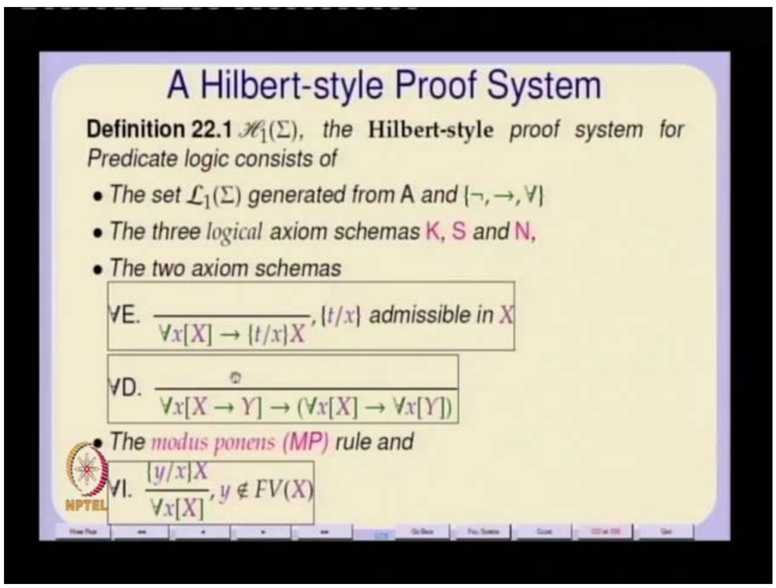

And, Hilbert had this there is small variation here So as usual so I will call this since its first-order I am calling it H1 and of course its parameterized on signature sigma. So, and then as we did for propositional logic will assume that there are only three operators. Or rather this universal quantifier is actually an infinite set of operators one for each variable. So, we will assume that there are only these two operators on the universal quantifier for all variables everything that you did in propositional logic. Since, predicates a just parameterized propositions and since the proof system of propositional logic required only pattern matching. And, logical truths was shown to be logical validity was shown to be pattern dependent in that. So, all the patterns which are tautologies in propositional logic can be directly imported into first-order logic. So, but now when we are talking about patterns we are talking about exact patterns so, even variable names should match in the patterns.

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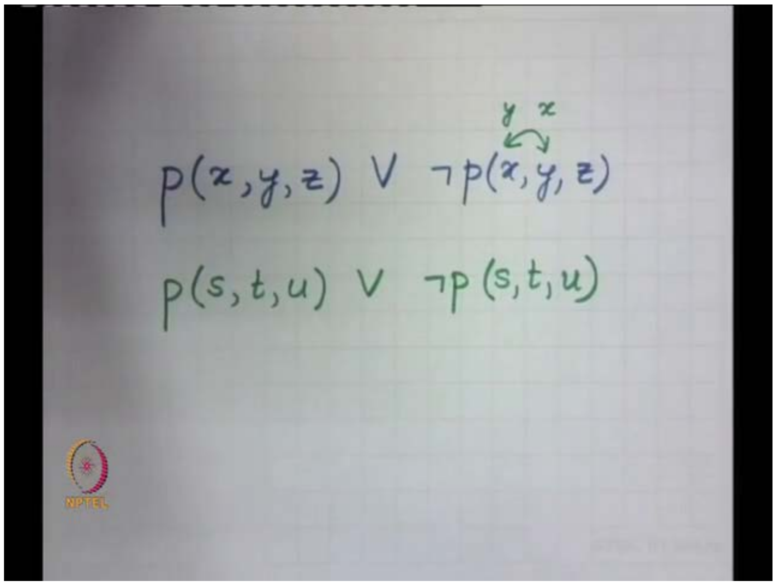

So, in particular what we are saying is I have some predicate of three variables. Then, this is a tautology where the three variables occur in exactly in the same places in both cases. So, for example we cannot just replace one of these variable you cannot just exchange these variables and claim that this is a tautology. So, if you look at any substitution of terms also the substitutions has to be an exact pattern match. So, that when you take three terms s, t and u let us say which essentially are substitutions in which the term s replaces x and t replaces y and u replaces z. Then, you have to take exactly the same kind of substitution. So, as the sequence of symbols should also be exactly the same in order to get exact propositional patterns so what your tautological. So, you the tautologies of propositional logic essentially form a skeletal framework of patterns with placeholders which you can import directly into first-order logic. And, therefore all the all the tautologies of propositional logic can be imported directly. So many books instead of saying K, S and N and MP they just say that to take all the tautology tautological structures of propositional logic. And, think of the atomic propositions there as variables to which you can perform substitution of predicates. So, which is why actually I use these new meta variable symbols like x and y and so on so forth it satisfies both needs that of providing skeletal structures and being able to substitute. So, since of course K,S,N and modus ponens together give me a complete axiomatization for propositional logic instead of importing all the tautological structures from propositional logic I, just use these three axioms K,S,N and modus ponens proof.

So, all propositional structures essentially form frameworks in which we can plug-in predicates instead of the proposition instead of the atomic propositions there. So, two each atomic proposition I can think of it is variable which can be replaced uniformly by a certain predicate. In which the terms also would be identical copies of predicates for every occurrence of the same variable. So, then so then every propositional tautology would therefore also be a predicate logic tautology would be logically valid in predicate logic. The other two the other axioms and rules inference are these. One is universal quantifier essentially means that it does not matter how I what term I substitute for a variable. Which, is universally quantified throughout the body of the quantified formula it is still true and that's preserves truth it preserves validity and it preserves satisfiability. So, the first axiom schema is that for all x in the structure capital X which might have x small x as a free variable. I, can replace all occurrences of small x in capital X by any term t provided of course capital X could be a skeletal structure in first-order logic. It, means it might have other quantifiers provided no variable of t gets captured by a quantifier in X.

So, that is what this admissibility I think I defined admissibility in some earlier lecture. Basically, you have to avoid capture of free variables because when you because free variables have a different meaning all together in some global context. And, they should not be confused with bound variables which have only a local meaning in local context and by coincidence happened to have the same name. So, you should not be able to capture free variables. So, in which case what of course if you cannot change the variables and t. But, what you can do is you can do an alpha conversion by changing the boundary variables in x in order to get equivalent predicates. So, that it makes t for x admissible in X. So, one thing is so I did not include this alpha conversion thing but because x not something that is there in most books. But, it should actually be there so I should have made it some x prime. Where x prime is alpha equivalent to x and t for x is admissible in x prime that would have been more accurate way of rendering this axiom schema. So, this in many in many case this is called universally instantiation but I am following my standard convention of naming which basically comes from Genson's natural deduction system and which is which is inductive on the structural formula.

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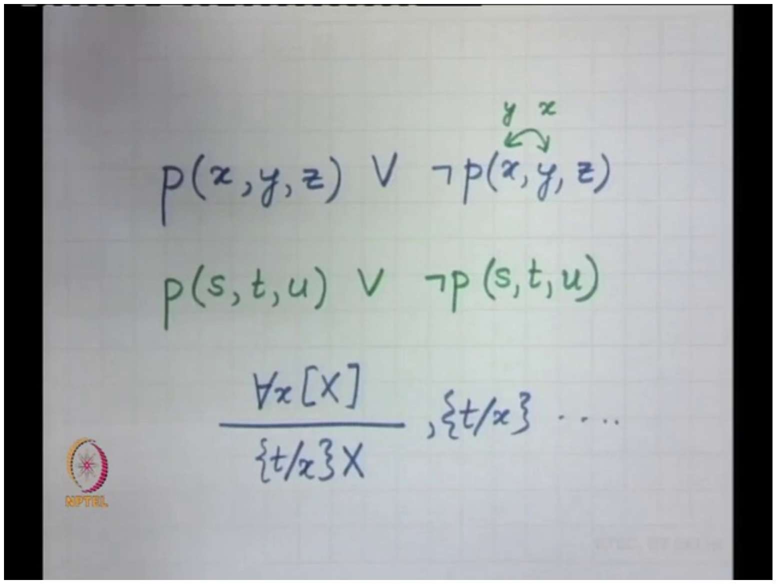

So, this is just a for all elimination rule it could equally well have been written as this where t for x is admissible it could have been written as a rule. And, in fact in a Genson natural deduction system could be written this way. But, the two things are equivalent but with a caveat which we will worry about. Because, the deduction theorem does not exactly go through in predicate logic is a problem in the sense that. Because, of the presence of variables there is we have to fine tune the reduction theorem for free variables. So, that it still continuous to hold and so, it holds in the sort of restricted fashion.

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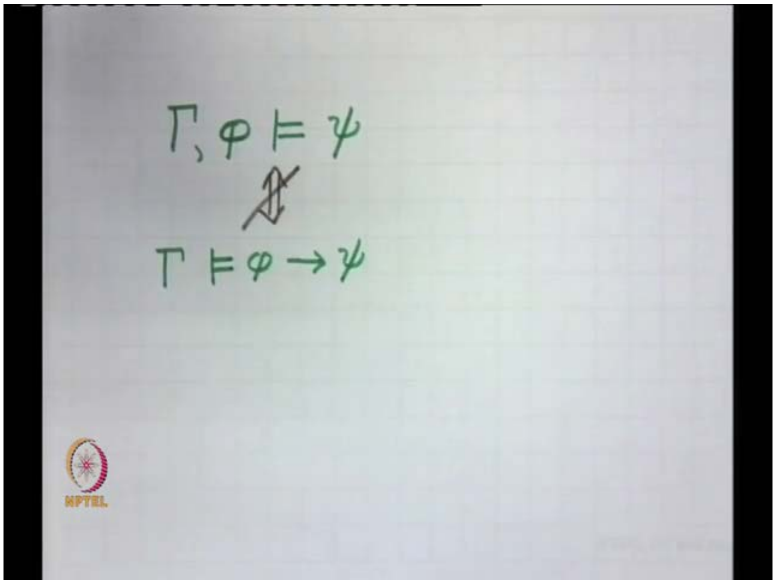

So, then what happens is that these two notions let me write it fresh so if I have gamma phi psi is a logical consequence of gamma union phi. And, phi arrow psi is a logical consequence of gamma these two are not equivalent notions first-order logic. And, it is precisely because of the fact that there might be free variables hanging around a, which are not quantified. However, this equivalence does hold for completely closed forms. So, which means if you do not have any free variables at all anywhere. Then, is equivalent to a proposition and then the deduction theorem actually holds. But, between the two also there exist some fine tuning which we can do to make in the presence of free variables certain restricted variations of deduction theorem can be made to hold.

And, so we will look at that in some detail in the proof of reduction theorem. But, basically now logical consequence cannot be replaced by arbitrarily by an implication I mean you cannot move formulae across without first-looking at certain restrictions. So, that is one thing so even though these two forms of for all elimination equivalent there is some it is a Hilbert found it better to treat it as an axiom. So, that it remains on the right hand side there is another one this. Essentially, says that the universal quantifier can be distributed over the arrow there is a variation of this which actually some people prefer.

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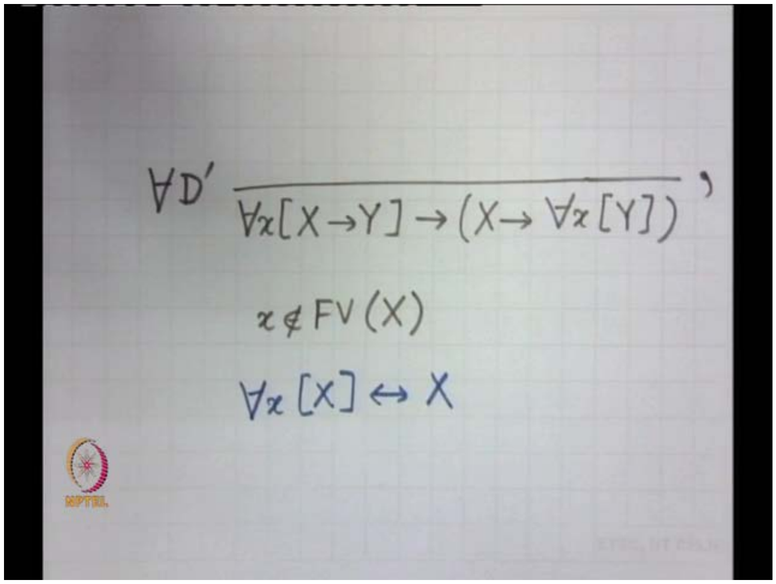

But, I have not found in any there is some books actually prefer this rule which I have call for all D prime. Which, essentially says that it is an axiom of the form for all x, X arrow Y arrow X arrow for all x Y. And, then there is a side condition which says that x does not belong to the free variables of x. But, the point is this if X does not belong to the free variables of X then it is possible to actually prove that for all x, X implies X and actually it is possible to prove this. So, therefore I preferred instead of having to have as few side conditions is possible. So, I am going with this which essentially says that I can distribute the universal quantifier across the arrow. But, this is not this cannot be a bi-conditional I mean what I mean there are lots of certainties of language here. So, here so what we are saying is if this is a if this left hand side is valid then, of course I can distribute the universal quantifier this way. But, if this right hand side is valid I cannot necessarily get the left hand side from that. Because, in particular the bound variable x here is restricted in scope to this body X. And, the bound variable x here is restricted to the scope of this body y and they need not be the same.

So, you cannot you may not be able to factor out the universal quantifier easily but you can distribute it definitely. So, the distributor law here works only one way. If, you try to distributed the other way then you will have to impose various conditions to ensure that somehow you can keep the identity of the bound variable the same in the two and you have to you have to do enough work to do that. So, this arrow is strictly union directional you cannot it is not possible to prove the reverse without putting in I will larger number of conditions. So, these rules have to be carefully designed. The other thing is, when do we the other thing this for all i the Gensen way of naming would just said that it is for all introduction. But, in the case of so what is the thing, What is a main point here? The main point here is that this substitution at the top s and this side condition y does not belong to the free variable of X. Essentially says that if I, have enough reason and by enough reason I means a proof that by replacing the free variable small x in capital X by any new variable Y. The proofs still holds then, the proof is independent of that symbol y. If, the proof is independent of that symbol y then that symbol y can be replaced by any other symbol also. And, if it is independent that means it is also independent of any valuation that y any value get might be given in its semantics. And, if it is independent of any valuation then it can be universally generalized I, mean this thing is very settle for reasons that will become clear when we do the corresponding existential rules and, there again will have to be careful. So, essentially what this whole thing says is what we actually do in a many of our proofs. We say let y be anything of this kind using just the symbol why we do the proof. And, then we say therefore it holds for all y therefore the universally generalized statement also holds. That, is an that is that last step is this universal generalization. Essentially says that I am using the symbol y throughout the proof but that's because of I need something to it is not actually dependent on the value of y it could have been any other symbol and the proof would still be correct.

And, would it would still go through and if that were the case then I can generalize on y. So, these are the rules actually and we will look at some. So, you can see that so one thing with quantifier set supposing you have some you have a statement which is quantified to be proven. We will do a very small example of that then we follow the in the Hilbert style or even in the Gensen style. Now, actually does not matter whether we talk about Hilbert or Gensen because Gensen just has four rules. Because, he uses the existential quantifier also but will derive those two those existential quantifier rules. And, so he has just the two elimination rules and the two introduction rules. But, basically what we are saying that this structure of any proof of a universal statement or of an existing let us just strictly universal statements. This, the structure of any proof of a universal statement goes through a quantifier elimination the first step is let x be any such thing the that is essentially eliminating this quantifier and putting some variable like y and then going through a proof.

And, then the last step is having shown that y remains y is not a free variable of the original statement X. And, therefore the proof is independent really on y therefore it can be generalized so the last step is an introduction of the quantifier. And, in all in any direct proof of universally quantified statement then this is this is the structure that we see. So, there is between this quantifier elimination and quantifier introduction is the scope of the proof. So, that sub-tree is entirely a sub-tree which cannot which has a certain which is the scope of that the free variable y that we introduce as a result of the quantifier elimination. And, you can have sub-trees within if you have many quantifiers nested quantifiers. Then, you would have nested sub-trees with more and more other symbols occurring. And, then you will be gradually so you take this entire proof tree you which has several quantifiers. Then, what you have our scopes defined by sub-trees for each of the individual variables.

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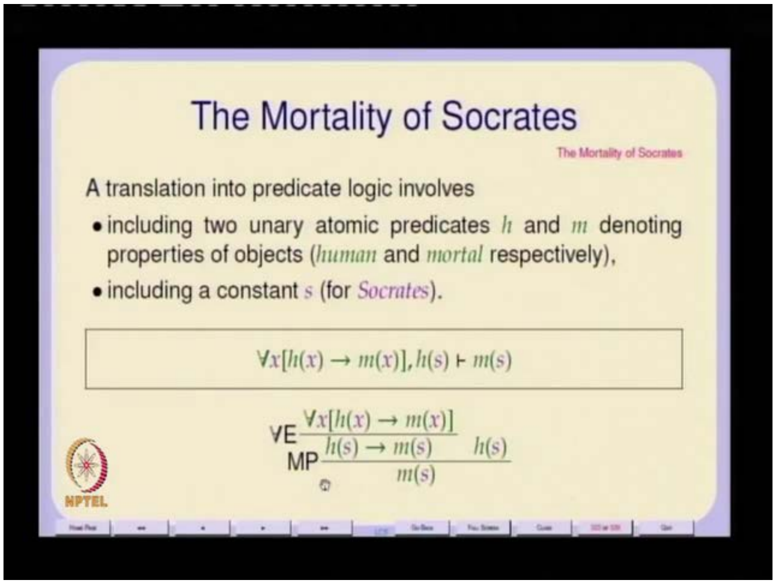

We will see that that is the last case structure but of course we have you got a basic application where to prove that Socrates.

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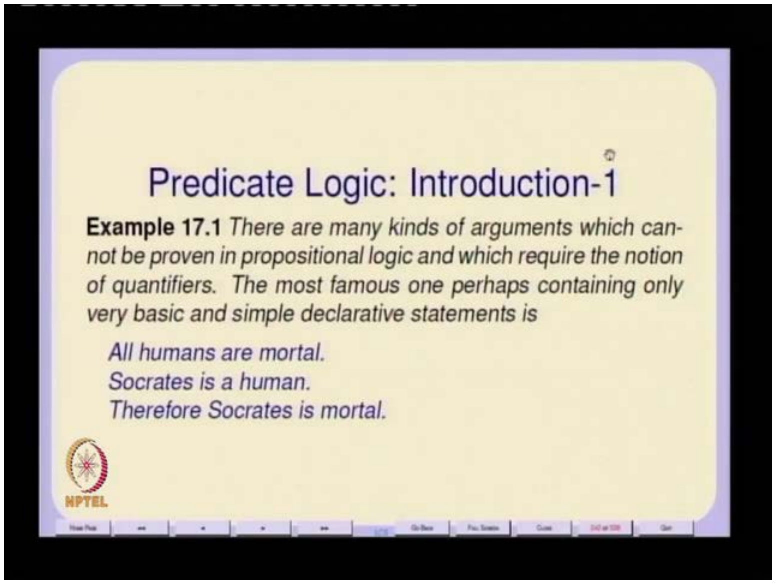

So, this so here this I mean all this so it is taken 23 lectures to come to this argument 22 lectures. So, let us do this argument and we dispose it out. But, of course what we need to do is take this verbal arguments. Now, we have an extra obligation, What we are saying is? We, gave all the semantics in terms of sigma algebra. So, we cannot just ignore all, that even if it is verbal argument like this we have to somehow specified. What, is the signature that we are considering? That is an obligation which by the way no logic book specifies. They give you this argument they give you the proof and then they are done with it they have not hold you why they went into signatures and structures and sub-structures and so on so forth but, I think we need to do it.

So, essentially what we are saying is that the signature consist of 2 unary predicates 1 that of the. So, there is some set of objects unspecified but that is not part of the signature that is a carrier that is a structure if you were to take a model. But, we have to specify the signature because we are using that in a first-order logic terms. And, basically what we are saying is a there is some unary atomic predicate called human. And, there is another atomic predicate called mortal. And, whatever mortally set of objects you might take as a model presumably some subset of it, has the property of being human some subsets of it presumably has the property of being mortal those subsets could be empty also. But, taken given non-empty set of objects in which these two properties can be defined. In, that signature we require the existences of a certain constant. So, the verbal the import of this verbal argument is there in that model structure it is a non empty structure that definitely exist a constant S standing for Socrates. And, now you are looking at this argument so, this argument of course it is just. So, now these quantify quantification arguments are not so easy to translate the way we could translate propositional arguments.

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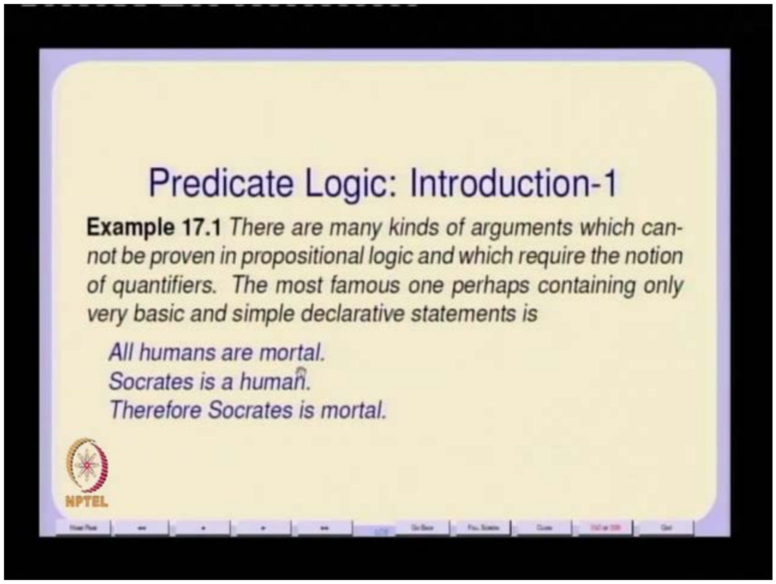

Now, you have to actually look deeper into the semantics of the sentence and then decide what is, the exact translation. So, for example all humans are mortal the only way to really translate it is that given that being human is some atomic predicate being mortal is another atomic unary predicate. If, something is human for every X it is if something if that x is human then that x is also mortal. And, so that is so it actually works out to be a conditional which is not obvious from the sentence structure. Because, now the sentence structure go beyond just the proposition and conjunction they go beyond AND, OR. And, of course the next premise was that Socrates human and therefore Socrates model. And, this the proof is just this and in this particular case we are of course the conclusion of the proof is about this constant S.

So, it is not a universally quantified statement. So, it is sufficient to just eliminate the quantifier. And, here when you eliminate the quantifier I am using quantifier eliminations specifically with the constant s if I did not use the constant s. If, I use some y or z or some such thing then, I would not able to apply my modus ponens is as simple as that. Because, there is no guarantee once a variable is free supposing replaced it to replaced it by h or z arrow m of z, z becomes a free variable of this predicate and, z cannot be replaced. So, you have to replace it by this constant S. So, in particular this quantifier elimination allows you to do that. And, then you can use modus ponens because now you have got pure propositions. And, now you are essentially using modus ponens on pure propositions or on pure propositional forms we will see another example here.

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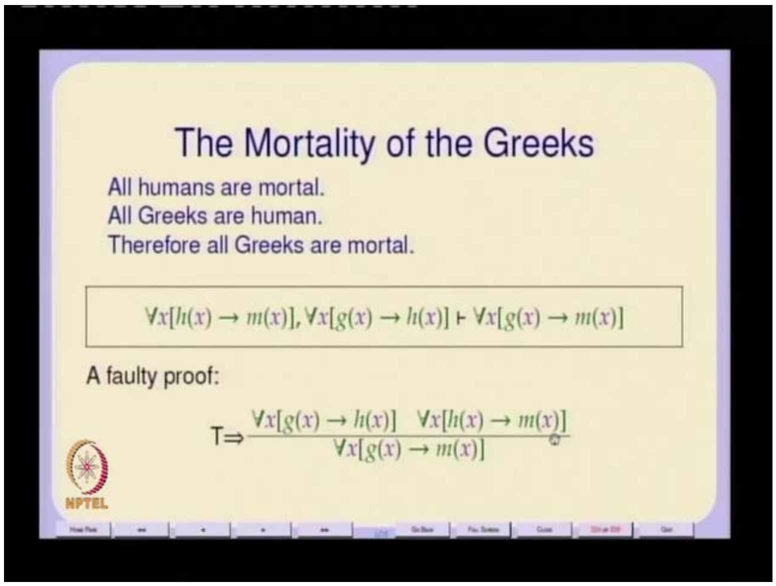

So here, let us look at The Mortality of Greeks so all humans are mortal all Greeks are human therefore all Greeks are mortal. So, this is an argument in which you actually have universal quantification everywhere. So, this by the way so what we can do is, so you now as I said now this is a faulty proof I mean basically you are using transitivity of that arrow. When, that arrow is deep inside some tree you cannot apply rules like that. Why there is a very simple reason why you cannot apply rules deep inside the abstract syntax trees.

Because, it is quite possible that you make an assumption supposing you are doing the proof the contradiction you make an assumption. And, then if you do these application of modus ponens deep inside a tree then you might actually come up with false conclusions. Which, do not preserve because eventually if it if that assumption lead to our contradiction then you can obviously prove anything so it does not matter. So, in any kind in any proof that involves quantification you cannot apply such rules like the transitivity of arrow directly. You, have to the transitivity of arrow is a propositional rule it is not a predicate calculus it is not a predicate logic rule. So, if it is has to be applied you have to first get propositions or at least propositional forms and, only then apply that. Which, means that in order to be able to apply this, you have to first eliminate the quantifiers. So, this is the faulty proof I mean I have seen this kind of steps in many logic proofs by students and, that is why I brought in here. This is, this is simply not acceptable the only time when it is acceptable to go deep inside a tree. And, replace some sub tree by another sub tree is when you can prove that those two sub trees are congruent. So, if there for example logical equivalence is the congruence relation. So, then you can actually do that replacement but, that is the only time you can do that you cannot do that for any other operator.

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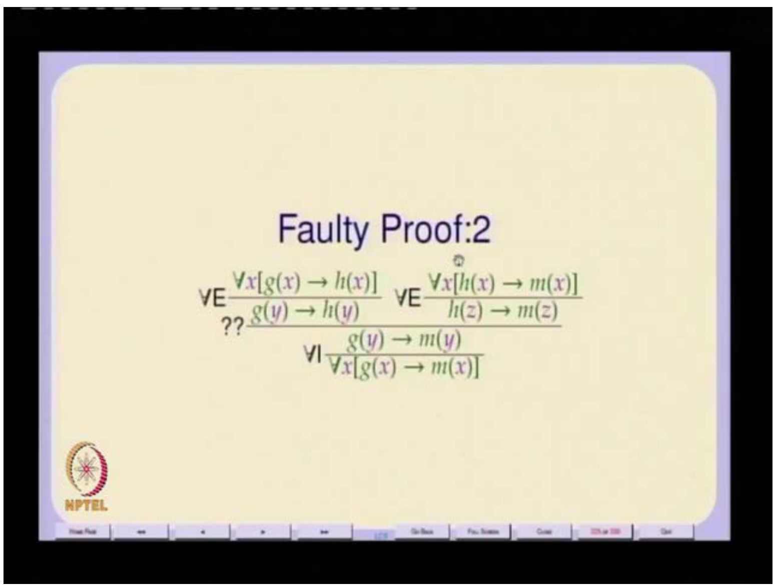

So, that is why the simple arguments are important here, is another Faulty Proof. So, here I eliminated this quantifier by using a variable y I, eliminated this quantifier by using a variable z. And, then somehow suspiciously I am using a transitivity rule which is not acceptable because as so these are not necessarily propositional say propositional forms is there then variables is there. But, as patterns this h of y is not the same as the patterns h of z because y and z are different variables. And, from your from our semantics of predicate logic it is clear that under certain valuations y and z might have different values in which case the truth is not guaranteed to be preserved by your by this kind of an assumption. So, this is so you cannot just arbitrarily substitute variables and prove therefore g y arrow m y and do this kind of fudging. This, is another mistake I, have often seen in logic proofs by students.

So, that is why I brought bring it to you all bring it to you all. So, you have to so the only correct way of doing it is to ensure that you use the same symbol. And, for both quantifier elimination and your quantifier elimination rule allows you to do that.

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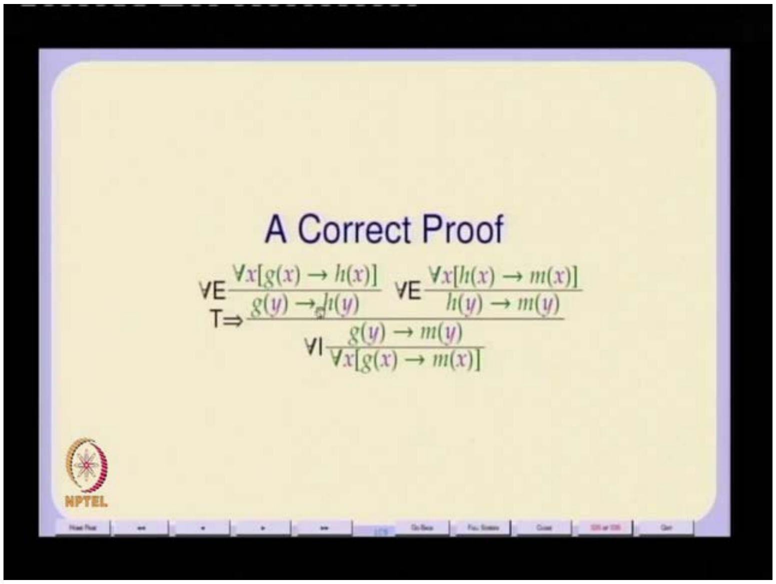

So, here is a Correct Proof I take these two I chose some variable y and I use that variable y in both quantifier eliminations both of them are universally quantified. So, I am actually free to use the same variable y and once I have got identical patterns I am actually free to use the transitivity of the arrow rule to get the g y arrow m y and, then I can generalize. So, this is how first-order logic proofs is, are going to go and that pattern matching is absolutely important. Because, what you had you what you are trying to show what you are trying to do is that notion of a syntactic proofs in syntactic. So, pattern matching is essential and when patterns have to match they have to match exactly. And, by being syntactic and you are what we are going to show in our soundness and completeness proofs is that, these rules preserve truths they preserve satisfiabilty they preserve validity. And, they cannot do that if free variables are lying around and a being even arbitrary valuations ye after all any propositional form with free variables will have might have different truth values.

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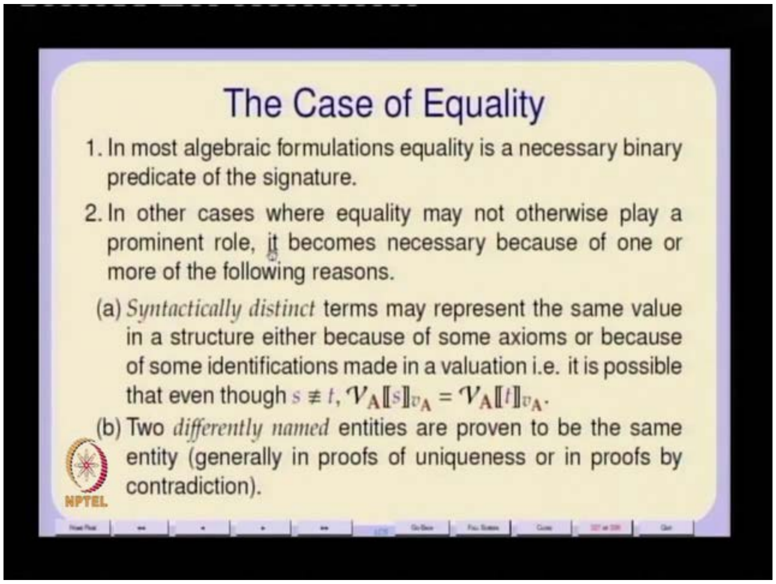

So, different values of the variables so you have to do that I said therefore I our language is two tiered. And, therefore our proof system is also two tiered the actually more to it our language is algebraic and in any form of algebra Equality plays a very important role. But it is not just the equality in an algebraic system equality relation in an algebraic system that is important. Equality plays other roles also and the thing is that there are various many different levels of the quality. One is that, in then you might have a term representation of some something like natural numbers for example. So, what you want to say is that even in the natural numbers its or they integers with successor and a predecessor functions.

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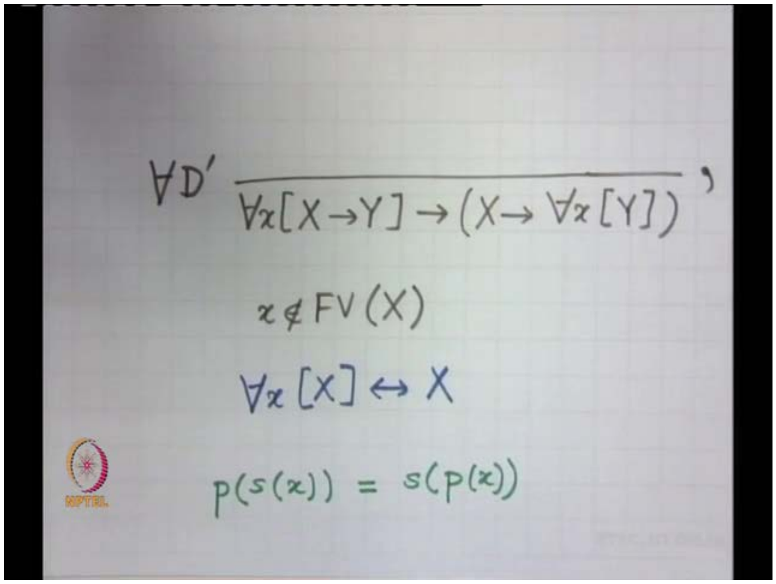

What, you are essentially saying is that the predecessor of the successor of any x for example is equal to the successor of the predecessor of any x of the same x. So, you do have identities like this and these two do not syntactically match. But, in any valuation in any model of the natural of the integers for any value of x the resulting values you get on the two sides of the same. And, in that sense they are equal. So, there is a possibility of syntactically different terms having the same meaning. Which, is when your asked which is also what happens when you are asked to write a write a program different people write different programs. But, all the programs might be equal in the sense that they give the same input output and relationship. So, there is an equality which comes because of that. The other thing is that it is just that sometimes in our proof by contradiction for example. Proof of uniqueness of something you must have done a lot of proofs of uniqueness. You, assume non-unique you start with an assumption that there were there are two different things so you name them differently.

And, then essentially you prove that the two names mean the same thing or they are the same thing this in that they are identically the same object. So, there again you have an equality so even if you did not have so, in fact what I am saying is actually in many of these cases even in the case of verbal arguments. What, might happen is? For example what might happen is you might they might be a quantified formula for all x for all y something involving x and y. And it is quite possible that this holds only for when x is equal to y when x and y are the same object take the standard case of let’s say an argument like I am formulating the argument on the flag. You, read peanuts comics so nobody other than Charlie Brown loves Charlie Brown conclusion there exist at least one x and one y.

Let us formulate it better nobody else loves Charlie brown Charlie brawn loves someone loves Charlie brown. Therefore Charlie brown loves himself I mean. So, if you were to take these as take the quantified predicates then you will have two variables two bound variables x and y. And, what you are essentially proving is that that is true only for the case when x equals y equals Charlie brown. Or at least when it is since its existential we are talking about x equals y equals Charlie brown.

But, if it is not existential it might be other things but the point is that you took two distinctly named variables. And, you essentially prove that they represent the same object their names are the same object. In which case those two variables are actually equal I mean this so you have equality occurring frequently in mathematical proofs. And, so even in verbal proofs actually we would may require equality. So this thing that it is so happened that this argument did not involve equality but in general if you have lots of verbal arguments. Then, equality is something that you would have to take as a primitive binary predicate to be used in your first-order tautology. So, that is so equality has this special meaning in all forms of mathematics and logical reasoning. And, of course you have got various kinds of equivalences.

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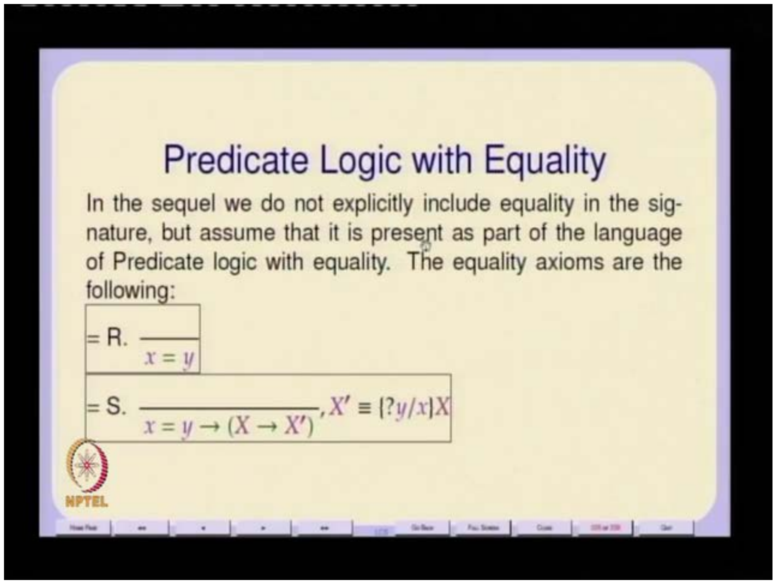

So, now so what we are doing is we introduce the third in between tier in our logical system with axioms of equality. So, these are the two axioms of equality it is amazing. So, that is there is something here, which I will explain and then will stop. Usually any equality of equivalence relation is defined in terms of those three axioms reflexivity, symmetry, transitivity. And, in the case of equality you saying that it is also congruence in an algebraic system I mean that so, substitutability. But, actually we require only these two axioms so, there is some is a question mark you see here, so this question mark says that. So, this is the substitutability property which is essentially comes from comes from Euclid that equals can be substituted everywhere for equals. But, the peculiar property of equality is that I am not obliged normally when we define substitutions we define the substitution as taking place uniformly like all free occurrences of x will be substituted by t.

If, you are looking at t for x on some term you are saying all free occurrences of x would be substituted by t will be replaced by t. This, question mark is essentially to say that you can choose some subset of those occurrences of X and replace them by y. And, that is the essential property of equality and congruence it I am not obliged to do a complete uniform substitution. I, can just choose to replace some occurrences by equal elements and I still get something that is equal. So, this question mark essentially is to denote that X prime is obtained from X by replacing 0 or more occurrences of small x by small y.

Student: Sir if that rule (Refer Time: 55:06) should be equal to

That X equals X I, made a mistake should this is a reflexivity rule. So, you just require reflexivity and substitutivity. And, then next time will show how the other properties can be by the way we have to correct this x equals to x.