Logic for CS

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Lecture - 21
Structures and Substructures

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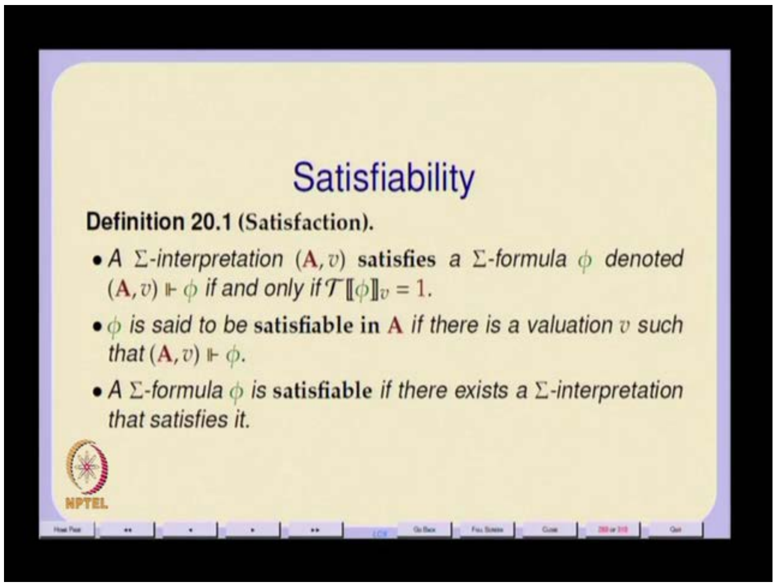

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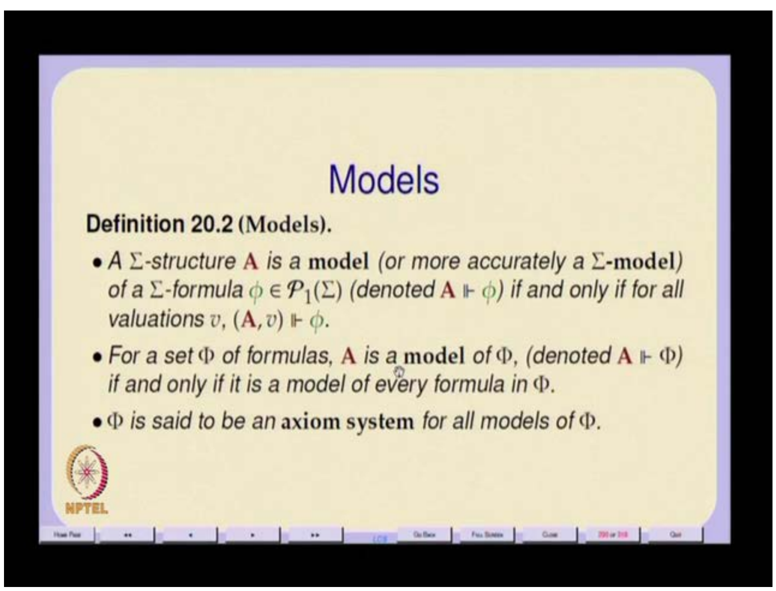

Let us look at models and satisfiability in validity for briefly. So, we look at the notion of the satisfiability we actually taken that so notion from proposition logic and generalized state the notion of a model. And, therefore from this notion of Model comes a notion of satisfiability also basically as a formula or set of formulae is satisfiable if and only if there is a model which satisfies the formula or the set of formulae.

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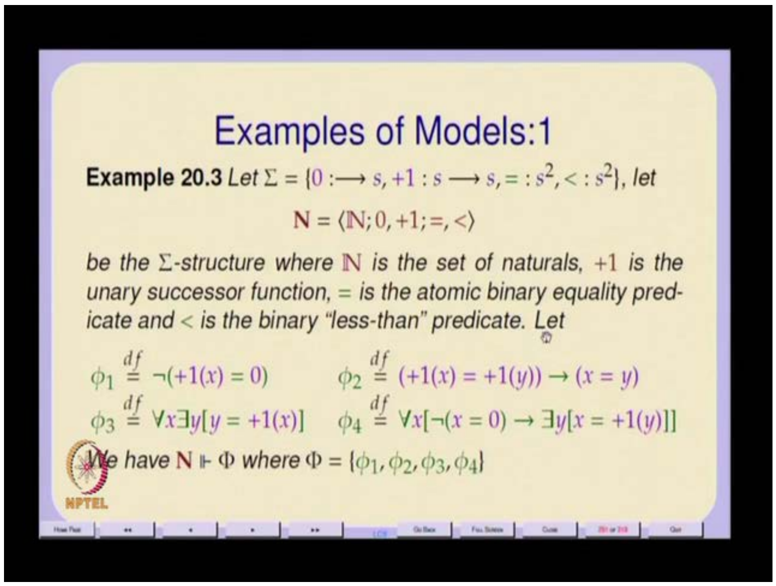

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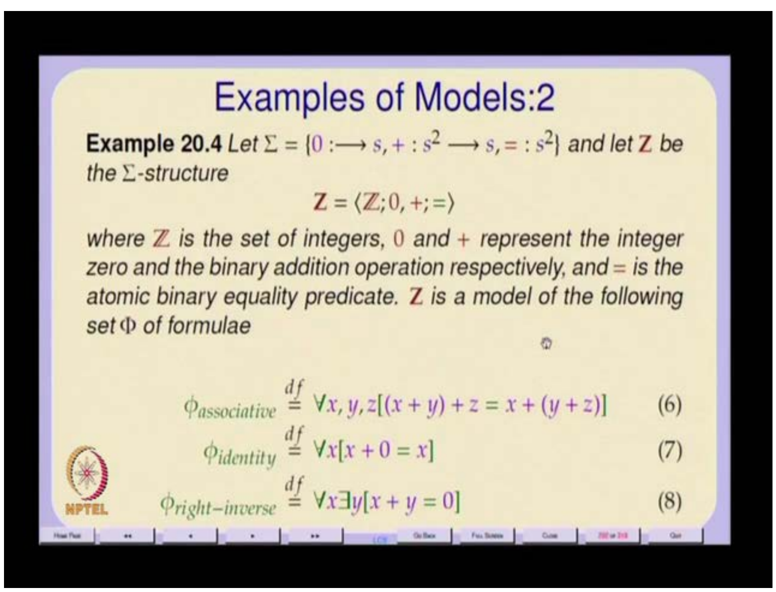

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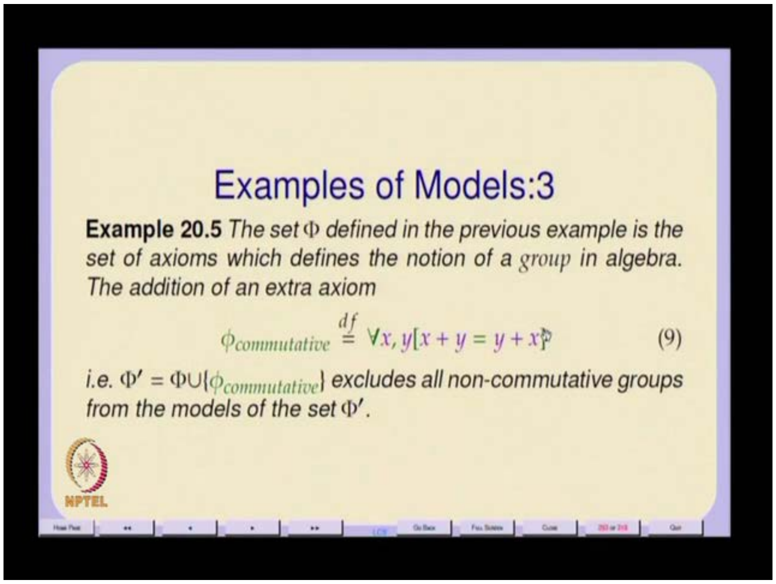

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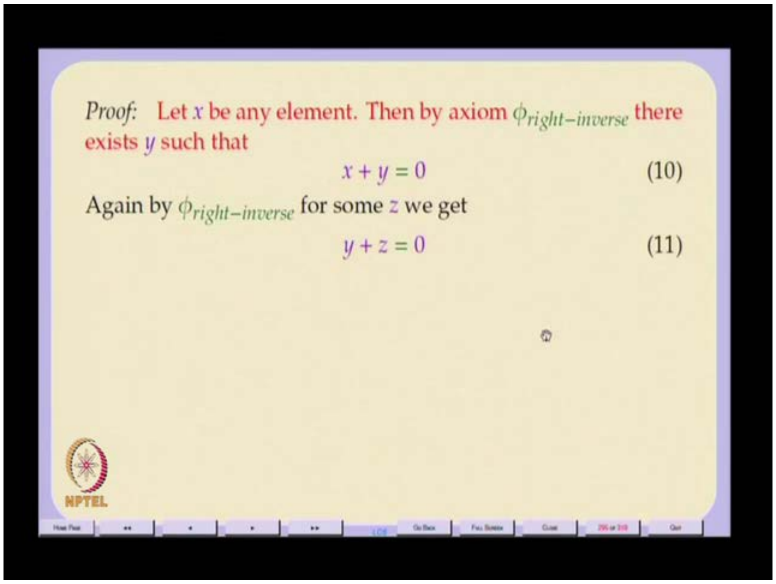

Then, simple examples and then we had these, notion of Logical Consequence which again is taken generalized from propositional logic.

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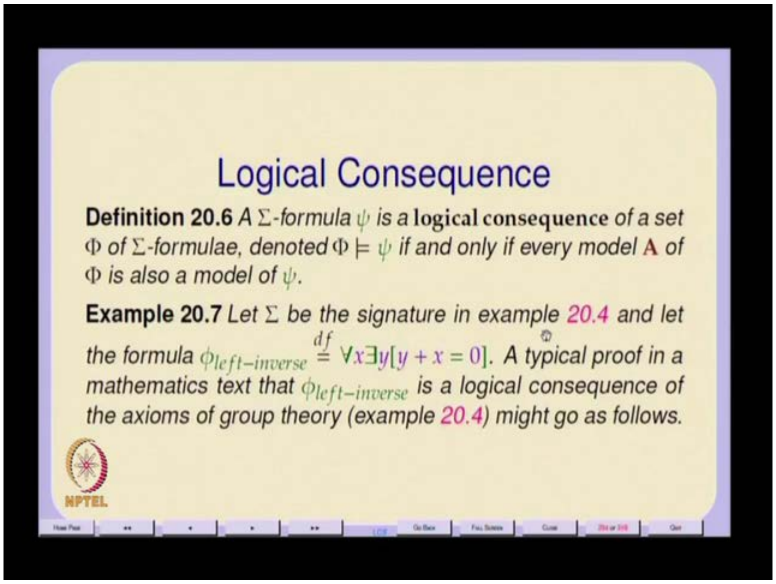

And then we had this notion of validity which is essentially of the generalization notion of a tautology in proposition logic and logically equivalent.

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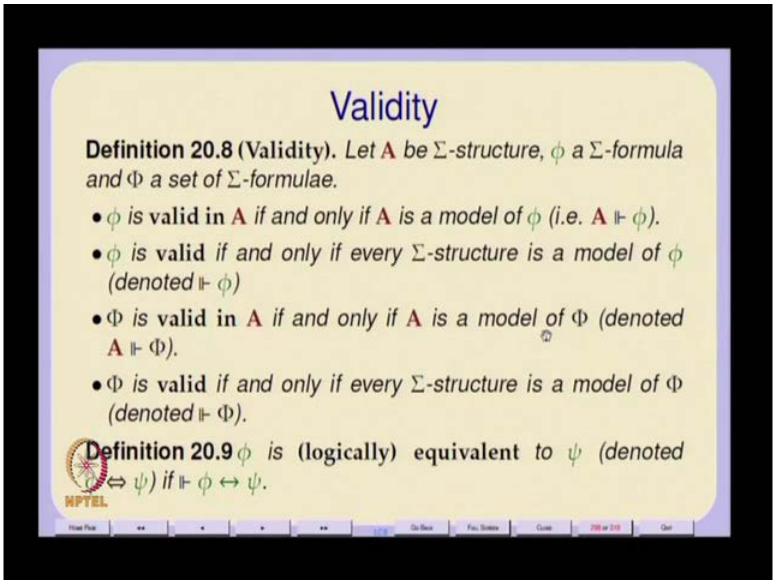

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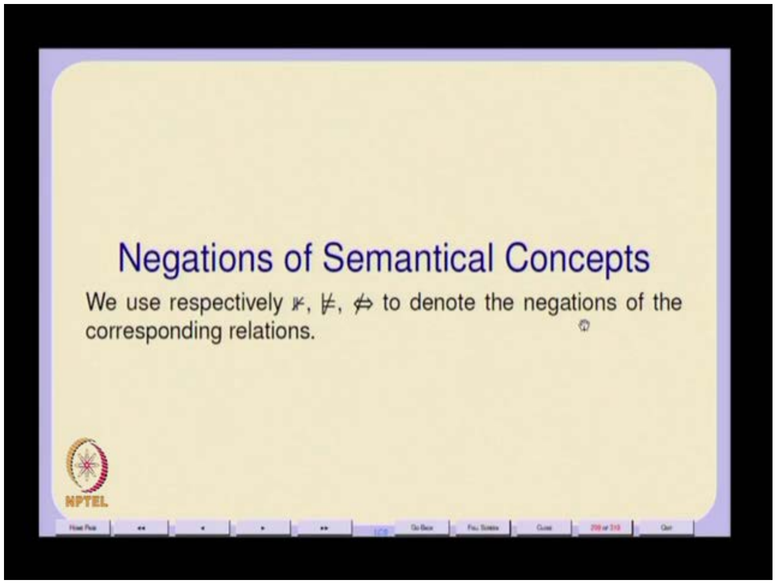

Which, is essentially a form of Validity. So, you have usual negations of semantical concepts and les look at or first order logic is of course parameterized on a signature. So, there is question of what happen when you have different kinds of signatures and some relationship between them. And, that is what we look at today. So, we look at structures and substructures and we will also lo at expansions and redacts briefly.

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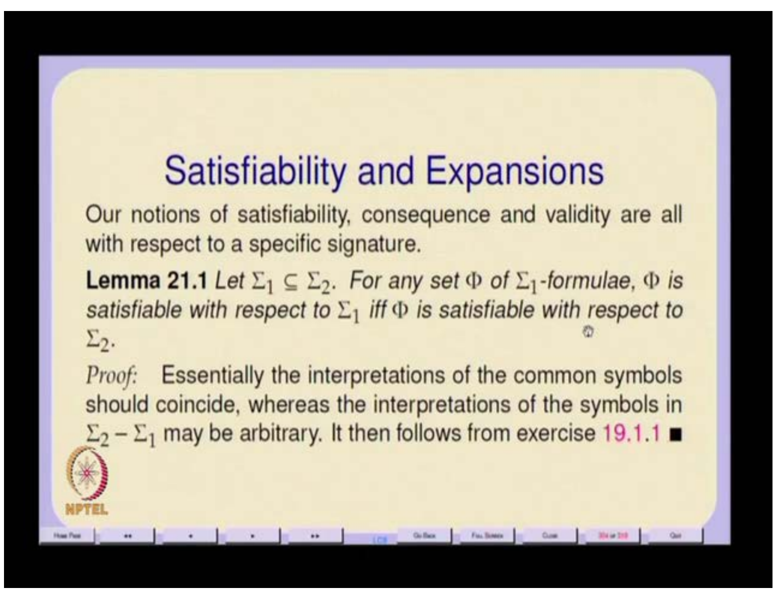

So, since that we are parameterized on specific signature we have to look at source supposing you have subsignature basically. So, sigma 1 is subsignature of sigma 2 that means that sigma 2 has all the elements of the signature sigma 1. And, we are looking at Satisfiability so for any set of formulae pi if it is satisfiable with respective sigma 1the smaller signature then pi is also satisfiable with respective sigma 2. Basically, what you are saying is that if it is satisfiable with respective sigma 1. Then the terms in the formula do not have any symbol which occurs in sigma 2 but not in sigma 1. So, satisfiability just gets carried over so the notion of satisfiability therefore is monotonic with respective the notion of a subsignature.

Basically what you are saying is that so you are essentially saying that the symbols that are common in the 2 signatures they will have the same interpretation as for as satisfiiability of pi is concerned. And in both the structure and for all other symbols are in sigma but not in sigma 1. You, can actually give them any interpretation you like and you can them some arbitrary interpretations and it is still follow. So that, is intuitively obvious so I will not grow through it, but otherwise there is an exercise in which you have to formulae prove this using semantics.

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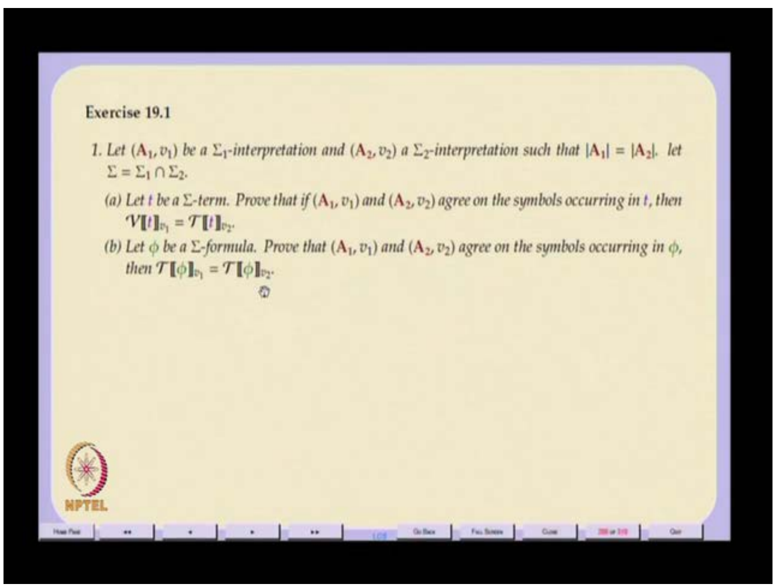

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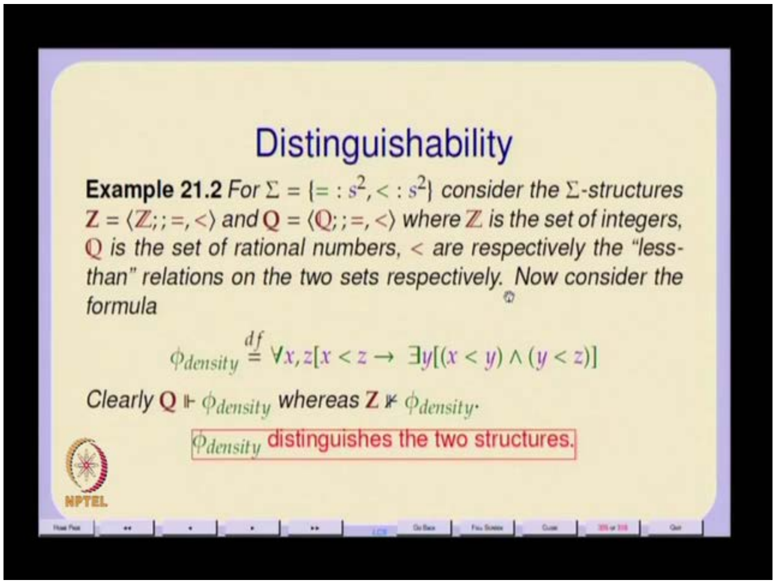

So, which are leave it as homework to do. Since, you are not doing in homework. So, there is this notion of Distinguishability also. So, what we can we can talk about a formula or set of formulae distinguish between two structures. Remember that we should always keep in mind the fact that these signatures and structures we are talking about can be quite abstract. And therefore, the only way of talking about, them that may be through a formal logical language like, first order logic. So, essentially looking at, the first order theory of those. So, we can talk about distinguishibility also using the formal language of the first order logic to distinguish between different structures. Essentially the proud so distinguishibility is has to do with distinguishibity from in the sense of properties.

So, we will see that we will come to that later so if you look at these 2 structures Which have the same signatures namely well take the integers without any operations. So, we will look at distinguishibilty of structures in terms of properties that they satisfy. And, so if you take these 2 structures integers without any operations and, we just the equality and the less than. So, the density property for example so we have less than relation in the density property essentially says that between any 2 rational numbers there exists a rational number. And, this density property for example is not obeyed by the integers given in any consecutive integers they are no integers between them. So, you we can think of this property as essentially distinguish these 2 structures. So, it is satisfied by 1 structure but not satisfied by other structure.

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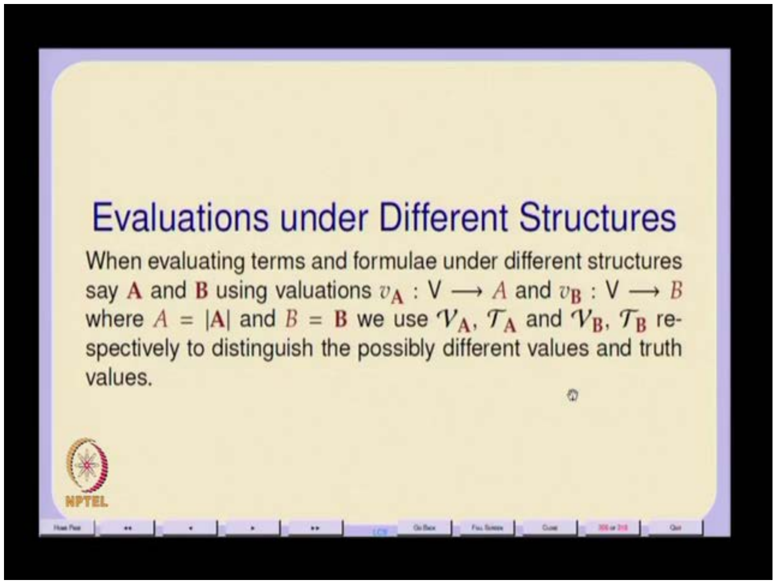

So, there is so will look at so occurs we require some small change in notation it will look at Evaluation and Different Structures. If, you had revaluating terms or truth of formulae and different structures will subscript and valuations or some truth functions so and so forth. By, the appropriate structure name so, this is one thing. And, now essentially we are ready to look at relationship between structures and notions. So, we already seen distinguishably it is quite possible 2 Structures are Isomorphic.

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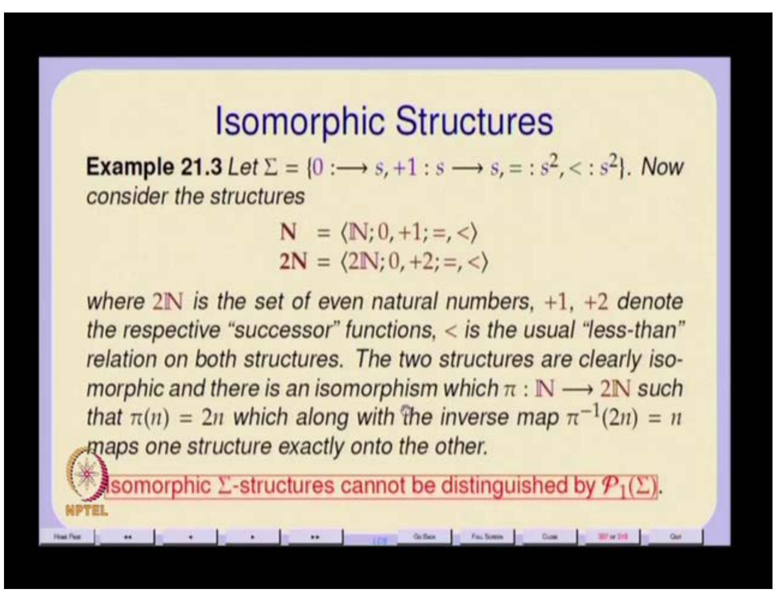

So, one thing is if you look at the naturals under the successor operation equality and less than it is actually Isomorphic to the set of even number under a successor operation. Which, is essentially adds to each number and with equality and less than. So, these two structures are isomorphic because, there is exists a 1 to 1 correspondence N goes to 2N between these carrier set. And, in this carrier set and preserves these corresponding operations. So, these two structures are isomorphic so in that sense as for as this signature is concern. If, you look at these two structures as being essentially structures with these common signatures. Then, what happens is that you take any formula with this with this signature. If, it is satisfied by 1 it will be satisfied with you other. So, the first limitation that we come across in the formulization of mathematical theories is that, first order formulization of mathematical theory will not be able to distinguish between isomorphic but distinct in structure. So, the two structures are isomorphic there is actually no first order formula which will distinguish them. And, that is something will be prove will prove next and that is very easy.

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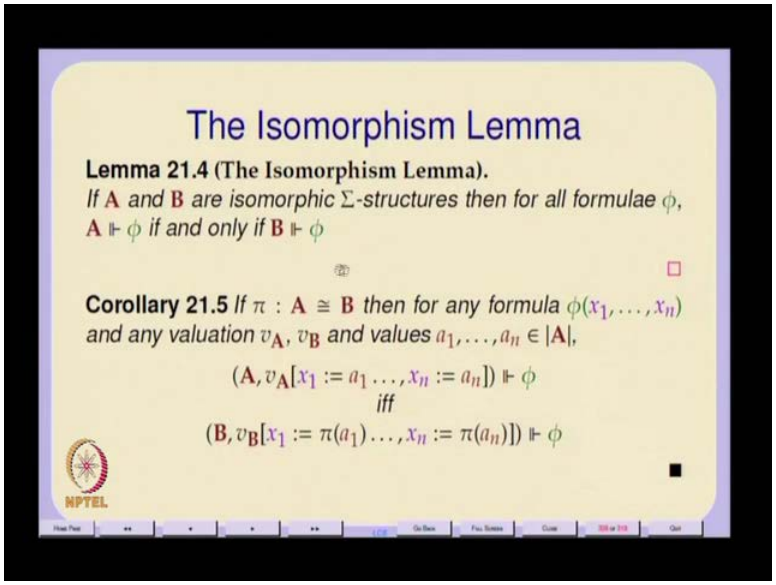

So, this is called Isomorphic Lemma so if A and B are isomorphic sigma structures for the same signature sigma then for all formulae pi A satisfies pi if and only if B is satisfies phi. Assume, that there is an there is a 1 to 1 correspondence between their carrier sets. Which, also preserves the operations and the relations over this function and pi. Note, that in the case of isomorphism if there is a pi from the carrier set A to the carrier set of B. Then, there is a pi inverse also a function which is also a 1 to 1 correspondence from the carrier set of B to carrier set of A. And, further pi composed with pi inverse gives you the identity on 1 set pi inverse composed with pi gives you on the other set. So, all these properties are satisfied by typical isomorphic right. Now, all that we are saying is firstly the first thing to realize is that.

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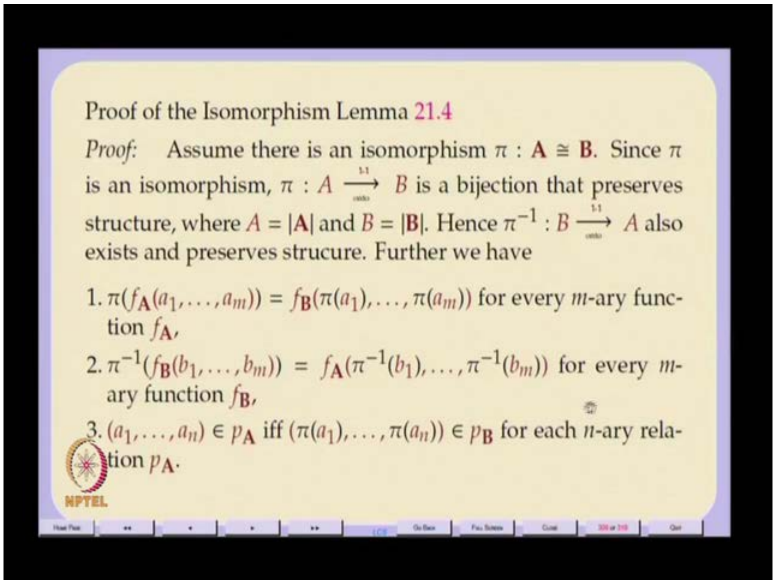

So, new just assume there is an isomorphism there is that is also an input which there is also in inverse function pi inverse which is 1 to 1 and on 2. And, of course these properties are all satisfies. So, you take any term in the structure A and apply pi to it corresponding term where pi works inside the corresponding function fB. So, you take the images of these arguments A1 to Am. Which, are pi 1 to pi m these equality would be would hold between the structures. And, similarly if you take any term in the in the structure B through pi inverse you can actually get a corresponding term in the structure A. And, the same also holds for these relation pA and pB.

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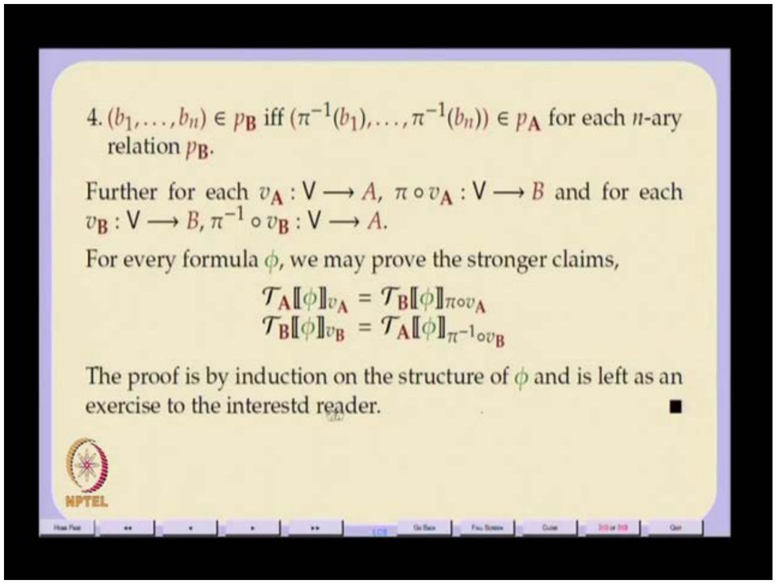

And, then what you can prove is this strong a claim and the other thing to realize is that in all these cases you are considering valuations. Whether you are considering valuations of terms or truth values of formulae the only variables that are important are the variables occur free in the formulae or in the term. So, for every formula phi so for every valuation VA there exists a corresponding valuation pi composed with VA. And, for every valuation VB there is corresponding a valuation pi inverse composed with VB. And, what you can show by induction on the structure of formulae is that the truth value of phi in the structure A under any valuation VA is equal to the truth value of phi in the B under the valuation pi composed with VA. Which, is a valuation in B.

And similarly, for the inverse case so for any valuation VB in the structure B the truth value of pi under that valuation is exactly the truth value of phi under pi inverse apply to VB. Which, gives you valuation in VA and it is easy to show this. So, you prove this by induction in the structure of the formula phi and it is left us an exercise. Remember that, the process of evaluating these you will also have to use at some point the valuation can functions VA and VB. But, the interest but the important thing is that your phi and pi inverse also carry through to the language of terms. So, in fact what happens is you have this so as you can actually prove that you will have to first prove that VA for any term t under the valuation VA equals VB of the same term T under the valuation pi composed with VA and, vice versa. So, that is something that will be important.

So, corollary of this is that essentially that for any formula phi which pre variable x1to xn I do not need to necessarily consider valuation separately. I need to I can consider any kind of valuation provided pi respects the value of the free variables. So, in fact as a as the corollary so the only things that are actually important or the free variables that are occur in the formula pi. And, so you take any valuation VA and you take any valuation VB, VA may not pi composed with may not be equal to pi composed with VB. However, you take the valuation VA for every for the variable x1 to xn. If, you preserve in the valuation VB the values pi of a 1 to phi of An it is sufficient. And, in fact you can show that this interpretation is a model of phi if and only if this interpretation a model of pi for all VA and VB.

So, that basically shows that all the valuation functions for all function for all variable which are outside the free variable set of these formula pi actually unimportant and does not matter. So, is there should the isomorphism should be preserved for that sub set of free variable which actually occurs in the formula. So, which means this now, this is true for every formula phi which means for any set formulae capital phi the structure A is a model of capital phi if and only if the structure B is a model of capital phi. So and this is true for any set of formulae phi capital phi then therefore means that using just first order formulae one cannot distinguish the isomorphic structures. The way could distinguish using the density formula one could distinguish between the rationales and the integers. For, example you cannot do the same thing for isomorphic structures.

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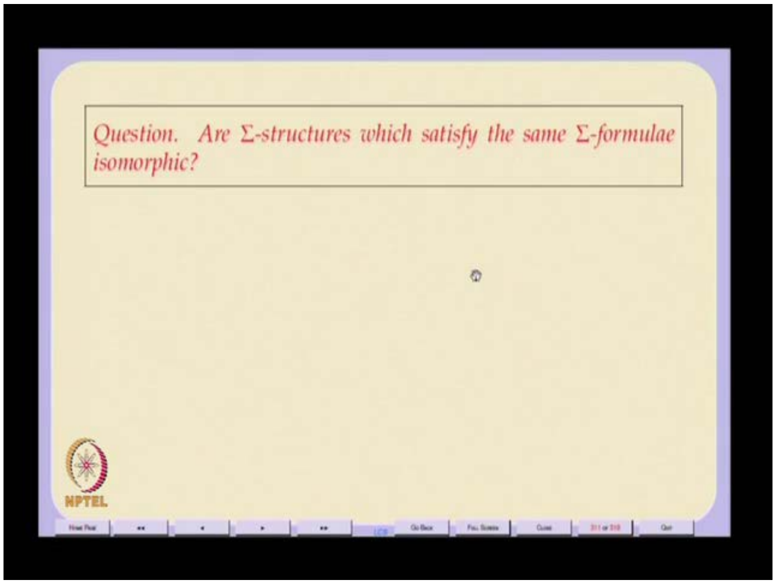

But, what this raises is another important question and this is the question. That has that was quite that is quite important and at we should answer it at some point. Supposing, I have 2 sigma structures which satisfy the same formulae then, are the necessarily isomorphic. So, this is the question that puzzled a lot of logicians actually in the initially years when first order logic was formalized a through various means. So, the answer actually lies in what are known as non standard model which will come to later. But, what it means is that the converse of the converse of isomorphism lemma does not necessarily out and that is an important thing.

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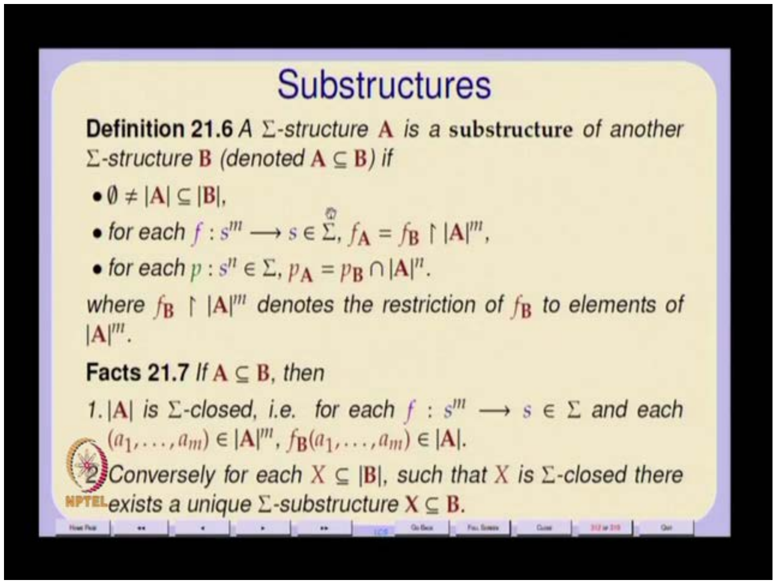

So, we look at that some point right, then we might look at structures are some out related through a Substructure relation. So, will say that a structure A is a substructure of another structure B where both of them have the same signature provided so, previously what we considered sub signatures. Now, we are considering substructure with the same signature. So and so a, typical example of course even number and the natural numbers. So, the even numbers are the sub structure of the naturals with the same signature in the example that we did. So, what we are saying is now will use for the sub structures will use the subset notation. Because, essentially what we are saying is that the domains are related by the sub set relation. So, the domain of the sub structure as a sub set of a is A sub set of the domain B. Or other the carrier set of this structure A is the sub set of the carrier set of the structure B.

But, of course one has to the notion of structure require something more there might be you might you cannot choose an arbitrary sub set. Because, the functions in your signature a may not be closed on the sub structure they might be closed on the structure B but, they may not closed on the structure A. For, example it is possible there is a function f which takes value A1 to An from A. But, the result of applying in this function on this might be some value which is in B minus A. So, there is when it comes the notion of the substructure it is not just enough for them to for the domains to be related by the subset relation. It, is also important that they be sigma closed. So, what this means is you do not really care much about, what happens in the B. But, in the case of A which is which is suppose the set A the domain of A. For, every function f for every emery function f the corresponding function in the structure A should be a restricted version of the function fB. So, what you are saying is that, essentially what you are saying is that fB. Supposing you restrict fB the domain of fB just to all the elements in the set A. Then, fA should not be undefined for any topple firstly and it should be closed underlay. So, you just so every restricted to this is fA so, what you are saying is so it has to be sigma closed in the sense that for each f, f should be completely closed on A. there should not be undefinedness by restricting your domain to A. And, the in the case of relation in the case of relation p actually the notion can be weaker. All that we are saying is you take those for any emery relation you take only those N topples which, are sub set of N topples in A. That, is the you have this restrictions so if you do the restriction you are sub structures should still be sigma closed.

And if, it is sigma closed then you have you say that A is sub structure of B. So, in this case it is important to realize for example if, you instead of the even numbers if you got to take the odd numbers with just equality less than or sums of thing. You, can have a successor function which again adds 2.But, the movement you take this signature to include an identity element like, 0 for addition. Or if, you even if you include the operation of addition in your signature then, what happens is you are odd numbers under addition are not sub structure of the naturals. On the other hand your even numbers still remain sub structure of the naturals. You can you some other kinds of things so we are using the fact that sum of 2 odd numbers and even number, How of the product of 2 odd numbers is an odd number? So, supposing you take the structure naturals with multiplication. Then the odd numbers, Do the form of substructure under multiplication of the naturals? And that is the question you can ask, They could form a substructure? No there is a problem there especially with 0 they will not form exact substructure because of the presence of 0 and the naturals where as there is no 0in the odd numbers.

Student: (Refer Time: 24:04)

If, you say odd numbers less 0 no then that is hack which will have to work out before we take any specific judgments. But, if I have to guess I would guess that there would still be a problem. So, what I am saying odd numbers do not I mean unless you can prove it odd numbers including 0 under multiplication unless you can prove this is actually a substructure by making its by and showing that it is sigma closed it is not clear.

Student: (Refer Time: 24:41) we can say that natural numbers are also not includes 0.

Because speaking we can say that natural numbers also will not include 0.

Student: (Refer Time: 24:56)

We, include that might be for CBSC mathematics it is not for me and then 0. There is distinguished element,

Student: (Refer Time: 2:06) why are you distinguishing between

No it is constant so actually what happens is supposing you think of the natural numbers as being generated let us through the through a successor operation. So, you can think of it as simple elementary language we generate. So, any generation process would require a seed a basis of the induction. And, they that basis of the induction would form the distinguished element. So, what you are saying is that the naturals can be generated by the successor operations starting with 0. The odd numbers would have to have a distinguished element 1 if you have if you just to if you restricted yourself to the odd numbers. You, can think of the odd numbers as being generated by a successor operation which gives which adds 2. But, starting from 1 0 and 1 can both be distinguish elements. But, it depends on what is your focus I mean your signature essentially gives you focus.

So, that is what you and the there is way of there is another thing also you take something that we did yesterday was this notion of a right inverse. So, what did we have we had that an universal existential formula. Which was let us go back to that look at this formula phi 3. So, look at this formula pi write inverse so this is an extremely weak formula that is if you are focus is on groups. If, you are talking about group theory and the group axioms then this is very weak formula on the other hand is pi identity is not weak formula. Because, I mean the question of 0 being also the left identity is provable in the theory of groups. So, that distinguished element 0 if it is right identity it is also left identity that is provable. And, therefore this identity axiom is not weak axiom. On the other hand the right inverse axiom even though we can prove that the right inverse is also the left inverse it is still weak axiom. Because, of the fact that in the case of groups your inverse is suppose to be is unique.

So, neither so even so, the phi left inverse is provable is phi right inverse. But, the question of uniqueness of an inverse of an, of x is not immediately prove I mean is not immediately capture by the axioms. So, in fact what happens is as a result if you are really looking at group theoretic axioms. Then, you would have to stringent this to a formula like this firstly you will have to include in this signature and inverse function.

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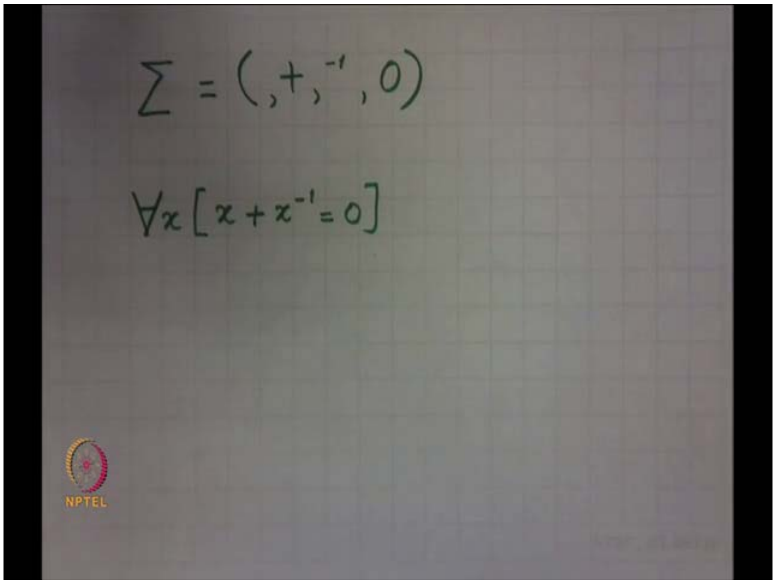

And, secondly you will have to strengthen this formula for all x, x plus x inverse equal 0 it would have to become a inverse of formula the possibly. And, which means what you are saying is that I had an original signature we did not have this inverse. And, now I am going to actually require this inverse this inverse also and only then I have essentially a group. Strictly, speaking or mathematics books we just say that the inverse is actually a derived function from the product operation of the group are not be in strictly in correct first order view point. Because, they are not ensuring the express ability this is this is a matter of expressive in the language. So, the expressiveness of the group theoretic properties cannot be done by these weak formulae it has to be done by a, strong formulae like this.

And, so which means actually you should restrict your product operation in an identity element just for monaids for group you should actually include the inverse operation also if you want in the signature. Those are something we will see, at this is like a sot of preview. So, the distinguish the existence if a distinguished element a constant often important in this case without that distinguished element you do not know the existence of an identity it is part of the axioms so you have it in the signature. And, similarly to express all group theoretic properties in first order logic all first order actually let us put it this way. All, first order group theoretic properties in first order logic first order first order theory of groups you required to also passiveness existence of the inverse operation. It, is not enough to just positiveness of the existence of the element for each element ay which access an inverse. We will come those mat I mean if you have a time we look at the notion of the express ability. But, those are certain important things in any logic I mean many properties think of this, way different valid formulae. The number of different properties could actually be accountably infinite where as the number of different formulae you have in a language. Which, has generation mechanism is only accountably infinite. So it is clear that any first order theory you will not expose all the properties that are you will not be able to express all the properties that that possibility exists for some mathematical theory. You, will require higher order logic but even in higher order logics you will always remain the movement its formal language you will remain within the domain of accountability.

So, if there are any uncountable number of properties which is quite likely there are uncountable there will be. Therefore, be an uncountable number of properties which is not be expressed in the language. So, that there are notions of expressiveness which are important. So, there are obvious limitations impose by compos ability. So, let us that was digression which we should and we should go back to the notion of structures right. So, we had isomorphic structures and we have this notion of substructures. And, so here we have the obvious examples on the substructures. Which, have already said the even numbers are under this operations are substructures of the naturals odd numbers may not be there are not closed under additions do not form a substructure of naturals.

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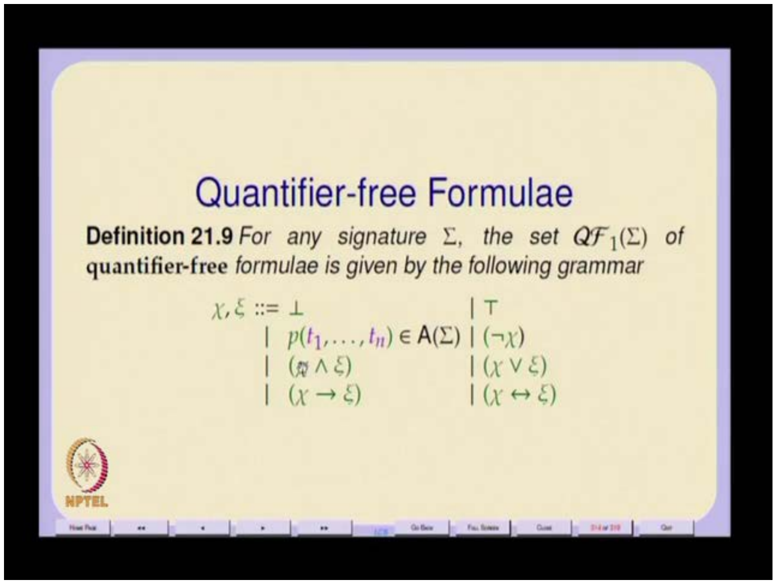

Let us look at the notion of a Quantifier- free Formulae. So, what happens in all if you look at the axiom set you have seen so far. Let us a group axioms is that the formulae all had an certain structure in which you had an sequence of quantifiers. And, then a body of the formula which was essentially quantifier free. Which, was free from quantifiers of course I have first order language does not put that restriction all formulae are not likely to be there. You, can have connectives proposition connectives mixed up with quantifiers can have other quantified inside when have other proposition in arbitrary raise.

So, we can think of since there is sufficient number of axioms. In fact most of the axiom systems we are talking about, will actually have the structure of a sequence of quantifiers followed by a body which is quantifier free. So, we can talk about quant the sub language of quantifier free formulae. As one which is just made up of the atomic formulae and the proposition connectives. So, this essentially forms let see the body of the formulae. So, this formulae for example is assuming that equality some atomic predicts this formula is quantifier followed by quantifier free formula so, it has body which is quantifier. So, for quantifier free formulae of course it is this was an easy lemma to show. Which, is that if A is a structure of B.

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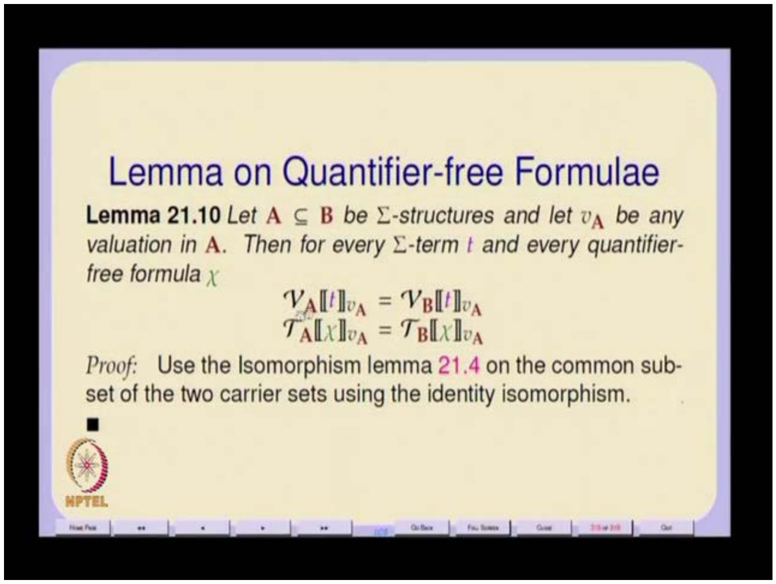

Then and VA is any valuation then of course the notion of so the values of terms of preserve under that valuation. And, of course when we have talking about a sigma term T and a Quantifier free formula if it can have valuation in A means that anyway all the variables get values only from A. And, they do not all the variables and T gets values from A they do not get values from B minus A. So, what happens this is the values are preserved and truth values are also preserved of quantifier free formulae.

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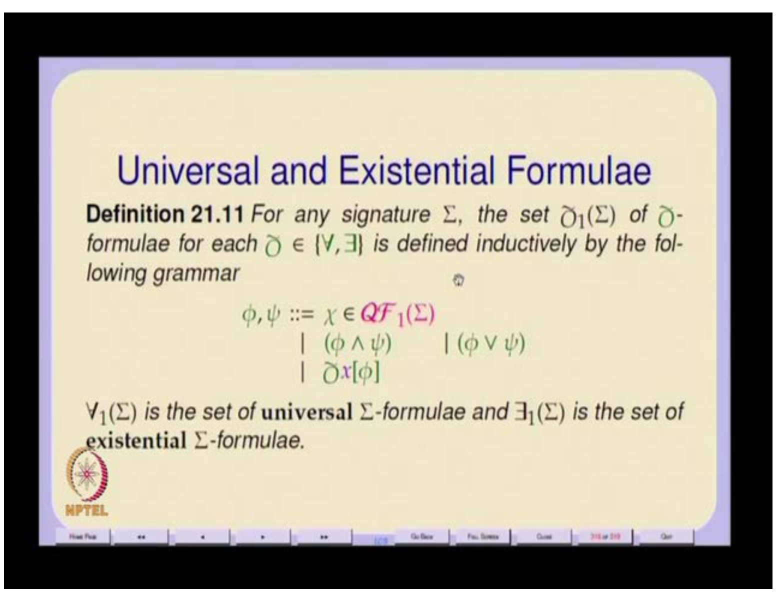

The reason for looking at quantifier free formulae is, look at the bodies first and, then look at certain classes of formulae. So, if you if you look at group theoretic axioms if I take the complete set of group theoretic axioms. Then, they have and if I include this that right inverse axiom then all the axioms are consistence of sequence of a universal quantifiers followed by quantifier free formula let they have the structure. So, the axioms of any algebraic systems are likely to be of that kind. So, if so the particular formula phi right inverse was it was the only formula which had mix of quantifiers. But, if you take a phi identity and phi associatively and you take this then all three of them. Which, actually define group the theory of groups first order theory of groups they all have formulas. Which, are essentially universal formulas right and so for any signature sigma the set so we can talk about universal formulas And, existence formula right I mean and mixtures of them.

So, here is a so take any quantifier free formula and you take combinations of and or. The important thing here is, I am not including negations because negations introduce arbitrarily as they could be negations inside each of these chi's may be. So, negations occurs inside already they already been pushed inside. So, that the negations do not appear outside and I am taking may be ands and ors as many I want. And, then I am closing with some quantifiers so I am closing with the same quantifier cube in each case. So, a cube formula is so you have universal formula will have sequence of universal quantifiers. And, an existence formula will have sequential of existence quantifiers. In the case of group theoretic axioms essentially you have universal formulae which, define the group theoretic axioms. So, you have the set of universal formulae, and existence formulae. Which, are sub languages of the languages of first order logic.

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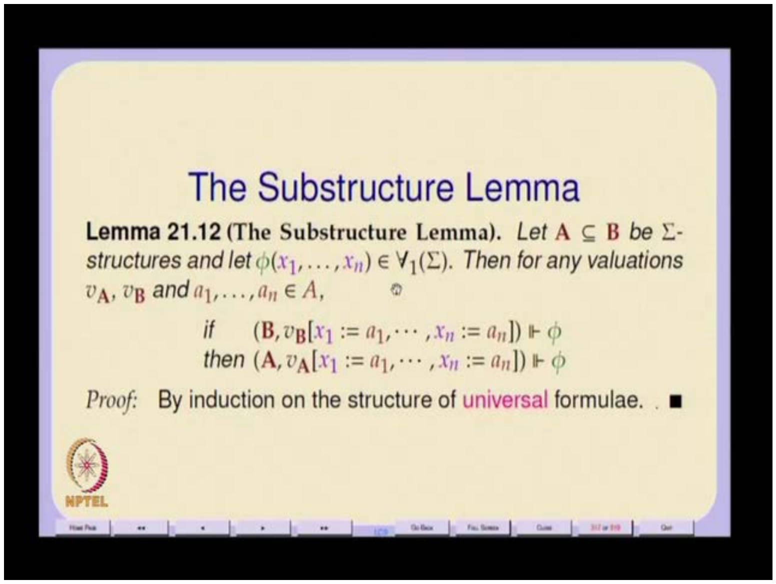

So, the sub structure lemma essentially says that if A is a sub structure of B and phi is some formula. Notice that universal formula could still have free variables I mean I mean not quantified all the variable that occur free in phi. So, it could I might quantified over sum of them only. So, if phi is a universal formula with free variables x1 to xn. Then for any valuation VA, VB and any values A1 to An belonging to the belonging to the domain of A of this of the smaller of putatively the smaller structure A. If, B for all those values satisfies phi then A would also satisfiable. So, what we are essentially saying is that property of sub structures can be preserved in a positive fashion. So, the notion of so anyway this is this is something that you can prove straight by induction on the structure of this universal formula. So, that is not serious issue but the importance of structure lemma. Essentially, says that the algebraic notion of a sub group of a group is preserve. So, all the properties of group restricted to values from the sub group or preserved in the sub group. You, take a lattice you take sub lattice, so lattice is also an algebraic system so the least of bound greater latest low bounds top and bottom element so and so forth. You, take sub lattice and if you restrict your property to the truth value of statement which uses only a values from that sub lattice. And, it is true in the larger structure then it is also true the smaller structure. And, so this is of course notice that this is this is not an if and only if it is one way. So, these are so what we have got today essentially a, formalizations for things like subgroups sub lattices sub monoids. And, for so that we are not completely so that we can actually move between within the same signature we can move between in some way. In certain cases in one direction from super structures to sub structures. And, in certain other cases like in the case isomorphism’s we can move both ways between structures and superstructures. These are some formalizations are required essential there necessary evils before we come up to the more interesting aspects of first order logic. (Refer Time: 41:51).