Logic for CS
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Module - 1
Lecture - 20
Models

I will, start with Models and in a certain sense this is where first-order logic as this is where logic should have initially begun in the sense that, if you look at logic as a essentially a description language for regress mathematics then essentially we your actual mathematics and there the description of mathematics begins essentially with first-order logic. So, and therefore it can be applied to the description of let us say things like axiomatic set theory or number theory and so on so forth. In a certain sense, today we will be looking at certain notion of models, which is and essentially look upon a mathematical theory as a certain model of a set of formulae and that is what we want to actually that is what the original purpose of logic as a language of formulization came up with. And, which historically of course it comes up from what is known as Hilbert's program which is most which actually dominated most of 20th century mathematics when in 1900 David Hilbert actually outlined what he called in the what has been what has come to be called as Hilbert's program. And, that was at essentially there is a lot of mathematics developed over 3000 years and it is not clear whether lots of new theorems have been proposed lots of new proofs have come up. And, lots of generalizations especially like in analysis and algebra and the important question that was bugging Hilbert at that time was that is all of mathematics somehow internally consistent.

And, Hilbert's program was designed a was essentially a question wanting the mathematicians to address this question of is all the mathematics we are developing actually consistent. Because, there is a danger that you can prove a lot of theorems if, you are internally inconsistent in fact you can prove practically anything you want if you are completely inconsistent. So, the overriding question was whether mathematics was internally consistent. And, therefore the overriding question was as a consequence of that, was to actually axiom attached mathematics in such a way that you could rewrite and reformulate all the branches of mathematics starting from some foundation in a non-circular fashion. And, therefore and then at least establish consistency in some way that was one thing. And, the second question of Hilbert's program was if you were to algebrize mathematics. How much of it can actually be done in some mechanical fashion essentially by a machine? How much of it can actually be automated? The what can you prove theorems in some automated fashion both the notions of so, both of these questions. Which, formed important parts of Hilbert's program of about 20 questions Hilbert of course, was a great all rounder kind of mathematician. So, his 20 questions address to the world congress mathematicians included problems from other areas like, arranging from number theory to algebraic geometry, through analytical and differential geometry all the way through you know Hilbert spaces and so on so forth. And, it also included things like Ramón hypothesis and the continuum hypothesis and so on and so forth. It included a lot of stuff and Hilbert as a mathematician was actually empowered because, of his all round contributions to various branches of mathematics starting from number theory to algebraic geometry to Hilbert spaces and to parts of physics.

So, he was peculiarly situated and able to address the question of whether this entire body of knowledge is consistent is it possible to give it a consistent formulation. So, that no inconsistencies can be shown firstly, is it consistent secondly can it be reformulated in such a way that no inconsistencies can be shown. And, thirdly the question is how much of it can actually be automatable? I mean is it possible at if you can automatize you can automate the if not the stating of theorems can you automate the proofs of theorems. So, can the question of proof be automated since a large amount of proving of an algebraic nature involved essentially symbolic processing the question of whether, it can be mechanically evaluated and proofs whether proofs can come mechanically was a question that bothered Hilbert. The, result of Hilbert's program and the questions he raised in 19 o 1 1900 or 19 o 1 all congress of mathematicians was actually, that it did guide the development of logic and mathematics essentially through the 20th century. So, and it gave raise to these notions like computability. So, one of the most important and landmark results was assuming a notion of mechanical evaluation. Without actually, having any formal model of it except what was available historically through let us say the works of Babbage and Pascal and Leibnitz except for that. So, there is a famous proof by Gödel which first showed that first-order logic the Hilbert proof system for first-order logic is essentially consistent and complete. And, when you apply it to something like number theory essentially through piano arithmetic. There, are propositions which cannot be decided by machine basically and that there are theorems.

So, he linked to the notions of consistency completeness and decidability in such a way that if number theory is consistent then it is not complete. And, by using a technique known as arithmatization he also showed that if, number theory is consistent. So, the consistency and completeness of first order number theory dependent on was dependent on whether the first-order logic that was used to describe number theory is itself consistent and complete. So, Gödel proved that first-order logic itself if you look at it independently of its application its axioms are consistent and complete. But, when you applied it to number theory to get what is known as first order number theory that is if it is consistent then it is not complete. And, by a notion of arithmatization of syntax he essentially showed that it is possible to have theorems in first order number theory. Which, cannot be which are not decidable by any mechanical means I mean. So, even before the notion of even before the formalization of exactly what is a mechanical means of computing Gödel could actually come up with this landmark kind of result. And, after that came the notion of formalizing the notion of computing from so the works of church basically the lambda calculus and combinatory logic. And, after that the Turing machine after that the proof that all these different mark of algorithms after that the proof post machines the proof later that all these machines all these models and machines are actually equivalent in their power of mechanical evaluation.

And, therefore so you can see that it has actually dominated a large part of a logic and computer science. And, it dominated mathematics to the extent that the fields like topology and analysis were formulated in some in an axiomatic fashion after Hilbert’s talk essentially.

So, the formulations of statistics by Colmo graph the formulations of topology by corotousky and so on so forth. As, axiomatic systems essentially came up in the 20th century and before that there were essentially desperately mathematics and it was not clear how they all repeated together. So, the notion of models therefore becomes important. So, this is model essentially will be a theory a mathematical theory for which you want to give some axiomatization for example. So, will look at that the notion of models.

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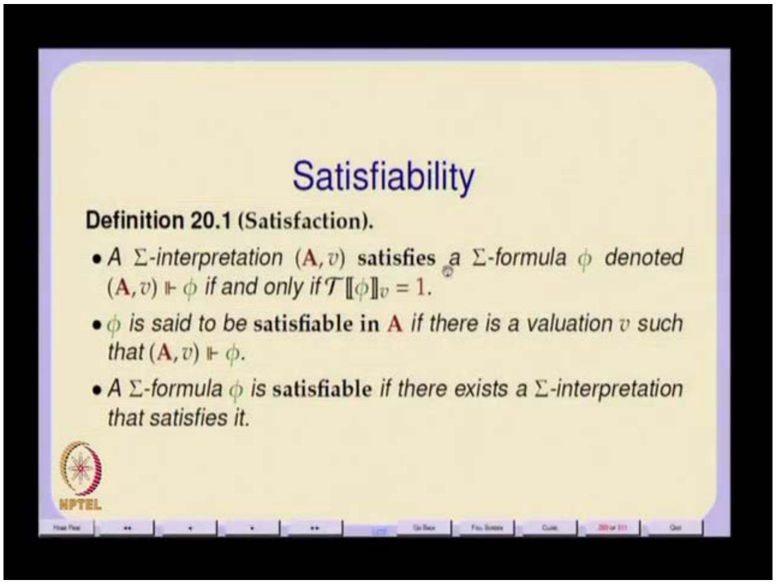

But, before that we have to look at basic things like Satisfiability. So, the notion of satisfiability is of course very simple in the case of first-order logic we are looking at sigma structure. And, is and a language of first-order logic for the signature sigma. So, and then you have the notion of a sigma interpretation which means you have a valuation for the variables along with the sigma structure. And, under that valuation you say that this structure A with the valuation v is that is an interpretation. This, interpretation satisfies a formula phi if and only if the truth value of this formula phi under that valuation v is 1. So, phi is said to be satisfiable in A if there is a valuation v which will make it true. And, a formula is satisfiable if there a sigma formula is satisfiable if there exists a sigma interpretation that satisfies it. So, which means a sigma structure A and, a valuation v it satisfies it.

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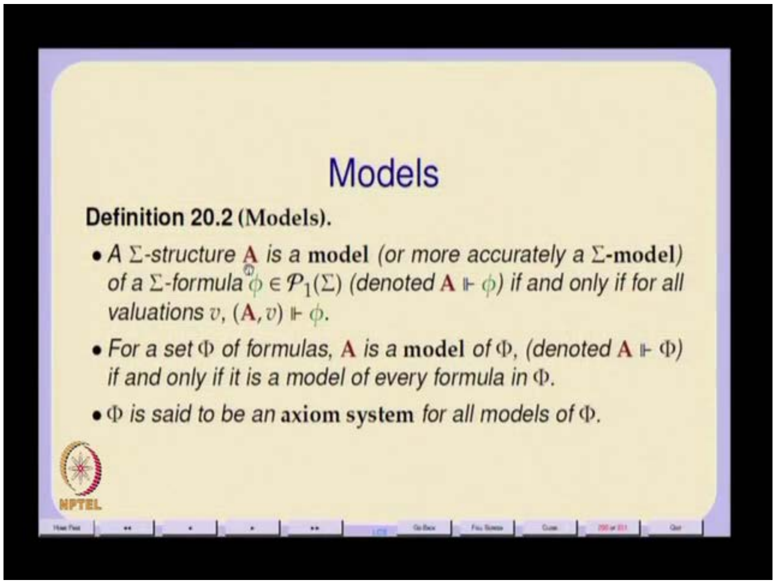

And, if it is so satisfied then, we will call this sigma structure A along with the valuation so we will say that this structure A is a model more accurately a sigma model. If, this formula phi is true for all possible valuations. So, in that sense phi acts as a property that is true of the entire algebra it is a statement of truth for so it is independent of valuation firstly. And, you are saying that so therefore this is this entire algebra is a model of this particular statement phi. So, we can of course instead of having a single statement phi we could have a set of statements capital PHI. And, we say A is a model of capital PHI and we denote it by this if and only if it is a model of every formula in capital PHI. So, by the way we just assume that this capital PHI is a non-empty set. And, then we can call this phi to be an axiom system for all models of phi. So, you can think of capital PHI as a set of all axioms which, define all those models for which all the formulae in phi are true independent of valuations or variables. So, that is the so the notion of a model is then so is then identified with the notion of an axiomatic system. So, what you are essentially saying therefore is that? If, I can find some models for a set of formulae phi. Then, I can think of this phi as essentially defining an axiom system for all those models.

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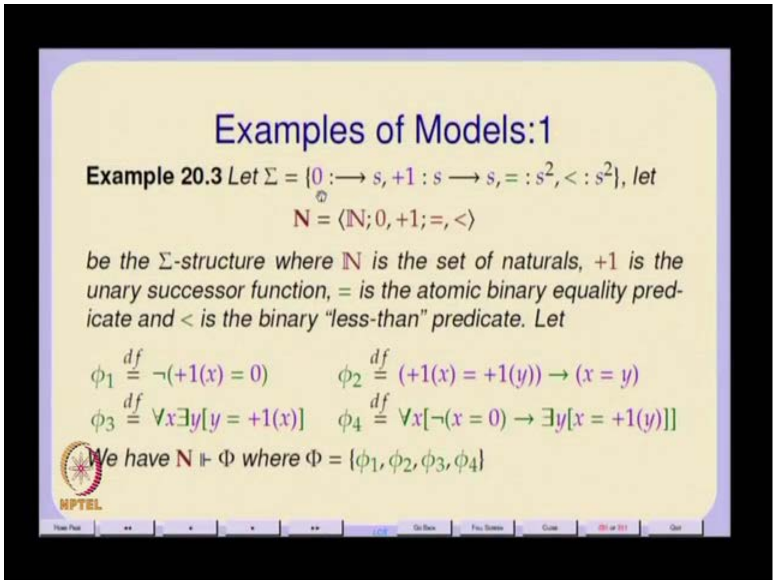

And, which actually is close to what happens in the way we describe most mathematical theories. So, here let us take this is actually an elementary subset of piano arithmetic. Basically, you have this set of natural numbers generated from 0 by a successor operation. So, here is sigma here is a signature sigma. And then, so in addition to these two operations there are these two relations one is the equality relation and other is less than relation so for example. So, this is in some sense elementary form of the piano arithmetic but will look at piano arithmetic later detail later the full axiom system. So, the notion of first order number theory as a first order theory of piano arithmetic is something will look at later. So, let us take this sigma structure and let us take the natural numbers under these operations and these relations. Then, you have these various formulae one. Here is, is formulae which says that essentially 0 is not the successor of any element I mean this. So, this first formula phi 1 would essentially be read as the successor of x cannot be equal to 0. And, of course we have x is a free variable here remember that. But, so this is a so will just keep that in mind will come to that what to do with free variables later. So, this essentially says if, I have a pattern like successor of anything it does not matter and that value cannot be equal to 0.

Here, we have that here we have one of the piano axioms actually. If, the successor of x is equal to the successor of y then x must be equal to y. Here, is some here is trivial statement for every x there is a y which is its successor. You can sharpen this statement make it more accurate like saying that there exists a unique y. But, unique y essentially means I will be adding another conjunct to it says for all z such, that z is equal to plus 1 of x y is equal to z. I mean that is how one would define uniqueness. But, anyway let us this is a simple example of a model and here is something which says that if x is not 0 then it must be the successor of some y.

So, these are the formulations of these so what is now intuitively clear though at the moment we do not have a proof systems so we cannot actually prove it. But, what is intuitively clear is that this structure N satisfies these four formulae phi 1 to phi 4. So, that is so N is therefore a model of these formulae. Notice that some of these formulae are free variables some of these formulae do not have free variables. So phi 3 and phi 4 do not have any free variables. So, they universally quantified. Where, is phi 1 and phi 2 have free variables and yet somehow the truth of phi 1 and phi 2 is independent of those free variables is independent of any valuation for those free variables that is important to realize. In fact that is the key to what most mathematistics do they do not actually explicitly put your put a quantifier anywhere in their axioms is just take the axioms with free variables. And, it is assumed that they are universally quantified and that is something that needs to be satisfied. And, that can be very easily justified.

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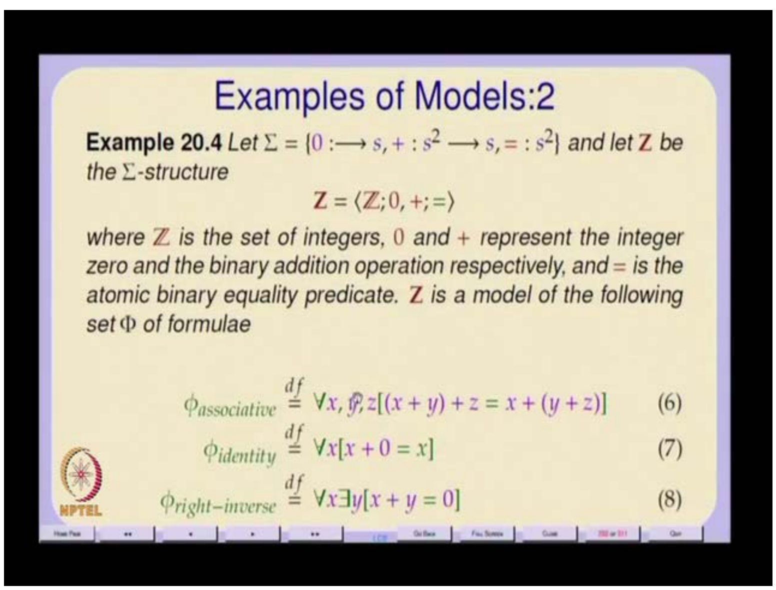

But, let us look at some other examples. So, here is a set of integers again with 0 and addition so you have so, I just I can actually take any set of operations and any set of relations that I am interested in. And, form a structure provided I have defined the carrier set in some way. So, this Z I have taken the set of all integers and I am interested especially in 0 and addition. And, this equality is something that keeps on coming it will keep on coming so that is often a very basic binary predicate that we have to use. In fact what we, will very what will very soon see is that what we require really is not just a first-order logic. But, the first-order logic with equality as a, basic predicate which should always be there. Essentially, what the reason for that becomes clear because in any algebraic system equality is a basic predicate and without equality it is very difficult to do any kind of manipulation. For, example two different syntactical expressions are not really equal syntactically but it is quite possible that they are equal in terms of values when you think of the under a under all valuations. For, example in which case the notion of equality is something that is, for more basic and we will need we will require that. So, but before we get on to first-order logic with equality lets first look at this.

So, I have essentially these are some group axioms. So, phi associativity basically just says for all x,y and z I think I said somewhere that this is a, short form for all x for all y for all z. And, use a single pair brackets so that I do not clutter up the notation with so many brackets so, then this is just a, associativity law. Now, here in this particular case I have of course universally quantified it on the entire free variable on all the variables that I use in the body the identity it should be more correctly called the right identity. But, let us call it the identity for all x, x to 0 equals x and then there is a right universe for all x there exists a y such that x plus y equals to 0 I mean. So, y is a right universe of x I mean that is important so this doesn’t say anything about uniqueness and so on so forth. So, if you take this set of integers under 0 and plus it is structure that is a model of these three axioms. So, that is essentially so that is an example. Now, of course when we are talking about group theory itself you might not be interested in the particular structures but you might be interested in general properties of the groups.

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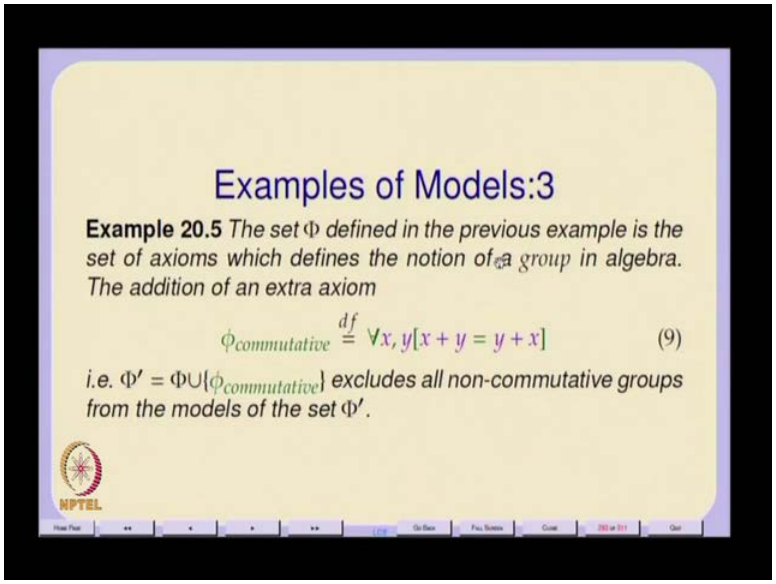

So, for example here is another example where I add a new axiom called the commutative axiom. Which, says that for all x and y,x plus y equals y plus x. So, now if you look at phi prime so phi let us say contained all these three formulae. And, phi prime contains everything and phi and also the commutativity. Then, what you are saying is this phi primes has is models only all commutative groups all abelian groups. And, their the non-abelion groups are excluded from it where as phi includes both commutative and non-commutative groups and as models.

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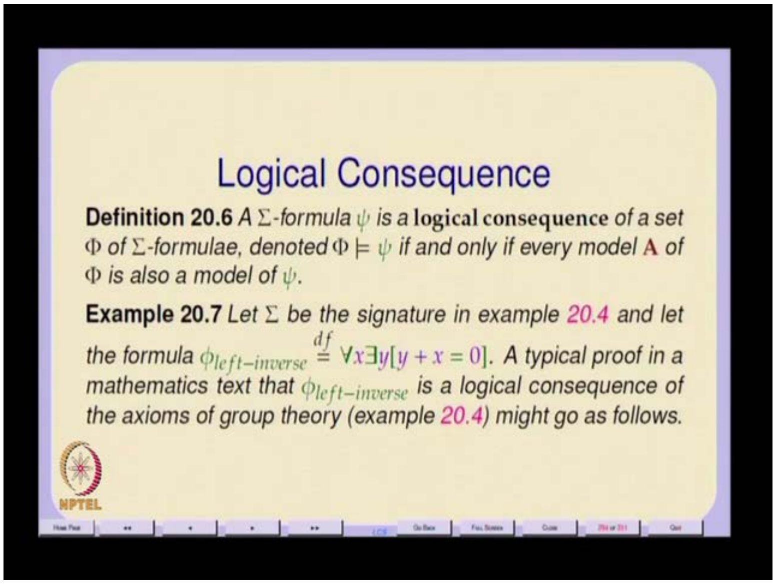

And, of course we have this notion of Logical Consequence as we did before a sigma formula psi is a logical consequence of a set phi of sigma formulae denoted phi prove psi is a logical consequence of phi if and only if every model A of phi is also a model of psi. So, now the notion of logical consequence has been imported straight from propositional logic as we can see. So, one question is that we had this notion of right universes and without worrying about, the an individual model of these axioms. I might want to show I might want to prove theorems that are true for all models of these axioms. That is, a way we work polymorphically in most of mathematics and computer science. And, so one possible logical consequence that you might want to prove is that every that right that they there are left universes also.

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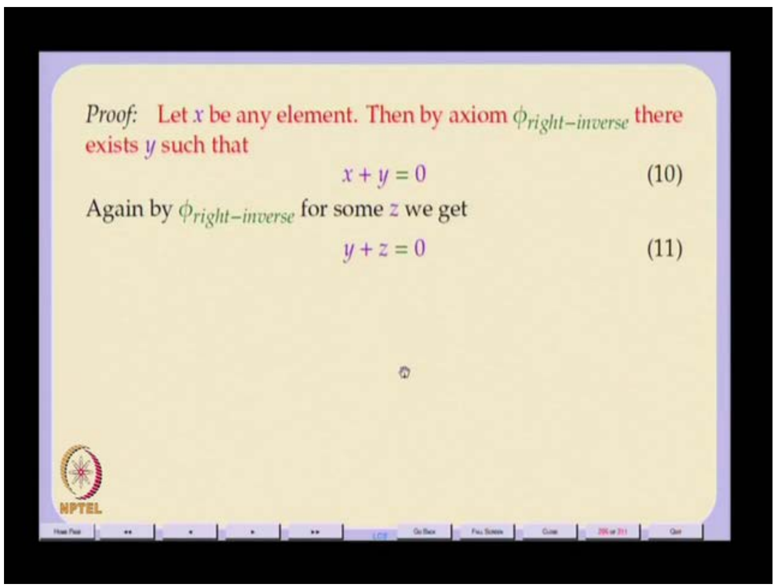

So, this formula phi left universe essentially says that every element x has a left universe y so, that y plus x is 0. And, the fact that it is a left universe that there is such a left universe usually can is is proven in any mathematics steps by assuming let x be some element. Then, by the axiom the right universe axiom we get that there is y such that x plus y is 0. And, again given that there is y again by the right universe axiom you know that there is some z I mean you cannot assume that x's you, only know about right inverses. So, you have to so y must be having a right universe I do not know what it is but, it is some z.

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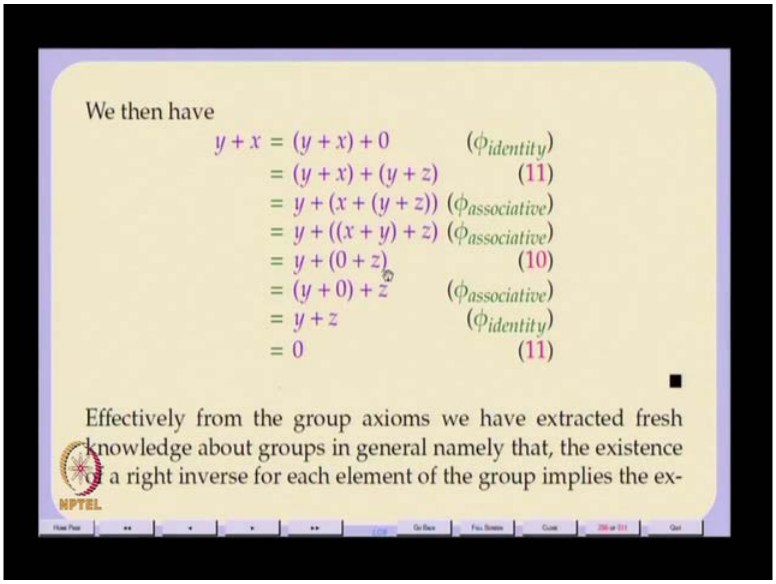

So, with these assumptions essentially what you can show is that y plus x gives you 0. And, in all these cases what we have used are only these the group axioms or these two which have been marked here and here throughout this. So, essentially so y plus x is 0 so what your essentially saying is that y is a left inverse of x.

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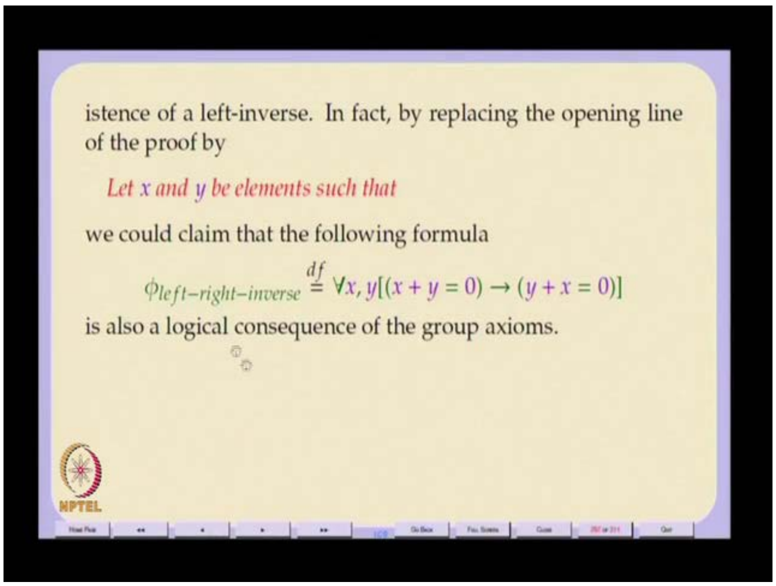

So, your left inverse formula actually is sort of proved by this proof. But, this proof actually is fairly general in the sense that, I can claim that every right inverse is also a left inverse. And, the proof for that is so you take the formula left right inverse. Which, essentially says that every right inverse is also a left inverse that is the formula for all x, y if x plus y is 0 then y plus x is 0. So, this can also be used the same proof except that this part in red has to be replaced by something like let x and y be elements such that x plus y equal 0. Then, by since there is a right inverse for every element there is some z such that y plus z equals 0. And, then this whole prove goes through and essentially shows that y is left inverse of x. So, this is how standard proofs go in mathematics. And, often what happens is that essentially the same proofs can hold for sharper statements in a certain sense this is a more this is a sharper statement than the statement that just says there exist left inverse right. Here, it says that every right inverse is also a left inverse and given that every element has a right inverse. You know how to find the left inverse because the same element is also a left inverse. So, its sense in a certain sense the sharper statement and this is also a logical consequence of the group axiom so, this is like a typical proof. So, what we are really looking at formulizing the notions of these proofs of such proofs in within first tautology.

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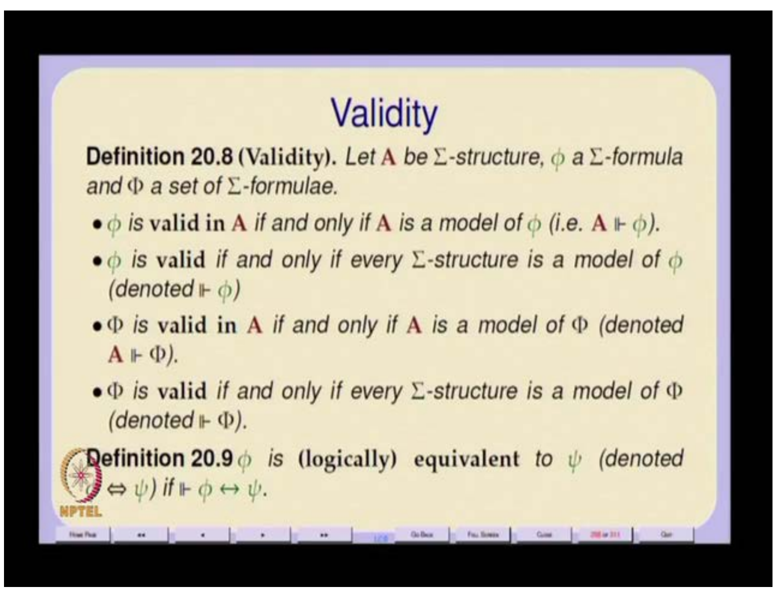

So, then there is a question of Validity which approximately is like tautologousness in propositional logic. But, not quite validity is slightly more general notion. So, let us take A to be some sigma structure and phi is some sigma formula. Then, we will say that phi is valid in A if and only if A is a model of phi so that is one thing. So, validity comes directly from the notion of a model phi is valid now this is where something closed to being a tautology comes in. Phi is valid if and only if every sigma structure is a model of phi. The reason it is goes beyond the notion of tautologousness is because, this phi could have free variables in it for one thing. Secondly, all are language is parameterized on the signature sigma so you are always talking about validity only in the context of a certain sigma it is not in that, sense it is the parameter sigma is always present. So, we would say so the notion of validity and we will so actually it should be denoted this way. But, most books in logic do not distinguish between this symbol and this symbol. But, I have tried to distinguish it because in one case you talking about models truth in models, in another case you are talking about logical consequence from a set of formulae. So, because of the fact that so notice that if, every sigma structure is a model of phi. Then, what you are essentially saying is that? Phi is also a logical consequence of the empty set of predicates.

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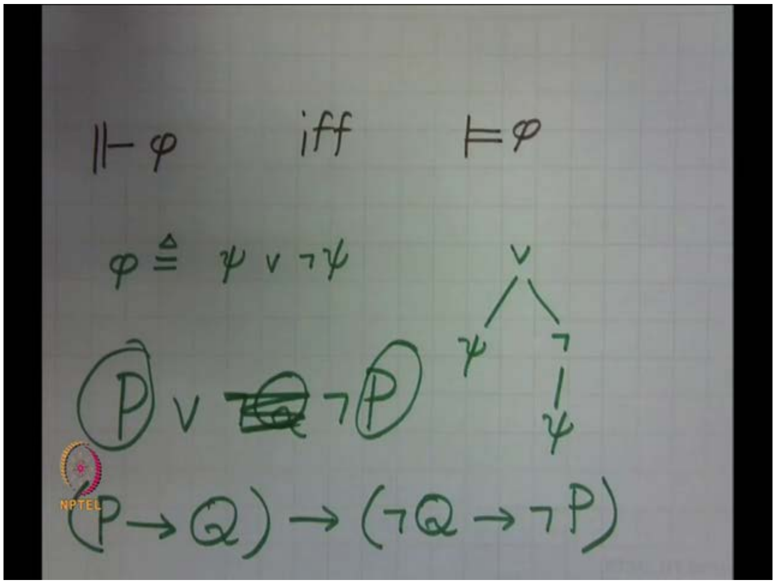

In that sense this is the same as this. And, because validity is a same notion so, anyway your parameterized on a signature sigma and, you are saying for every sigma structure phi is every sigma structure is a model of phi which, means that phi is true of every sigma structure regardless of valuations. So, in a certain sense it is a logical consequence of no assumptions so it is true for a all sigma structures without any assumptions and without an independent of validation. So, it is actually a logical consequence of an empty set of assumptions. So, most books would actually use this symbol for validity they would have also used this for models. Though, there are the types on the left hand sides are different in the case of logical consequence you are talking of the left hand sides are consisting of a set of formulae as assumptions. Whereas, in the case of a model they are some other structures, there are color difference between green and brown and that is a difference. So, however you have to watch out for that confusion but once you realize that they are using the same they overloading the same symbol for two different concepts. And, they are overloading the same symbol for two different concepts precisely for this reason because, the notion of validity in the two cases will coincide therefore you should not get confused by it.

So, now what holds for a single formula is extended to sets of formulae? So, we will say that capital PHI which is a set of formulae is valid in A. If, and only if A is model of all the formulae in phi. So, for example all groups are models of the group axioms which we have seen before. And, all Boolean groups are models of the group axioms including the commutativity axiom. So, in that sense all these statements of commutativity, associativity, right inverse and identity are true in all those structures with that signature. So, we are valid in all structures with their signature and hence they are valid essentially in all groups. So, the set phi is valid if and only if every sigma structure is a model of phi. And, this is a closest we come to having a set of tautologies so, this portion corresponds to phi this lower case phi being a tautology. So, in particular you if phi has the form has the shape of a propositional tautology. So, for example if phi has even though it might have free variables if, phi has the same shape as a propositional tautology and by the same shape I mean let us, say is of the form some psi or naught psi. So, this is a certain I am not really interested in what size I am not interested in whether, psi is got free variables or not all I am saying is the shape of this formula viewed as a tree for example is essentially like this.

Where, psi itself might be some tree all formulas of the shape are propositional tautologies and in first-order logic they all will be valid too. Because, it they you take the only restriction is that you have to somehow reconcile the terms since there it might have three variables it might have terms. You, have to reconcile the terms with some structure sigma but except for that it they all these formulae will also be valid regardless of whether they are free variables or not. So, validity extends propositional tautology to signatures arbitrary signatures. So, now once you have validity of course you can also talk about logical equivalence. We, will say that phi is logically equivalent to psi and will use the same notation phi is equivalent to psi. If, this phi by conditional psi is valid for all sigma mod a sigma structures. So, the importance of studying a propositional logic is essentially that now you can take all propositional tautologies.

So, you take any propositional tautology like this and what you can think of this as essentially defining skeletons into which you can plug-in first order formulae appropriately. So, if you take something with more than 1 like let, us say if you take a tautology like this then, basically what we are saying is for capital P and capital Q. You can substitute by uniformly by any first tautology formulae phi and psi may be. And, what you still get is a tautology so tautologousness propositional logic will be preserved as validity in first tautology. So, the only constrained is that the internal terms inside those phi and psi should confirm to the signature given that is it except, for that there are no other logical reasons. So, that is why so now we can import basically all propositional tautologies as skeletal structures on to which first order formulae may be hung uniformly.

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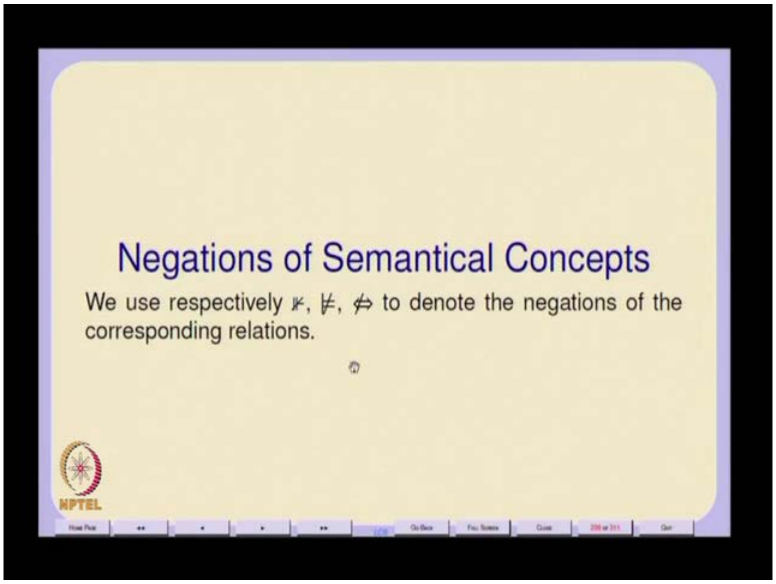

So, will take the appropriate slash to sense as the Negations of the Semantical Concepts basically invalid or not equivalent and so on and so forth.

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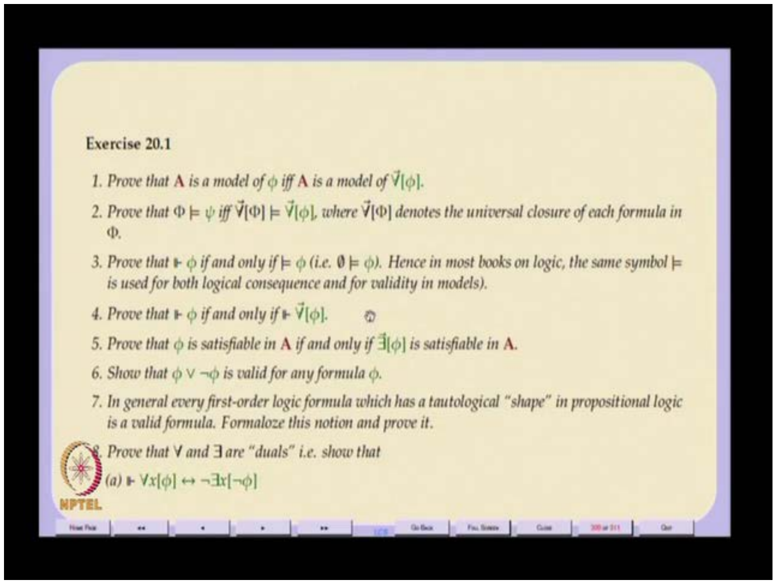

Now, I have got a lot of exercises for you to do of each there is this four. Can, you see this four point? So, this 4 essentially says that you take.

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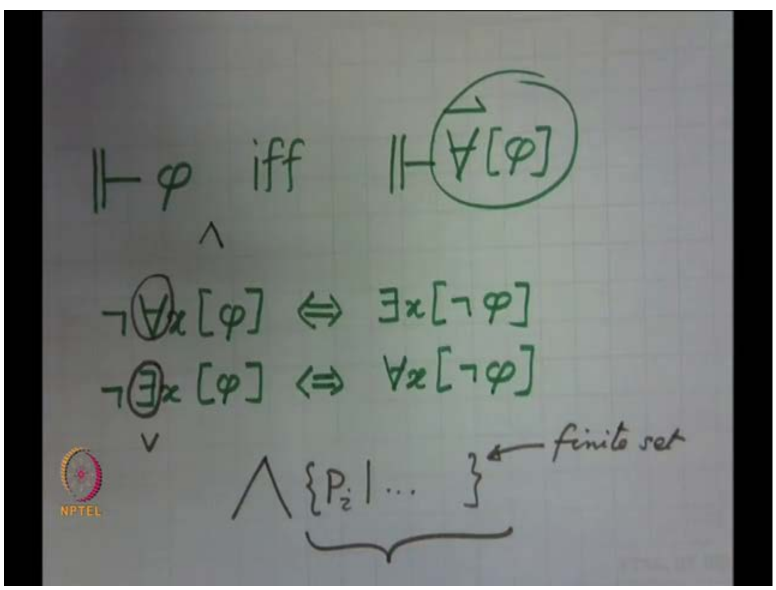

So, you take any formula phi this is valid if and only if its universal closure is valid. What is a universal closure of phi? You take all the free variables of phi and you put a universal quantifier for all of them so that this one has no free variables. And, this essentially the fact that validity of phi is independent of free variables means that. If, you universally close it also that it remains valid so, validity is preserved under universal closure. Which, is a good justification for why a mathematics books do not put these quantifiers before the axioms and they just take the axioms its free variables. But, yet they are essentially saying that they are considering only all the structures in which these axioms are valid. So, they are valid with free variables if and only if there universally quantified versions are also valid and so and therefore it is perfectly consistent. And, of course the same thing whole of sets of formulae I should have probably put this. I should have change the order of these two but anyway. And, the other question three essentially identifies these two concepts. So, you can but you can prove all this from an underline semantical model. The, other important things that you have to prove is that the existential and universal quantifiers are duals of each other.

So, they satisfy an extended De Morgan's law which essentially means that naught of for all x phi is logically equivalent to there exist x naught of phi. And, similarly naught of there exist x phi is logically equivalent to for all x naught of phi. So, one way of so this is essentially like so in a certain so this universal quantifier is really like a and extended to a possibly infinite number of indices. Similarly, this is essentially like or extended to possibly to an infinite number of indices. So, the fact that and is commutative and associative allows you to define a big AND in propositional logic. But, you can take a big AND of various propositions only for a finite set so, this has to be a finite set. What for all allows you to do is that when your models are infinite sets like integers or rational's or real's. Then, what this allows you to do is it essentially allows you to state an infinite conjunction in a finite sentence. So, that is you can think of the quantifiers intuitively. As, a dealing with finitary representation for an infinite conjunction or infinitary disjunction. And, so it is a one obvious thing supposing you have a sentence a universally or an existentially quantified sentence. For, which has only finite which has only finitary models then everything in that finitary model so, every first order logic sentence can be translated into an appropriate propositional sentence there with except for the occurrence of free variables. So, every closed first order logic formula can be expressed in terms of purely propositional logic formula. So, one thing is you could think of one simple example you could think of is supposing I take the theory of Boolean's. So, finitary model which just is a carrier set with two values so take this itself value your sigma. I mean so where is that? So, we had something we had the entire algebra specified some way. That, was in the notion of truth's. Here, is a signature.

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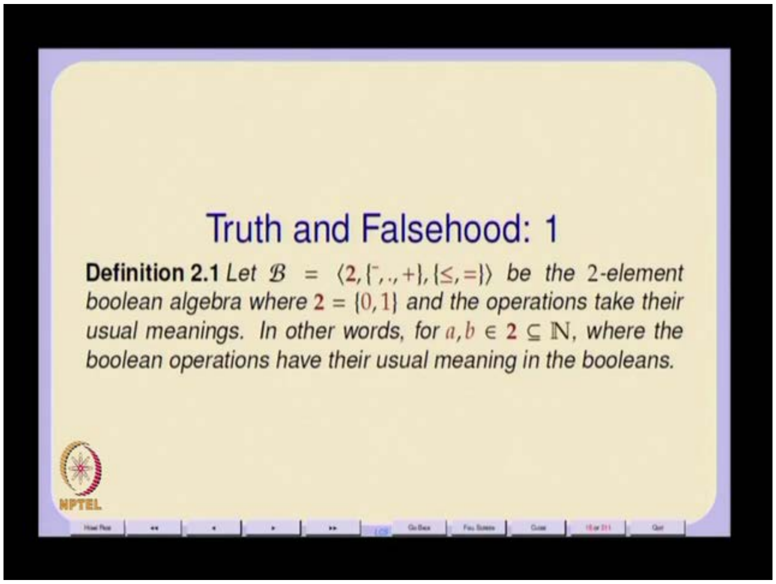

So, you take this signature of Boolean's which has just a two element carrier set all expressions in this can be evaluated and they will evaluate only to do this to one of these two values. You take a first order logic of Boolean algebra. So, I mean after our signatures are arbitrary there is absolutely no reason why we cannot define a first order logic of Boolean algebra. See if, you take a first order logic of Boolean algebra. What you can actually do is? You can take all universally quantified sentences in the first order logic of Boolean algebra. And, translate them into purely propositional sentences by replacing all the variables by their values for 0 and 1 they whatever are your violate versions of 0 and 1. Because, also the terms that has two constants 0 and 1 so there will be a violate 0 and a violate 1. And, we replace it and you have purely propositional statements describing Boolean algebra.

So, for finitary models for finite models all your universe universally quantified formulae and your existentially quantified formulae will reduce to pure propositional formulae with variables being replaced by all the possible values from that carrier set that is it. And, so that is so you can think of therefore one thing so the idea of parameterizing propositions to get predicate is to allow for the fact that we want finite we want finitary representation of sentences for formulae or facts which, might be infinitary in nature. And, so at the level when you are dealing with only finite models at least theoretically it is not easy to do first order logic it sufficient to just use propositional logic itself. So, that is a piece of in sight which most books do not give you actually and unless let, us proceed further. So, that is logical validity your semantical concepts.

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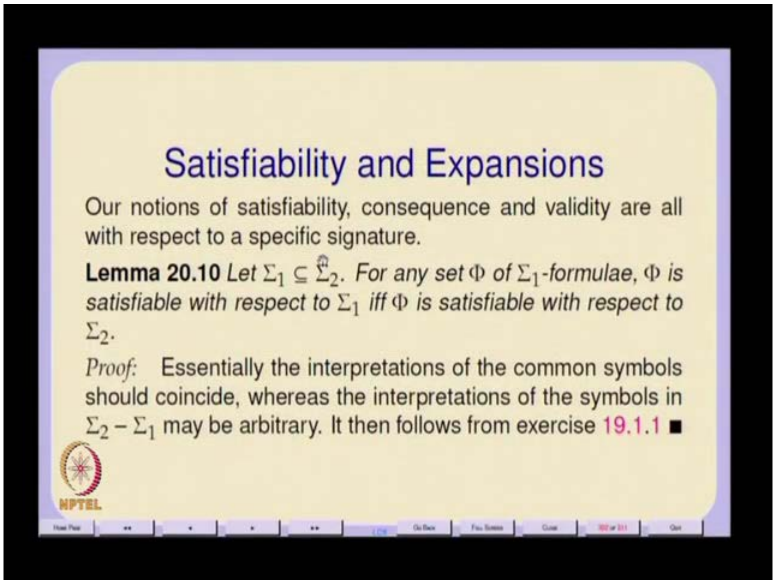

So, of course all the concepts so far were parameterized on a signature that signature was important. And, now the question is how do you deal with different kinds of signatures which might have some relationship among them? So, we had this notion of an Expansion and a reduct a somewhere. So, supposing one signature is contained in another. So, we took for example the signature of integers or natural numbers just 0 and successor. It is quite possible to expand that signature to include addition multiplication actually, the movement you want to do first order number theory. You, would require addition and multiplication because, the interesting questions like divisibility and primality and so on rely on addition and multiplication and may be subtraction and division. So, at least addition and multiplication would be there so then want you would do is you would want to expand the signature sigma to a larger signature. Let, us say sigma prime in this case let us say sigma 1 and sigma 2.

So, if you expand the signature sigma 1 to a signature sigma 2. And, for any set phi of sigma 1 formulae phi is satisfiable with respect to sigma 1 if and only if phi is satisfiable with respect to sigma 2. So, I mean the important thing here, what you are saying is. By expanding the signature all the interesting facts that had previously with the smaller signature continue to hold in the expanded signature. Notice that phi 1 this set phi consists sigma 1 formulae so, that means it does not contain any of the operators in sigma 2 which, are not there in sigma 1. So, this the version of satisfiability essentially means that you do not satisfiability is preserved under expansion of the signature. And, so you can just take the interpretations of common symbols according to sigma 1. And then for sigma 2 minus sigma 1 we can take any interpretation you can take any arbitrary interpretation it does not matter. And, then you can prove that satisfiability is preserved. So, we can expand signatures like that.

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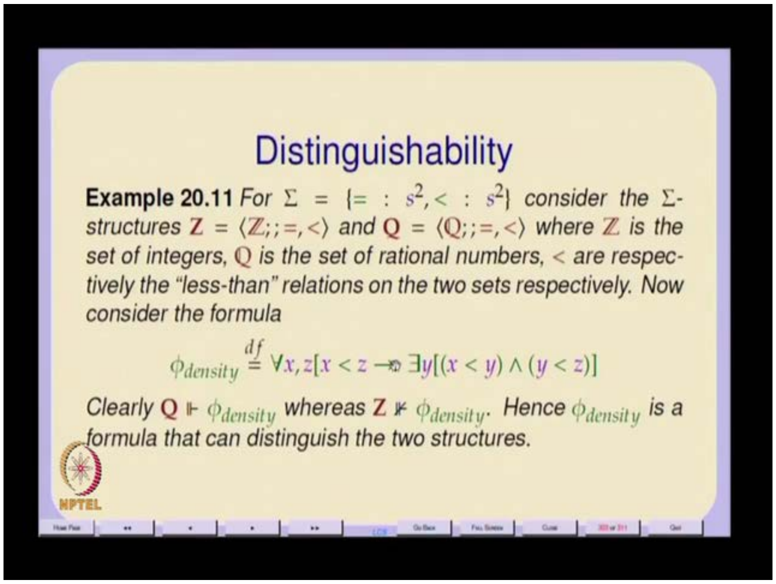

The, other thing is Distinguishability. So, let us we can use first order logic formulae as a method of distinguishing two models with the same signature. So, let us take. Let, us not have any operations let us just have the equality relation and the less than relation. So, there and so take the integers under equality and less than take the rational under equality and less than. So, there is a formula there is a first order logic way of expressing that the set of rational's is dense. What does it mean? Between any two distinct rational numbers there exist another rational number. You take this so this is expressed just this way for all x and z. If, x is less than z then there exist y such that x is less than y and y is less than z so, that is a density axiom. So, density axiom let us say or density formula. And, it is clear that in the case of the integers this is not valid. So, in a certain sense this sentence this is a sentence because universally quantified it is has no free variables. This sentence is a distinguishing formula between these two possible models the rational numbers under equality and less than and the integers under equality and less than. On the other hand this cannot be distinguishing formula between the Rational’s and the real’s because, both of them are dense for example. So, in the case of rational’s and real’s in order to distinguishing rational’s and real’s you will require some other kind of formula. Which, and that formula is more complicated you know because, the real’s are actually limits of infinite sets of rational’s. So, to formulate that in first order logic will take some effort but, let us not get into it. But, basically you can think of witness formulae which can distinguish between two possible models having the same signature. So, Q so phi density is valid in Q whereas, it is not valid in Z.