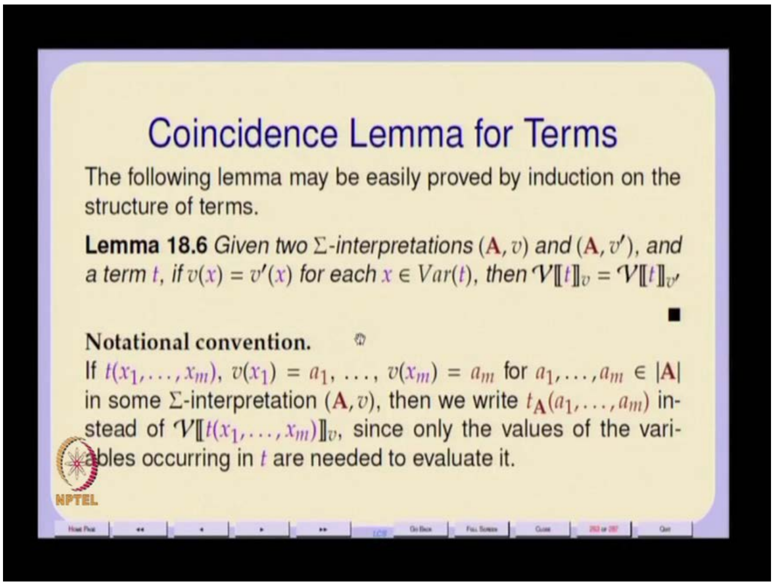
Logic for CS
Prof. Dr. S. Arun Kumar
Department of Computer Science and Engineering
Indian Institute of Technology, Delhi

Lecture - 19
Substitutions

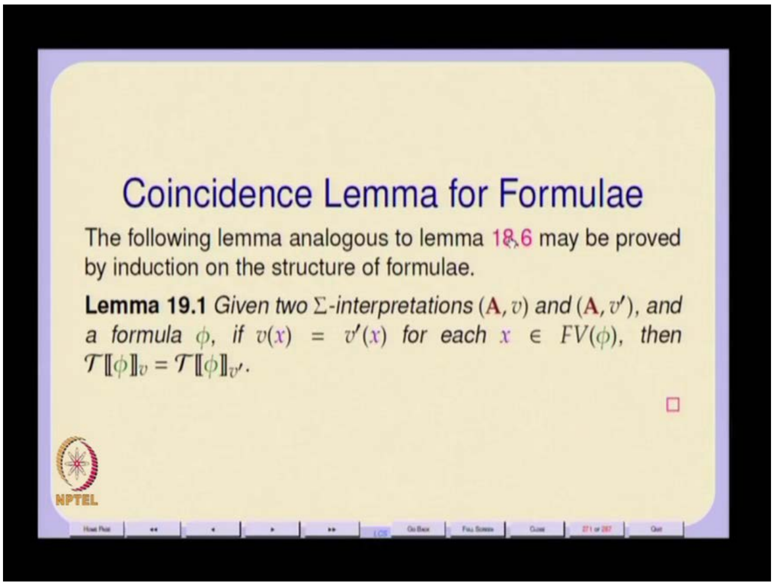
Semantics of predicate logic and we also had this notation of variants. And then we have now to introduce substitution. And what I said at the end of the last lecture was that there is an interaction between substitutions and variant valuations which essentially connect up the expressivity of substitutions with the kinds of variation you can get in the valuation. So, that is semantic to syntactic connection which is important.

(Refer Slide Time: 01:13)



Before we actually go into substitutions there is we had this coincidence lemma for terms, right, which essentially said that for any two valuations if the variables in a term are the same give the same value in both the valuations, then the meaning of the term is the same in both valuations, right.

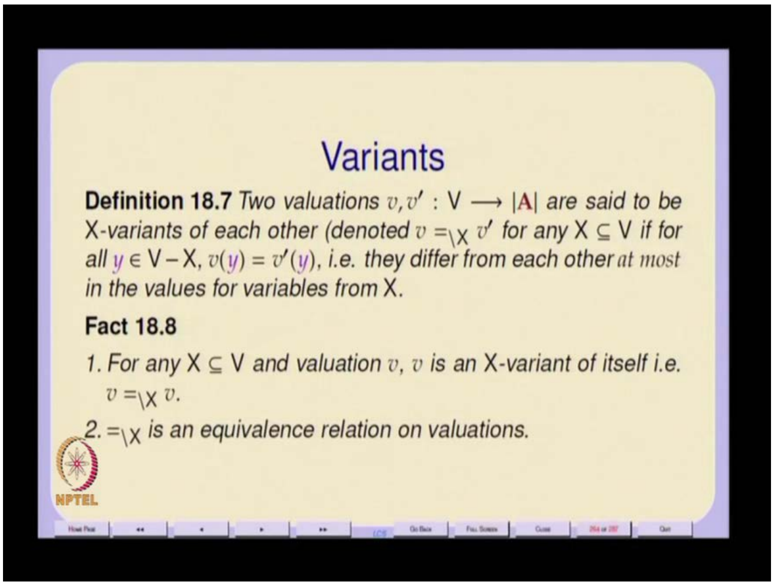
(Refer Slide Time: 01:40)



So, there is something further that. So, we can do a similar coincidence lemma actually for formulae and we need to do that. So, essentially what we are saying is that given two valuations v and v prime and a formula phi if the values of all the free variables in the formula. So, here there is at a restriction in the case of terms we did not have any bound variables; in the case of formulas we do have free variables.

If for all the free variables in the formula phi, if both valuations match; then of course, the truth values of the two of the formula under both valuation will also be equal, right. So, this actually brings us to something which I mean. So, an actual proof requires some kinds of extinction of the notation of variant.

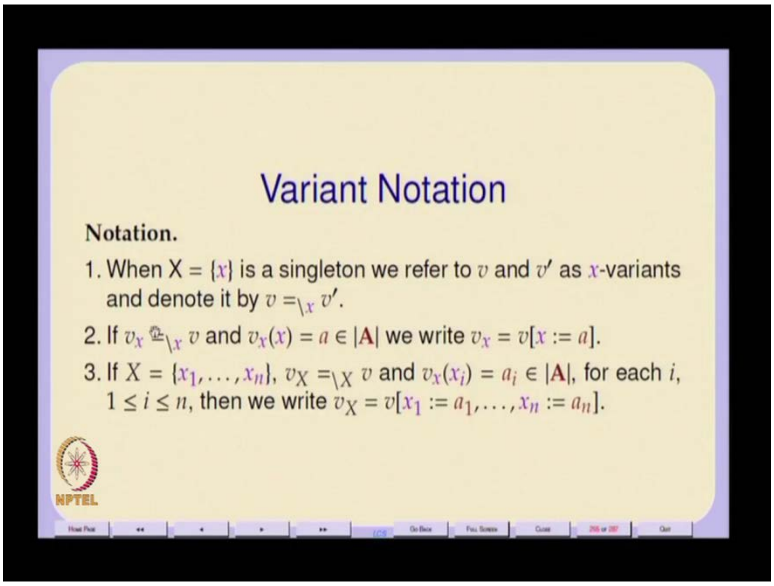
(Refer Slide Time: 02:41)



So, one thing is that we can actually extend; we had this notation of the of an x variant where x is some subset of the variables. So, v and v prime are x variants if for every variable other than the variables in x, they have the same values, right. So, now it is also possible. So, the other thing I said was that everywhere valuations is actually an x variants of itself for all subsets x actually.

So, this actually can be formulized as an equivalence relation which I will denote by this; equal to back slash x, back slash is like set subtraction. So, basically it stands for expect x, yeah; e is equal expect for x is an equivalence relation on valuations.

(Refer Slide Time: 03:31)

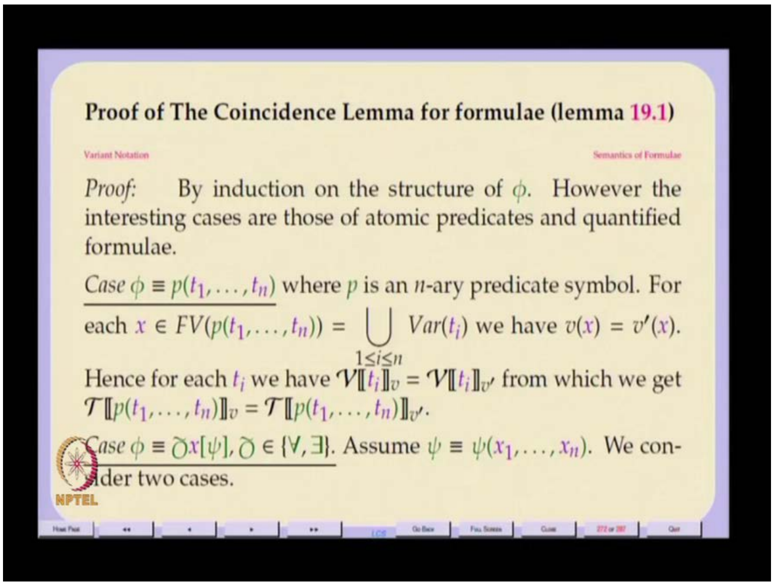


And then of course, we will use this instead of saying every time that something is an x variant of another valuation; we will just use this equivalence. Since, anyway the notation of a variant is symmetric because of the equivalence we can just use that, right. Now of course, what happens is that when we say that for example, v x is an x variant of v then what we are saying is that v x and v coincide for all variables other than x in which case sometimes the important question is this.

Okay, so what does v x gives to x, right. So, we could specify v x essentially as an assignment of various values which distinguish it from v. So, essentially this notation v x equals v x 1 assigned a 1 x and assigned a n essentially says that well v x and v match for all other variables expect possibly for these variables in this set x, and for those variables in the set x these are the values that v x needs.

So, you are saying that v x varies from v only in these values of these variables, yeah. So, we will be using these notations. So, whatever I am doing today is going to be quite regress, but surprisingly even then some of the most regress books on logic do not actually treat it in this regress fashion. They tend to gloss over it which I think is not suitable if you are going to actually program these things.

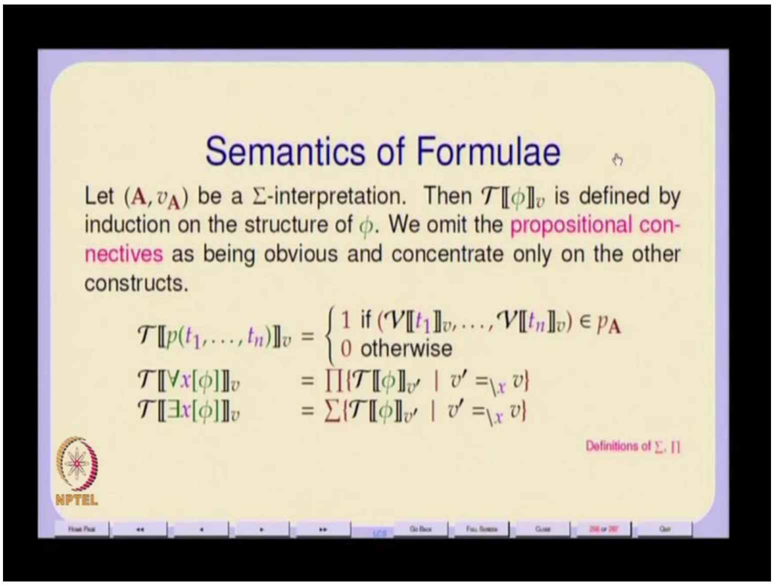
(Refer Slide Time: 05:26)



So, we will go back to. So, this coincidence lemma essentially it says that. So, if for two valuations v and v prime which are x variants of each other are which coincide for each value each free variable of the formula phi, then the truth values of the formula under the two valuations is the same, right. So, this is something that you have to prove by induction on the structure of the formula phi.

In most cases we will just forget about the proposition connectives because they are very easy and trivial and it only add to the t d m. So, the most interesting cases are that of the cases of the bases when phi is an atomic formula and when phi is a quantified formula, right.

(Refer Slide Time: 06:27)



In this it is probably necessary to keep in mind that we have defined the semantics of these constructs like this. I think I have corrected the slides here. So, this is the corrected version yeah. So, then obviously, what we are saying is that. So, since your language is such that the language of predicates builds up on the language of terms. So, it is natural that what we will do is in evaluating this formula phi for some atomic predicate p t 1 to t n, we require the coincidence lemma for terms which we had previously proven, yeah where was that? Here

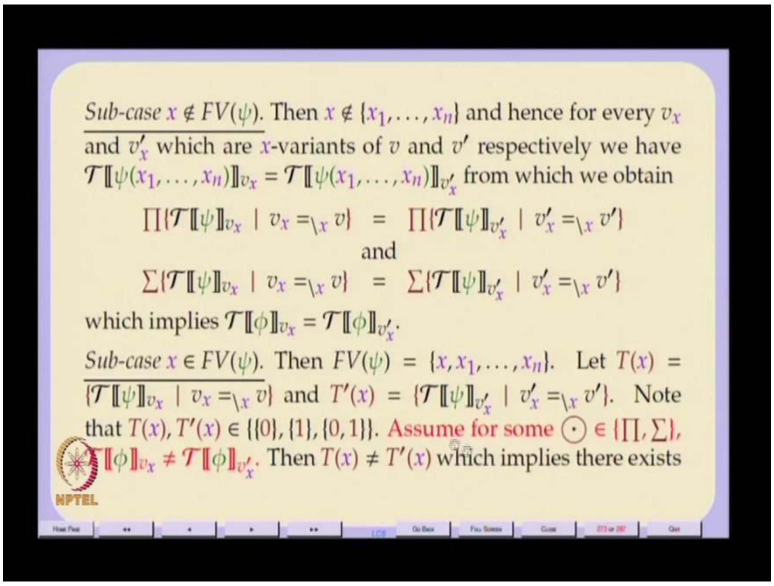
So, this was the coincidence lemma for terms, and now we use that essentially to claim that each of these terms t 1 to t n would have the same value under both valuations. And therefore, the truth values of the atomic predicate under both valuations would be identical would be equal. The only thing interesting really about the case of the quantifier is the fact there are bound variables, right. So, we have to take that into an account.

And the fact that it is in some sense a summation or a product depending upon the quantifier over a possibly large set of variants over the bound variable, right, valuation variants over the bound variable. So, that is what might slightly complicate matters. So, let us assume that x 1 to x n are the free variables of this quantified variable. I am using this inverted q because to me actually it does not matter whether I am considering the universal quantifier or the existential quantifier, because of the fact that what I am going to do is I am not going to worry about these whether it is product or sum. Instead I am going to worry about these sets.

So, supposing I can prove for both the valuations v and v prime that this set is equal for both v and v prime; that two sets that I obtained without this product or sum are equal then it follows that the truth value that will also be equal. So, it is necessary to prove that these two that under v and v prime the sets obtained for the truth values are the same, right. So, that is what we will do. So, that is why it does not matter to me which quantifier I am using; many books actually consider the two cases individually and go through some painful detail, but actually it is not necessary if you consider it as the set of all truth values for all possible x variants of the body of the formula.

And if you can prove that under both v and v prime the two sets obtained are equal in terms of the set of truth value, then you are essentially done because then whichever quantifier you are talking about, you can take the product or sum and it will be preserved under equivalent. So, this case as we have we assumed that x 1 to x n are the free variables of the body of this quantified formula, right; this inverted q x side, right.

(Refer Slide Time: 10:20)



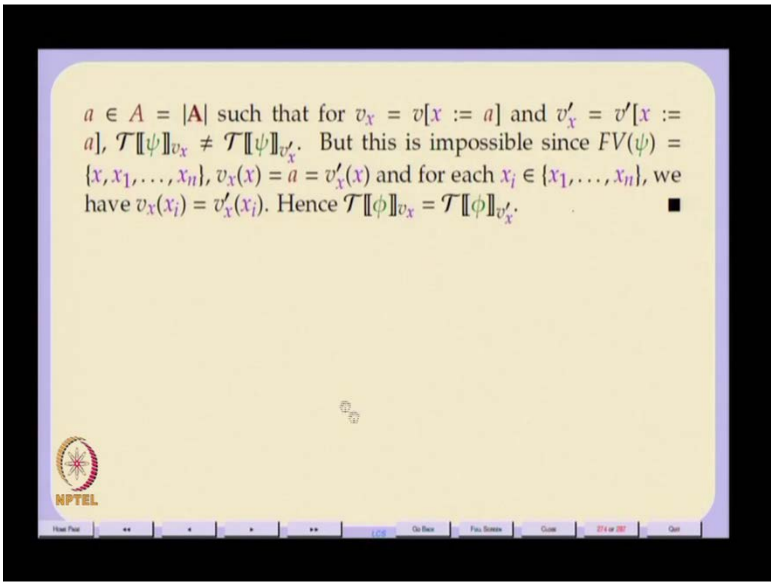
So, there are two possibilities either the bound variable x is a free variable of the body of the formula or it is not. So, this simpler case is when the bound variable is not a free variable of the body of the formula. So, x is not a free variable of psi, then which means that x does not belongs to this set x 1 to x n. And hence for every v x and v x prime which are x variants of v and v prime it follows that you are anyway not interested in anything other than x 1 to x n. And therefore, these two sets will actually be the same.

So, regardless of what quantifier you are talking about you will get this equality which implies that the truth values for phi under both the variations will be the same. In the case when x is a free variable of psi, then the free variables of psi are x, x 1 to x n. So, there is an extra free variable in psi, and now we have to consider the various x variations, right. So, let me take this set t of psi v x such that v x is an x variant of v; let me take this entire set for all possible x variations of v and similarly for all possible x variation of v prime.

And what I am essentially going to show is that these two sets t x and t prime x are equal. And once I have shown that basically it does not mean matter whether I am talking about an existential or a universal quantifier; the result will always have to be the same if they are uniformly applied. So, the other thing of course to realize is that finally, after all those x variations are considered, even if that is in infinite domain; finally, your sets t x and t prime x can only be one of each of them can be one of three possible sets. Either just the singleton set zero or the singleton set one or the set containing both zero and one; that is it.

There is no other possibility; especially since where the empty set is not a possibility because we are not considering empty domains, right. So, supposing for some. So, we will prove this by contradiction. So, assume that for one of these two operators' product or sum this thing is not equal to this. Then that means that these two sets are not equal. Remember one thing; if you assume that these two truth values are equal, it does not imply that the corresponding sets t x and t prime x are equal. But if these two are not equal, then it is definitely sure that the two sets t x and t prime x are unequal, and that comes from these three possibilities for t x and t prime x.

(Refer Slide Time: 13:38)



So, if these two are not equal then it definitely means that t x is not equal to t prime x which means that there exist a distinguishing element in the domain A such that the two x variants v x. So, that for this x variant v x and v prime x where x is assigned the value A, the two truth values for psi are different; otherwise, it is not possible, right. But this is impossible because of the fact that you are assigning x the same value A and they coincide on all other free variable x 1 to x n.

And therefore, what it means is that. So, therefore, these two must. So, it essentially means it there is a contradiction, and therefore, these two sets actually must be equal. Therefore, what it means is that these two sets this assumption that these two are unequal is false which means that essentially these two sets are equal, right.

Student: Now it will be like it will exist in A belonging to x square?

No no, see look at it. T x and t prime x both are drawn from this set of values, right. Now if t x is not equal to t prime x, then consider this. So, there are three possibilities for t x and three possibilities for t prime x. So, the two sets are different. So, which means out of those nine possibilities, three other possibilities of equality have been ruled out and there are the other six possibilities, right. Without loss of generality assume that t x just consist of the singleton set zero.

And if t x is not equal to t prime x, then t prime x either consist of the singleton set one or the set containing both zero and one. In either case what you are saying is that there is a distinguishing element A for which t x gives the value zero whereas t prime x gives you a value of one. There is at least one distinguishing element. This same kind of argument can be taken for all the other possibilities. So, if t x is the set zero, one and let us say t prime x is the set just containing one, then how did that zero come in t x? There is a distinguishing element A for which t prime x gave you one whereas for t x gave you zero, right.

So, for all those other possibilities of inequality there is a distinguishing element in each case, and that distinguishing element I am calling A. And basically what I am saying is that for x being assigned that distinguishing element, the truth value of psi must be different in the two cases v and v prime, right. So, without actually elaborating on the six possible different cases, I am just using the inequality itself to reason about the inequality of truth values, yeah. So, there is a distinguishing element, yeah?

Student: Sir, so v x and v prime x are both x variants of two different valuations?

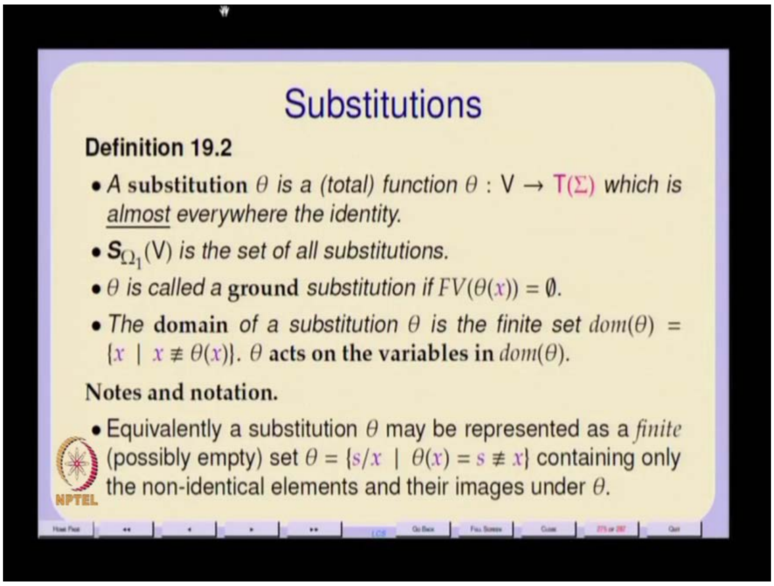
Yeah. They are x variants of two different valuations, but the two different valuation coincide on the variables x 1 to x n and x 1 to x, x are the only free variables of the formula fact; that is the point; that is the coincidence. So, we are showing that therefore, even if they vary on all other variables it does not matter really. The truth values would have to coincide; that is what we are trying to say, yeah. So, this essentially says that therefore, the assumption that the two sets t x and t prime x are different, it is false; therefore, the two sets must be equal. So, now it does not really matter whether I am talking about product or sum if the two sets are equal then if you apply there will be preserved under the same operation anyway, whichever it might be. So, therefore, it is it is not necessary to take quantifiers individually.

The thing about these proves is that in some of the books they actually consider only the case when the formula is true or only when the case when the formula is false. Usually only when the formula is true; they basically say the formula is true in this depend only if it is true in that which by implication says that it is false in this if and only if it false in that. And then they argue for truth, and then they consider the case of product and the summation individually. But what I am saying is it is not necessary to; for the purpose of reasoning if you consider this inequality and the fact that this inequality is based on whether they both have an element that is not common; that is all.

We are looking for an element that is not common which creates an inequality, and therefore, we are looking for a distinguishing element in the model in this semantics which will clearly distinguish for the inductive step and we are reasoning on that basis. So, we are not making or we are not doing out detail case analysis because that requires six cases. It is possible to just reason without doing a detailed case analysis in this fashion. So, therefore, essentially for both formulas and terms it is clear that if two different valuations coincide on all the free variables of the formulas and all the variables of that terms, then the two valuations yield the same results for both formulas and terms.

There is a certain sense in which this does not matter if you consider the whole thing of first order predicate logic from a purely universal algebra framework because then all that you are talking about are term algebras. So, predicates are also terms and terms are also terms of a two level algebra, right; that is all there is to it. But if you are looking at the predicates and the terms as are belonging to different levels, it is necessary to consider this notion of coincidence and prove it explicitly especially when there is the question of bound variables, because you do not know then what might affect the valuations. So, next we will come to essentially this notion of syntactic substitutions, and syntactic substitutions are something we all are familiar with in programming languages pattern matching and so on and so forth.

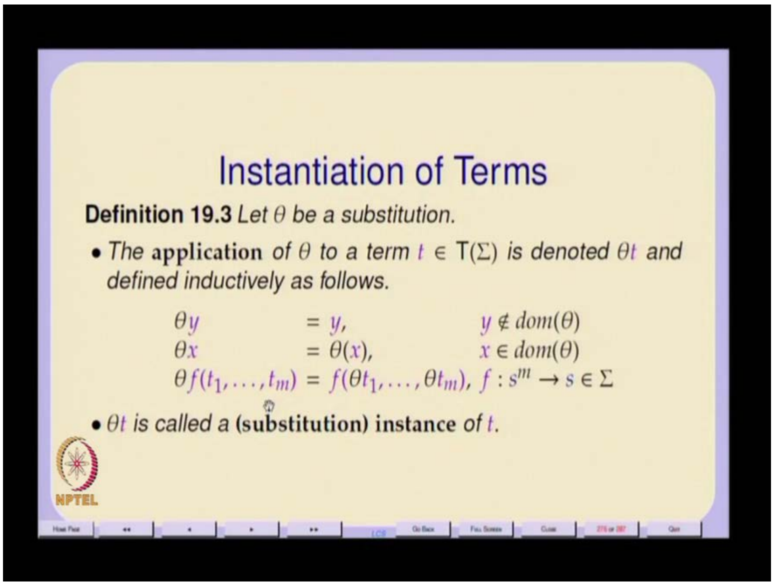
(Refer Slide Time: 21:30)



But to specify that in some details as far as I am concerned a substitution is a total function from variables to terms and which is almost everywhere the identity and by almost everywhere the identity what I mean is it is identity for every variable expect for a finite subset of the variables, right. So, we will think of for any signature omega 1, we will think of s omega 1 v as a set of all possible substitution. We will say that this substitution theta is a ground substitution if the free variables of theta; actually it should be the variables of theta applied to x are empty, yeah for every x.

For any substitution theta since only a finite number of the variables are being replaced by terms, we can also express it as a set in this fashion s for x. So, I am replacing x by the term s. So, this has to be read as s for x. So, where of course, s is not identically equal to x. So, when you specify that s is not identically equal to x, your essential saying that this set has to be a finite set if theta is a syntactic substitution. And the domain of the substitution is only those variables which are actually being replaced by other terms by distinct other terms, yeah, non-identical elements.

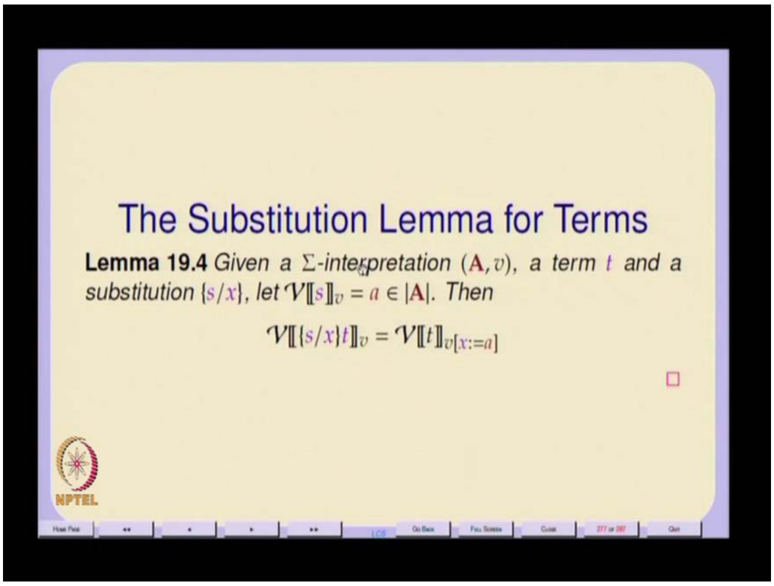
(Refer Slide Time: 23:22)



And for any substitution you can apply a substitution to a term and that can be defined by induction on the structure of the terms. So, you take for any variable which is outside the domain of the substation, basically there is nothing to be replaced. So, the effect of the substitution is to leave the variable unchanged, and for any variable that is in the domain of the substitution the effect is to replace it by the corresponding term.

So, if s for x is the replacement then theta applied to x will give me s, right, and for any complex term you essentially push theta down into the term. So, if you think of these terms as abstract syntax trees and you are applying theta from the top, essentially it filters down till it reaches the leafs which are variables and then some of those variable are replaced by sub trees; s is also a tree replaces by new tree. So, that is what, and then for any term t theta t will be called as substitution instance of t; very often we just say an instance of t.

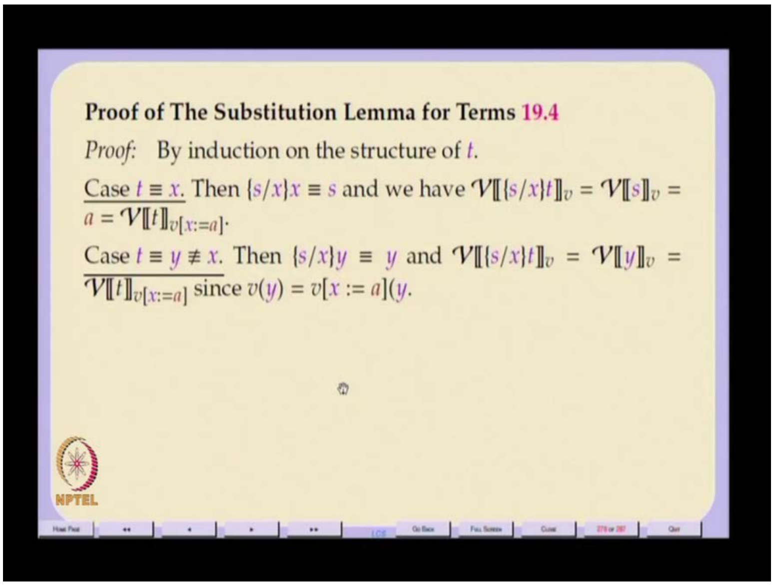
(Refer Slide Time: 24:45)



The same kind of definition it can also be applied to predicates, but of course, now we have to worry about bound variables, right. But before that there is a substitution lemma for term which is like in a certain sense the complement of the coincidence lemma for terms, right. So, essentially what it says is and moreover this substitution lemma essentially relates the notion of synthetic substitution to the notion of valuation and variation in valuations. So, this essentially says that supposing I have a term t and a substitution of a single variable x by a term s. And if I have a valuation v under which this s evaluates to some element A, then by substituting s for x in t and evaluating it in the same valuation is exactly the same as taking an x variant of that valuation and not doing the substitution where you ensure that that x variant of that valuation gives x the value A, right.

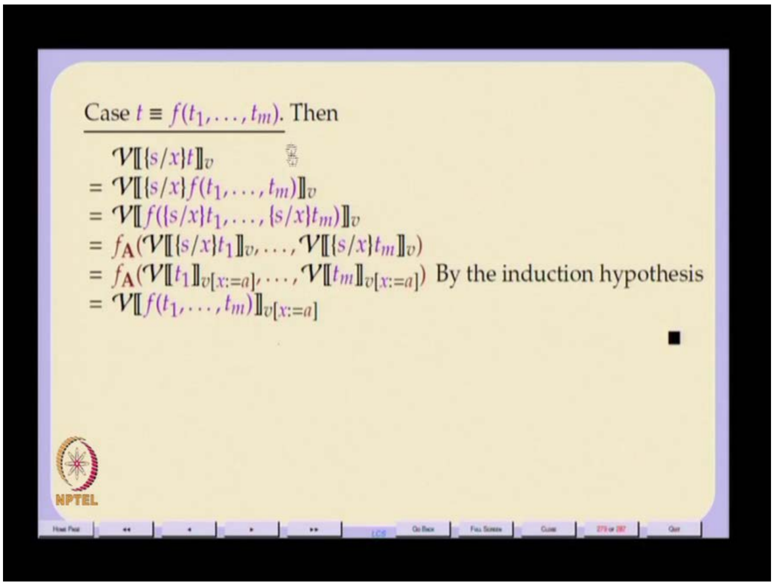
So, the colors here should make it clear that, therefore, for the terms at least there is an equally expressive way of looking at substitution as changes as variation in valuations. And in fact, finetary variations in valuation, right; that means, that in a certain sense the notion of substitution is not just powerful; it is also expressive in the semantics. That is essentially the moral of the story if you like.

(Refer Slide Time: 26:59)



And this is something that can also be proved by induction on the structure of terms and there is a more or less straightforward proof. So, we just have to consider the case when there are two cases for variables; one when the variable itself is x, the other when the variable is something other than x, and from that that it will follow. There is some problem here which I do not know which I have to correct.

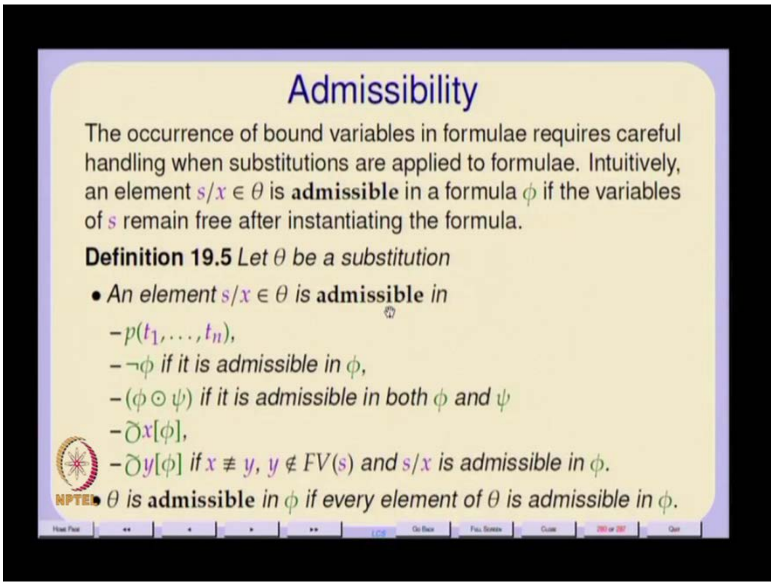
(Refer Slide Time: 27:31)



But in the case of a complex term you can actually use the induction hypothesis. So, here is where we are using. So, from our semantics of valuations what we do now is that for every function symbol f, there is a corresponding function in the semantics which is this brown f a right. And then both our substitution filters through and distributes through the syntax tree are to each of the sub terms. And therefore, here I can use the induction hypothesis to claim that for these t 1 to t m, the effect of substituting s for x in each of them and evaluating each of them in the valuation v is the same as not performing the substitution, but doing the valuation in an x variant where x is assigned the value A.

And once you have that, then your semantics also allows you to pull back and come to this form, after all the meaning of this. So, this term would have it is a backward substitution, yeah. So, is that clear. So, our notion of substitution for terms is at least fairly powerful in the sense that it can be reflected in the semantic of valuations, and that is the semantic notion.

(Refer Slide Time: 29:23)



So, the notion of bound variables in the case of formulae complicates matter. In fact, what does it complicate? It complicates a notion of substitutions. So it is always a good idea to actually separate that complication out and that is what I am going to call admissibility, not something that most people use, but several books use this term of admissibility, right. So, essentially for those of you who have done programming languages and things like the lambda calculus, one of the thing that we have to worry about when doing either beta reduction or a substitution is that one should ensure the there is no capture of free variables.

So, this admissibility is essentially the concept which ensures the there is no capture of free variables. So, we would say that s for x is admissible on some formula phi, if phi does not contain any variable in a quantifier fashion such that the effect of replacing s with x might capture that variable. So, basically what we are saying is the bound variables inside phi should be different from the variables that occur in the term s, right. Because the effect of substitution is percolated to write down and distribute through all the branches of the syntax tree, and it should not happen per chance that one of the variables in s happens to have the same name as the variable in a quantifier. And therefore, becomes bound by shear accident if you like.

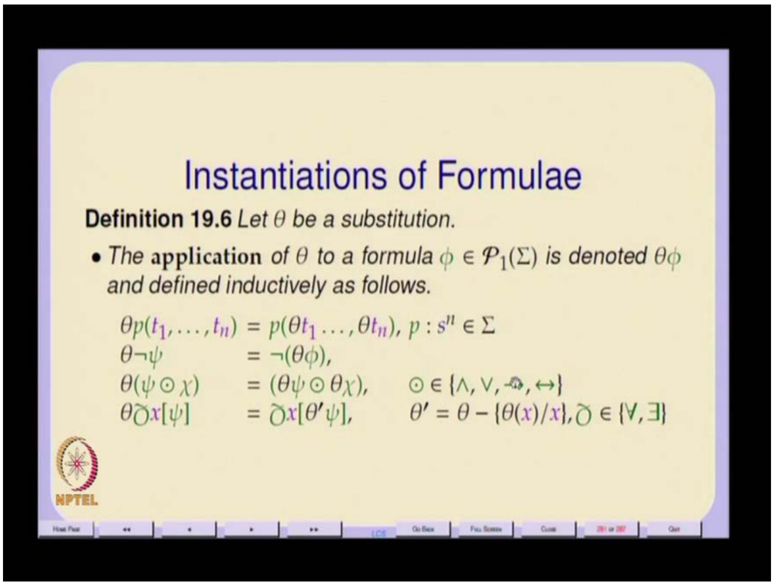
So, the idea is that intuitively and semantically the variables in s that occur in s are meant to be different from the variables that occur in the bindings of quantifiers inside any formula phi. I mean the two things are separate and therefore, they should remain separate. So, we would say that s for x is admissible if there is no variable in s which can get captured by a quantifier in phi when you perform the substitution, right, and this notion of admissibility can also be defined in a structurally inductive fashion in this. Basically what we are saying is that s for x is admissible in every atomic predicate because there are no bound variables in an atomic predicate.

And it is in compound predicate which is with propositional connectives, basically the admissibility goes down to the sub formulize, right. In the case of the quantifier there are now two cases. So, you are substituting s for x and if x is been quantified, then the effect of the substitution is to leave the formula unchanged, because effectively this substitution is only for free variables. And since x is a bound variable this substitution has no effect. Basically x is not a free variable of this entire formula; therefore, the effect of that substitution is to leave the formula unchanged. On the other hand, if you had a quantified formula of the form q y phi, then the admissibility also percolates down where y is different from x.

So, one thing is that admissibility requires that y should not be a variable of s because otherwise there is a possibility of capture of a free variable by the binding quantifier y. And the other thing is that s for x should be admissible in the sub formula phi, right. So, there is an inductive way of specifying admissibility. There is of course, one other thing that usually happens in the case of substitutions in the lambda calculus. And in fact, it is applicable also in predicate logic though I have not actually mentioned it and, that is this notion of alpha conversion.

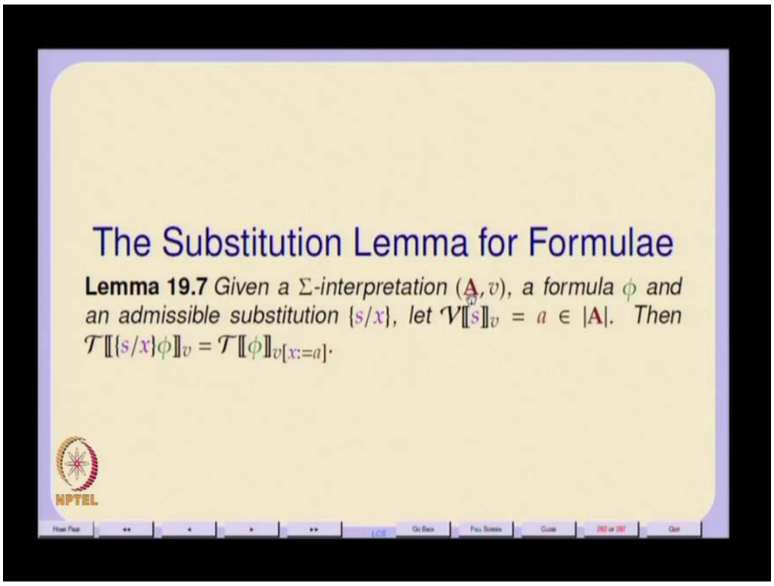
So, it is possible to disambiguate all the variables by renaming bound variables uniformly so that you have some unique names for all the bound variables, and the bound variables do not clash with the free variables. So, by doing an alpha renaming you can actually reduce the confusion, and therefore, make a substitution admissible. So, of course, full substitution theta is admissible in phi if and only if every element in theta is admissible in phi.

(Refer Slide Time: 35:05)



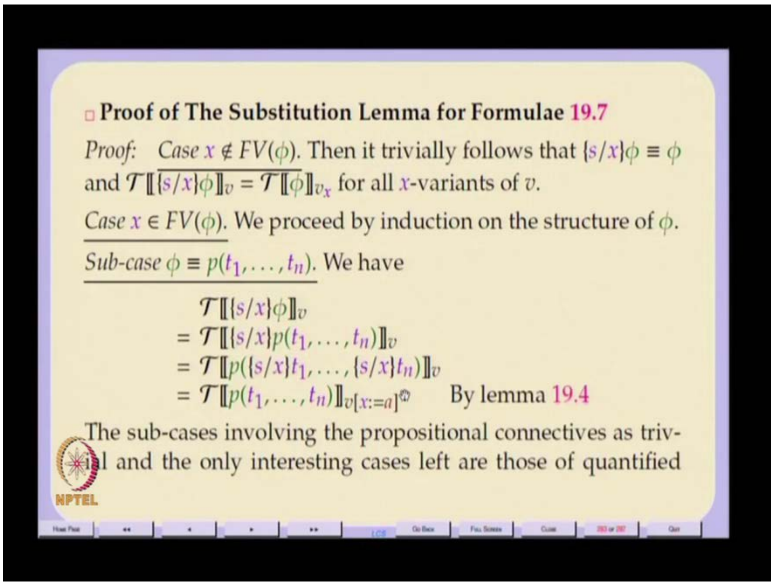
And when you apply the substitution to a formula it works exactly like in the case of terms; however, it is also defined by structural induction. However, what we need to worry about is the case of bound variables. And supposing there is a term s for x in this substitution theta and you have a quantified formula q x of psi, then what you are essentially saying is that you should not substitute x anywhere inside this formula. Because all occurrence of x inside this formula are bound to this quantifier, and therefore, are different term any occurrence of x which might occur in theta, right. And so, which means that I remove that s for x from theta it is a finite set and I remove that and the resulting substitution theta prime is what I apply into the body of this quantifier formula. So, that is the only complication that this gives us.

(Refer Slide Time: 36:27)



So, now we have a substitution lemma for formulae which works almost exactly like the substitution lemma for terms, and here we are essentially talking about truth values. And we are saying that of course, there is this notion of admissibility is of course important again. So, given an admissible substitution s for x and assume that s under the valuation v has a value A. So, this substitution s for x on phi under a valuation v can be captured without the substitution but with an x variant where x is assigned the value A, yeah.

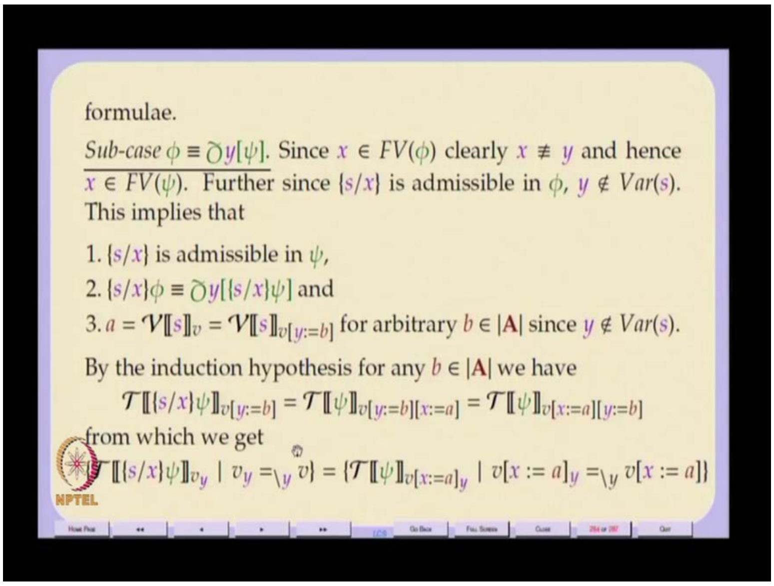
(Refer Slide Time: 37:21)



So, the proof actually is quite similar. Here again the interesting cases are only the quantifier cases, right. But before we get on to that there is one case when this. So, we are looking at s for x applied to a formula phi, and of course, if x is not a free variable of phi, then it is clear that all x variants give the same value to the free variables of phi. And therefore, the truth value under the substitution is the same as a truth value for phi under the x variant for all x variants. So, it does not actually matters, since x does not occurs in it. Since x is not a free variable, the effect of this substitution is to leave the formula unchanged.

And since x is not a free variable of phi, it does not matter what x variant you consider of the variation, you will always get the same truth value. So, having disposed of this case you basically have this case of x being a free variable of phi in which case you have to proceed by induction on the structure of phi. So, again here in the case of the atomic predicates you push the substitution down, and then you use the induction hypothesis essentially to say that this works this way; I am sorry the coincidence lemma because you come down to the terms, right.

(Refer Slide Time: 39:16)



So, use the coincidence lemma essentially to make this claim, and the sub cases involving the propositional connective are just tedious. So, the only interesting cases are quantified formally and here again we will not consider the two quantifiers separately. We will consider them together which means again we are looking at sets of truth values that are generated by variations of a valuation. However, what does complicate matters is that x is a free variable of phi which means that x is not the same as this y. S for x is admissible in phi which means that y does not belong to the variables of a. So, there is not possibility of capture of free variables.

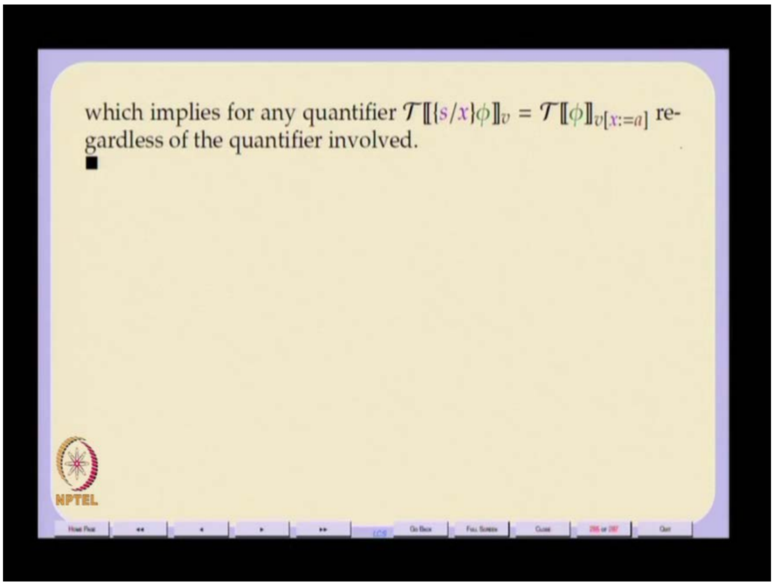
Now all this implies firstly, that s of x is admissible to the body and s of x phi because s for x is admissible I can push s for x inside the scope of this bound variable y. And given this A equals the value of s under v, it is clear that since y does not occur in s anywhere any y variation of v that you might consider leaves the value of s unchanged, right. So, for arbitrary b belonging to the domain, since y does not belong to the variables of s, the value of s does not change or does not vary with the variation in the value of y, right. So, for all y variants of v this will give you the same value A ,and then by the induction hypothesis for any b belonging to A we have this; here is some certainty that you might want to look at.

So, essentially I am looking at y variants all y variants. So, here I am taking all b belonging to A. So, take any arbitrary b belonging to A and consider a y variant of v, the effect of this substitution on psi under this y variant is the same as not doing the substitution and taking an x variant of that y variant, right. That is the important thing, and of course, we know that x and y are different distinct variables. So, one of the things you can do is you started off with an original valuation v, and now you can commute you can permute these two variations.

So, you can think of this same variant. So, you can think of this as a y x variant of v; any y x variant of v is also an x y variant of v and that is what. So, by permuting these two assignments I am essentially getting this. And this can be done only because of the fact that y and x are different variables. Otherwise there is a certain sequentiality and the assignment which is very much like assignments in programs in imperative languages which will have to be respected, but this permutation can only be done because x and y are distinct. And since this is true for all v what it means is that the entire set for all y variants of v and I can actually enclose it by the set brackets, right.

So, these two sets are equal. So, for all y variants of v this set that is evaluated gives me some essentially either a singleton set zero or a singleton set one or a set consisting of both the elements zero and one. And here I am considering the x variants and I am considering a y variant of this x variant. So, this is what is; why is the subscript of v of x assigned A, right. So, this is a y variant of this x variant. And of course, these two y variants are equal, yeah. This is all metasyntactic manipulation.

(Refer Slide Time: 44:24)



And what this gives us therefore is. So, this implies since the two sets are equal, therefore, it does not matter what quantifier I am talking about as long as I use the same quantifier on both sides I get inequality, and this is the substitution. So, the moral of the story is that basically substitutions can be captured in the semantics and more importantly semantical variations can be captured through substitutions. After all ultimately in the description of an abstract mathematical theory, what you are trying to say is you are trying to say that I want to explain or present in some language the formalities of the theory. And therefore, I will be able to present all those semantical variations also by syntactic substitutions, yeah.

So, this is fairly involved and it is actually not there in any book that I have seen except in cursory ways in some books they talk about validity within the question of validity. So, many of these things that we have used here are actually hidden deep in some other proofs because they did not prove them earlier like as a substitution lemma or as a coincidence lemma. But what I thought was that it is better to actually specify it right from the start. So, I will stop here now, and then we can see that the semantics of first order logic is getting to be a little stressful, yeah. So, we will stop here now and we will continue again later.