Logic for CS
Prof. Dr. S. Arun Kumar
Department of Computer Science
Indian Institute of Technology, Delhi

Lecture - 16
The Hilbert System : Completeness

From yesterday's lecture there the hilbert system changing the terminology to soundness rather than consistency. So, that we reserve consistency for a set of segments and we think of system is being sound and which is a which is fairly well known which is a fairly failures word for prove system in general. So, might have just used that. So, we are talking and of course, the main reason for that is that you know it is a its possible to prove inconsistent things if you are set of assumptions inconsistence. So, the soundness of a theory does not does not necessary mean you cant prove any inconsistencies right if you make an inconsistencies set of assumptions you can actually prove inconsistencies things which is fine that is way its important to define the soundness.

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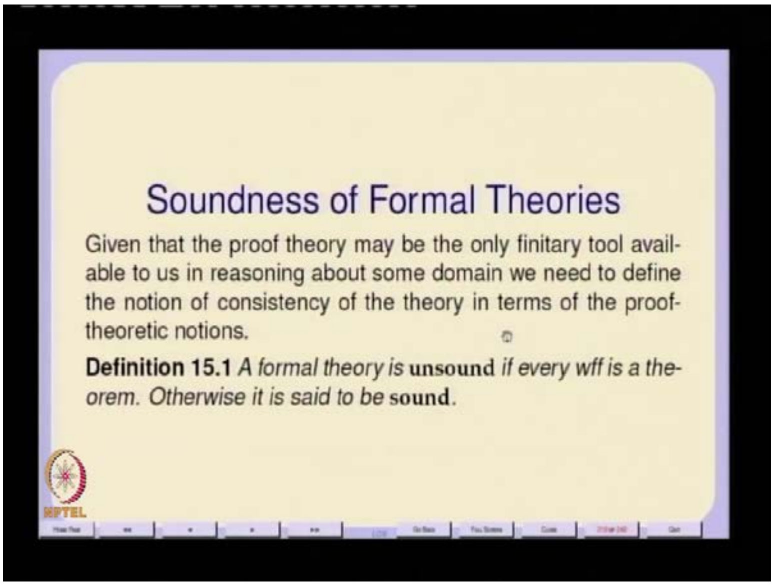

As being one in which you cannot prove all possible set things I mean or rather not all possible sentences are theorems formal theorems so; that means, given an empty set of assumptions whatever you can prove should not be the entire language that is. So, that is an notion of soundness of a formal theory and lets quickly go through it. So, basically we showed that every instance of every axiom schema's and the Hilbert system is a proctology and more rule actually preserves property of tautologousness right of right.

So, so which and since we know that not all formulae in the language I will not our tautologies says it may shows that the Hilbert system is sound right. So, so as I said the proof of soundness it just means that showing each of the axiom schema's the actually is a kind of a template for a class of tautologies following a certain pattern right and and proof that modest pronouns tautologousness is also something that is easily done.

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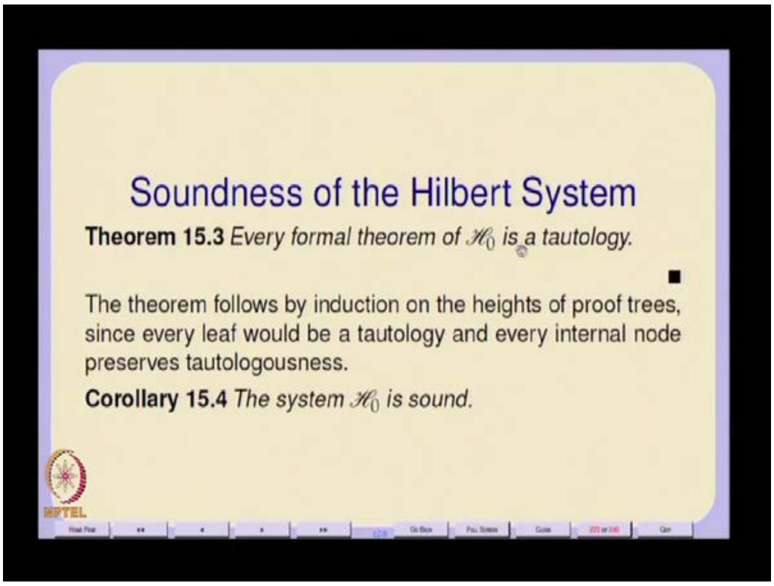

So, and from the soundness of the Hilbert system we also get that every formal theorem; that means, every statement that you can prove without any assumptions is a tautology and therefore, the system is sound and now lets actually look at the completeness.

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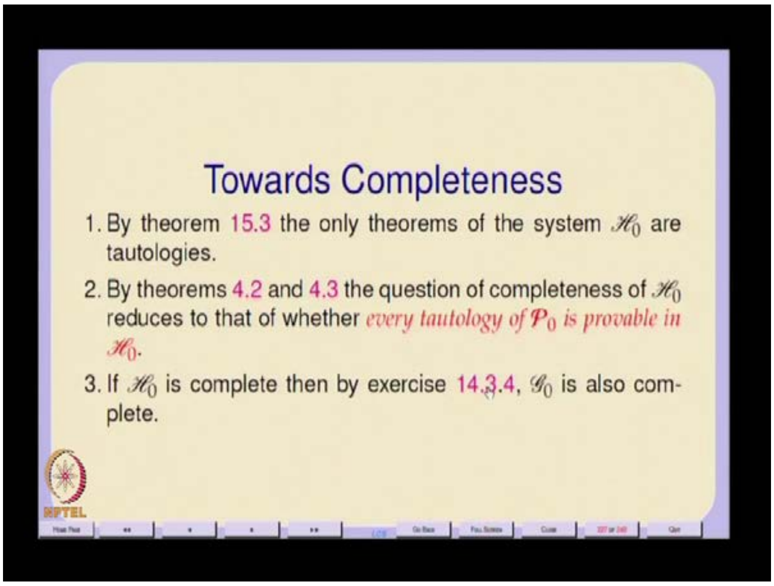

So, what I said was. So, one thing is that the only formal theorems are the system are tautologies and, but we need for for completeness we need to show that every tautology is. In fact, provable in the system right.

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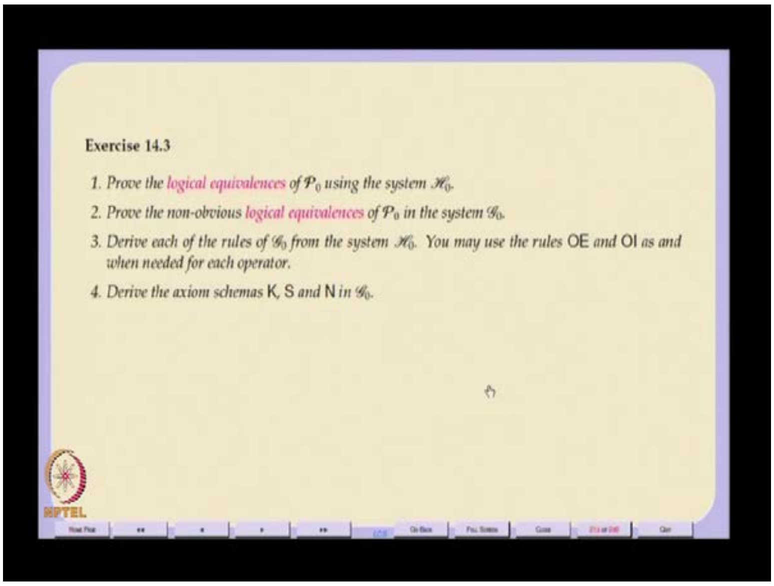

So, and then a consequence of that is that is the hilbert system is complete then by this exercise in which you derive the axiom schemas k s and n in the gensis system if you floow the gensisi system is also complete right.

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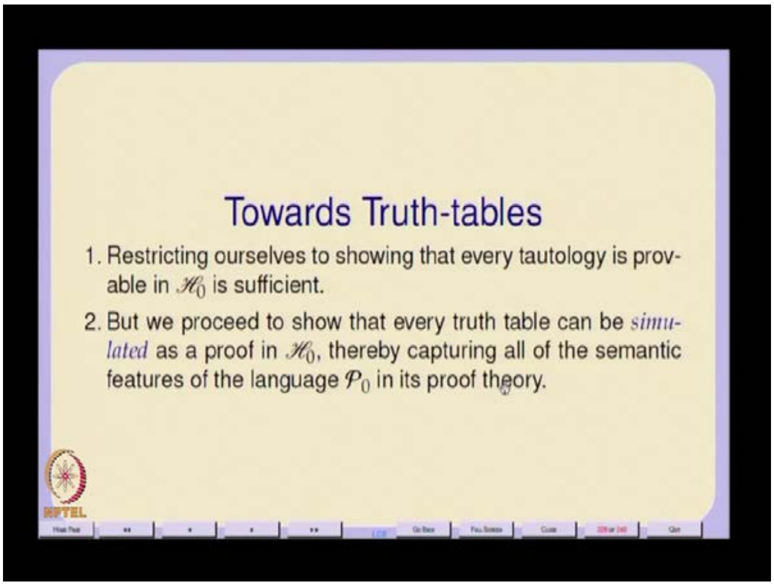

And what I said was we will we were yesterday we were actually looking at simulating the truth table and in that proof of the simulation of the truth table they were quite a few errors. So, first thing I thought I would do is correct this those errors and the next thing I thought was I will actually go through the proof of completeness because its non trivial.

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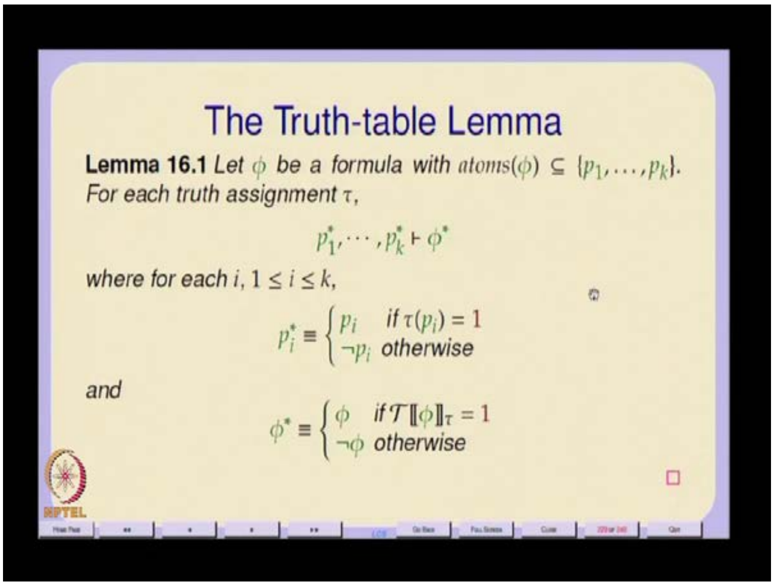

So,. So, will stimulate a truth table. So, the simula. So, the truth table lemma as I call it essentially says that you take for any formula with whose atoms are contained in this set and some finite set p one to p k for every truth assignment tau there is formal deduction from the assumptions p I star the formula phi star where p I star and phi star are defined as being p I and phi if the truth assignment tau gives truth value one otherwise the negation of p a and p I and phi here ok.

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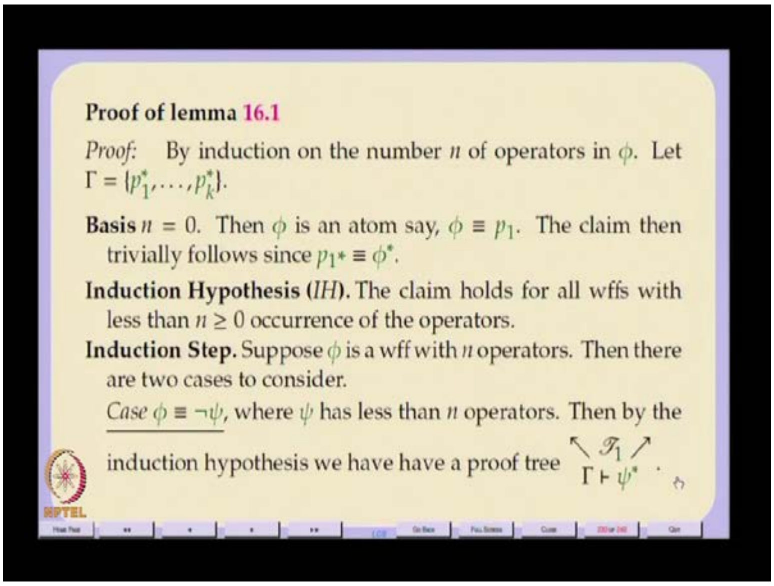

So, this. So, as I said. So, we are going to this proof is by induction on the number n of operators in the formula phi. So, and the set of assumptions the set gamma which is. So, essentially we are taking some particular row of the truth table and p one star to p k star of the set of assumptions if that number of operators and phi is the zero then phi must be an atom and therefore, it trivially follows that p one star is syntactically the same as phi star and therefore, and that is provable right.

Part of the monotonocitic considerations and that is provable and any anywhere you can prove p r o p without any assumption and then by the deduction theorem you can move the left p to the assumption and therefore, you are proved proved this. So, assume this induction hypothesis that for all the claim holds for all well found formulas of which have less than n operators and we take this inductions step. So, the induction step of course, is by case analysis on this structure of the formula. So, let us assume that phi is a formula with n operators and then we have two cases to consider essentially one is where phi is of the form naught of some sie where sie has less than n operators.

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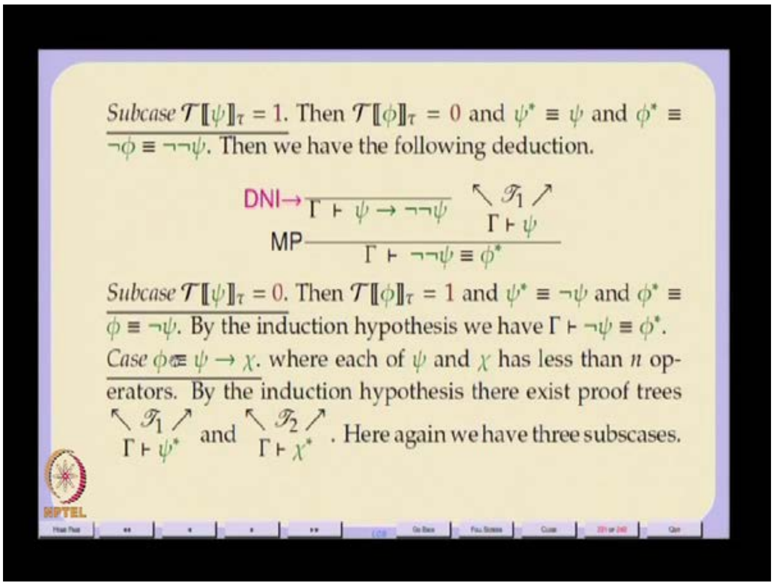

And the other case is when phi is of the form some sie arrow kie where each of sie and kei has less than n operators right. So, the case when phi is not sie then we by the induction hypothesis we actually have a proof tree t one where from assumption gamma you can actually prove sie star whatever that is yeah. So, then of course, when you have two sub cases one is with the truth assignment sie is one or zero. So, in each case what happens is if. So, if the truth assignment to sie is one then phi must be zero because phi is not sie and therefore, sei star is the same as sie and phi star is naught phi which is the same as naught of naught of sie and then this this deduction essentially shows that from gamma you can prove phi phi star yeah this prove tree essentially shows this given that so.

Actually. So, so we actually had gamma proves sie star, but I have written gamma proves sie here because we know sie star is the same as sie right in the case when sie is been assigned zero by the is assigned false by the truth assignment then t of phi must be true I mean phi must be true therefore, sie star is not sie and sie star is tee same as phi. So, it is also naught not sie by the induction hypothesis we have this and and the proof is clear in the case when phi is of the form sie arrow phi we have essentially three sub cases when sie is assigned false clearly it does not matter what kei is when kei is assigned true clearly it does not matter what sie is the only other case third case is when sie is assigned false.

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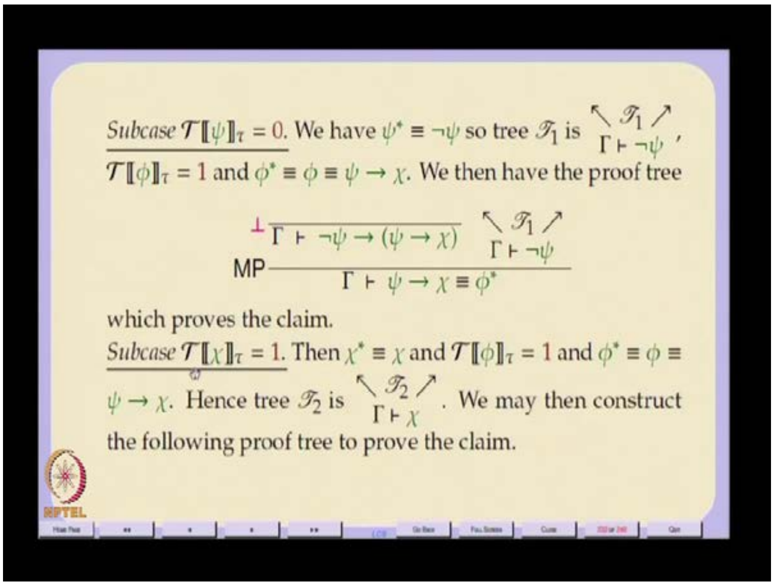

And kei is assigned true the I am sorry when sie is assigned true and kei is assigned flase and that is the important case.

So, in each of these cases. So, in the case when sie is assigned false and it does not matter what kei is we have sie star as naught sie so; that means, our induction hypothesis says there is a proof tree t one of this form and phi must be therefore, b true which means phi star is the same as phi which is which must be syntactically sie arrow kei. So, now, we can we do not have this proof tree this is one of the exercises.

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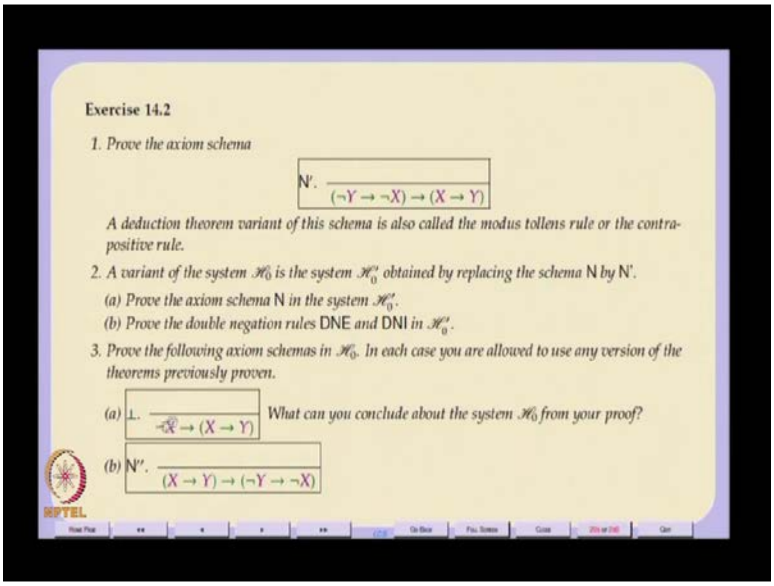

You can see as you can prove naught x arrow x arrow y right. So, that and you can actually apply this ah this derived inference rule and you can claim that from gamma you have proved naught sie arrow sei arrow kie and you already have naught sie here by modest pronence therefore, you have sie arrow kie which is the same as phi star in the case when kei has been assigned one then kei star is the same as kei phi is been assigned one and therefore, phi star phi star is the same as phi. So, phi star is the same as sie arrow kei.

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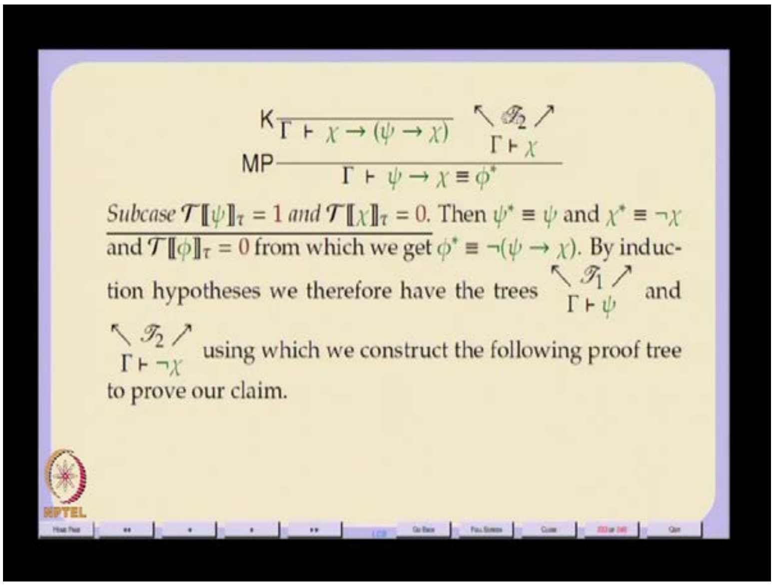

So, we have this proof tree proving kie by the induction hypothesis and then we can use that proof of tree starting with the axiom k where you have kei arrow sei arrow kei and you have proved kei arrow in this prrof tree and therefore, you have sie arrow kie which is the same as phi star yeah the last sub case epsilon sie has been assigned one and kie has been assigned zero clearly phi is not true since phi is of the form sie arrow kie right.

So, which means that. Firstly, sie star is the same as sie kei star is not kei and phi star is naught of sie arrow kei. So, we have these proof trees t one and t two by the induction hypothesis which from the same assumptions gamma prove respectively sie star which is sie and kei star which is naught kei.

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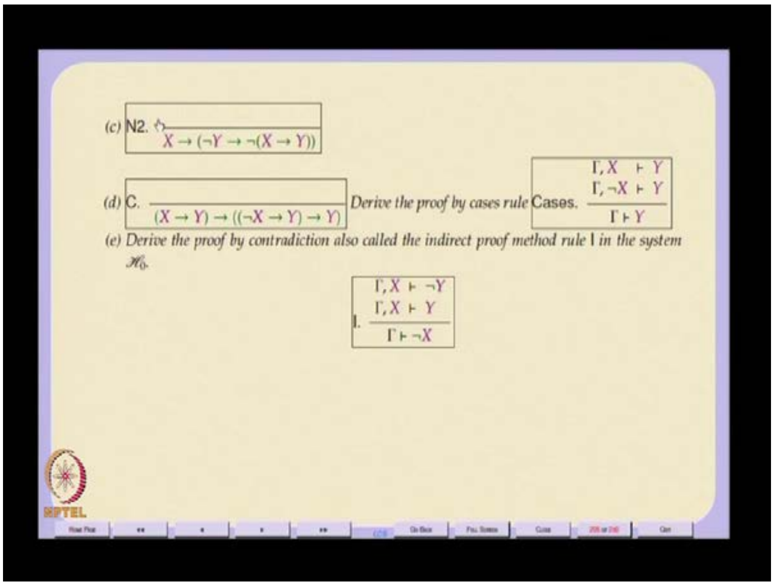

And now what we do is we have this derived rule here which says x arrow naught y arrow naught of x arrow y right if you proved soy can use this derived rule along with t one.

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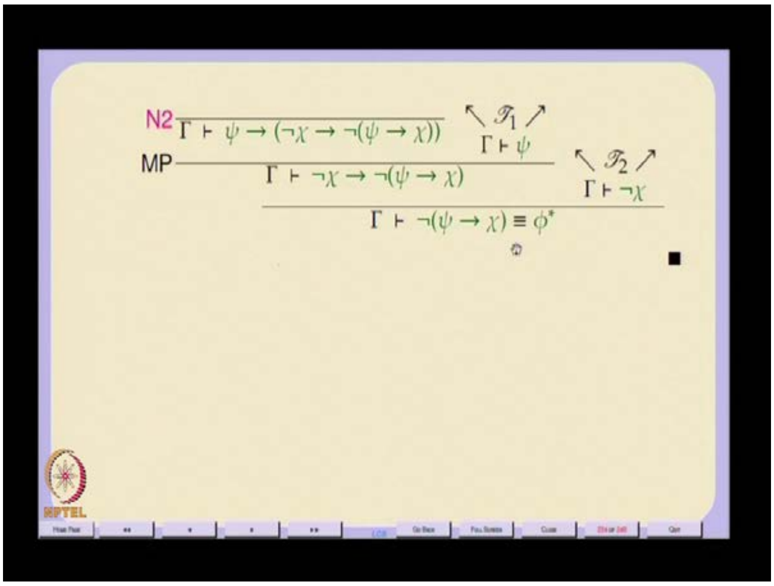

Which proves sie apply modest pronece to get naught kie arrow naught of sie arrow kei and of course, you have by the induction hypothesis this proves tree tree t two with the co9nclusion naught kei. So, you can again apply them modest pronence rule to get naught of sie arrow kei which is the same as phi star right. So, those those exercises called were actually important in order to show that you you can simulate every row of a truth table by by a proof by a deduction from assumptions.

Ah from the assumptions which are the truth values essentially of the atoms of the truth table right.

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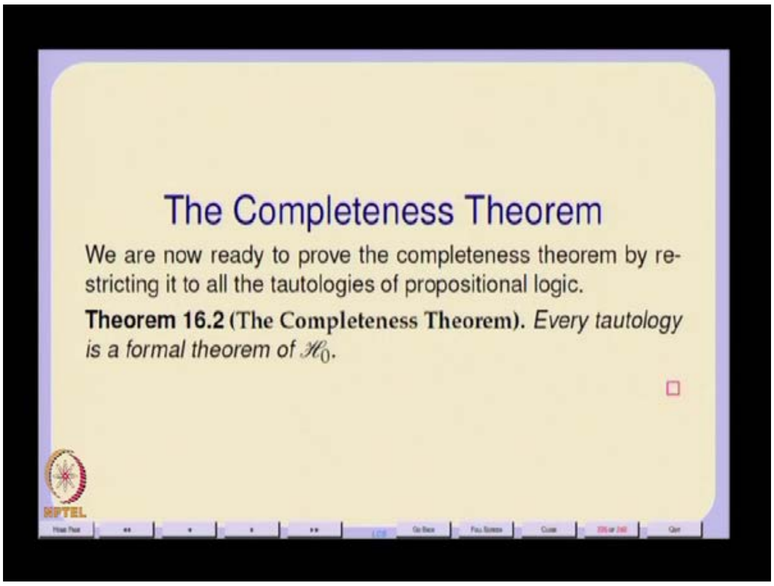

So, now. So, the actual completeness theorem has to show that every tautology in l naught is a formal theorem of the system h naught and though I thought it was though I initially felt it was sort of like obvious if you simulated the entire truth table and you have intituary clearly captured all the schematics. So, that intuition is fine, but the formal proof of this theorem still requires the fact that you are not allow to have any assumptions right you are talking about every assumption every the truth table has assumptions right the atoms the truth assignments to the atoms are still assumptions in every proof. So, is it is it clear from their you can actually get down to proving every proctology and that step is not as I originally thought it was and this is a and it is also very beautiful proof as you can see. So, will go through this proof right.

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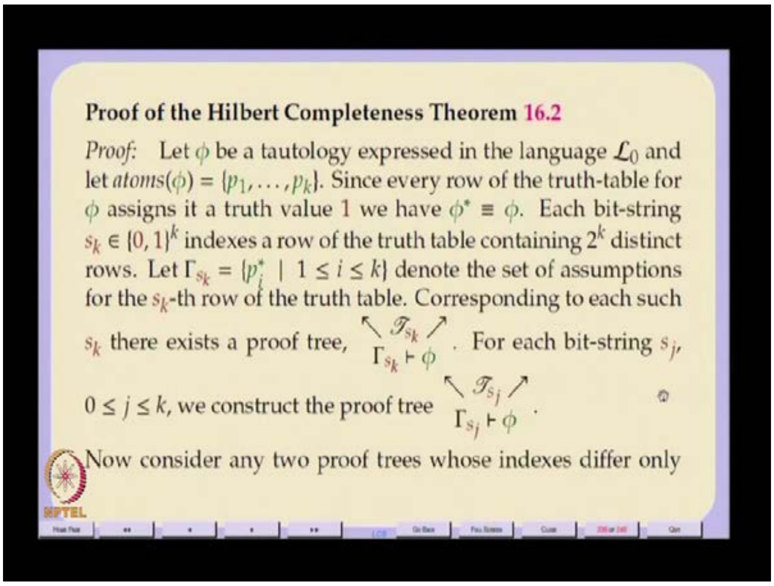

So, what we do have is that every row of the truth table cannot simulated by a proof where the truth values of the atoms from the assumptions of that proof right. So, what is. So, what I will do is. So, we will encode each row of the truth table with a standard by a bit string of if if we are talking about atoms then each row of the truth table has essentially k columns one corresponding to each atom take the k atoms in the order p one to p k and take the truth assignment in that row as essentially a a k length bits string right. So, now, what we are saying is. So, the back. So, the truth table lemma essentially says that for every such k length bit string s k I have a proof tree and we have to proof that every tautology is a formal theorem. So, assume phi is a tautology which means the truth table assigns the truth value true to in all rows of truth table to phi. So, phi star in this case by the truth table lemma is just phi itself.

So, now what this means is that corresponding to each row of the truth table I have proof tree a t s k where the assumptions are gamma s k. So, I am using the same encoding to to specify what is the truth assignment to each of the atoms with those atoms I can proof phi and that is this is this is given from the truth table lemma. So, for each now what we are going to prove is that you take any bits string s j where j is the length less than or equal to k we are going to construct the proof tree t s j such that gamma s j proves phi. So, in the case when j is equal to k of course, we have already have these proof trees, but what we are going to do is we are going to systematically contract the assumptions till j becomes zero when j becomes zero essentially you got a gamma s zero s zero is just a empty string that is gamma epsilon and essentially that that is empty set of assumptions otherwise s j for j less than or equal to k essentially gives you the truth assignment for each of the atoms p one to p j and if and assuming that you have already removed the atoms p j plus one to p k right. So, so. So, so. So, there are two raised to k rows on the truth table and we have two raised to k different proof trees one for each string s k there are two raised to k strings bits strings of length k right. So, what we are going to do is we are going to contract this bit strings still you come to the empty bit string right. So, this is what. So, in order to do that consider two bit strings which differ only in the last bit in the right most bit. So, we. So, assume that you you. So, we have already got t s k right for all for all the case.

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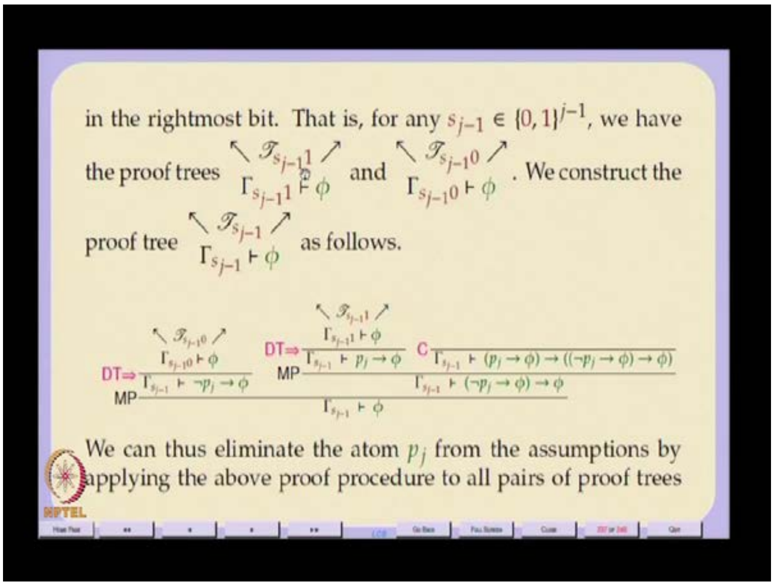

So, you take essentially. So, starting from any proof tree t s j you take s j minus one zero and s j minus one one. So, you got proof trees t s j minus one zero and t s j minus one one which proves from the assumptions gamma s j minus gamma s j minus one one it proves phi and from gamma s j minus one zero it proves phi right. So, this is a this is a this is a j length bit string this is also a j length bit string where s j minus one is j minus one length bit string and then you have had have a one all zero. So, essentially we pair wise we choose the two bit strings which which are identical who have identical j minus one prefixes, but differ only in the j’th bit right. So, what we. So, then what we are going to construct is a proof tree t s j minus one from these proof trees. So, this is very simple. So, this is how we are going to go about it. So, you take t s j minus one one which proves from gamma s j minus one it proves phi and you apply essentially the deduction theorem. So, that you move the j’th assumption to the right.

So, when you move the j’th assumption to the right you get p j arrow phi right in the case of one you get p j arrow phi in the case of zero you get naught p j arrow phi right and then we have this exercise of proof by cases right which shows that which says that x arrow y and naught x arrow y implies y essentially that is what it says. So, this is a proof by cases and. So, we essentially apply this proof by cases right. So, when you apply this proof cases and for these two conclusions p j arrow phi and naught p j arrow phi you get that gamma s j minus one proves phi right. So, what we have done is we started with two raised to k different proof trees and we halve it by taking proof trees pair wise and creating proof trees two raised to k minus one proof trees with assumptions only from p one to p k minus one and this procedure of removing the right most assumption is very general is absolutely general. So, you can do this systematically one by one. So, we start with two raised to k proof trees halve it down to two raised to k minus one by removing the assumption p k and naught p k then you remove the assumption p k minus one and naught p k minus one s one and till you will actually obtain this a singe monolithic proof tree right and this procedure is absolutely this procedure is absolutely general and this proof by cases works as generally right.

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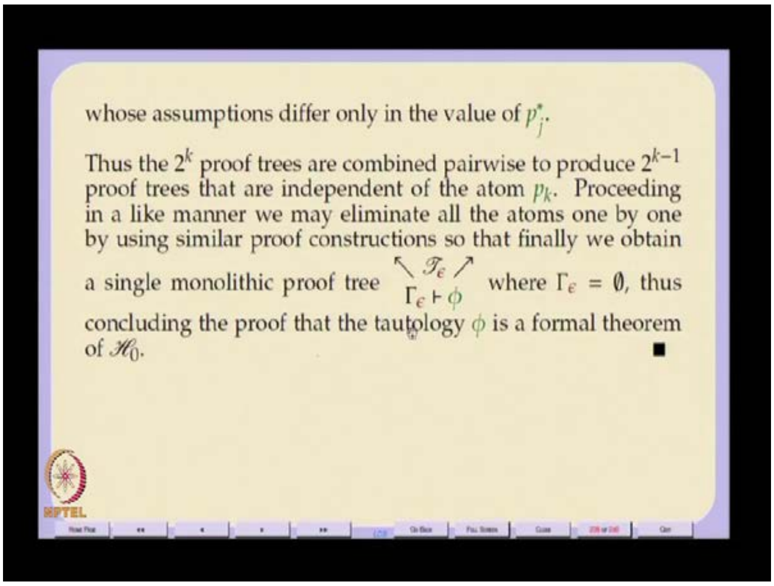

So, this one monolithic proof tree essentially gamma epsilon is empty and therefore, you have proved that starting with the assumption phi is a tautology can actually be the proven right. So, so this is how this proof goes and this proof cannot be replicated the moment you have you have a system where the schematics is infinite tree right and that is precisely what is going to happen when we do first order logic right. So, this. So, which means we when we when we come to first order logic there are certain things that we will not have any access too one is the notion of the truth table or at least a finite truth table any logical notion of a truth table there with probably be infinite tree depending o0n the model that you are looking at depending on what you are trying to aximotize. Secondly, there is. So, which means that this this proof is actually you need to this this construction and therefore, this proof will not such a proof cannot be extended easily to first order logic or anything which is which has an infinite tree schematically nature which might generate infinite infinite trees. So, on. So, right and you can see that this proof is also very different from the proof of completeness that we use to the tabular system right was there is question was not really of validity, but of set un staisfiability or satisfiability right and. So, you have different kinds of proofs and it is a good idea to actually look at. So, it is a it is a good idea to look at various different kinds of proofs where as the tabular method of completeness proof might actually be applicable even to other other systems which are infinite tree in nature, but this kind of proof will no0t be applicable then right.

So,. So, there is there is actually one more concept which I have not done, but which should which I will mention, but I will not proceed further with it and that is a very important concept, but I do not think we have the either the time or the where without to actually deal with it.

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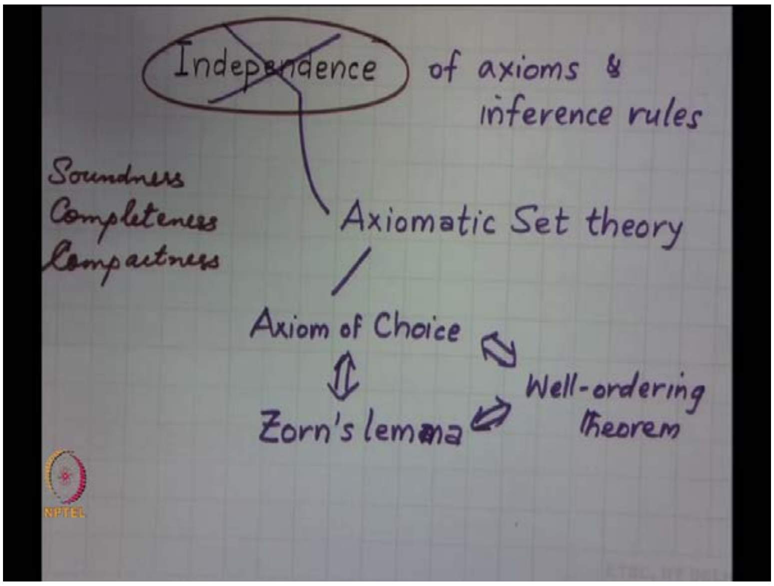

And that is the independent of axioms. So, there. So, we we had we had several important concepts in in a logic. So, we of course, we have already dealt with three important concepts one is of course, I mean besides soundness completeness compactness. So, these are the this are the three important these are the three important concepts there is fourth important concept called independence and that independence has to do with the question of. So, we have this notion of derived rules and derived operators h we also had this notion of adequacy and function completeness right. So, our functional completeness essentially said that every any operator that you might define in propositional logic can be expressed in terms of whatever the operators are well not actually it is a its enough to just have is adequate to just have the operators arrow and not right.

So, in that sense the operators any operator that you might define for propositional logic is not functionally independent of the operators arrow and naught this a similar and analogous notion is that of independence of axioms and inference rules right. So, one important thing that therefore, at which needs to be shown especially in a minimal system the hilbert system was suppose to be a minimal system is that the the two concepts I mean the the the axiom schemas of the hilbert system are mutually independent it it is not possible to take any sub set of those axiom schemas and inference rules and prove the rest as in a derived fashion this this independence is actually fairly a hard concept to prove, but its an important concept for example, its not a concept that is easily applicable to something like natural deduction which is not meant to be a minimal system, but if you take a minimal system then you actually have to prove the independence of the axioms and the proofs of independence are general not not are not very easy. So, I will not actually go into them, but independence is as an impart is an important concept from the point of view of logical theory for other reasons.

So, yesterday I took the example of the parallel postulate right. So, what we what are what are we saying now I can remove uclid's parallel postulates and put another parallel postulate which says that there are no parallel lines basically and still get a consistent geometrical theory I can still have a sound geometrical theory. So, in that sense the parallel postulate is sort of independent of other axioms of uclid right and this independent notion also have comes up when you when you do something like axiomatic set theory right. So, one of the one of the most boolean proofs of independence was this. So, I do not know how much of axiomatic set theory you know. So, its possible to take tree set theory as an axiomatic theory very much like the wave you have defined within the domain of a stardology. So, which we have in the end yet done and one of the important things about axiomatic set theory is this notion of the axioms of choice which actually you have you have probably seen a different fashion because the axiom of choice is actually equivalent to other thing one is on's lemma and and the other is the well ordering theorem right yeah. So, so. So, and actually in any kind of in any kind of in in a large number of proofs we use a notion of a well ordering the fact that there are no infinite descending chains right and this. So, what for a long time it was assume that the axiom of choice which which essentially is required for which is required for any kind of proof that you have only a finite sequence or it is always possible to construct from an infinite set some finite from total ordering and. So, on and. So, forth all those things required this notion of either the axiom of choice or they require a form of well ordering or they require ones lemma, but what Pual in the sixtees well after axiomatic set theory was well established was that this axiom of choice is actually a independent of the other axioms of set theory. So, which means its possible to remove the axiom of choice and have a completely separate set of set theory separate axiom system for set theory or non standard set theory in which the axiom of choice may not be applicable the moment axiom of choice may not be applicable it also means you may not you may not use any kinds of well ordering you van not use ones lemma and. So, on. So, forth right and in that sense you still have a you can this the axioms of set theory still allow enough leaving to adsorbed some of their axiom instead of the axiom of choice because axiom of choice itself is independent of time right, but this concept of independence otherwise in a first of logic is actually quite hard to understand where it ahs come up with some phenomenally interesting results on formal theories. So, I will not. So, this is a the concept that I will not touch on we will proceed straight to first order logic yeah and. So, today I will not late to start that right will I think will stop here today