Indian Institute of Science
Bangalore

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<u>Course Title</u>

Finite element method for structural dynamic
And stability analyses

Lecture – 09
FRF-s and damping models - 1

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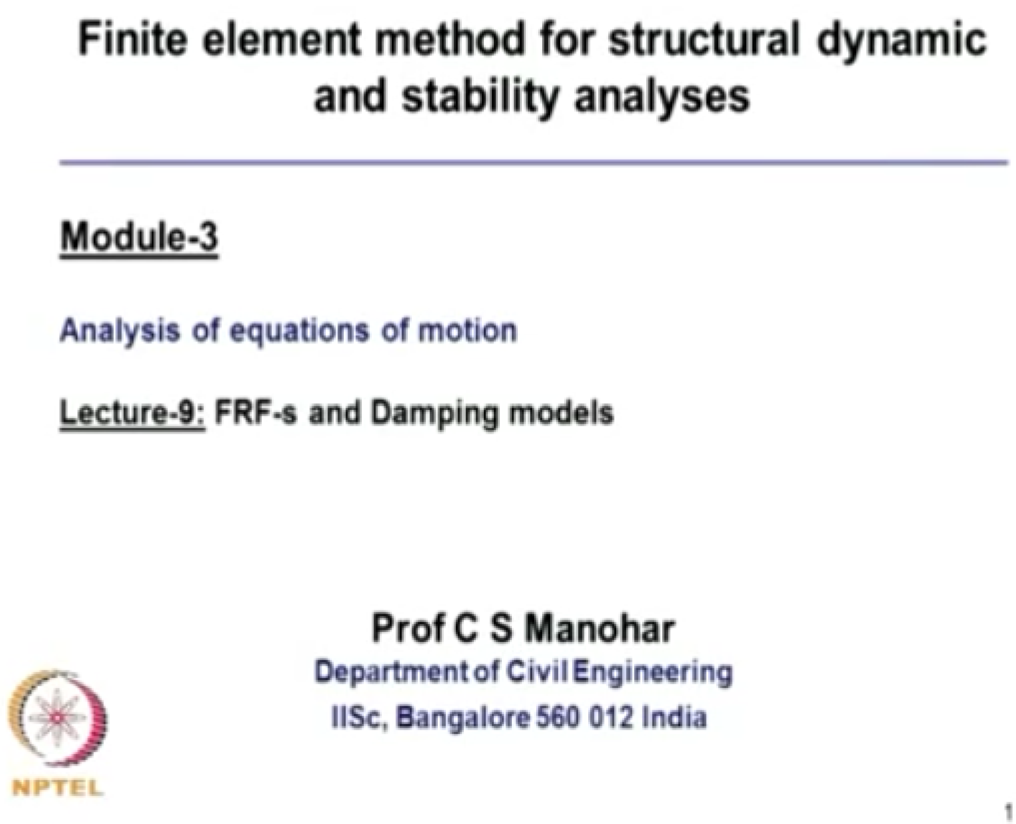
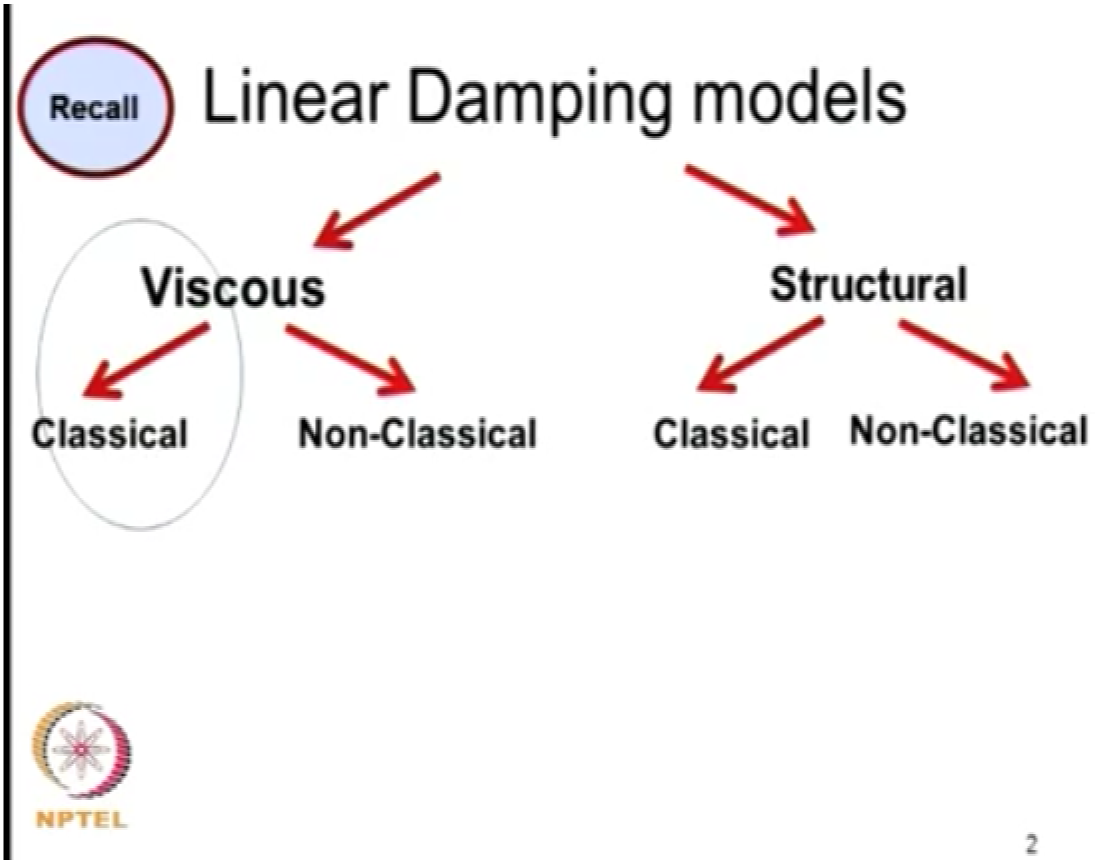
We have been discussing modeling of frequency response functions and the role played by



damping models in doing that exercise, so we are dealing with linear damping models and we



broadly classify the damping models as being viscous or structural and within each of these groups we have classical and non-classical models. The classification into viscous and structurally something to do with the way the energy dissipated in a cycle behave as a function of frequency, and the classification into classical and non-classical depends upon whether the un-damped normal modes uncouple the equation of motion or not. In a classical damping model the un-damped normal modes diagonalize the damping matrix, whereas in non-classical damping models that won't happen.

Viscously damped MDOF system with s-th dof driven by an unit harmonic force

Recall

$$M\ddot{X} + C\dot{X} + KX = F\exp(i\omega t)$$

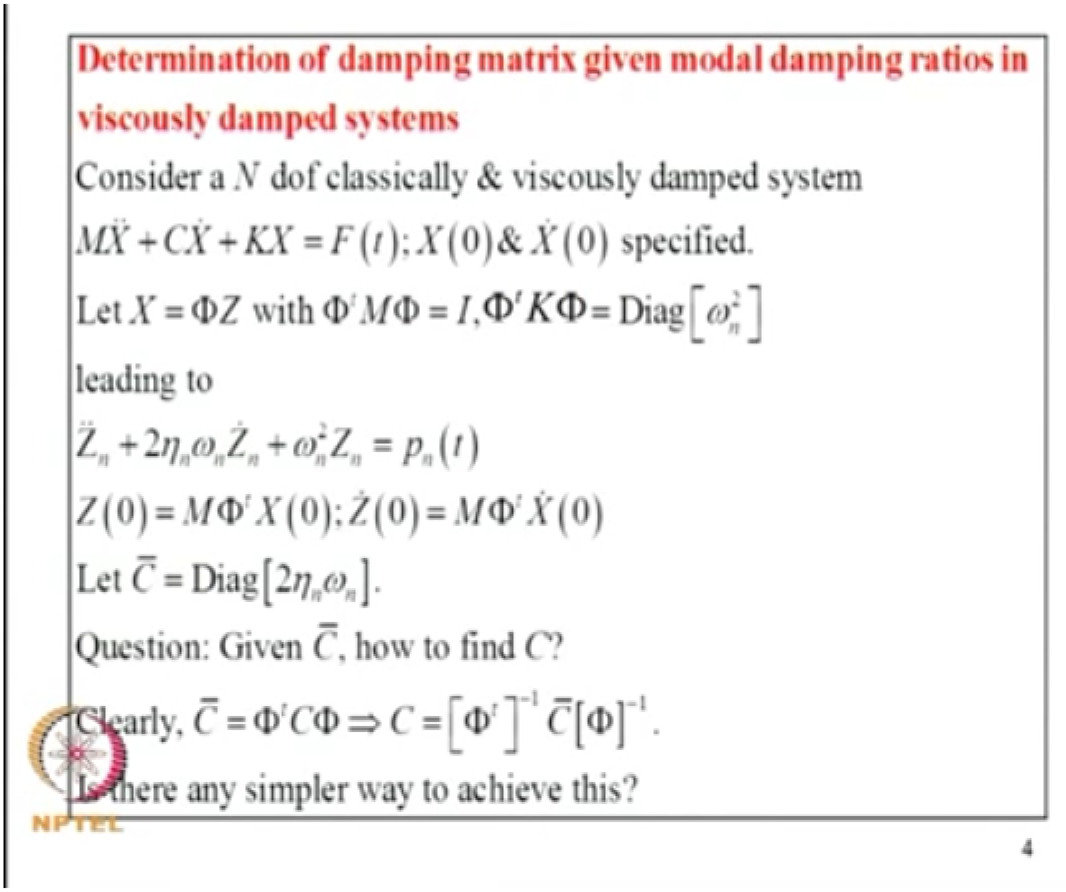
$$F' = \{0 \quad 0 \quad \cdots \quad 1 \quad \cdots \quad 0 \quad 0\}$$

- $[H(\omega)] = \left[-\omega^2 M + i\omega C + k\right]^{-1}$
- $[H(\omega)] = \left[\sum_{n=1}^{N^* \le N} \frac{\Phi_{rn}\Phi_{sn}}{(\omega_n^2 - \omega^2 + i2\eta_n\omega_n\omega)}\right]$

$$H_{jk}(\omega) = -\frac{1}{\omega^2 M_{jk}^R} + \sum_{r=m_1}^{m_2}\left[\frac{{}_rA_{jk}}{\omega_r^2 - \omega^2 + i2\eta_r\omega_r\omega}\right] + \frac{1}{K_{jk}^R}$$

So we discussed the problem of finding frequency response function for viscously damped multi degree freedom system, so if we drive the S degree of freedom by a unit harmonic force we have shown in the previous lecture that the matrix of frequency response functions can be evaluated either directly by inverting this matrix, the so called dynamic stiffness matrix or we can evaluate in terms of un-damped normal modes and we get this as a summation.

Now we also discuss few issues about truncation of this series so if we omit the first M1 modes, M1-1 modes and the last M2 + 1 to the N number of modes this will be the kind of representation and these two terms here and here provide the corrections for omitting the contributions from normal modes, which have not been included in this summation so we discussed this in the previous lecture.



Now we will address few points before we move on to discussion on non-viscously damped systems, so let us ask the question how do we determine damping matrix given modal damping ratios in viscously damped systems, so let's consider a N degree of freedom classically and viscously damped system the governing equation will have this form F (t) is excitation and X (0) and X dot (0) are the specified initial conditions. So we introduce the transformation X = Phi Z, where Phi is the matrix of un-damped normal modes which have been normalized so that Phi Transpose M Phi is I, and Phi Transpose K Phi is the diagonal matrix of the eigenvalues which are the squares of natural frequencies.

Now this leads to set of uncoupled equations in the new coordinate system which is a family of single degree freedom systems and we discussed how to evaluate the initial conditions without inverting the Phi matrix. Now the C bar be the diagonal matrix of the model bandwidths or these terms 2 ETA N, Omega N. Now the question is if I provide C bar how do I get C, you will see later in the course that we may like to integrate these equations directly and in which case I need C matrix whereas the damping modal may be in terms of modal damping ratios, so the question we are asking is given the modal damping ratio is how do we construct a special damping model, this is a modal damping model I want a special damping modal, so clearly C bar is Phi Transpose C Phi, so if you have a full square matrix for the modal matrix you can directly evaluate this by inverting, so you need to invert 2 matrices, Phi Transpose and Phi so in principle this is possible but in practical situations, first of all we don't like to invert matrices, if we can help avoiding that we would be happy to do so. Secondly more importantly we will not be evaluating Phi as a square matrix, we will not be evaluating all the modes that are possible in a given discretized system, we may be focusing on only first few modes in which case Phi will be a rectangular matrix, the number of rows will be equal to degree of freedom but number of columns will be equal to the number of modes that you want to include in the analysis, so in which case using this formulary will not be feasible.

$$I = \Phi^t M \Phi$$

Post multiply by Φ^{-1}

$$\Rightarrow \Phi^{-1} = \Phi^t M \Phi \Phi^{-1} = \Phi^t M$$

Premultiply by $\left[\Phi^t\right]^{-1}$

$$\left[\Phi^t\right]^{-1} = \left[\Phi^t\right]^{-1} \Phi^t M \Phi = M \Phi$$

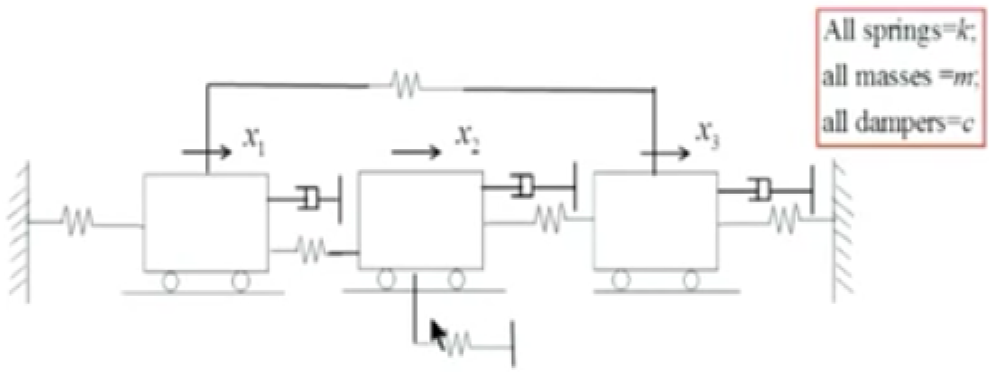
$$C = \left[\Phi^t\right]^{-1} \bar{C} \left[\Phi\right]^{-1}$$

$$\Rightarrow C = M \Phi \bar{C} \Phi^t M$$

So now how do we proceed, so what we can do is there is a simple way of overcoming this problem suppose if I consider the orthogonality relation I as Phi transpose M Phi, now I can post multiply by Phi inverse, so Phi inverse will be Phi transpose, I'm post multiplying Phi transpose M Phi, Phi inverse, now Phi Phi inverse is identity matrix so therefore Phi inverse is Phi transpose M. Similarly if I pre multiply by Phi transpose inverse I will get this which is M Phi, so for Phi inverse and Phi transpose inverse I can use these representations and I can get the damping matrix without inverting the fee matrix or its transpose, so this is a simple way of evaluating the damping matrix.

Treatment of systems with repeated natural frequencies

$$M=\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}; K=\begin{bmatrix} 3k & -k & -k \\ -k & 3k & -k \\ -k & -k & 3k \end{bmatrix}; C=\begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}$$



$$\begin{vmatrix} 3k-m\omega^2 & -k & -k \\ -k & 3k-m\omega^2 & -k \\ k & -k & 3k-m\omega^2 \end{vmatrix} = 0 \Rightarrow \omega_1^2 = \frac{k}{m}; \omega_2^2 = \omega_3^2 = \frac{4k}{m}$$

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Now another question is what happens if natural frequencies repeat, it is not very unusual if structure is symmetric in a geometric sense the eigenvalues can repeat, for example a square plate if you look at normal modes what is a normal mode in X Direction would also be a normal mode in Y direction, suppose I am talking about a square plate suppose all round simply supported if one of the mode shapes in this direction is this the same mode shape will also be possible in the other direction, so therefore the same eigenvalue there will be two eigenvectors so how do we deal with such situations, so we will return to this later but we will now consider a simple example artificially constructed example where we can see that the eigenvalues repeat, so the system has three masses and a set of springs so the data is that all springs are having same value K, all masses have the same value M, and all dampers have same matrix C.

So now the mass matrix will be this, stiffness matrix will be this, and damping matrix will be this, so the characteristic equation will be given by the determinant of K - M omega square = 0 which is this, if we solve this characteristic equation we see that first natural frequency is K/M, and the second and third natural frequency square it will be 4K/M, so the second and third eigenvalues repeat.

$$\omega_1^2 = \frac{k}{m}$$

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = 0$$

$$\text{Let } R_1 = 1 \Rightarrow R_2 = 1 \; \& \; R_3 = 1$$

$$\Phi_1^t = [1 \quad 1 \quad 1]$$

$$\omega_2^2 = \frac{4k}{m}$$

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = 0$$

$$\Rightarrow R_1 + R_2 + R_3 = 0$$

$$\text{Let } R_1 = 1, R_2 = 1 \Rightarrow R_3 = -2$$

$$\Phi_2^t = [1 \quad 1 \quad -2]$$

$$\omega_3^2 = \frac{4k}{m}$$

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = 0$$

$$\Rightarrow R_1 + R_2 + R_3 = 0$$

$$\text{Let } \Phi_3^t = [1 \quad a \quad -(1+a)]$$

How to select a? Any a would do.

Select a such that $\Phi_2^t M \Phi_3 = 0$.

$$[1 \quad 1 \quad -2] \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ -1-a \end{Bmatrix} = 0$$

$$\Rightarrow \Phi_3^t = [1 \quad -1 \quad 0]$$

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Now how do we handle this in evaluation of eigenvectors and in calculating the response, so let us find the eigenvectors suppose if omega 1 square is K/M the first eigenvalue then to determine eigenvectors I go back to the original eigenvalue, statement of the eigenvalue problem and put omega 1, omega S, omega 1 square and I get this equation. Now in this since eigenvector has a free-floating constant, that means any constant multiple of an eigenvector is also an eigenvector, so there is a non-uniqueness to the extent of a scalar multiplier we can start by assuming that R1 is 1, and it implies by writing this equation that R2 and R3 would also be 1, you can verify that so because 2 – 1 – 1 is 0, so you can see that they satisfy that, so the first eigenvector can be written with this normalization as 1, 1, 1. For the second eigenvalue that is omega 2 square by 4 KM, I write the equation for the eigenvector, we see that the equation is R1 + R2 + R3 = 0, so there are infinitely many solutions that are possible but what we do is we select deliberately only two of, only one of them so that we retain the orthogonality property, this eigenvalue repeats this will be the equation for the next eigenvalue also, so let us do here let R1 be 1, and R2 be 1, so then R3 would become – 2, so Phi 2, the second eigenvector will be of this form.

Now let us come to the third eigenvector the governing equation is same as what it was here because eigenvalue is the same, so R1 + R2 + R3 = 0. Now what I do is I select now the third eigenvector so that it is orthogonal to both first and second eigenvector so I assume that Phi 3 transpose is 1A and - 1 + A so then we can see that R1 + R2 + R3 is 0 so A is a free-floating constant, so how do we select A we can select any A as far as this equation is concerned this is true for any A, but what we do is we select A such that Phi 2 transpose M Phi 3 is 0, so if we

$$\Phi = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -2 & 0 \end{bmatrix}$$

Consider equation of motion given by

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} + \begin{bmatrix} 3k & -k & -k \\ -k & 3k & -k \\ -k & -k & 3k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ p(t) \end{Bmatrix}$$

$$x = \Phi z \Rightarrow$$

$$\ddot{z}_1 + \frac{c}{m}\dot{z}_1 + \omega_1^2 z_1 = p(t)$$

$$\ddot{z}_2 + \frac{c}{m}\dot{z}_2 + \omega_2^2 z_2 = -2p(t)$$

$$\ddot{z}_3 + \frac{c}{m}\dot{z}_3 + \omega_3^2 z_3 = 0$$

$$z(0) = \Phi^t M x(0); \dot{z}(0) = \Phi^t M \dot{x}(0)$$

impose that condition I get A as 0, and the third eigenvector is this, so if we now consider the modal matrix to be 1 1 1, 1 1-2, 1 -1 0, then we will return to this equation with a forcing on the one of the degrees of freedom. And now if I impose this transformation and use the orthogonality relation for this Z1, Z2, Z3, I get these three equations, which are again couple, you know uncoupled and these are the generalized forces that we have to use, so we can solve this problem and construct back the solution using X = Phi Z, okay so that means if eigenvalue is repeat we can construct eigenvector so that the required orthogonality property is still possessed by the calculated eigenvectors.

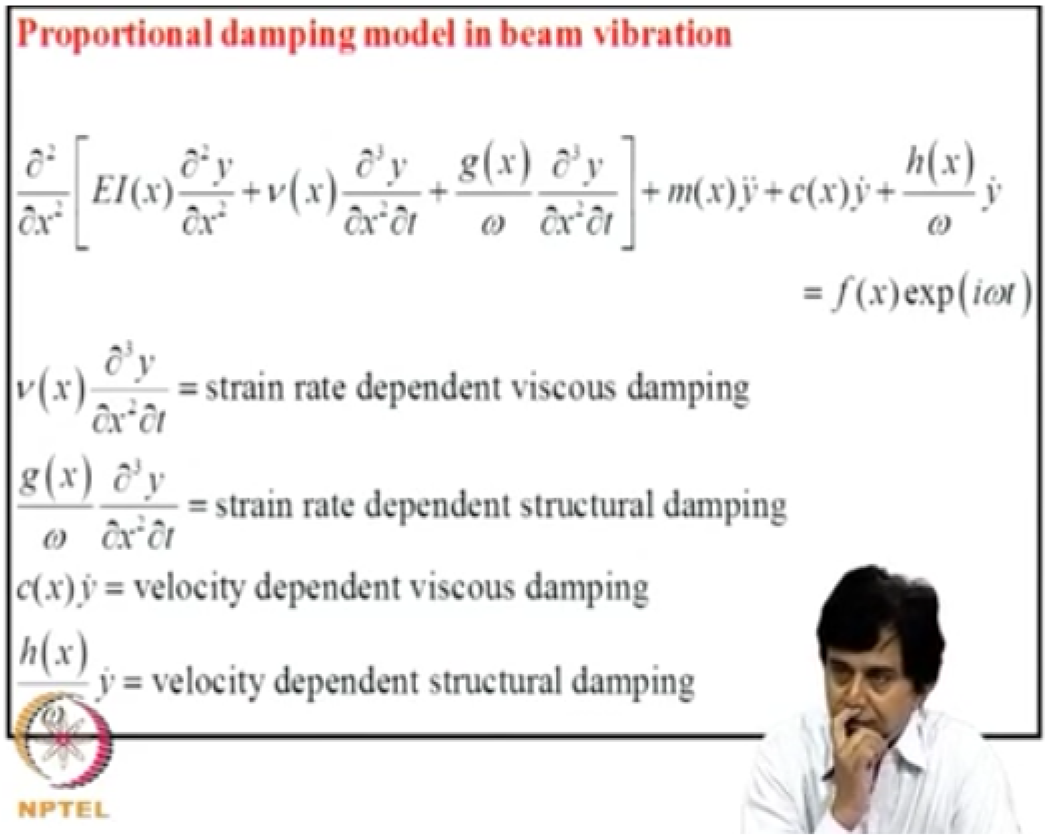
Remark

Consider a N dof system.

Let the n^{th} natural frequency repeat r number of times $(2 \leq r < N)$.

The eigenvectors associated with the repeated eigenvalues can be selected (through a process of orthogonalization) to be orthogonal to the structure M matrix (and hence also orthogonal to the K matirx).

So a remark can be made at this stage if we consider N degree of freedom system if the nth natural frequency repeat R number of times, where R can be between 2 and N the eigenvectors associated with the repeated eigenvalues can be selected through a process of orthogonalization to be orthogonal to the structure mass matrix and hence also the orthogonal to the structure stiffness matrix, okay, so this orthogonalization procedure you have to implement if you find that your system has repeated eigenvalues, so this helps us to uncouple the equation.



Now we talked about proportional damping and non-proportional damping models in the context of discretized system it may be useful to visit the equation for the beam vibration and see what is the meant by mass proportional damping and stiffness proportional damping as far as the beam equation is concerned, so it turns out that if you consider the beam equation we can add, we have added now here 4 terms, this is the one, this is second, this is third, and this is fourth, which represents energy dissipation properties. If these energy dissipation properties are absent the traditional equation is Dou square / Dou X square EI, Y double prime + MY double prime, Y double dot = F. Now this term Nu(x) into Dou cubed Y / Dou X square Dou T is actually damping which depends on strain rate, and it is viscous damping. The second term is strain rate dependent structural damping because you see here in the denominator I have put omega to ensure the required energy dissipation characteristics as a function of frequency, and this C into Y dot is a traditional velocity dependent viscous damping, so this doesn't depend upon strains, it only depends on the displacements.

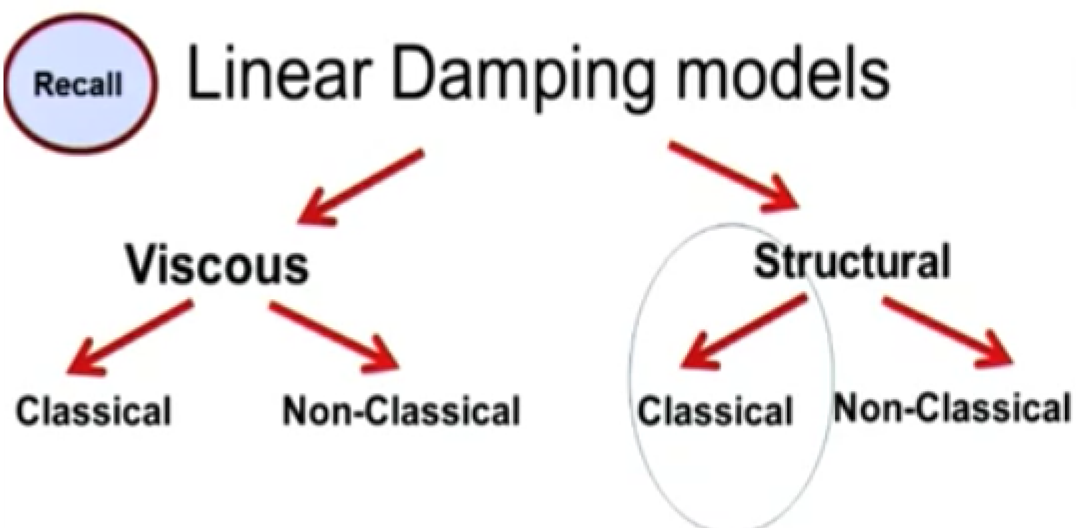
$$\nu(x)\frac{\partial^3 y}{\partial x^2 \partial t} = \nu_0 EI(x)\frac{\partial^3 y}{\partial x^2 \partial t} \quad \text{(stiffness proportional viscous damping)}$$

$$\frac{g(x)}{\omega}\frac{\partial^3 y}{\partial x^2 \partial t} = g_0 \frac{EI(x)}{\omega}\frac{\partial^3 y}{\partial x^2 \partial t} \quad \text{(stiffness proportional structural damping)}$$

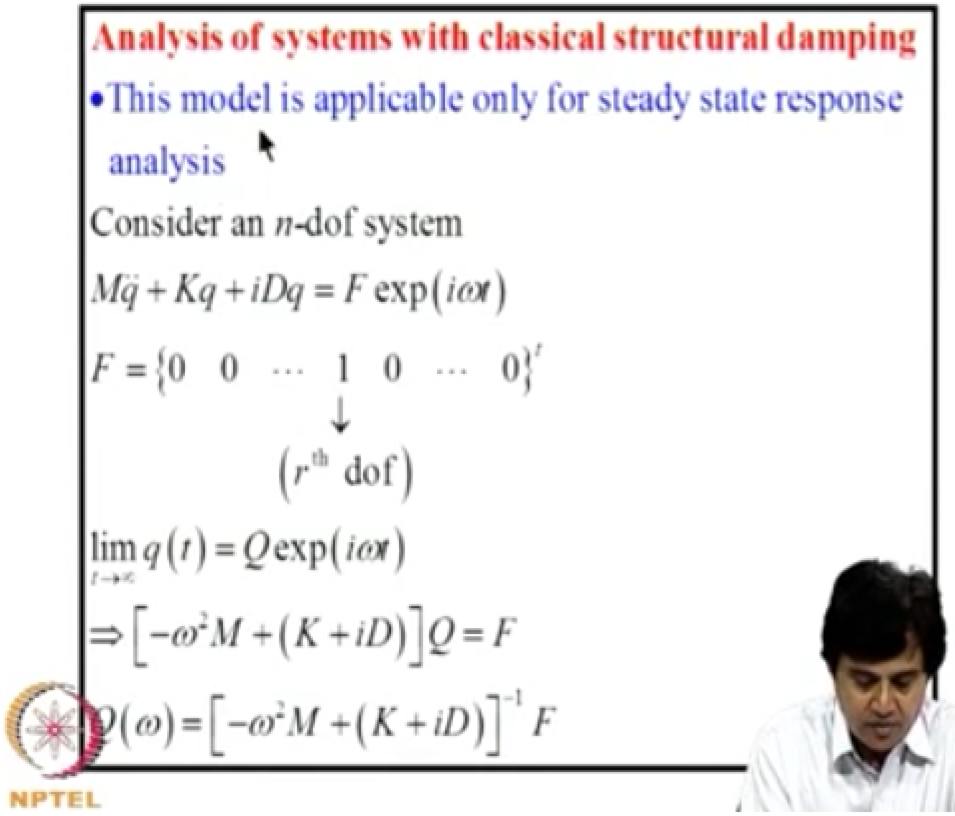
$$c(x)\dot{y} = c_0 m(x)\dot{y} \quad \text{(mass proportional viscous damping)}$$

$$\frac{h(x)}{\omega}\dot{y} = h_0 \frac{m(x)}{\omega}\dot{y} \quad \text{(mass proportional structural damping)}$$

Now this term is the velocity dependent structural damping, so this is how the damping represent terms appear if I were to write the original partial differential equation. Now if I make now this Nu (x) to be proportional to the flexural rigidity then we get stiffness proportional viscous damping, if I make G(x) to be proportional to EI, I get stiffness proportional structural damping, similarly if C is made proportional to mass M(x) I get mass proportional viscous damping, and so finally if I make H(x) to be proportional to M(x) then I get velocity dependent structural damping. So when we discretize these equation we get MX double dot + CX dot + KX = F(t), but it is useful to have this insight at the original PD level, so we know what we are doing at the original PD level when we assume different alternative models for damping, okay.

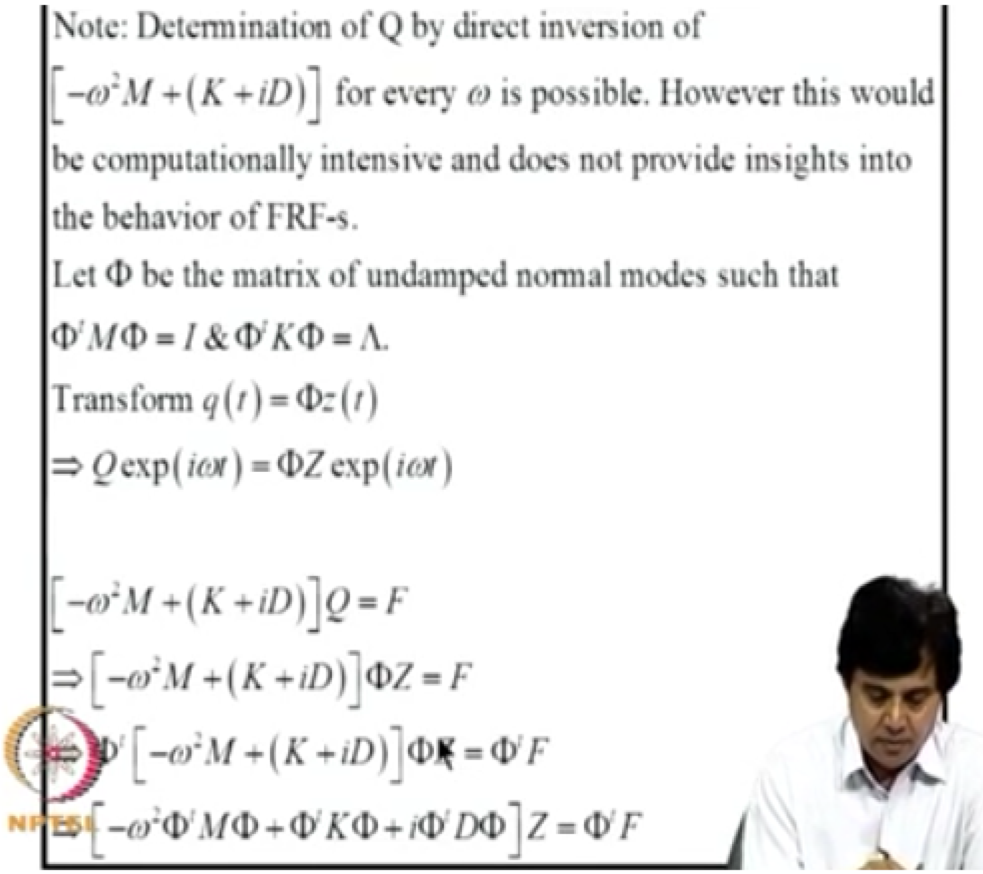


Now let us now move on to other damping models, so we have finished discussing viscous classical damping systems, systems with viscous and classical damping, now we will move on to structural damping models and begin by discussing classical damping models, so how do we



discuss this, so again when we are whenever we are talking about structural damping I want to emphasize that such models are applicable only for steady state response analysis there you cannot do transient analysis, no time demand analysis is applicable, so we consider N degree of freedom system with harmonic forcing, so this equation can be used only for the purpose of calculating frequency domain response, it is not a differential equation in with T as independent variable, although it is made out to be like that so while interpreting this equation you should only use this only to find the frequency response function.

Now let R-th degree of freedom be driven by an unit harmonic excitation, now we are interested in steady state, suppose in the steady state the response becomes harmonic Q E raise to I omega T, the system is linear it is being driven harmonically and we are considering steady state, so in steady state the system response at a driving frequency in a harmonic manner, so if I now substitute this into the governing equation i get this as a dynamic stiffness matrix this into Q is F, so if you are interested in finding the matrix of receptances or the frequency response function you can invert this matrix directly for every omega, okay, so these are direct solution to the given problem, but we are interested in finding a solution which employs normal modes and I want to write this as a summation, because if I use this approach for every value of omega that I'm interested I need to invert to this matrix which can be computationally demanding, so



the note is that determination of Q by direct inversion of dynamic stiffness matrix for every omega is possible, however this would be computationally intensive and does not provide insights into the behavior of a frequency response, it is characterless description.

Now let Phi be the matrix of un-damped normal modes such that it is mass normalized and Phi transpose K Phi is a diagonal matrix of square of the eigenvalues, now we will make the transformation Q as Phi Z, so the usual steps if I now find the QE raise to omega T will be this and this will be, if I now substitute into this equation I will get in frequency domain this equation and if I am pre multiply by phi transpose I get this. Now if I take Phi transpose and Phi insight I'll get - omega square, Phi transpose M Phi + Phi transpose K Phi + I into Phi transpose D Phi Z is the generalized force, Phi transpose F.

$$\left[-\omega^2\Phi^t M\Phi+\Phi^t K\Phi+i\Phi^t D\Phi\right]Z=\Phi^t F$$

Suppose, that D is classical, that is, $\bar{D}=\Phi^t D\Phi$ is a diagonal matrix, it follows

$\left[-\omega^2 I+\Lambda+i\bar{D}\right]Z=\Phi^t F$ with $\left[-\omega^2 I+\Lambda+i\bar{D}\right]$ being a diagonal matrix.

$$\Rightarrow Z_k=\frac{\sum_{j=1}^{n}\Phi_{jk}F_j}{\omega_k^2-\omega^2+i\bar{D}_k}=\frac{\Phi_{rk}}{\omega_k^2-\omega^2+i\bar{D}_k}$$

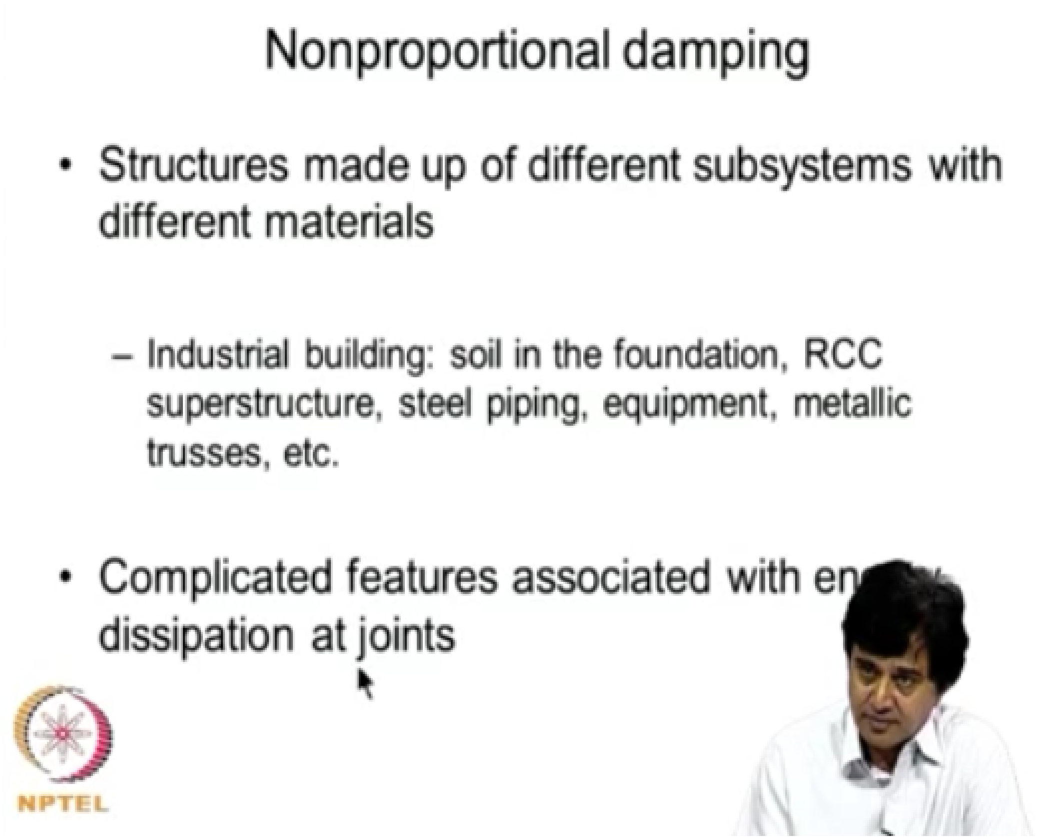
$$q(t)=\Phi z(t)$$

$$q_j(t)=\sum_{k=1}^{n}\Phi_{jk}z_k(t)=\sum_{k=1}^{n}\frac{\Phi_{jk}\Phi_{rk}}{\omega_k^2-\omega^2+i\bar{D}_k}\exp(i\omega t)=\alpha_{jr}(\omega)\exp(i\omega t)$$

$$\alpha_{jr}(\omega)=\sum_{k=1}^{n}\frac{\Phi_{jk}\Phi_{rk}}{\omega_k^2-\omega^2+i\bar{D}_k}=\alpha_{rj}(\omega)$$

Now suppose D is classical by that what I mean, the same this Phi transpose D Phi is diagonal, that means un-damped normal mode diagonal as a damping matrix, if that is possible then it follows let us call D bar as Phi transpose D Phi, so this will be Phi transpose M Phi is I, that is how we are normalized, Phi transpose K Phi is a diagonal matrix of square of eigenvalues which is capital Lambda, and this ID bar into Z is Phi transpose F, so this matrix - omega square I + Lambda + ID bar is a diagonal matrix now, right, so it's inversion is a simple task so if I right now this in the scalar form and look at the K-th term I get this as the, you know, expression for ZK, so this can be, since only one of the ordinates is being driven here, this summation collapses to a single term is a Phi RK, so this is ZK.

Now if I go back to Q it is Phi Z, so if I’m interested in QJ, we have to sum over all the modes I get this expression, so this quantity is the receptance as we have seen and this is given by this where we are now getting this without inverting a matrix as a summation, okay. Now this is symmetric and not Hermitian as I have been pointing out so this is the receptance function. So this completes the formulation for FRF for classically damped and structurally damped systems.



Now let us move on to questions about treatment of non-proportional damping, first we can ask the question where do you expect to get non-proportionally damped systems, typically whenever we have structures made up of different subsystems with different materials we can expect that the resulting damping matrix will be non-proportional, for example in an industrial building there will be soil in the foundation and there will be RCC superstructure there could be steel piping and equipment, metallic equipment, and there could be a metallic truss, so on and so forth, so the entire structure consisting of soil, superstructure, the secondary systems and the trusses etcetera consist of several materials, okay, several subsystems, each subsystem made up of a different material, so for this type of systems we can expect that the damping in the system would be not classical, that means un-damped normal mode will not uncouple the equation of motion. So there would be other complicated features associated with energy dissipation at joints like you know you may have welded joint, bolted joints and so on and so forth, so how does energy dissipate at the joints, so this also could contribute to damping being non-proportional.

$C = \sum_{i=1}^{N_s} C_i$; N_s =number of subsystems with different materials

C_i = Contribution to the structural damping matrix from the *i*-th subsystem.

Let us assume that Rayleigh's proportional damping model is valid within each subsystem.

$$\Rightarrow C = \sum_{i=1}^{N_s} \left(\alpha_i M_i + \beta_i K_i \right)$$

Clearly, even for this case

$$\Phi^t C \Phi = \sum_{i=1}^{N_s} \left(\alpha_i \Phi^t M_i \Phi + \beta_i \Phi^t K_i \Phi \right)$$

would not be a diagonal matrix.

Questions

What is the mathematical framework to uncouple the equation of motion?

Are there any simplifications possible so that the damping remains classical and yet the same time we take into account the fact the the structure is made up subsystems with different materials?

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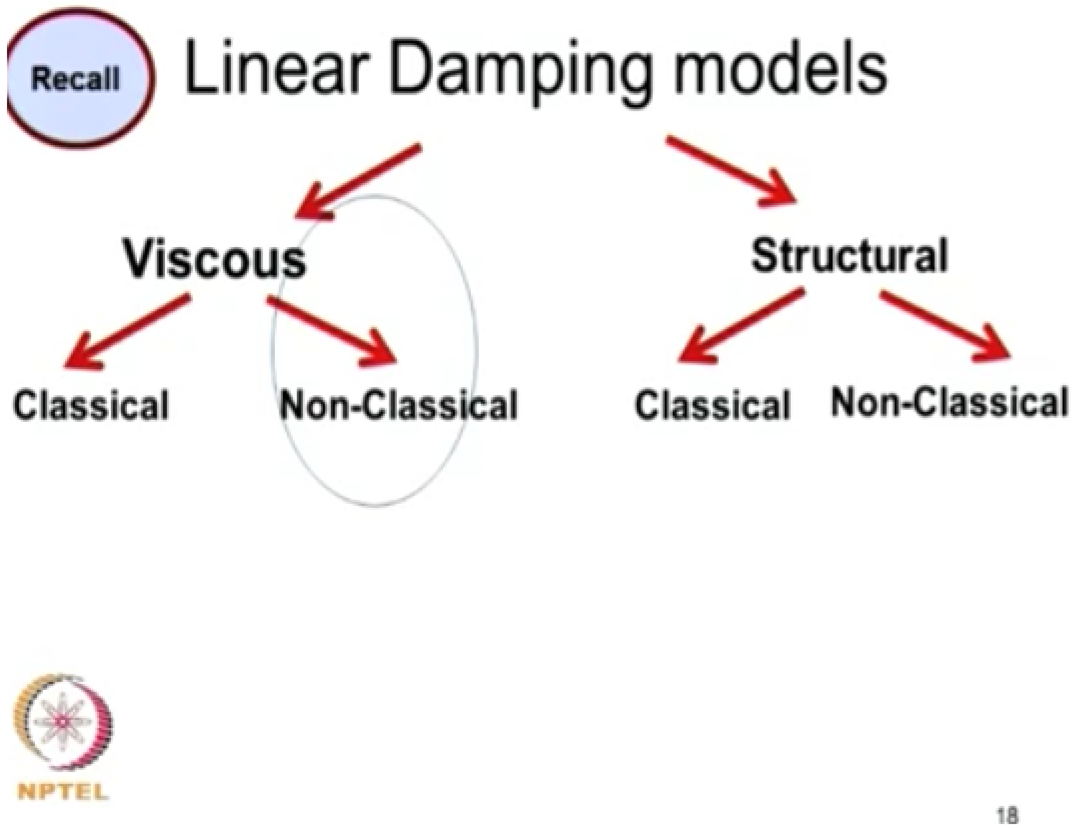
So if we now consider a system made up of NS subsystems, the system damping matrix can be viewed as summation I = 1 to NS CI, where NS is number of subsystems with different materials, CI is a contribution to the structural damping matrix from the I-th subsystem to the C matrix. Now if I assume that Rayleigh's type of model is applicable for each of the subsystem, okay I can write C as alpha A MI + beta I KI, right, this is I-th material, then even for this case if I now compute Phi transpose C Phi I will get this, on the right hand side I get Phi transpose MI Phi + Phi transpose KI Phi, but we know that Phi transpose M Phi is diagonal, it doesn't mean that Phi transpose MI Phi will be diagonal, so the resulting C matrix would still be non-diagonal, so now this leads to a few questions one of that is what is the mathematical framework to uncouple the equation of motion in such cases, should we use direct integration or direct analysis in frequency domain by inverting the dynamic stiffness or is there a mode superimposition type of representation possible.

Then are there any simplification possible so that the damping remains classical and yet at the same time we take into account the fact that the structure is made up of subsystems with different materials, okay, so these two questions we will now consider and try to find answers to these questions, so I will start by first considering viscous damping and we will focus on



non-classical damping, so the topic now is analysis of systems with non-classical viscous

Analysis of systems with nonclassical viscous damping

Preliminaries: sdof systems

- Undamped system

$$m\ddot{x} + kx = 0; x(0) = x_0, \dot{x}(0) = \dot{x}_0$$

$$\Rightarrow x(t) = \exp(st) \Rightarrow ms^2 + k = 0 \Rightarrow s = \pm i\sqrt{\frac{k}{m}} = \pm i\omega$$

$$\Rightarrow x(t) = x_0 \cos\omega t + \frac{\dot{x}_0}{\omega}\sin\omega t = X_0 \cos(\omega t - \Theta_0)$$

- Damped system (under damped system)

$$m\ddot{x} + c\dot{x} + kx = 0; x(0) = x_0, \dot{x}(0) = \dot{x}_0$$

$$\Rightarrow x(t) = \exp(st) \Rightarrow ms^2 + cs + k = 0 \Rightarrow s = -\eta\omega \pm i\omega_d$$

$$\Rightarrow x(t) = \exp(-\eta\omega t)\left[x_0 \cos\omega_d t + \frac{\dot{x}_0 + \eta\omega x_0}{\omega_d}\sin\omega_d t\right]$$

$$= \bar{X}_0 \exp(-\eta\omega t)\cos(\omega_d t - \bar{\Theta}_0)$$

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damping, so we can start with a preliminary ideas on single degree freedom system, suppose I have an un-damped single degree freedom system I know that if you consider free vibration the equation MX double dot + KX = 0, and if you assume solution to be of this form this is admissible provided S satisfy this equation, where S is plus minus square root K/M, so it is plus minus I omega, so the solution itself will be of this form this harmonic at frequency omega with the square root K/M, this is well known.

Now if we assume that system is damping, system has damping and it is under damped then we again consider this equation and consider these roots to be complex conjugates this is what happens when damping is less than critical, so in this case the solution will be of this form, okay X naught and X naught dot are the initial conditions this is the decaying, the term contributes the decay of the solution and as you can see ETA is sitting here. Now this again I can write it in this form X naught bar exponential E raise to - ETA omega T cos omega T - theta naught bar. So if you now compare this solution for un-damped system and this solution for the damped system, we can see that, we can see that there are couple of things that change, the frequency is now a damped natural frequency it is affected by damping, but a change is marginal, but the amplitude is now modulated by an exponentially decaying function.

Remarks

Undamped system

$$s = \pm i\sqrt{\frac{k}{m}} = \pm i\omega \;\; \& \;\; x(t) = X_0 \cos(\omega t - \Theta_0)$$

- Characteristic roots are pure imaginary
- Motion is periodic.
- Motion is sinusoidal at system natural frequency.

Damped system (under damped system)

$$s = -\eta\omega \pm i\omega_d \;\; \& \;\; x(t) = \bar{X}_0 \exp(-\eta\omega t)\cos(\omega t - \bar{\Theta}_0)$$

- Characteristic roots are complex with nonzero real parts
- Motion is aperiodic.
- Motion is exponentially decaying sinusoidal at damped natural frequency.

What happens in MDOF systems?

So for un-damped systems the characteristic roots are pure imaginary, motion is periodic and motion is sinusoidal at system natural frequency. For a damped system characteristic roots are complex with nonzero real parts and these roots are complex conjugates, motion is a periodic, and motion is exponentially decaying sinusoidal at damped natural frequency, okay, this is for single degree freedom system.

Damped MDOF systems

Consider an n-dof system

$$M\ddot{q}+C\dot{q}+Kq=f(t);q(0)=q_0;\dot{q}(0)=\dot{q}_0$$

We augment this equation with an identity and rewrite as

$$M\dot{q}-M\dot{q}=0$$

$$M\ddot{q}+C\dot{q}+Kq=f(t)$$

$$\Rightarrow\begin{bmatrix}0 & M\\ M & C\end{bmatrix}\begin{Bmatrix}\ddot{q}\\ \dot{q}\end{Bmatrix}+\begin{bmatrix}-M & 0\\ 0 & K\end{bmatrix}\begin{Bmatrix}\dot{q}\\ q\end{Bmatrix}=\begin{Bmatrix}0\\ f(t)\end{Bmatrix}$$

Introduce a new state vector y, forcing vector F and structural matrices A and B as

$$y=\begin{Bmatrix}\dot{q}\\ q\end{Bmatrix},F=\begin{Bmatrix}0\\ f(t)\end{Bmatrix},A=\begin{bmatrix}0 & M\\ M & C\end{bmatrix}\&B=\begin{bmatrix}-M & 0\\ 0 & K\end{bmatrix}$$

Note:

- $A^t=A;B^t=B$
- A and B are not positive definite.

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Now we can ask the question what happens to multi degree freedom systems in presence of damping, do we get a similar features in our response. So let's begin that discussion, we will consider N degree of freedom system let Q be the vector of displacements so the equation of motion will be MQ double dot + CQ dot + KQ is F(t) with prescribed initial conditions.

Now for reasons that would become clear soon we augment this equation with an identity, that is MQ dot - MQ dot = 0, and this is a given governing equation, so we write now these two equations together in a matrix form as shown here, so you can see here 0 into Q double dot + MQ dot - MQ dot into 0Q is 0, that is your first equation, the second equation is a governing equation, so we are not altered the mathematical nature of the problem, we have rewritten the governing equation in a slightly different way.

Now I will call now this vector Q dot Q as the state of the system at time T and denote it by Y, so I call Y as Q dot Q, and this 0F(t) I call it as forcing vector. Now I will introduce two notation, this matrix I will call as A, and this matrix as B. Now with these notations we can make few observations we can see that this A and B matrices are symmetric, okay, A transpose is A, B transpose is B, but they are not positive definite, okay, there are negative terms here and so on and so forth.

The governing equation can be written as

$A\dot{y} + By = F(t); y(0) = y_0$

Remarks:

- A and B are non-diagonal
- The above equation represents a set of $2n$ coupled first order ODE-s which is equivalent to the set of n 2^{nd} order ODE-s in the configuration space.
- The above representation is called the state space representation with $y(t) = \begin{Bmatrix} \dot{q}(t) \\ q(t) \end{Bmatrix}$ representing the system state at time t.
- As before we seek to find a transformation $y = Tz$ which uncouples the $2n$ set of equations.

Now the governing equation therefore can be now be written as AY dot + BY = F(t), so the original set of governing equation in the configuration space was a set of N coupled second order differential equations. In the state space the equations are now, there are 2N equations and they are all first order equation, so I have now 2N coupled first order equation, A and B are now the new structural matrices they don't have the interpretation of being mass and stiffness and damping and so on and so forth, nor there is any energy is associated with A and B matrices, so they have some abstract meaning, and A and B are non-diagonal they are symmetric. Now the question we can ask is can we uncouple the set of first order differential equation, now this representation as I mentioned is called state space representation and this Y which is Q dot Q is called the system state at time T.

Now as before what we want to do is we want to introduce a transformation Y = TZ, now T is a 2N/2N matrix, Z is a 2N cross 1 new coordinate system, and our objective is to find T so that upon making this transformation in the Z coordinate system the equations become uncoupled, how to find this capital T? For that we will again start with the free vibration problem AY dot +

Consider the equation governing the free vibration

$$A\dot{y} + By = 0$$

Seek the solution on the form $y(t) = R\exp(\alpha t)$

$$\Rightarrow [\alpha AR + BR]\exp(\alpha t) = 0$$

$$\Rightarrow BR = -\alpha AR$$

This represent a generalized algebraic eigenvalue poroblem.

The characteristic equation is given by $|B + \alpha A| = 0$

There would be $2n$ roots which are complex valued.

Associated with these complex roots there would be $2n$ complex valued eigenvectors.

Note: $BR = -\alpha AR \Rightarrow B^*R^* = -\alpha^* A^* R^* \Rightarrow BR^* = -\alpha^* AR^*$

$\Rightarrow$ If (α, R) is an eigenpair $\Rightarrow (\alpha^*, R^*)$ is also an eigenpair.

That is, eigenpairs appear as complex conjugates.

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BY = 0, and now what we will do is we will seek the solution in the form RE raise to alpha T, okay alpha is complex, is not pure imaginary as we used to do for un-damped system, it is now a complex number with nonzero real part and nonzero imaginary part. Now if I substitute into this the eigenvalue problem that I will get will be BR = - alpha AR, okay E raise to alpha T cannot be 0 for all T therefore the term inside the bracket should be 0 that leads to this eigenvalue problem. So this is a generalized eigenvalue problem, it's an algebraic eigenvalue problem and the characteristic equation is given by determinant of B + alpha A must be equal to 0. Now this will lead to a 2N order polynomial, and it will have 2N roots which will be complex valued.

Now we cannot rank order them because complex number cannot be ordered so we simply list them as 2N eigenvalues associated with each of these eigenvalues there will be a complex-valued eigenvector, now we can make one observation if you now consider BR = - alpha AR and take conjugation on both sides I will get B star, R star = - alpha star, A star, R star, but A and B are real valued, so B star is same as B, A star is same as A, so that would mean it will be BR star = - alpha star AR star, so what does it mean, if alpha, R is an Eigen pair, its conjugate is also an Eigen pair, that means eigenvalues and Eigen pairs appear as complex conjugate pairs, okay. And the order of this equation is always E1, because I start with the N degree of freedom system and the state space model has two end degrees of freedom, so this will be always true.

Let us denote the eigensolutions as follows

Eigenvalues : $\alpha_1, \alpha_2, \cdots, \alpha_n, \alpha_1^*, \alpha_2^*, \cdots, \alpha_n^*$

Eigenvectors: $R_1, R_2, \cdots, R_n, R_1^*, R_2^*, \cdots, R_n^*$

Recall $y = \begin{Bmatrix} \dot{q} \\ q \end{Bmatrix}$

Let $q(t) = \phi \exp(\alpha t) \Rightarrow \dot{q}(t) = \alpha\phi\exp(\alpha t)$

$$\Rightarrow y = \begin{Bmatrix} \dot{q} \\ q \end{Bmatrix} = \begin{Bmatrix} \alpha\phi \\ \phi \end{Bmatrix} \exp(\alpha t) \Rightarrow R = \begin{Bmatrix} \alpha\phi \\ \phi \end{Bmatrix}$$

$$\Rightarrow R_k = \begin{Bmatrix} \alpha_k\phi_k \\ \phi_k \end{Bmatrix}$$

Define the matrix

$$\Psi = \begin{bmatrix} R_1 & R_2 & \cdots & R_n & R_1^* & R_2^* & \cdots & R_n^* \end{bmatrix}$$

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Now let us denote the eigenvalues as alpha 1, alpha 2, alpha N then their conjugates alpha 1 star, alpha 2 star, alpha N star, similarly for the eigenvectors I will write it as R1, R2, RN and then their conjugates R1 star, R2 star, RN star, now let us try to understand the nature of an eigenvector, so to do that lets recall that Y is made up of Q dot and Q, suppose now the system is vibrating in its normal mode the solution for displacement alone can be written as Phi into E raise to alpha T, now Q dot will be there for alpha Phi exponential alpha T, so if I now write the state vector in the eigenvector, when the system is vibrating its normal mode, it will be of the form alpha phi, phi into E raise to alpha T, because Q dot is this, and Q is this, so I can write in this form, so that would mean each of the eigenvector will be of this form eigenvalue multiplied by phi and phi, okay, so the K-th eigenvector is of the form alpha K into phi K and phi K.

Now what I do is I define a modal matrix capital Sai, where I will now assemble these vectors R1, R2, RN and their conjugates in this order. So I have this Sai, now for each of these R1, R2 I will write it as alpha 1 Phi 1 for R1, so alpha 2 Phi 2 for R2, and their conjugates will appear in the next set of terms here, so now if I introduce N/N matrix phi with phi 1, phi 2, phi n as

$$\Psi = \begin{bmatrix} R_1 & R_2 & \cdots & R_n & R_1^* & R_2^* & \cdots & R_n^* \end{bmatrix}$$

$$\Rightarrow$$

$$\Psi = \begin{bmatrix} \alpha_1\phi_1 & \alpha_2\phi_2 & \cdots & \alpha_n\phi_n & \alpha_1^*\phi_1^* & \alpha_1^*\phi_1^* & \cdots & \alpha_1^*\phi_1^* \\ \phi_1 & \phi_2 & \cdots & \phi_n & \phi_1^* & \phi_2^* & \cdots & \phi_n^* \end{bmatrix}$$

$$\text{Let } \Phi = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \end{bmatrix} \ \& \ \Lambda = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \alpha_n \end{bmatrix}$$

$$\Psi = \begin{bmatrix} \Phi\Lambda & \Phi^*\Lambda^* \\ \Phi & \Phi^* \end{bmatrix}$$

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columns and capital Lambda as a diagonal matrix of the N eigenvalues, then you see that the modal matrix in this problem has this structure, it will be of the form phi lambda phi, phi star, lambda star, phi star, so it is important to recognize that this structure is present in the modal matrix.

Orthogonality relations

Consider r^{th} and s^{th} eigenpairs

$$BR_r = -\alpha_r AR_r \qquad (1)$$

$$BR_s = -\alpha_s AR_s \qquad (2)$$

Premultiply (1) by R_s^t and (2) by R_r^t

$$R_s^t BR_r = -\alpha_r R_s^t AR_r \qquad (3)$$

$$R_r^t BR_s = -\alpha_s R_r^t AR_s \qquad (4)$$

Transpose both sides of (4)

$$R_s^t B^t R_s = -\alpha_s R_s^t A^t R_r$$

Since $A^t = A$ and $B^t = B$ we get

$$R_s^t BR_s = -\alpha_s R_s^t AR_r \qquad (5)$$

Substract 3 and 5

$$\Rightarrow (\alpha_s - \alpha_r) R_s^t AR_r = 0 \Rightarrow R_s^t AR_r = 0 \text{ for } r \neq s$$

$$\Rightarrow R_s^t BR_r = 0 \text{ for } r \neq s$$

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Now we can now talk about orthogonality relations, so let us consider R-th and S-th Eigen pairs, so the eigenvalue problem for R-th eigenvalue will be BRR is - alpha R, ARR, similarly for S I get this equation, so lets name these equations as 1 and 2, I will pre multiply 1 by RS transpose, and 2 by RR transpose, so I get these equations, so first one is RS transpose BRR is minus alpha R, RS transpose ARR. The next equation is RR transpose BRS - alpha SR, RR transpose ARS, now we will transpose both sides of 4, I will get RS transpose B transpose RS, and - alpha S RS transpose A transpose RR.

Now the benefit of writing the original equation in the specific form that we chose, now pays dividends here we know that A and B are symmetric, so A transpose is A, B transpose is B, so

Damped MDOF systems

Consider an n-dof system

$$M\ddot{q}+C\dot{q}+Kq=f(t);q(0)=q_0;\dot{q}(0)=\dot{q}_0$$

We augment this equation with an identity and rewrite as

$$M\dot{q}-M\dot{q}=0$$

$$M\ddot{q}+C\dot{q}+Kq=f(t)$$

$$\Rightarrow\begin{bmatrix}0 & M\\ M & C\end{bmatrix}\begin{Bmatrix}\ddot{q}\\ \dot{q}\end{Bmatrix}+\begin{bmatrix}-M & 0\\ 0 & K\end{bmatrix}\begin{Bmatrix}\dot{q}\\ q\end{Bmatrix}=\begin{Bmatrix}0\\ f(t)\end{Bmatrix}$$

Introduce a new state vector y, forcing vector F and structural matrices A and B as

$$y=\begin{Bmatrix}\dot{q}\\ q\end{Bmatrix},F=\begin{Bmatrix}0\\ f(t)\end{Bmatrix}=A=\begin{bmatrix}0 & M\\ M & C\end{bmatrix}\&B=\begin{bmatrix}-M & 0\\ 0 & K\end{bmatrix}$$

Note:

- $A^t=A;B^t=B$
- A and B are not positive definite.

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if you go back when we wrote this additional equation the way we converted this N second order equations to 2, and first order equation there are other ways of doing it, I will come to that later, this way of doing ensures that these coefficient matrices are symmetric, okay, so that is helpful you know to consider orthogonality relation, so now since the A and B are symmetric this equation can be written in this form, so this is equation 5.

Now if you subtract 3 and 5, I get alpha S - alpha R is RS transpose ARR = 0, so I can demand that RS transpose ARR is 0 whenever R is not equal to S, if alpha R = alpha S I can still insist that RS and RR can be chosen so that this relation holds this is what I showed in the case of a repeated eigenvalue. Now once this is 0 for R not equal to S you can go back to any of these equations, for example here and if right hand side is 0 for S not equal to R, left hand side will also be this, so automatically I get this, so in the matrix form we can rewrite this, but before that we can normalized eigenvalues so what we can do is we can select the normalization constant so that RS transpose ARS is 1, for S = 1, 2, 2n, so in that case what happens it will be RS transpose ARR will be the chronica delta, and RS transpose BRR will be - alpha R into chronica delta.

$R_s^t A R_r = 0$ for $r \neq s$

$R_s^t B R_r = 0$ for $r \neq s$

Select the normalization constant such that $R_s^t A R_s = 1; s = 1, 2, \cdots, 2n$

$\Rightarrow R_s^t A R_r = \delta_{rs}$ & $R_s^t B R_r = -\alpha_r \delta_{rs}$

In terms of the representation $\Psi = \begin{bmatrix} \Phi\Lambda & \Phi^*\Lambda^* \\ \Phi & \Phi^* \end{bmatrix}$ the orthogonality relations read as

$$\Psi^t A \Psi = I$$

$$\Psi^t B \Psi = -\begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda^* \end{bmatrix}$$

Note

If we write $\Lambda_r = -\eta_r \omega_r + i\omega_{dr}$ we get

$$\eta_{kr}(t) = \Phi_{kr} \exp(-\eta_r \omega_r) \exp(i\omega_{dr} t)$$

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So now in terms of the modal matrix that we showed while before the orthogonality relation with this normalization in place will be Sai transpose A Sai is I, and Sai transpose B Sai is this diagonal matrix. The first block has lambda and the next block is, its conjugate and the off diagonal terms are zeros. Now I can write the R-th eigenvalue in this form - ETA R omega R plus this, if we indeed do that we can see that the K-th coordinate in R-th mode will have this type of representation, so this is quite similar to the free vibration format of the free vibration solution for a single degree freedom system that we saw, so there is a connection that I want you to notice.

Forced response analysis

$$A\dot{y} + By = F(t);\ y(0) = y_0;\ F(t) = \begin{Bmatrix} 0 \\ f(t) \end{Bmatrix}$$

$$y(t) = \Psi z(t)$$

$$\Rightarrow A\Psi\dot{z} + B\Psi z = F(t)$$

$$\Rightarrow \Psi^t A\Psi\dot{z} + \Psi^t B\Psi z = \Psi^t F(t)$$

$$\Psi^t F(t) = \begin{bmatrix} \Lambda^t\Phi^t & \Phi^t \\ \Lambda^{*t}\Phi^{*t} & \Phi^{*t} \end{bmatrix} \begin{Bmatrix} 0 \\ f(t) \end{Bmatrix} = \begin{Bmatrix} \Phi^t f(t) \\ \Phi^{*t} f(t) \end{Bmatrix}$$

$$\Rightarrow I\dot{z} - \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda^* \end{bmatrix} z = \begin{Bmatrix} \Phi^t f(t) \\ \Phi^{*t} f(t) \end{Bmatrix}$$

$$\text{Let } z = \begin{Bmatrix} u \\ v \end{Bmatrix} \Rightarrow \begin{matrix} \dot{u} - \Lambda u = \Phi^t f(t) \\ \dot{v} - \Lambda^* v = \Phi^{*t} f(t) \end{matrix}$$

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Now let's look at force response analysis, so I have now AY dot + BY is F(t) with these initial conditions and this is a forcing vector, so I make this substitution, Y is Sai Z, and I put it back in the original equation I get this equation. Now if I pre multiplied by Sai transpose I get this, now Sai transpose F let us first look at it, Sai transpose is this and F is this, so this will be then since there is a 0 here the structure of the generalized forcing vector will be this. Now Sai transpose SI is I and Sai transpose B Sai is this matrix and this is Z and this is a forcing vector. So I will get now for each of the Z uncoupled equation but we can write Z as 2 vectors U and V, if I write like this you can see that these 2 equations can be written this 2N equations can be written in terms of a set of 2, set of 2N uncoupled equations as shown here, one for U and one for V.

Case - 1 Let the r^{th} dof be driven harmonically.

$$\Rightarrow f(t) = \{0 \quad 0 \quad \cdots \quad \underset{\uparrow}{1} \quad 0 \quad \cdots \quad 0\}^t \exp(i\omega t)$$

r^{th} dof

$$\dot{u} - \Lambda u = \Phi^t f(t)$$

$$\dot{v} - \Lambda^* v = \Phi^{*t} f(t)$$

$$\Rightarrow \dot{u}_k - \alpha_k u_k = \sum_{s=1}^{n} \Phi_{sk} f_s(t) = \Phi_{rk} \exp(i\omega t)$$

$$\Rightarrow \lim_{t \to \infty} u_k(t) = \frac{\Phi_{rk}}{i\omega - \alpha_k} \exp(i\omega t)$$

Similarly, $\lim_{t \to \infty} v_k(t) = \dfrac{\Phi^*_{rk}}{i\omega - \alpha^*_k} \exp(i\omega t)$

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Now let’s consider the case when R-th degree of freedom is the driven harmonically by E raise to I omega T, okay, now the equation for U and V will be of this form, so F of, now if you right now for F(t) the summation, this summation if you expand since only one coordinate has a nonzero value this summation collapses to a single term, so in the ST tends to infinity UK(t) will be given by this, and similarly ST tends to infinity VK will be given by this, okay, now if I

$$y(t) = \Psi z(t) \Rightarrow$$

$$\begin{Bmatrix} \dot{q} \\ q \end{Bmatrix} = \begin{bmatrix} \Phi\Lambda & \Phi^*\Lambda^* \\ \Phi & \Phi^* \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

$$q = \Phi u + \Phi^* v$$

$$q_j(t) = \sum_{k=1}^{n} \Phi_{jk} u_k(t) + \Phi^*_{jk} v_k(t)$$

$$\lim_{t \to \infty} q_j(t) = \left(\sum_{k=1}^{n} \frac{\Phi_{rk}\Phi_{jk}}{i\omega - \alpha_k} + \frac{\Phi^*_{jk}\Phi^*_{rk}}{i\omega - \alpha^*_k} \right) \exp(i\omega t)$$

$$\Rightarrow \alpha_{jr}(\omega) = \sum_{k=1}^{n} \frac{\Phi_{rk}\Phi_{jk}}{i\omega - \alpha_k} + \frac{\Phi^*_{jk}\Phi^*_{rk}}{i\omega - \alpha^*_k} = \alpha_{rj}(\omega)$$

put back into the original coordinate system Y will be Sai Z, so Q dot Q is this, so Q will be Phi U + Phi star V, so if I now right QJ in this form we can see that ST tends to infinity I get the expression for the frequency response function in terms of the normal modes and the eigenvalues and their conjugates, so this is the transfer function in terms of the model description, in the model domain, again we can see that this function is this is a receptance which is symmetric but it is not Hermitian.

Case - 2 Let the r^{th} dof be driven by an impulse.

$$\Rightarrow f(t) = \{0 \quad 0 \quad \cdots \quad 1 \quad 0 \quad \cdots \quad 0\}^t \delta(t)$$

r^{th} dof

$$\dot{u} - \Lambda u = \Phi^t f(t)$$

$$\dot{v} - \Lambda^* v = \Phi^{*t} f(t)$$

$$\Rightarrow \dot{u}_k - \alpha_k u_k = \sum_{s=1}^{n} \Phi_{sk} f_s(t) = \Phi_{rk} \delta(t)$$

$$\Rightarrow u_k(t) = \Phi_{rk} \exp(\alpha_k t)$$

Similarly, $v_k(t) = \Phi^*_{rk} \exp(\alpha_k^* t)$

Now let us consider the other case where R-th degree of freedom system is driven impulsively, this is a viscously damped system therefore I can talk about response in time domain also, so I will get now the response I mean I get the equation for U and V in this form, so the equation for UK will be this, and equation for VK will be this, now you can see here that U and V will form a conjugate complex, conjugate pair, so VK(t) will be this and UK(t) is this therefore VK is nothing but UK conjugate, so now if I use those relations we can go back to Y coordinate

$$y(t) = \Psi z(t) \Rightarrow$$

$$\begin{Bmatrix} \dot{q} \\ q \end{Bmatrix} = \begin{bmatrix} \Phi\Lambda & \Phi^{*}\Lambda^{*} \\ \Phi & \Phi^{*} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

$$q = \Phi u + \Phi^{*} v$$

$$q_j(t) = \sum_{k=1}^{n} \left(\Phi_{jk} u_k(t) + \Phi_{jk}^{*} v_k(t) \right)$$

$$= \sum_{k=1}^{n} \left[\Phi_{jk} \Phi_{rk} \exp(\alpha_k t) + \Phi_{jk}^{*} \Phi_{rk}^{*} \exp(\alpha_k^{*} t) \right]$$

$$\Rightarrow h_{jr}(t) = 2\,\mathrm{Re} \sum_{k=1}^{n} \Phi_{jk} \Phi_{rk} \exp(\alpha_k t)$$

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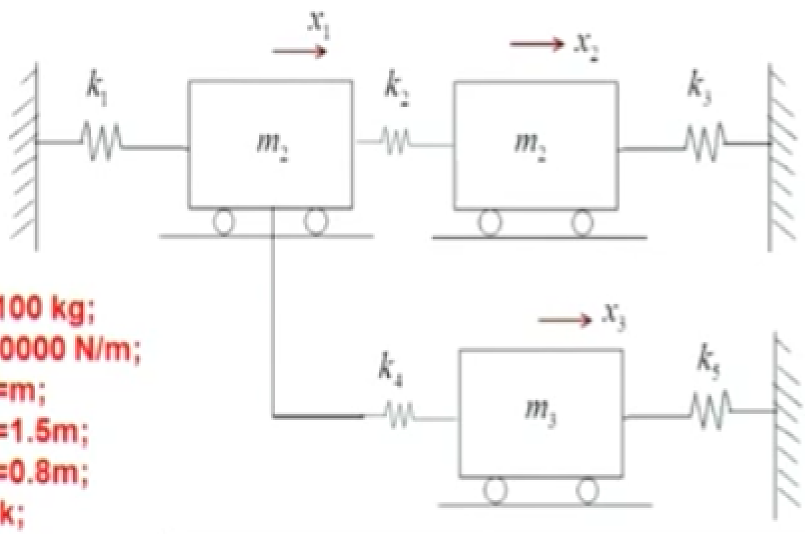
system, Y is Sai Z, and for Sai I'm writing this and if I use the fact that U and V are conjugate pairs, now the impulse response function will be in terms of this, okay, again this is impulse response function is obtained in terms of the eigenvectors and eigenvalues, and it is real valued here, okay, although the eigenvalues and eigenvectors are complex valued. We can quickly go

Numerical illustration



m=100 kg;
k=10000 N/m;
m1=m;
m2=1.5m;
m3=0.8m;
k1=k;
k2=[illegible]*k;
k3=1.5*k;
k4=2*k;
k5=2.5*k;

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1+k_2+k_3 & -k_2 & -k_3 \\ -k_2 & k_2+k_3 & 0 \\ -k_3 & 0 & k_4+k_5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = 0$$

through a numerical illustration so let's consider 3 degree freedom system, configured as shown here and these are the numerical values and if I write the equation of motion for this system I get for the un-damped free vibration this is the equation, so we can quickly do the calculations

Model 1 Proportional damping matrix

$$\eta_1 = 0.01, \eta_2 = 0.03, \eta_1 = 0.025$$

$$C = \begin{bmatrix} 101.6259 & -62.7423 & -22.4115 \\ -62.7423 & 101.7491 & -22.1904 \\ -22.4115 & -22.1904 & 100.1737 \end{bmatrix}$$

Model 2 Nonproportional damping matrix

$$C = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 175 & 0 \\ 0 & 0 & 200 \end{bmatrix}$$

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because the simple problem we will consider two alternative models, one is proportional damping matrix with damping ratio specified for the 3 modes to be 0.01, 0.03 and 0.025, this is one model, and associated with this, this will be the C matrix which we can compute following the procedure that I outlined.

And the second model is a non-proportional damping matrix model where off diagonal, you see this almost similar to this except that the off diagonal terms are 0 and the diagonal terms are nearly you know of the same order as this, this 100, 175, 200. Now for the un-damped system

$$M=\begin{bmatrix}100 & 0 & 0\\ 0 & 150 & 0\\ 0 & 0 & 80\end{bmatrix};K=\begin{bmatrix}55000 & -30000 & -20000\\ -30000 & 45000 & 0\\ -20000 & 0 & 45000\end{bmatrix}$$

$$\Phi=\begin{bmatrix}0.0578 & -0.0355 & -0.0735\\ 0.0623 & 0.0452 & 0.0272\\ 0.0323 & -0.0842 & 0.0660\end{bmatrix};\Lambda=\begin{bmatrix}114.65 & 0 & 0\\ 0 & 457.15 & 0\\ 0 & 0 & 840.70\end{bmatrix}$$

$$\Phi^t M\Phi=\begin{bmatrix}1.0000 & -0.0000 & 0.0000\\ 0.0000 & 1.0000 & 0.0000\\ 0.0000 & 0.0000 & 1.0000\end{bmatrix};\Phi^t K\Phi=\begin{bmatrix}114.65 & 0 & 0\\ 0 & 457.15 & 0\\ 0 & 0 & 840.70\end{bmatrix}$$

$$\text{Let } C=\begin{bmatrix}101.6259 & -62.7423 & -22.4115\\ -62.7423 & 101.7491 & -22.1904\\ -22.4115 & -22.1904 & 100.1737\end{bmatrix}\Rightarrow\Phi^t C\Phi=\begin{bmatrix}0.2141 & 0.0000 & 0.0000\\ 0.0000 & 1.2829 & 0\\ 0.0000 & 0 & 1.4497\end{bmatrix}$$

$$\eta_1=0.01,\eta_2=0.03,\eta_3=0.025;\omega_1=10.7074,\omega_2=21.3812,\omega_3=28.9948\text{ rad/s}$$

$$\left(-\eta_n\omega_n\pm i\omega_n\sqrt{1-\eta_n^2}\right)_{n=1}^{3}$$

$$=(\ -0.1071\pm 10.7068i\quad -0.6414\pm 21.3716i\quad -0.7249\pm 28.9857i)$$

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this is a mass, this is stiffness, after we insert the numerical values, we can show that the un-damped normal modes this is the model matrix, and this is the matrix of eigenvalues. You can see that Phi transpose M Phi is the identity matrix, Phi transpose K Phi is the lambda matrix which is same as this, squares of the eigenvalues or the eigenvalues themselves squares of the natural frequencies.

Now if I now consider C, that is the first case where C is proportional we can see that Phi transpose C Phi is a diagonal matrix, okay, so the system has these 3 damping ratios and these 3 natural frequencies and these are the complex you know roots of the characteristic equation which we get as this, for this problem.

$$A=\begin{bmatrix} 0 & 0 & 0 & 100.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 150.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 80.0000 \\ 100.0000 & 0 & 0 & 101.6259 & -62.7423 & -22.4115 \\ 0 & 150.0000 & 0 & -62.7423 & 101.7491 & -22.1904 \\ 0 & 0 & 80.0000 & -22.4115 & -22.1904 & 100.1737 \end{bmatrix}$$

$$B=\begin{bmatrix} -100 & 0 & 0 & 0 & 0 & 0 \\ 0 & -150 & 0 & 0 & 0 & 0 \\ 0 & 0 & -80 & 0 & 0 & 0 \\ 0 & 0 & 0 & 55000 & -30000 & -20000 \\ 0 & 0 & 0 & -30000 & 45000 & 0 \\ 0 & 0 & 0 & -20000 & 0 & 45000 \end{bmatrix}$$

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Now what I will do now is even for this case where damping is proportional we can see now if I approach the formulation through A and B matrices so what happens will I get the same answer or will I get something different, we should expect to get the same answer. Suppose for the MCK damping matrices, MCK matrices that we have chosen this will be the A matrix, this will be the B matrix, you can see that now this is 6 by 6, these two are 6 by 6 matrices.

$$\Psi = \begin{bmatrix} -0.9827 + 0.0173i & -0.9827 - 0.0173i & -0.4109 + 0.0105i & -0.4109 - 0.0105i & -0.8394 - 0.0873i & -0.8394 + 0.0873i \\ 0.3635 - 0.0064i & 0.3635 + 0.0064i & 0.5229 - 0.0134i & 0.5229 + 0.0134i & -0.9058 - 0.0942i & -0.9058 + 0.0942i \\ 0.8831 - 0.0155i & 0.8831 + 0.0155i & -0.9751 + 0.0249i & -0.9751 - 0.0249i & -0.4686 - 0.0487i & -0.4686 + 0.0487i \\ 0.0014 + 0.0339i & 0.0014 - 0.0339i & 0.0011 + 0.0192i & 0.0011 - 0.0192i & -0.0074 + 0.0785i & -0.0074 - 0.0785i \\ -0.0005 - 0.0125i & -0.0005 + 0.0125i & -0.0014 - 0.0244i & -0.0014 + 0.0244i & -0.0080 + 0.0847i & -0.0080 - 0.0847i \\ -0.0013 - 0.0304i & -0.0013 + 0.0304i & 0.0025 + 0.0456i & 0.0025 - 0.0456i & -0.0041 + 0.0438i & -0.0041 - 0.0438i \end{bmatrix}$$

$$\Psi^t A \Psi = \begin{bmatrix} -1.0490 - 12.2875i & -0.0000 + 0.0000i & -0.0000 + 0.0000i & -0.0000 + 0.0000i & 0.0000 + 0.0000i & -0.0000 + 0.0000i \\ -0.0000 - 0.0000i & -1.0490 + 12.2875i & -0.0000 - 0.0000i & -0.0000 - 0.0000i & -0.0000 - 0.0000i & 0.0000 - 0.0000i \\ -0.0000 + 0.0000i & -0.0000 - 0.0000i & -1.3892 - 12.4563i & -0.0000 + 0.0000i & -0.0000 + 0.0000i & 0.0000 + 0.0000i \\ -0.0000 + 0.0000i & -0.0000 - 0.0000i & -0.0000 - 0.0000i & -1.3892 + 12.4563i & 0.0000 - 0.0000i & -0.0000 - 0.0000i \\ 0.0000 + 0.0000i & -0.0000 - 0.0000i & -0.0000 + 0.0000i & 0.0000 - 0.0000i & 7.4217 - 39.1573i & 0.0000 + 0.0000i \\ -0.0000 + 0.0000i & 0.0000 - 0.0000i & 0.0000 + 0.0000i & -0.0000 - 0.0000i & 0.0000 - 0.0000i & 7.4217 + 39.1573i \end{bmatrix}$$

$$\frac{\Psi^t B \Psi}{100} = \begin{bmatrix} -3.5692 + 0.2150i & -0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 - 0.0000i & 0.0000 - 0.0000i & 0.0000 + 0.0000i \\ -0.0000 - 0.0000i & -3.5692 - 0.2150i & 0.0000 + 0.0000i & 0.0000 - 0.0000i & 0.0000 - 0.0000i & 0.0000 + 0.0000i \\ 0.0000 - 0.0000i & 0.0000 + 0.0000i & -2.6710 + 0.2170i & -0.0000 & 0.0000 + 0.0000i & 0.0000 - 0.0000i \\ 0.0000 - 0.0000i & 0.0000 - 0.0000i & -0.0000 & -2.6710 - 0.2170i & 0.0000 + 0.0000i & 0.0000 - 0.0000i \\ 0.0000 - 0.0000i & 0.0000 - 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & -4.1846 - 0.8366i & 0.0000 + 0.0000i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 - 0.0000i & 0.0000 - 0.0000i & 0.0000 - 0.0000i & -4.1846 + 0.8366i \end{bmatrix}$$

Note: $\Psi^t A \Psi$ is diagonal but not an identity matrix.

$\Rightarrow \Psi^t B \Psi$ is diagonal but diagonal entries are not equal to eigenvalues.

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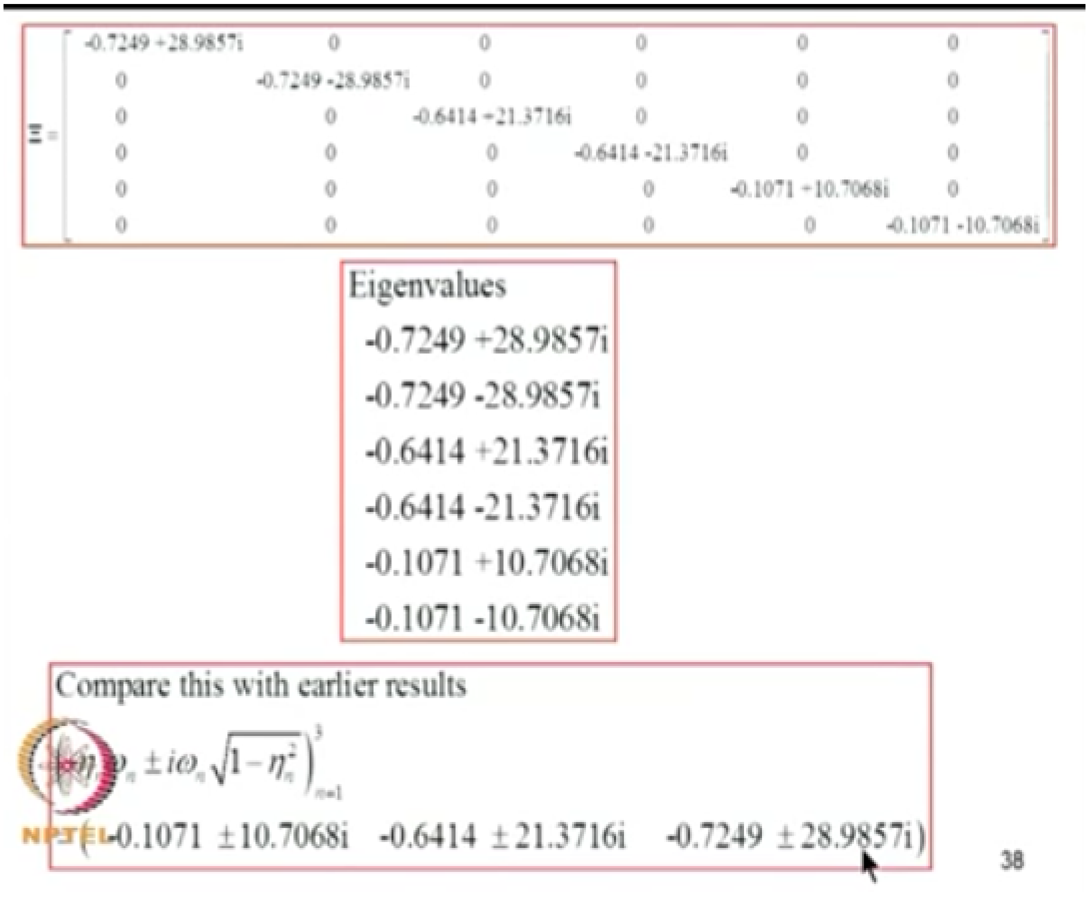
Now there are too many numbers here but you need to follow a few details, this Sai is the 6 by 6 complex value eigenvector matrix, so you can see that they appear as conjugate pairs, this is not ordered in the way that I showed, this is output of eigenvalue solver, so I am just reporting the way we got the results. Now you can see here this is one eigenvector, its conjugate appears here, this is the next eigenvector its conjugate is here, okay, and this is this so what I mean to say is this is not in this form, okay, it doesn't matter.

Now you can quickly see that Sai transpose SI is a diagonal matrix, Sai transpose B Sai is also a diagonal matrix, you must notice that Sai transpose A Sai is diagonal but it is not an identity matrix it has not been normalized to be a diagonal matrix, and sorry an identity matrix. Similarly Sai transpose B Sai is diagonal, but diagonal entries are not equal to the eigenvalues, okay so that you should notice.



Now the matrix of eigenvalues are shown here, and this has a square matrix and as a column it is shown here, so that you can see here that they are appearing as complex conjugates. Now if you now compare this with what the results that we got earlier, we see that we are getting the same results, for example the first pair of roots here is same as these roots, the next pair is this which is this, third pair is this, this is this, so what does it mean? That means the system is classically damped, so it doesn’t really matter which approach you take you will get the same roots.

Now how do we see the nature of eigenvalue? If you look at now the eigenvectors for the un-damped case we can see that for this eigenvalue all the 3 degrees of freedom are in perfectly in phase, whereas here this is negative, this is positive, therefore the phase difference is 180 degrees and similarly this is again phase difference is 180 degrees, so in un-damped free vibration all points vibrate harmonically either perfectly in phase or perfectly out of phase, there is no other phase difference is possible okay, the phase differences will come, occur because of damping but in classically damped system the question that we should ask is what happens to the phase difference, so if we ask that question if you look at this Sai matrix the question about phase difference is not, it cannot be easily answered, so what we do is we normalize this eigenvector so that I fix the third entry here to be 1, so you recall that the modal

Scale the modal matrix

$$\Psi = \begin{bmatrix} -0.7249 + 28.9857i & -0.7249 - 28.9857i & -0.6414 + 21.3716i & -0.6414 - 21.3716i & -0.1071 + 10.7068i & -0.1071 - 10.7068i \\ 0.2681 - 10.7216i & 0.2681 + 10.7216i & 0.8163 - 27.1980i & 0.8163 + 27.1980i & -0.1155 + 11.5530i & -0.1155 - 11.5530i \\ 0.6514 - 26.0478i & 0.6514 + 26.0478i & -1.5222 + 50.7182i & -1.5222 - 50.7182i & -0.0598 + 5.9768i & -0.0598 - 5.9768i \\ 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 + 0.0000i & 1.0000 - 0.0000i \\ -0.3699 + 0.0000i & -0.3699 - 0.0000i & -1.2726 - 0.0000i & -1.2726 + 0.0000i & 1.0790 - 0.0000i & 1.0790 + 0.0000i \\ -0.8986 - 0.0000i & -0.8986 + 0.0000i & 2.3732 + 0.0000i & 2.3732 - 0.0000i & 0.5582 - 0.0000i & 0.5582 + 0.0000i \end{bmatrix}$$

$$\text{Angle}(\Psi) = \begin{bmatrix} 91.4325 & -91.4325 & 91.7191 & -91.7191 & 90.5730 & -90.5730 \\ -88.5675 & 88.5675 & -88.2809 & 88.2809 & 90.5730 & -90.5730 \\ -88.5675 & 88.5675 & 91.7191 & -91.7191 & 90.5730 & -90.5730 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 180.0000 & -180.0000 & -180.0000 & 180.0000 & -0.0000 & 0.0000 \\ -180.0000 & 180.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 \end{bmatrix}$$

$$\text{Recall: } \Psi = \begin{bmatrix} \Phi\Lambda & \Phi^*\Lambda^* \\ \Phi & \Phi^* \end{bmatrix}$$

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matrix is of this form so the lower half of this is on displacements and the upper half is on velocities, so I make the first displacement to be 1, that is a normalization you know I have the freedom to do that because there is an arbitrary scaling factor.

Now consequently the eigenvectors will have this appearance, now if I look at angle associated with this, if I find the angle of that I get this, so now you focus on the lower half of it I see that the I, the phase angles are either perfect out of phase or perfect in phase, so the way the eigenvalues are organized you can see that the third eigenvalue here is the first eigenvalue in the un-damped system so you can see that all points are in perfectly in phase, for this eigenvalue this is perfectly out of phase with the other 2 degrees of freedom, right, so this is because we are using classical damped, classically damped systems, now let us go to the non-proportional damping matrix, so the un-damped normal mode is this, now if I simply find phi

Model 2 Nonproportional damping matrix

$$C=\begin{bmatrix} 100 & 0 & 0 \\ 0 & 175 & 0 \\ 0 & 0 & 200 \end{bmatrix}$$

Undamped normal modal matrix

$$\Phi=\begin{bmatrix} 0.0578 & -0.0355 & -0.0735 \\ 0.0623 & 0.0452 & 0.0272 \\ 0.0323 & -0.0842 & 0.0660 \end{bmatrix}$$

$$\Phi^t C\Phi=\begin{bmatrix} 1.2220 & -0.2556 & 0.2980 \\ -0.2556 & 1.9027 & -0.6370 \\ 0.2980 & -0.6370 & 1.5419 \end{bmatrix}$$

$\Rightarrow$ Undamped normal modes do not uncouple C matrix.

transpose C phi, this won't be a diagonal matrix, that is why I call this is non-classical. First you should convince that we are dealing with a non-classical damping and so this is how we could quickly verify that.

$$A = \begin{bmatrix} 0 & 0 & 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 0 & 150 & 0 \\ 0 & 0 & 0 & 0 & 0 & 80 \\ 100 & 0 & 0 & 100 & 0 & 0 \\ 0 & 150 & 0 & 0 & 175 & 0 \\ 0 & 0 & 80 & 0 & 0 & 200 \end{bmatrix}$$

$$B = \begin{bmatrix} -100 & 0 & 0 & 0 & 0 & 0 \\ 0 & -150 & 0 & 0 & 0 & 0 \\ 0 & 0 & -80 & 0 & 0 & 0 \\ 0 & 0 & 0 & 55000 & -30000 & -20000 \\ 0 & 0 & 0 & -30000 & 45000 & 0 \\ 0 & 0 & 0 & -20000 & 0 & 45000 \end{bmatrix}$$

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Now let us form the A and B matrices, this is again 6 by 6 matrices here and perform the eigenvalue analysis, this is a Sai matrix as thrown out by the eigenvalue solver without any adjustment of the normalization you'll see Sai transpose ASI is again diagonal, Sai transpose B

$$\Psi = \begin{bmatrix} -0.9211 + 0.0789i & -0.9211 - 0.0789i & -0.3907 - 0.0166i & -0.3907 + 0.0166i & 0.8786 - 0.0508i & 0.8786 + 0.0508i \\ 0.3408 - 0.0361i & 0.3408 + 0.0361i & 0.4953 - 0.0287i & 0.4953 + 0.0287i & 0.9483 - 0.0517i & 0.9483 + 0.0517i \\ 0.8273 + 0.0119i & 0.8273 - 0.0119i & -0.9268 + 0.0732i & -0.9268 - 0.0732i & 0.4900 - 0.0434i & 0.4900 + 0.0434i \\ 0.0036 + 0.0317i & 0.0036 - 0.0317i & 0.0000 + 0.0183i & 0.0000 - 0.0183i & -0.0094 - 0.0816i & -0.0094 + 0.0816i \\ -0.0016 - 0.0117i & -0.0016 + 0.0117i & -0.0024 - 0.0231i & -0.0024 + 0.0231i & -0.0099 - 0.0881i & -0.0099 + 0.0881i \\ -0.0003 - 0.0285i & -0.0003 + 0.0285i & 0.0053 + 0.0431i & 0.0053 - 0.0431i & -0.0067 - 0.0455i & -0.0067 + 0.0455i \end{bmatrix}$$

$$\Psi' A \Psi = \begin{bmatrix} -1.7215 - 10.7118i & 0.0000 - 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \\ 0.0000 + 0.0000i & -1.7215 + 10.7118i & 0.0000 - 0.0000i & 0.0000 - 0.0000i & 0.0000 - 0.0000i & 0.0000 - 0.0000i \\ 0.0000 + 0.0000i & 0.0000 - 0.0000i & -2.2837 - 11.0592i & -0.0000 - 0.0000i & -0.0000 - 0.0000i & -0.0000 - 0.0000i \\ 0.0000 + 0.0000i & 0.0000 - 0.0000i & -0.0000 + 0.0000i & -2.2837 + 11.0592i & -0.0000 + 0.0000i & -0.0000 + 0.0000i \\ 0.0000 + 0.0000i & 0.0000 - 0.0000i & -0.0000 - 0.0000i & -0.0000 + 0.0000i & -9.9029 - 42.1062i & -0.0000 - 0.0000i \\ 0.0000 + 0.0000i & 0.0000 - 0.0000i & -0.0000 - 0.0000i & -0.0000 + 0.0000i & -0.0000 + 0.0000i & -9.9029 + 42.1062i \end{bmatrix}$$

$$\frac{\Psi' B \Psi}{100} = \begin{bmatrix} -3.1162 + 0.4163i & -0.0000 + 0.0000i & 0.0000 - 0.0000i & 0.0000 + 0.0000i & 0.0000 - 0.0000i & 0.0000 - 0.0000i \\ -0.0000 - 0.0000i & -3.1162 - 0.4163i & 0.0000 - 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \\ 0.0000 - 0.0000i & 0.0000 - 0.0000i & -2.3850 + 0.3826i & -0.0000 - 0.0000i & -0.0000 + 0.0000i & -0.0000 + 0.0000i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & -0.0000 + 0.0000i & -2.3850 - 0.3826i & -0.0000 - 0.0000i & -0.0000 - 0.0000i \\ 0.0000 - 0.0000i & 0.0000 + 0.0000i & -0.0000 + 0.0000i & -0.0000 - 0.0000i & -4.5624 + 0.8015i & -0.0000 - 0.0000i \\ 0.0000 - 0.0000i & 0.0000 + 0.0000i & -0.0000 + 0.0000i & -0.0000 - 0.0000i & -0.0000 + 0.0000i & -4.5624 - 0.8015i \end{bmatrix}$$

Note: $\Psi' A \Psi$ is diagonal but not an identity matrix.

$\Psi' B \Psi$ is diagonal but diagonal entries are not equal to eigenvalues.

Sai is diagonal, so then again Sai transpose ASI is not an identity matrix, and Sai transpose B Sai although it is diagonal, the diagonal entries are not eigenvalues, good. Now this is the

$$\Xi = \begin{bmatrix} -0.7695+28.9675i & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.7695-28.9675i & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.9527+21.3692i & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.9527-21.3692i & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.6111+10.6916i & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.6111-10.6916i \end{bmatrix}$$

Eigenvalues

-0.7695 +28.9675i
-0.7695 -28.9675i
-0.9527 +21.3692i
-0.9527 -21.3692i
-0.6111 +10.6916i
-0.6111 -10.6916i

Undamped natural frequencies

$\omega_1 = 10.7074$ rad/s

$\omega_2 = 21.3812$ rad/s

$\omega_3 = 28.9948$ rad/s

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eigenvalues written as a square matrix, diagonal matrix, I can write it as a column vector and I get these as eigenvalues, okay.

Now this are damped system, so now un-damped natural frequencies are shown here, see you can see now there is a slight change in this number 10.691, it is 10.7074 etcetera, okay, now

Scale the modal matrix

$$\Psi = \begin{bmatrix} -0.7695 + 28.9675i & -0.7695 - 28.9675i & -0.9527 + 21.3692i & -0.9527 - 21.3692i & -0.6111 + 10.6916i & -0.6111 - 10.6916i \\ 0.0705 - 10.7416i & 0.0705 + 10.7416i & -1.5122 - 27.0962i & -1.5122 + 27.0962i & -0.6964 + 11.5344i & -0.6964 - 11.5344i \\ 3.2688 - 25.7276i & 3.2688 + 25.7276i & 3.8910 + 50.7065i & 3.8910 - 50.7065i & -0.1589 + 5.9840i & -0.1589 - 5.9840i \\ 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\ -0.37[illegible] + 0.0074i & -0.3706 - 0.0074i & -1.2623 + 0.1270i & -1.2623 - 0.1270i & 1.0790 + 0.0035i & 1.0790 - 0.0035i \\ -0.8905 - 0.0892i & -0.8905 + 0.0892i & 2.3601 - 0.2873i & 2.3601 + 0.2873i & 0.5587 - 0.0171i & 0.5587 + 0.0171i \end{bmatrix}$$

$$\text{Angle}(\Psi) = \begin{bmatrix} 91.5217 & -91.5217 & 92.5527 & -92.5527 & 93.2715 & -93.2715 \\ -89.6239 & 89.6239 & -93.1943 & 93.1943 & 93.4553 & -93.4553 \\ -82.7591 & 82.7591 & 85.6120 & -85.6120 & 91.5211 & -91.5211 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 178.8544 & -178.8544 & 174.2530 & -174.2530 & 0.1838 & -0.1838 \\ -174.2808 & 174.2808 & -6.9407 & 6.9407 & -1.7504 & 1.7504 \end{bmatrix}$$

$$\text{Recall: } \Psi = \begin{bmatrix} \Phi\Lambda & \Phi^*\Lambda^* \\ \Phi & \Phi^* \end{bmatrix}$$

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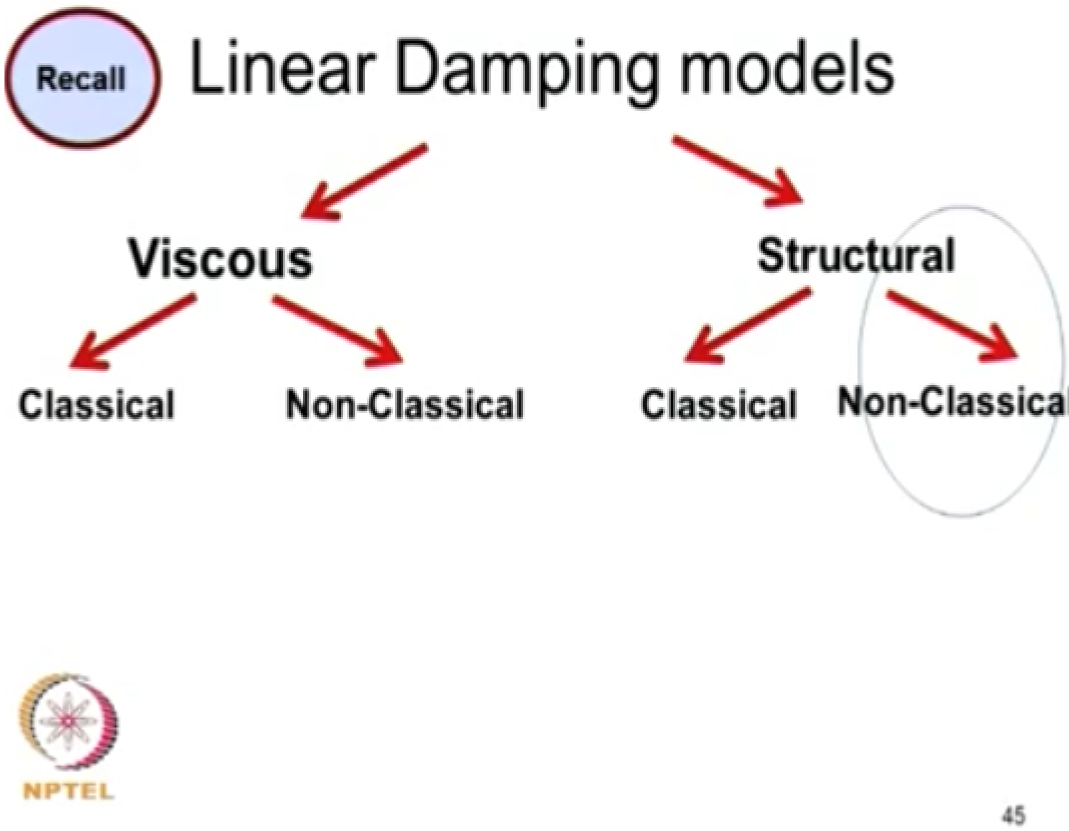
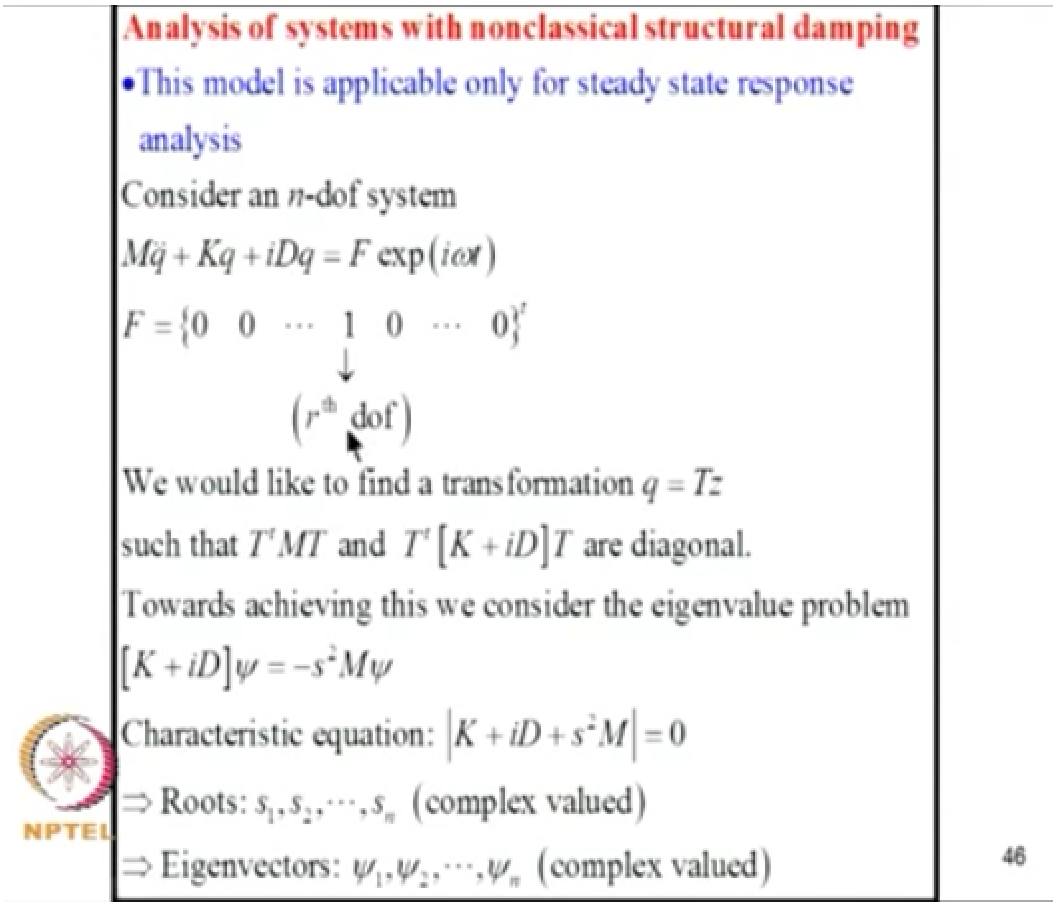
again I normalize the first degree of freedom I will arbitrarily make it as 1, okay and find the angles now, the moment you find angles you see something interesting happening, the phase differences are now nearly 0, but slightly changing, this is -0.18 degree to 1.75 degrees, whereas here it is not perfectly out of phase instead of being 180 degrees it is 178 degrees, okay, so this is how the damping influences the normal modes, that means the all points on the structure will now be executing a decaying harmonic oscillation, but different points will not be in perfectly in phase or perfectly out of phase in non-classically damped system, in classically damped systems again in free vibration all points will be executing decaying exponentially decaying harmonic motions, but the motions will be either perfectly in phase or perfectly out of phase, but here there is a phase difference you know other than perfect in phase or perfect out of phase there is a slight modification to this.

This is how the damping manifest itself in the eigenvectors, so if you are doing an experiment and if you measure normal modes you will always measure damped normal modes, because there is no magic switch to remove damping in an experimental work, in a computational work you can put C = 0 and find out the eigenvalues very easily but that counterpart of that in experimental work is not possible, so you will have to deal with damped normal modes if you are doing an experiment, so whenever you measure normal mode there are methods for that you will see that there will be a slight phase differences in a lightly non-proportional damped system this phase angles will be clearly close to 0 or 180 degrees.



Now I have now discussed non-classical viscous damping system, now I will extend this formulation to structural damping with non-classical damping, so we consider the N degree of freedom system, this is the structural damping term which is the complex part of the stiffness



and again I drive the R-th degree of freedom harmonically this is the only case that we can consider there is no transient analysis possible here, now we need to now find a transformation Q = TZ so that T transpose MT, and T transpose K + IDT are diagonal, okay. Now this has the form of MQ double dot + KQ = 0, un-damped free vibration but K is complex valued, so the normal mode should be orthogonal I mean the transformation matrix should be orthogonal to the complex valued stiffness matrix, so that is the novel feature here, so what we do? We towards achieving this we consider the eigenvalue problem K + ID Sai is - S square M Sai, okay I cannot call it as a free vibration problem here, it is a simply a mathematical statement of an eigenvalue problem because you cannot talk of free vibration for this problem, so the characteristic equation is given by this and we have N complex-valued roots and associated eigenvectors.

Now you should notice now that these eigenvalues will not appear as complex conjugates they're N complex numbers similarly eigenvectors are N complex eigenvectors, and N need not be an even number, okay so this all the eigenvectors I assemble in Sai matrix and we will normalize that so that Sai transpose M Sai is Sai, and Sai transpose K + ID Sai is a diagonal

$$\Psi = [\psi_1 \quad \psi_2 \quad \cdots \quad \psi_n]$$

$$\Psi^t M \Psi = I \ \& \ \Psi^t [K + iD] \Psi = \Lambda = \begin{bmatrix} -s_1^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & -s_n^2 \end{bmatrix}$$

$$M\ddot{q} + Kq + iDq = F\exp(i\omega t)$$

$$q = \Psi z$$

$$\Rightarrow M\Psi\ddot{z} + [K + iD]\Psi z = F\exp(i\omega t)$$

$$\Rightarrow \Psi^t M\Psi\ddot{z} + \Psi^t [K + iD]\Psi z = \Psi^t F\exp(i\omega t)$$

$$I\ddot{z} + \Lambda z = \Psi^t F\exp(i\omega t)$$

$$\ddot{z}_k - s_k^2 z_k = \sum_{j=1}^{n} \Psi_{jk} F_j \exp(i\omega t) = \Psi_{rk}\exp(i\omega t)$$

$$z_k = \frac{\Psi_{rk}}{-\omega^2 - s_k^2}\exp(i\omega t)$$

$$q_j(t) = \sum_{k=1}^{n} \Psi_{jk} \frac{\Psi_{rk}}{-\omega^2 - s_k^2}\exp(i\omega t) = \left[\sum_{k=1}^{n} \frac{\Psi_{jk}\Psi_{rk}}{-\omega^2 - s_k^2}\right]\exp(i\omega t)$$

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matrix, okay, you can prove the orthogonality using the arguments which I have now used several times, you consider R and S pair and you know to go through the few steps of calculation you can show that these orthogonality relations are true here.

Now you come to the force vibration problem, you make the substitution Q = Sai Z, substitute back I get this, pre multiplied by Sai transpose I get this. Now Sai transpose M Sai is I, and Sai transpose K + ID Sai is another diagonal matrix, that is a diagonal matrix of complex value eigenvalues as shown here, so the equation for ZK will be of this form, so I can find amplitude in steady state to be this, so go back to the Q coordinate system I get this and this is my expression for the problem, this SK is now a complex-valued number, SK square, so again we

$$M = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 150 & 0 \\ 0 & 0 & 80 \end{bmatrix}; K = \begin{bmatrix} 55000 & -30000 & -20000 \\ -30000 & 45000 & 0 \\ -20000 & 0 & 45000 \end{bmatrix}$$

$$\Phi = \begin{bmatrix} 0.0578 & -0.0355 & -0.0735 \\ 0.0623 & 0.0452 & 0.0272 \\ 0.0323 & -0.0842 & 0.0660 \end{bmatrix}; \Lambda = \begin{bmatrix} 114.65 & 0 & 0 \\ 0 & 457.15 & 0 \\ 0 & 0 & 840.70 \end{bmatrix}$$

$$\Phi^t M \Phi = \begin{bmatrix} 1.0000 & -0.0000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 \end{bmatrix}; \Phi^t K \Phi = \begin{bmatrix} 114.65 & 0 & 0 \\ 0 & 457.15 & 0 \\ 0 & 0 & 840.70 \end{bmatrix}$$

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can consider a simple example we will have the same M and K and this is an un-damped normal mode matrix, and this is un-damped natural frequency is phi transpose M phi is diagonal, phi transpose K phi is this, so for this data now we will introduce a proportional structural damping model and a non-proportional structural damping model.

Case of proportional damping matrix

$D = 0.03K; K^* = K + iD$

$$\Phi^t D\Phi = \begin{bmatrix} 3.4394 & 0 & 0 \\ 0 & 13.7147 & 0 \\ 0 & 0 & 25.2209 \end{bmatrix}$$

$\Rightarrow$ Undamped normal modal matrix uncouples the equation of motion.

$$\Psi = \begin{bmatrix} 1.0000 + 0.0000i & 0.9268 - 0.0000i & 0.4214 + 0.0000i \\ -0.3699 - 0.0000i & 1.0000 - 0.0000i & -0.5363 + 0.0000i \\ -0.8986 - 0.0000i & 0.5173 - 0.0000i & 1.0000 + 0.0000i \end{bmatrix}$$

$$\Psi^t M\Psi = 100\begin{bmatrix} 1.8513 + 0.0000i & 0.0000 - 0.0000i & 0.0000 - 0.0000i \\ 0.0000 - 0.0000i & 2.5730 - 0.0000i & -0.0000 + 0.0000i \\ 0.0000 - 0.0000i & -0.0000 + 0.0000i & 1.4089 + 0.0000i \end{bmatrix}$$

$$\Psi^t(K + iD)\Psi = 10^5\begin{bmatrix} 1.5564 + 0.0467i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \\ 0.0000 + 0.0000i & 0.2950 + 0.0088i & -0.0000 + 0.0000i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.6441 + 0.0193i \end{bmatrix}$$

NPTEL

Note: $\Psi^t M\Psi$ is diagonal but has not been normalized to be an identity matrix

So let us start with case of proportional damping matrix, so I will take D to be 0.03 times K, so K star will be K + ID, and Phi star D phi will be this, okay, this is a mere transposition, it is not a conjugation + transposition, it is mere transposition that is also important to note. Now see un-damped normal modal matrix uncouples the equation of motion in this case, Sai is this, you can see phi transpose D phi, where phi is un-damped modal matrix it uncouples the damping matrix, so i can take Sai to be this, Sai transpose M Sai is this, sai transpose K + ID Sai is also this, that means even if I find complex valued Sai, the uncoupling takes place. So now we will

Scaled Ψ

$$\Psi = \begin{bmatrix} 1.0000 & 1.0000 & 1.0000 \\ -0.3699 - 0.0000i & 1.0790 + 0.0000i & -1.2726 + 0.0000i \\ -0.8986 - 0.0000i & 0.5582 + 0.0000i & 2.3732 + 0.0000i \end{bmatrix}$$

$$\text{Angle}(\Psi) = \begin{bmatrix} 0 & 0 & 0 \\ -180.0000 & 0.0000 & 180.0000 \\ -180.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

Eigenvalues for the damped system $\left([K + iD]\psi = -s^2 M\psi\right)$

$\left[-s_k^2, k = 1,2,3\right]$

840.70 + 25.22i

114.65 + 3.44i

457.16 + 13.71i

Eigenvalues for the undamped system $\left(K\phi = \omega^2 M\phi\right)$

$\left[\omega^2 \text{ in } (\text{rad/s})^2\right]$

114.6479, 457.1553, 840.6968

NPTEL

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scale the first eigenvector element of the eigenvector to be 1 again, again you see if I find the angle of Sai the phase differences are either 180 degrees or 0, so the points are still perfectly in phase or perfectly out of phase, because this is proportionally damped system.

So now if I find eigenvalues for the damped system, the eigenvalues will be of this form, okay now un-damped system if you recall the eigenvalues obtained are this, so if you compare these two for a kind of notations we have used the real part of eigenvalues will be the natural frequencies here, okay, these 3 terms appear as a 3 natural frequencies and these are the terms associated with damping, so this is a classical structural damping. Now let’s consider a non-proportional damping,

Case of nonproportional damping matrix

$$D = \begin{bmatrix} 1000 & 0 & 0 \\ 0 & 1250 & 0 \\ 0 & 0 & 2000 \end{bmatrix}; K^* = K + iD$$

$$\Phi^t D \Phi = \begin{bmatrix} 10.2766 & -3.9647 & 2.1324 \\ -3.9647 & 18.0069 & -6.9842 \\ 2.1324 & -6.9842 & 15.0498 \end{bmatrix}$$

$\Rightarrow$ Undamped normal modal matrix does not uncouple the equation of motion.

$$\Psi = \begin{bmatrix} 0.9994 - 0.0006i & 0.9173 - 0.0003i & 0.4121 + 0.0177i \\ -0.3697 + 0.0048i & 0.9900 + 0.0100i & -0.5236 + 0.0097i \\ -0.8974 - 0.0316i & 0.5116 - 0.0170i & 0.9772 - 0.0228i \end{bmatrix}$$

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so again I consider D to be this and K star is K + ID, quickly we can verify phi transfer D phi is not diagonal, so it is not a classically damped system, so now I can do a complex eigenvalue analysis and I get this as Sai matrix, so we can verify Sai transpose M Sai is this, Sai transpose K Sai is this, these are diagonal but Sai transpose M Sai is not identity this is not, this is again diagonal but diagonal entries are not the eigenvalues.

$$\Psi' M \Psi = 100\begin{bmatrix} 1.8472 + 0.0387i & 0.0000 + 0.0000i & -0.0000 - 0.0000i \\ 0.0000 + 0.0000i & 2.5206 + 0.0152i & -0.0000 - 0.0000i \\ -0.0000 - 0.0000i & -0.0000 - 0.0000i & 1.3440 - 0.0363i \end{bmatrix}$$

$$\Psi' [K + iD] \Psi = 10^5\begin{bmatrix} 1.5521 + 0.0603i & 0.0000 - 0.0000i & 0.0000 - 0.0000i \\ 0.0000 - 0.0000i & 0.2890 + 0.0276i & -0.0000 - 0.0000i \\ 0.0000 - 0.0000i & 0.0000 + 0.0000i & 0.6152 + 0.0076i \end{bmatrix}$$

Note: $\Psi' M \Psi$ is diagonal but has not been normalized to be an identity matrix.

Scaled Ψ

$$\Psi = \begin{bmatrix} 1.0000 & 1.0000 & 1.0000 \\ -0.3699 + 0.0046i & 1.0792 + 0.0113i & -1.2672 + 0.0780i \\ -0.8979 - 0.0321i & 0.5577 - 0.0183i & 2.3646 - 0.1570i \end{bmatrix}$$

$$\text{Angle}(\Psi) = \begin{bmatrix} 0 & 0 & 0 \\ 179.2883 & 0.6004 & 176.4784 \\ -177.9503 & -1.8834 & -3.7992 \end{bmatrix}$$

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So I will scale now the eigenvector so that the first element is 1, I get this as a scale Sai matrix and if I now compute the angle of this you see now I get phase difference of about 2 degrees and less than 1 degree here and 0.6 degrees and 1.8 degrees and about 4 degrees and here, so this is a manifestation of the role played by damping on eigenvectors.

Eigenvalues for the damped system $\left([K+iD]\psi=-s^2M\psi\right)$

$\left[-s_k^2, k=1,2,3\right]$

840.56 + 15.05i

114.70 + 10.28i

457.24 + 18.01i

Eigenvalues for the undamped system $\left(K\phi=\omega^2M\phi\right)$

$\left[\omega^2 \text{ in } (\text{rad/s})^2\right]$

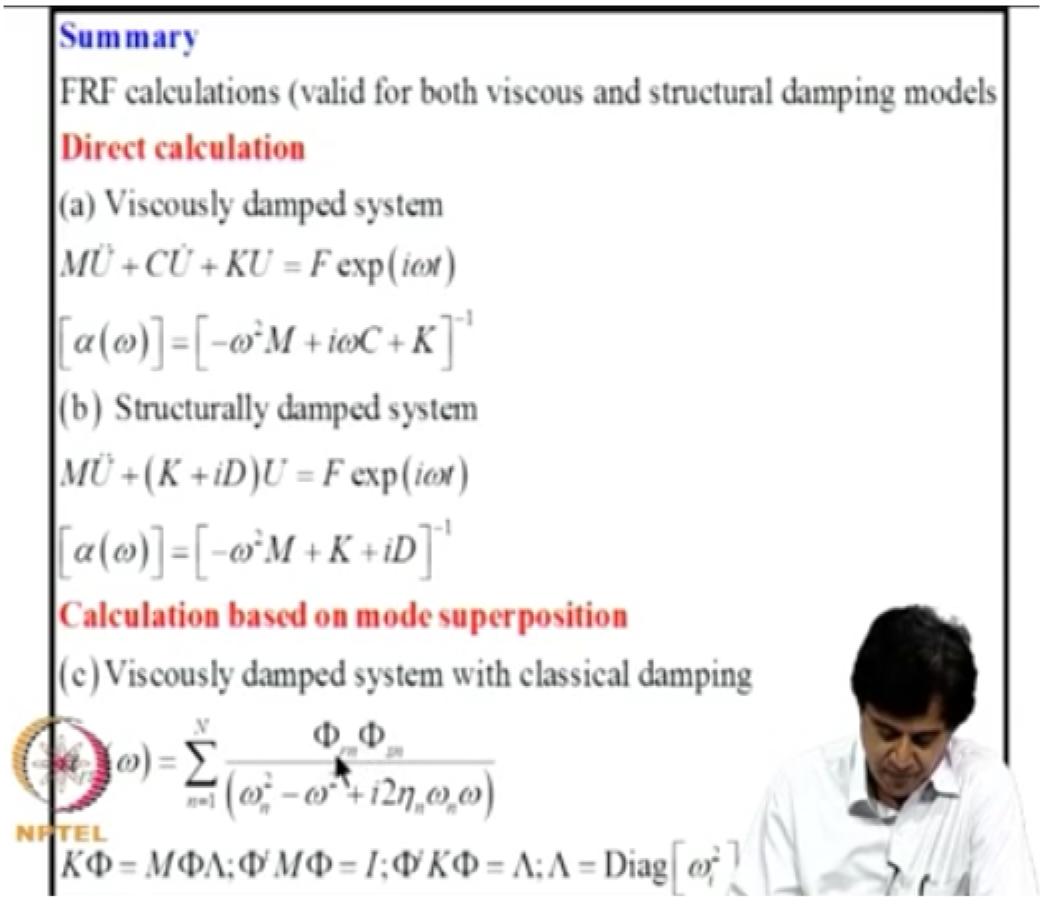
114.6479

457.1553

840.6968

NPTEL

Now you can look at now the eigenvalues for the damped system I get this, you can see here the real parts are again close to the un-damped square of the natural frequencies, but they are slightly different because of influence of damping, okay, so now if we can quickly summarize



now what we have done is we have evaluated frequency response functions for both viscous and structurally damped systems by direct calculation for viscously damped system the FRF matrix is given by this, for structurally damped system the FRF matrix is given by this, but if you want a calculation based on modal superposition then you have to find the appropriate modal coordinates for that, so if we talk about viscously damped system with classical damping we have shown that it is given in terms of these un-damped eigenvectors and natural frequencies like this, for similarly for structurally damped system is classical damping again with the un-damped normal modes we get these solutions.

Calculation based on mode superposition (continued)

(d) Structurally damped system with classical damping

$$\alpha_{jr}(\omega)=\sum_{k=1}^{n}\frac{\Phi_{jk}\Phi_{rk}}{\omega_k^2-\omega^2+i\bar{D}_k}$$

$$K\Phi=M\Phi\Lambda;\Phi^t M\Phi=I;\Phi^t K\Phi=\Lambda;\Lambda=\mathrm{Diag}\left[\omega_i^2\right]$$

(e) Viscously damped system with nonclassical damping

$$\alpha_{jr}(\omega)=\sum_{k=1}^{n}\frac{\Phi_{rk}\Phi_{jk}}{i\omega-\alpha_k}+\frac{\Phi^*_{jk}\Phi^*_{rk}}{i\omega-\alpha_k^*}$$

$$B\Psi=-\alpha A\Psi;\Psi=\begin{bmatrix}\Phi\Lambda & \Phi^*\Lambda^*\\ \Phi & \Phi^*\end{bmatrix};\Psi^t A\Psi=I;\Psi^t A\Psi=\mathrm{Diag}\left(\Lambda\ \Lambda^*\right)$$

$$\Lambda=\mathrm{Diag}\left(\alpha_1\ \alpha_2\cdots\alpha_n\right)$$

(f) Structurally damped system with nonclassical damping

$$\alpha_{jr}(\omega)=\sum_{k=1}^{n}\frac{\Psi_{jk}\Psi_{rk}}{-\omega^2-s_k^2}$$

$$\left[K+iD\right]\psi=-s^2M\psi;\Psi^t M\Psi=I;\Psi^t\left[K+iD\right]\Psi=\mathrm{Diag}\left(-s_1^2\ \ -s_2^2\cdots-s_n^2\right)$$

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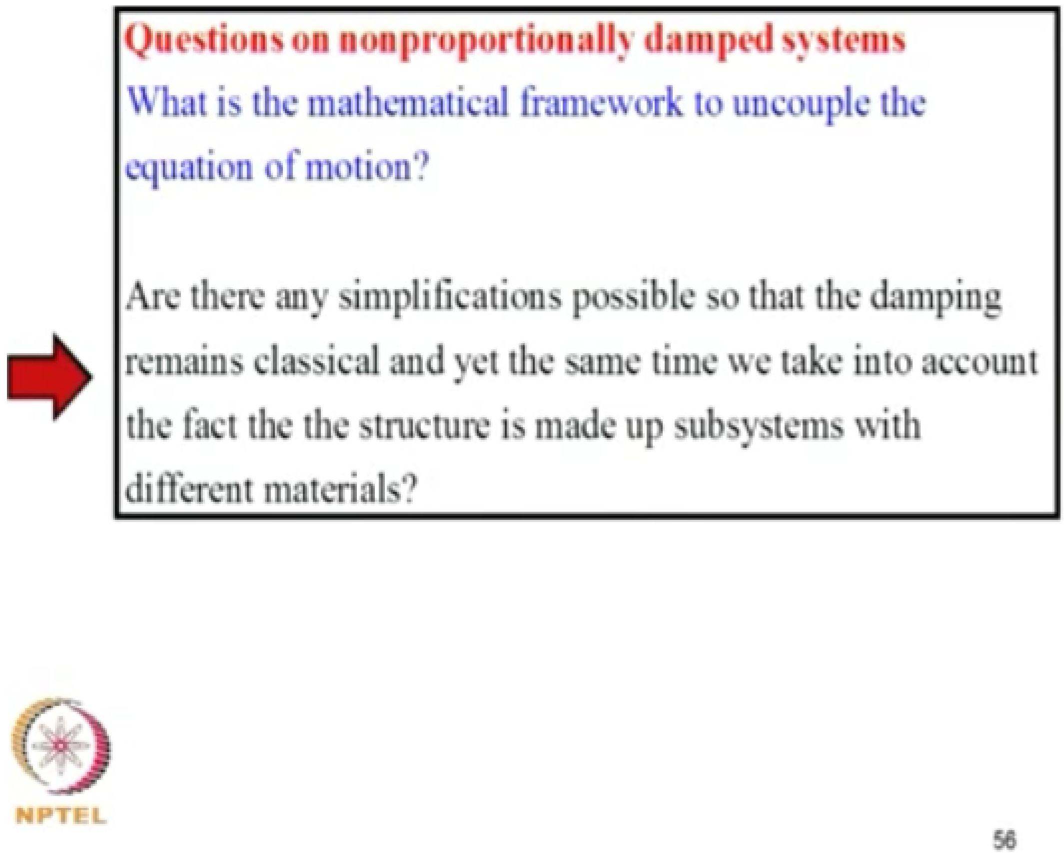
For viscously damped system with non-classical damping we go to that A and B matrices and we get now the frequency response function in terms of complex valued eigenvectors and complex valued eigenvalues this is how, for structurally damped system with non-classical damping the one that we did lost again this is summation in terms of complex valued Eigen pairs, so this is direct calculation, this all these are through mode superposition, so this is what basically I intended to demonstrate.



Now I also posed this question, we posed basically two questions when we are talking about non-proportional damped system, the first one was what is the mathematical framework to uncouple the equation of motion, this we have answered now. The other question that is remaining is are there any simplification possible so that a damping remains classical and at the same time we take into account the fact that structure is made up of subsystems with different materials so this takes us into damping model known as material damping model, we will consider that in the next lecture. So we will conclude this lecture at this point.

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