Indian Institute of Science
Bangalore

NP-TEL
National Programme on
Technology Enhanced Learning

<u>Course Title</u>

Finite element method for structural dynamic
And stability analyses

Lecture – 06
FE Modelling of Planar structures

By
Prof. CS Manohar
Professor
Department of Civil Engineering

Indian Institute of Science,
Bangalore-560 012
India

In this lecture we will continue discussing about dynamics of planar structures, so what we will do in this lecture is, we will try to solve a few problems.

Finite element method for structural dynamic and stability analyses

<u>Module-2</u>

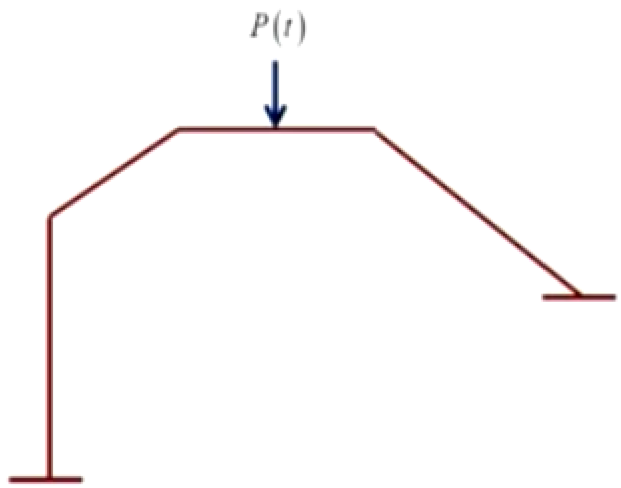
Finite element analysis of dynamics of planar trusses and frames

<u>Lecture-6</u>: FE modeling of planar structures: problem solving session

Prof C S Manohar
Department of Civil Engineering
IISc, Bangalore 560 012 India

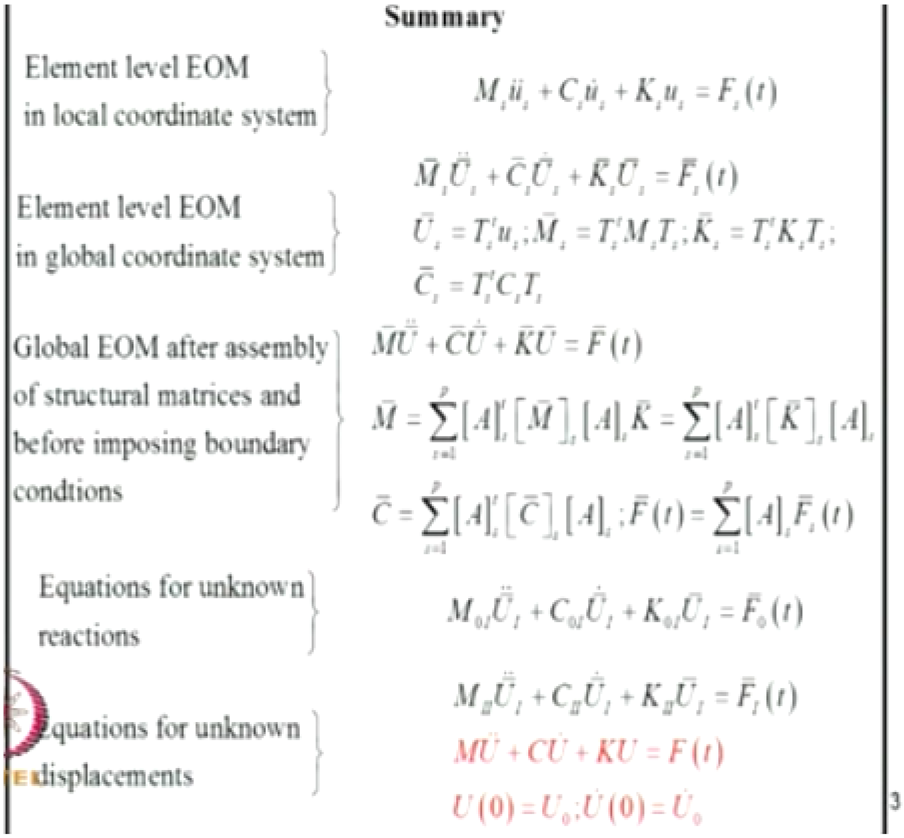
NPTEL

Recall



NPTEL

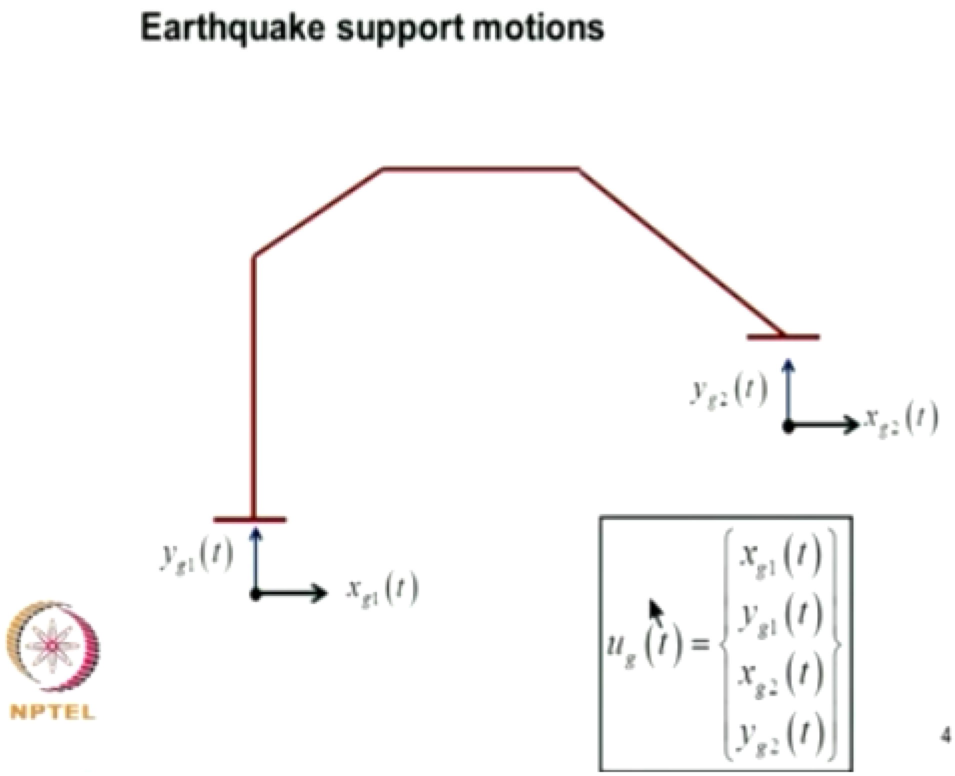
Before that we can quickly recall what is that we have done, we are trying to develop methods that we can use to analyze frames like this, so some of the features of this frame is that different members are oriented in different directions, and both bending and axial deformations coexist and there is a force that is acting in one of them members and it has to be expressed in terms of equivalent nodal forces and so on and so forth.



So now the steps that we adopt to deal with this problem is first of all we discretize this structure into subdomains, and this is S = 1, 2, and so on and so forth, P elements what we do is for each of the element we use the structural mass damping and stiffness matrices and the force vector, and we write the element level equation of motion for each of the elements. Then transform these element level equation of motion and express it in the global coordinate system for each of the elements, here we would need the coordinate transformation matrix T for that, TS then we assemble this we would have done for all the elements and then we assemble all the structural matrices and force vector for each of these elements to obtain the global equation of motion before imposing boundary conditions they will have disappearance.

Next what we do is we partition the displacement vector into those components which are specified either to be zero or nonzero they are the boundary conditions on displacements and the remaining displacement degrees of freedom which are to be determined, so at places where we, displacements are specified that we will have reactions so the unknowns will be the unknown displacements and the unknown reactions and this set of equations provide the necessary number of equations to obtain those unknowns.

Now the equation for unknown reactions take this form where M0I, K0I etcetera are the partition matrices obtained from the global structural matrices, then the equation for unknown displacement has this form and as we agreed we will write this equation in this form, we will drop all the superscripts bars etcetera, and we will write this equation which is the equation for unknown displacements and in this form and call this equation MU double dot + CU dot + KU = F(t) as a governing equation of motion for the system. So the strategy would be to solve this equation first, find out all the displacements and then go back and find reactions.



Now in the previous lecture I considered the case where the support displacements where specified to be 0, but in many problems such as in earthquake engineering problems we will have situations where the support displacements would be specified to be nonzero and a time history would be available for that, so for this frame suppose the two supports are subjected to this displacement XG1(t), YG1(t) and XG2(t) and XG2(t), the subscript G refers to ground, so the support displacement vector now UG(t) will have these 4 components, so Q is 4 for this case.

Global structure equation of motion

$$\bar{M}\ddot{\bar{U}} + \bar{C}\dot{\bar{U}} + K\bar{U} = \bar{F}(t)$$

Boundary condtions

$$\bar{U}(t) = \begin{Bmatrix} \{u_g(t)\}_{q\times 1} \\ \{\bar{U}_I\}_{(N-q)\times 1} \end{Bmatrix} = \begin{Bmatrix} \text{Specified support motions} \\ \text{Unknown nodal displacements} \end{Bmatrix}$$

NPTEL

Now the global equation of motion for this system would be again in this form and we will partition now the displacement vector into Q components which are specified support displacements, and the remaining N – Q components which are the unknown nodal displacements, so this partitioning of the displacement vector automatically implies a partitioning of mass, damping and stiffness matrices and that partitioning is in this form. So the forcing vector accordingly also will get partitioned and it will have Q components which are reactions which are not known, and N- Q components which are the applied equivalent nodal forces, if there is no external force acting on the system other than the support motions then this FI (t) would be 0.

$$\Rightarrow \bar{M} = \begin{bmatrix} M_{00} & M_{0I} \\ M_{I0} & M_{II} \end{bmatrix}; \bar{K} = \begin{bmatrix} K_{00} & K_{0I} \\ K_{I0} & K_{II} \end{bmatrix}; \bar{C} = \begin{bmatrix} C_{00} & C_{0I} \\ C_{I0} & C_{II} \end{bmatrix};$$

$$\bar{F}(t) = \begin{Bmatrix} \bar{F}_0(t) \\ F_I(t) \end{Bmatrix} = \begin{Bmatrix} \text{Unknown reactions} \\ \text{Applied equivalent nodal forces} \end{Bmatrix}$$

Global equation of motion

$$\begin{bmatrix} M_{00} & M_{0I} \\ M_{I0} & M_{II} \end{bmatrix} \begin{Bmatrix} \ddot{U}_g \\ \ddot{\bar{U}}_I \end{Bmatrix} + \begin{bmatrix} C_{00} & C_{0I} \\ C_{I0} & C_{II} \end{bmatrix} \begin{Bmatrix} \dot{U}_g \\ \dot{\bar{U}}_I \end{Bmatrix} + \begin{bmatrix} K_{00} & K_{0I} \\ K_{I0} & K_{II} \end{bmatrix} \begin{Bmatrix} U_g \\ \bar{U}_I \end{Bmatrix} = \begin{Bmatrix} \bar{F}_0(t) \\ F_I(t) \end{Bmatrix}$$

Equations for unknown reactions

$$M_{0I}\ddot{\bar{U}}_I + C_{0I}\dot{\bar{U}}_I + K_{0I}\bar{U}_I + \left\{ M_{00}\ddot{U}_g + C_{00}\dot{U}_g + K_{00}U_g \right\} = \bar{F}_0(t)$$

Equations for unknown displacements

$$M_{II}\ddot{\bar{U}}_I + C_{II}\dot{\bar{U}}_I + K_{II}\bar{U}_I = \bar{F}_I(t) - \left\{ M_{I0}\ddot{U}_g + C_{I0}\dot{U}_g + K_{I0}U_g \right\}$$

Number of equations$=q + (N - q) = N$

Unknowns: q reactions $\left(\bar{F}_0(t)\right)$ and $(N - q)$ displacements $\left(\bar{U}_I(t)\right)$.

6

Now the governing equation of motion accordingly can be written in this form, so the first row of this equation provides the equation for the unknown reactions, and the second row provides the equation for the unknown support displacements, so you can see here that the right hand side here consists of equivalent nodal forces due to external forces acting on the structure and these 3 additional terms provide the forces on the system created due to the support displacements, so this is support acceleration, this is support velocity, and this is support displacement, so the number of equations here again Q reactions and N - Q unknown displacements and that is provided by these, this n number of equations.

Reduced structure equation of motion

$$M_{II}\ddot{\bar{U}}_I + C_{II}\dot{\bar{U}}_I + K_{II}\bar{U}_I = \bar{F}_I(t) - \{M_{I0}\ddot{U}_g + C_{I0}\dot{U}_g + K_{I0}U_g\}$$

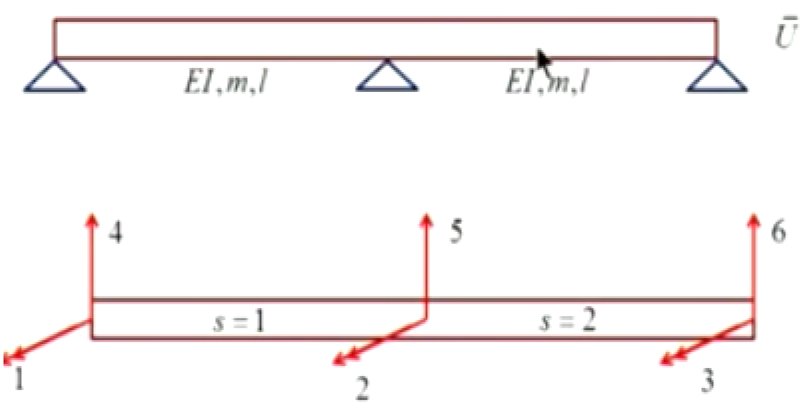
Or simply, the governing equation of motion

$$M\ddot{U} + C\dot{U} + KU = F(t)$$

$$U(0) = U_0; \dot{U}(0) = \dot{U}_0$$

Now the reduced structure equation of motion can be written in this form, this is the equation for the unknown displacements and simply by omitting these bars and subscripts we write this as the equation of motion for the system, this F (t) has again the external forces acting on the system equivalent nodal forces due to that and the forces due to the support displacement.

Example



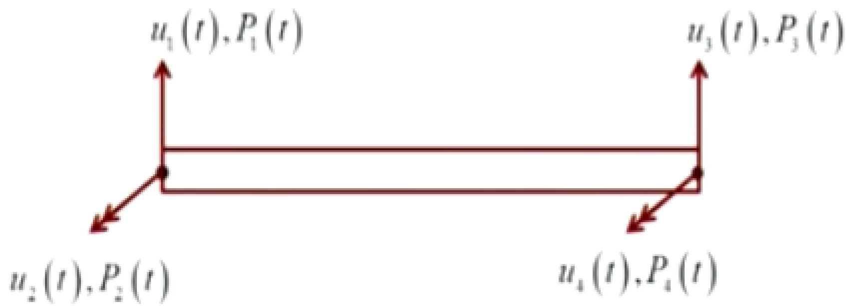
$$\bar{U}(t) = \begin{Bmatrix} U_1(t) \\ U_2(t) \\ U_3(t) \\ U_4(t)=0 \\ U_5(t)=0 \\ U_6(t)=0 \end{Bmatrix}$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}; A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

NPTEL

8

Now what we will do now is we will consider a set of examples to which would bring out different facets of what we have been discussing till now, so let’s start with the case of a 2 span beam which is simply supported, we will discretize this problem using 2 element, S = 1 to 2, and the objective is to make a model for the system so as to, so that we can evaluate the natural frequencies of the system and more shapes, so we name the elements as 1 and 2, and these are the degrees of freedom, so the numbering of these degrees of freedom follows a pattern so we have arranged the degrees of freedom in this form U1, U2, U3 which are the unknown rotations at the supports appear at the top 3 slots here, and U4, U5, U6 which are support translations at the supports which are all 0, specified to be 0 appear in a cluster in the second half of this vector.



$$M = \frac{ml}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \; \& \; K = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

9

So in order to assemble the structural matrices we need to form those A matrices, A1 is given here, A2 is given here, so this follows a node numbering scheme that we have adopted here. So this sorry, a single element just to quickly recall this is how the structural matrices are obtained, so this is the node numbering here is 1, 2, 3, 4 as shown here, 1, 2, 3, 4 but this numbering would not agree with this, so this A matrices help us to deal with that.

$$s=1: M=\frac{ml}{420}\begin{array}{cc} \begin{matrix}4 & 1 & 5 & 2\end{matrix} & \\ \begin{bmatrix}156 & 22l & 54 & -13l\\ 22l & 4l^2 & 13l & -3l^2\\ 54 & 13l & 156 & -22l\\ -13l & -3l^2 & -22l & 4l^2\end{bmatrix} & \begin{matrix}4\\1\\5\\2\end{matrix}\end{array}$$

$$s=2: M=\frac{ml}{420}\begin{array}{cc} \begin{matrix}5 & 2 & 6 & 3\end{matrix} & \\ \begin{bmatrix}156 & 22l & 54 & -13l\\ 22l & 4l^2 & 13l & -3l^2\\ 54 & 13l & 156 & -22l\\ -13l & -3l^2 & -22l & 4l^2\end{bmatrix} & \begin{matrix}5\\2\\6\\3\end{matrix}\end{array}$$

$$M=\frac{ml}{420}\begin{array}{cc} \begin{matrix}1 & 2 & 3\end{matrix} & \\ \begin{bmatrix}4l^2 & -3l^2 & 0\\ -3l^2 & 8l^2 & -3l^2\\ 0 & -3l^2 & 4l^2\end{bmatrix} & \begin{matrix}1\\2\\3\end{matrix}\end{array}$$

10

Now for the first element the mass matrix is this, and for the second element the mass matrix is this, now we could use the method based on using the A matrices and assemble the matrices, but if you are doing the problem with pen and paper it may be possible to simply write down the governing equation of motion by inspection, so that is what I will illustrate in this example. So for the first element the numbering scheme is 4, 1, 5, 2, so we write on this matrix at the top, these node numbers and also along the side like this. For the second element it is 5, 2, 6, 3, what is that I mean? The numbering here is for 4, 1, 5, 2 and 5, 2, 6, 3 so now to get the structure matrices 1, 2, 3 are the unknown degrees of freedom, so to write this matrix what we do is we inspect these two matrices, where all for example to fill up this slot here, I look for places where I get 1, 1, for example I get 1, 1 here and so this 4L square comes and occupy this slot.

$$s=1: M=\frac{ml}{420}\begin{array}{c} \begin{array}{cccc} 4 & 1 & 5 & 2 \end{array} \\ \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \end{array}\begin{array}{c} \\ 4 \\ 1 \\ 5 \\ 2 \end{array}$$

$$s=2: M=\frac{ml}{420}\begin{array}{c} \begin{array}{cccc} 5 & 2 & 6 & 3 \end{array} \\ \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \end{array}\begin{array}{c} \\ 5 \\ 2 \\ 6 \\ 3 \end{array}$$

$$M=\frac{ml}{420}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 4l^2 & -3l^2 & 0 \\ -3l^2 & 8l^2 & -3l^2 \\ 0 & -3l^2 & 4l^2 \end{bmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}$$

NPTEL

Now 2, 1 appears here 2, 1 – 3L square comes here, there is no 3, 1 element anywhere so this stays as 0.

Now to get for example 2 2, 2 2 appears where, one place it appears here, the next place it appears here and we have to add these two so this two added will come here and appear as A del square, so similarly 3 2 is – 3L square, where do you get that? 3 2 is – 3L square, it is here, so 3 2 is – 3L square which is here that comes here. Similarly 3 3, 3 3 appears only in the second matrix which is 4L square, which is 4L square, so this is how we can construct the mass matrix associated with the unknown displacement degrees of freedom.

$$
\begin{array}{cc}
 & \begin{array}{cccc} 4 & 1 & 5 & 2 \end{array} \\
s=1: K = \dfrac{EI}{l^3} & \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \begin{array}{c} 4 \\ 1 \\ 5 \\ 2 \end{array}
\end{array}
$$

$$
\begin{array}{cc}
 & \begin{array}{cccc} 5 & 2 & 6 & 3 \end{array} \\
s=2: K = \dfrac{EI}{l^3} & \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \begin{array}{c} 5 \\ 2 \\ 6 \\ 3 \end{array}
\end{array}
$$

$$
\begin{array}{cc}
 & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
K = \dfrac{EI}{l^3} & \begin{bmatrix} 4l^2 & 2l^2 & 0 \\ 2l^2 & 8l^2 & 2l^2 \\ 0 & 2l^2 & 4l^2 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}
\end{array}
$$

The same exercise can be done for the stiffness matrix, so I write the stiffness matrix for the first element and again write the node numbering as 4, 1, 5, 2, and for the second element 5, 2, 6, 3 this is the element, and when we add again we follow the same logic 1 1 is 4L square that appears here, 2 1 is 2L square that appears here, 3 1 is not an element that appears here either here or neither, here nor here so it is 0, 2 2, so 2 2 is 4L square + 4L square that is 8L square, so you can fill up this, so the governing equation of motion therefore at the end of this exercise

$$\frac{ml}{420}\begin{bmatrix} 4l^2 & -3l^2 & 0 \\ -3l^2 & 8l^2 & -3l^2 \\ 0 & -3l^2 & 4l^2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 4l^2 & 2l^2 & 0 \\ 2l^2 & 8l^2 & 2l^2 \\ 0 & 2l^2 & 4l^2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = 0$$

$$\lambda = \frac{ml^4\omega^2}{420EI} \Rightarrow \begin{bmatrix} 4 & -3 & 0 \\ -3 & 8 & -3 \\ 0 & -3 & 4 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \lambda \begin{bmatrix} 4 & 2 & 0 \\ 2 & 8 & 2 \\ 0 & 2 & 4 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}$$

$$\Rightarrow \lambda = 2/7, 1, 6$$

$$\Rightarrow \omega_1 = 10.95\sqrt{\frac{EI}{ml^4}}, \omega_2 = 20.49\sqrt{\frac{EI}{ml^4}} \;\&\; \omega_3 = 50.20\sqrt{\frac{EI}{ml^4}}\,(\text{rad/s})$$

$$\omega_1^{\text{Exact}} = 9.87\sqrt{\frac{EI}{ml^4}}, \omega_2^{\text{Exact}} = 15.41\sqrt{\frac{EI}{ml^4}} \;\&\; \omega_3^{\text{Exact}} = 39.48\sqrt{\frac{EI}{ml^4}}\,(\text{rad/s})$$

Errors: 10.94% , 32.98%& 27.15%

would be the mass matrix which is 3 by 3 matrix into the acceleration vector plus the 3 by 3 stiffness matrix U1, U2, U3 so the degrees of freedom here or 1, 2, 3.

Now in order to find the natural frequencies we need to perform the eigenvalue analysis to facilitate that I introduce a parameter lambda which is ML4 omega square by 420EI, so with this notation the eigenvalue problem that we need to solve reads as shown here, so this is the K matrix after this the matrix resulting from the stiffness matrix, this is a matrix resulting from the mass matrix, so we can do this eigenvalue analysis and we can show that the 3 eigenvalues are 2 / 7, 1 and 6, so once I find these 3 eigenvalues I can go back to this formula and write the equations for omega 1, omega 2, and Omega 3, so these are numbers that are shown here so this factor square root EI by ML to the power of 4 appears in all of this, and the scaling the multiplier for this for the first mode is 10.95, this is 20.49. this is for 50.20, it is possible to obtain exact solution to the first 3 natural frequencies for this problem, so if we do that exercise we find that the exact first natural frequencies 9.87 into this factor, the second exact natural frequency is this 15.41 into this factor and third one is 39.48 into this factor.

Now if we compute the errors we see that the error in finding first natural frequency is 10.94%, second one is 32.98% percent, and this third one is about 27%, now if you recall in the previous lecture we had studied a propped cantilever and a single span simply supported beam and this

$$\omega_1 = 20.49\sqrt{\frac{EI}{ml^4}};\ \omega_1^{\text{Exact}} = 15.41\sqrt{\frac{EI}{ml^4}}$$

Error: 32.98%

$$\omega_1 = 10.95\sqrt{\frac{EI}{ml^4}}\ \&\ \omega_2 = 50.20\sqrt{\frac{EI}{ml^4}}\ (\text{rad/s})$$

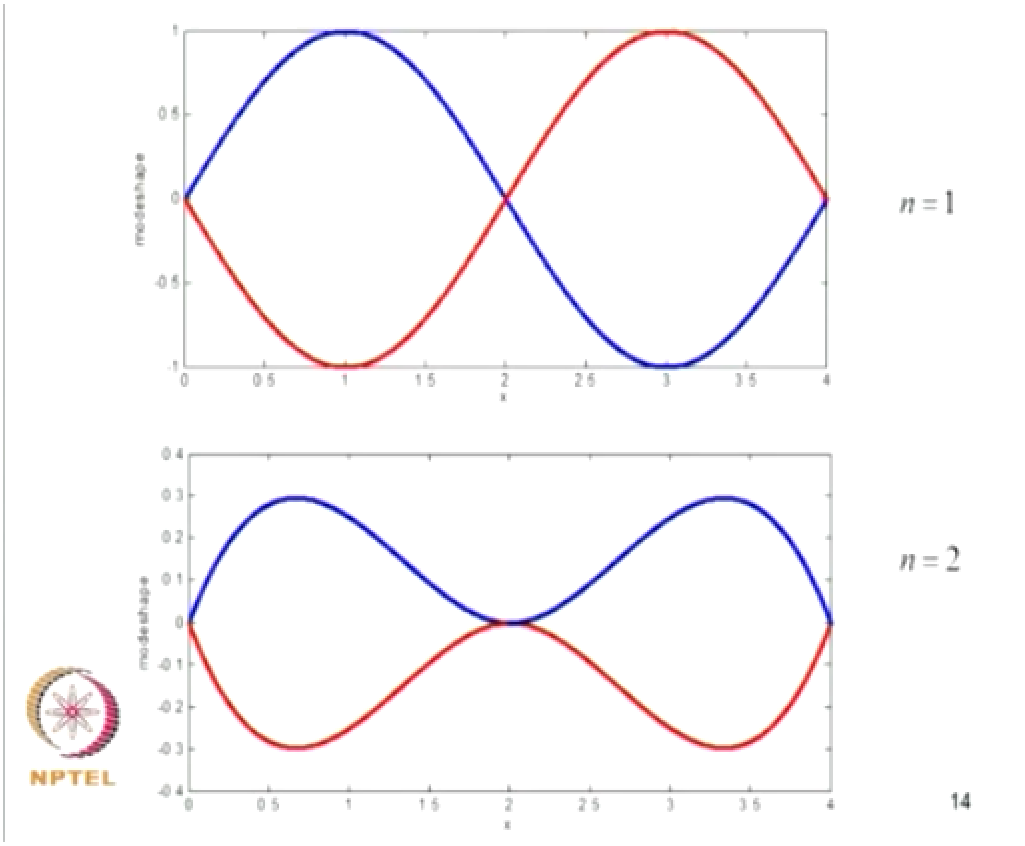
$$\omega_1^{\text{Exact}} = 9.87\sqrt{\frac{EI}{ml^4}}\ \&\ \omega_2^{\text{Exact}} = 39.48\sqrt{\frac{EI}{ml^4}}\ (\text{rad/s})$$

Errors: 10.94% & 27.15%

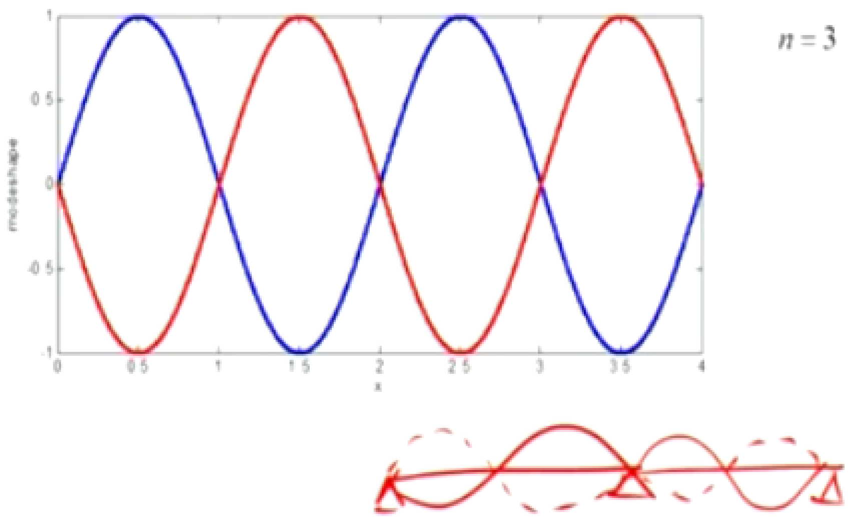
NPTEL

13

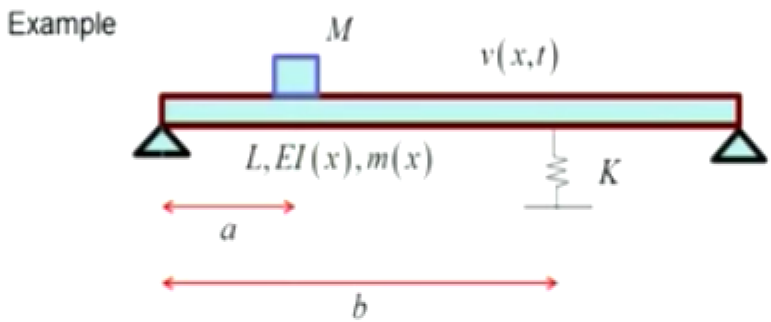
was a 1 degree freedom system and I had got the first natural frequency as this and this had shown about 33%. Now this was approximated the simply supported beam was approximated as a 2 degree freedom system and we had got these estimates of natural frequencies with these errors. Now if you inspect the results you will see that the numbers that we are getting 10.95, 20.49, 50.20 are also appearing in these solutions 20.49 for example appears for propped cantilever, and 10.95 and 50.20 appear for simply supported beam, why is that so? It is not a coincidence because if we now look at the mode shapes for N = 1 you can show that the mode



shape that we get for the 2 span beam will be like this, so the structure vibrates, so when the 2 span beam vibrates when this beam goes down, this point goes down, this point goes up, so they are perfectly out of phase, so we get this type of oscillations, so blue is one snapshot, red is another snapshot. The second mode on the other hand will be the first, when the point on the first span is going down, the point on the second span is also going down, so the first span behaves as if it is one propped cantilever, and the second span behaves as if it is another propped cantilever, so these mode shapes, the mode shape for the 2 span beam is composed of the first mode for propped cantilever that is this beam, and the first mode for the propped cantilever that is this beam, so therefore the second frequency for the continuous beam happens to be the first frequency of a propped cantilever.



The third mode on the other hand will be the second mode for a simply supported beam the second mode shape will be like this, so that is what we are seeing for the 2 span beam, for the 2 span beam the third mode consists of, the second mode of the individual spans, so that is why we get this and therefore the third natural frequency of the 2 span beam happens to be the second natural frequency of the single span beam, so all these errors therefore you know the numbers that we are getting or matching for that reason.



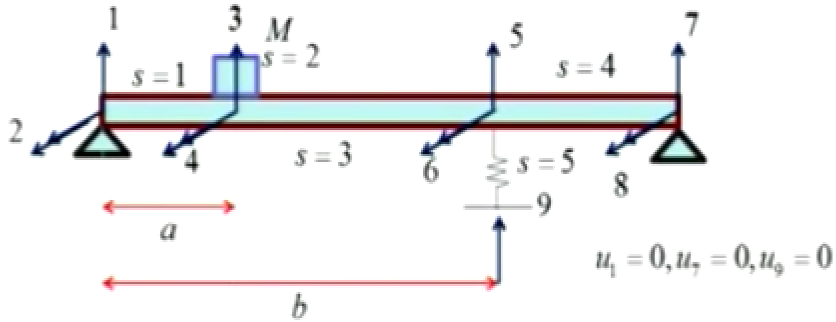
L=3 m; E=210 GPa; ρ=7800 kg/m^3; $a = L/3$; $b = 3L/4$

Beam cross section (rectangular): width=0.1 m; depth=0.2 m

$M = \frac{mL}{2}$; $K = \frac{2EI}{L}$

NPTEL

Now we will consider some more examples, suppose we consider now a single span beam which carries a point mass M and a discrete spring K, so if you recall this problem we have studied in one of the previous lectures using Rayleigh-Ritz method, now let us try to analyze the same problem using the finite element method, so these are the system parameters that we had used earlier in the illustration so we will continue the same numbers, so for the purpose of developing finite element model what I do is I divide the beam I will introduce, so I will



$s = 1$

$l = a$

$$K = \frac{EI}{l^3}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$$M = \frac{ml}{420}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

NPTEL

$s = 2$

$$K = \begin{array}{c} 3 \\ [0] \end{array} 3 \qquad M = \begin{array}{c} 3 \\ [M] \end{array} 3$$

introduce a node here, a node here, a node here, and a node here, so this will be made up of 3 beam elements. 1 mass element and 1 spring element so this is where also I should have another node, so I have S = 1 which is this beam, S = 2 is the element which is this mass, S = 3 is this spring, S = 4 is this beam, S = 5 is this spring, so there are 5 elements so now we can construct the mass and stiffness matrix for each of these elements, the local and global coordinates here coincide therefore we need not worry about any transformation, so for the first element length will be A, so stiffness matrix is this, mass matrix is this, the node numbering scheme is 1, 2, 3, 4 because this is 1, 2, 3, 4 so that is this.

Now for the second element which is only the mass the stiffness matrix is 0, the mass matrix is a single entry who's value is this point mass M.

$s = 3$

$l = b - a$

$$K = \frac{EI}{l^3}\begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \end{array}\begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \qquad M = \frac{ml}{420}\begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \end{array}\begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$s = 4$

$l \to l - b$

$$K = \frac{EI}{l^3}\begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \end{array}\begin{array}{c} \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \qquad M = \frac{ml}{420}\begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \end{array}\begin{array}{c} \\ 5 \\ 6 \\ 7 \\ 8 \end{array}$$

$$K = k\begin{array}{c} \begin{array}{cc} 5 & 9 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} \\ 5 \\ 9 \end{array} \qquad M = \begin{array}{c} \begin{array}{cc} 5 & 9 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}\begin{array}{c} \\ 5 \\ 9 \end{array}$$

NPTEL

18

For the third element that is this, the length is B – N, that is B – A, and K and M matrix remains structurally the same, but now the node numbering scheme is 3, 4, 5, 6, so this is 3, 4, 5, 6, 3, 4, 5, 6, for the next element that is S = 4 the length will be L – B, that is L - B and the node numbering scheme is 5, 6, 7, 8 so I have 5, 6, 7, 8,

$$K = \begin{array}{c} \begin{matrix} \quad 2 \quad & \quad 3 \quad & \quad 4 \quad & \quad 5 \quad & \quad 6 \quad & \quad 8 \quad \end{matrix} \\ \begin{bmatrix} \frac{4EI}{a} & -\frac{6EI}{a^2} & \frac{2EI}{a} & 0 & 0 & 0 \\ -\frac{6EI}{a^2} & \frac{12EI}{a^3}+\frac{12EI}{(b-a)^3} & -\frac{6EI}{a^2}+\frac{6EI}{(b-a)^2} & -\frac{12EI}{(b-a)^3} & \frac{6EI}{(b-a)^2} & 0 \\ \frac{2EI}{a} & -\frac{6EI}{a^2}+\frac{6EI}{(b-a)^2} & \frac{4EI}{a}+\frac{4EI}{b-a} & -\frac{6EI}{(b-a)^2} & \frac{2EI}{b-a} & 0 \\ 0 & -\frac{12EI}{(b-a)^3} & -\frac{6EI}{(b-a)^2} & K+\frac{12EI}{(b-a)^3}+\frac{12EI}{(l-b)^3} & -\frac{6EI}{(b-a)^2}+\frac{6EI}{(l-b)^2} & \frac{6EI}{(l-b)^2} \\ 0 & \frac{6EI}{(b-a)^2} & \frac{2EI}{b-a} & -\frac{6EI}{(b-a)^2}+\frac{6EI}{(l-b)^2} & \frac{4EI}{b-a}+\frac{4EI}{l-b} & \frac{2EI}{l-b} \\ 0 & 0 & 0 & \frac{6EI}{(l-b)^2} & \frac{2EI}{l-b} & \frac{4EI}{l-b} \end{bmatrix} \end{array}$$

NPTEL

19

so this is the stiffness matrix, this is a mass matrix. The last element which is a spring element the stiffness matrices K 1 – 1 - 1 and 1, and the node numbering is 5 and 9, so that is what the spring is massless, so mass matrix is identically equal to 0. So now we are ready to assemble so what are the unknown degrees of freedom, I have 2, 3, 4, 5, 6, and 8, okay, so now if I do that I have 2, 3, 4, 5, 6, and 8 are the unknown degrees of freedom, so I will have to pick now elements say 2 2, 3 2, 4 2, 5 2, 6 2, 8 2, etcetera by inspecting these matrices, so careful inspection of that leads to this assembled stiffness matrix you can check if this is right then similarly the mass matrix can be constructed by following the same procedure, so we now get the 6 by 6 mass and stiffness matrix, this system has how many degrees of freedom? 1, 2, 3, 4, 5 and 6, so I have the 6 by 6 mass and

$$
M = \frac{m}{420}\begin{bmatrix}
4a^3 & 13a^2 & -3a^3 & 0 & 0 & 0 \\
13a^2 & 420\dfrac{M}{m}+156b & -22\left[a^2-(b-a)^2\right] & 54(b-a) & 13(b-a)^2 & 0 \\
-3a^3 & -22\left[a^2-(b-a)^2\right] & 4\left[a^3+(b-a)^3\right] & 13(b-a)^2 & -3(b-a)^3 & 0 \\
0 & 54(b-a) & 13(b-a)^2 & 156(l-a) & -22\left[(b-a)^2-(l-b)^2\right] & -13(l-b)^2 \\
0 & 13(b-a)^2 & -3(b-a)^3 & -22\left[(b-a)^2-(l-b)^2\right] & 4\left[(b-a)^3+(l-b)^3\right] & -3(l-b)^3 \\
0 & 0 & 0 & -13(l-b)^2 & -3(l-b)^3 & 4(l-b)^3
\end{bmatrix}
$$

stiffness matrix and if I put all the numerical values I have got these numbers, again this

$$
K = 10^8 \times \begin{bmatrix}
0.5600 & -0.8400 & 0.2800 & 0 & 0 & 0 \\
-0.8400 & 2.5402 & -0.3024 & -0.8602 & 0.5376 & 0 \\
0.2800 & -0.3024 & 1.0080 & -0.5376 & 0.2240 & 0 \\
0 & -0.8602 & -0.5376 & 4.9357 & 0.9557 & 1.4933 \\
0 & 0.5376 & 0.2240 & 0.9557 & 1.1947 & 0.3733 \\
0 & 0 & 0 & 1.4933 & 0.3733 & 0.7467
\end{bmatrix}
$$

$$
M = \begin{bmatrix}
1.4857 & 4.8286 & -1.1143 & 0 & 0 & 0 \\
4.8286 & 364.3714 & 4.5964 & 25.0714 & 7.5446 & 0 \\
-1.1143 & 4.5964 & 4.3875 & 7.5446 & -2.1763 & 0 \\
0 & 25.0714 & 7.5446 & 115.8857 & -8.1714 & -2.7161 \\
0 & 7.5446 & -2.1763 & -8.1714 & 3.5286 & -0.4701 \\
0 & 0 & 0 & -2.7161 & -0.4701 & 0.6268
\end{bmatrix}
$$

21

remains to be verified by you, if this is right, then we can launch the eigenvalue analysis and if I

Natural frequencies (rad/s)
272.474
1122.210
3382.572
5331.485
11269.945
22615.047

Recall

Rayleigh Ritz approximation

Estimates of the natural frequencies in rad/s

$N=1$	270.309				
$N=2$	266.024	1127.975			
$N=3$	265.983	1127.917	2960.057		
$N=4$	265.867	1124.862	2960.056	4624.599	
$N=5$	265.846	1123.404	2960.0545	4549.745	7535.873

22

do that I get the estimates of first 6 natural frequencies as per this model to be these numbers.

Now if you recall as I already said this problem was a tackle earlier using relatives approximation in which you use the global trial function made up of sinusoidal sin N Phi X (l) type of functions and if we used only one term we got 270.309 as the first natural frequency and as we started increasing the number of elements this number seem to converge to about 265, similarly the second mode was around 1127 or it came to 1123 and so on and so forth, so if we now compare these numbers with these numbers we see that the first 2 natural frequencies are reasonable, but the remaining ones are of the mark, so this is one of the common features that will start seeing in finite element modeling that if you have a say N degree of freedom approximation to the system about 10% of the natural frequency that you compute could be expected to give reasonable answer, this is a thumb rule, it is not a theorem, roughly about 10% of the modes can be taken to be reasonably accurate.

Modal vector and orthogonality relations

$$\Phi = \begin{bmatrix} 0.0562 & 0.0805 & -0.2638 & 0.4609 & 0.7593 & 0.6137 \\ 0.0451 & 0.0251 & 0.0091 & -0.0020 & -0.0189 & -0.0394 \\ 0.0237 & -0.0712 & 0.2837 & -0.1137 & 0.5165 & 0.5543 \\ 0.0325 & -0.0730 & -0.0436 & -0.0041 & -0.0417 & 0.0770 \\ -0.0361 & 0.0281 & -0.1893 & -0.3755 & 0.1780 & 0.9059 \\ -0.0471 & 0.1366 & 0.2363 & 0.3512 & -0.6766 & 1.5053 \end{bmatrix}$$

$$\Phi' M \Phi = I$$

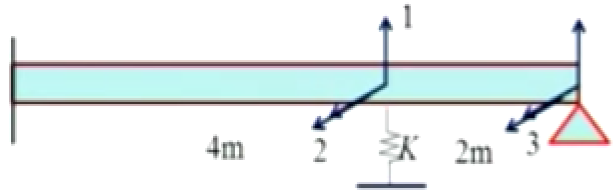
$$\Phi' K \Phi = 10^8 \times \begin{bmatrix} 0.0007 & 0.0000 & 0.0000 & -0.0000 & -0.0000 & -0.0000 \\ 0.0000 & 0.0126 & -0.0000 & 0.0000 & -0.0000 & 0.0000 \\ 0.0000 & -0.0000 & 0.1144 & 0.0000 & -0.0000 & -0.0000 \\ -0.0000 & 0.0000 & 0.0000 & 0.2842 & -0.0000 & -0.0000 \\ -0.0000 & -0.0000 & -0.0000 & -0.0000 & 1.2701 & -0.0000 \\ -0.0000 & 0.0000 & -0.0000 & -0.0000 & -0.0000 & 5.1144 \end{bmatrix}$$

NPTEL

23

Now associated with these natural frequencies we also have the model vectors I have given this for sake of completion and we have verified the orthogonal T relation Phi transpose M Phi I, Phi transpose K Phi is a diagonal matrix made up of the system squares of the system natural frequencies, so this completes the solution of this problem, basically we were aiming to find the estimates of natural frequencies.

Exercise



$EI = 10^8\ \text{Nm}^2$

$m = 100\ \text{kg/m}$

$\dfrac{8K}{EI} = 5$

- Formulate the structure mass and stiffness matrices
- Find natural frequencies and modal matrix.

Partial answers

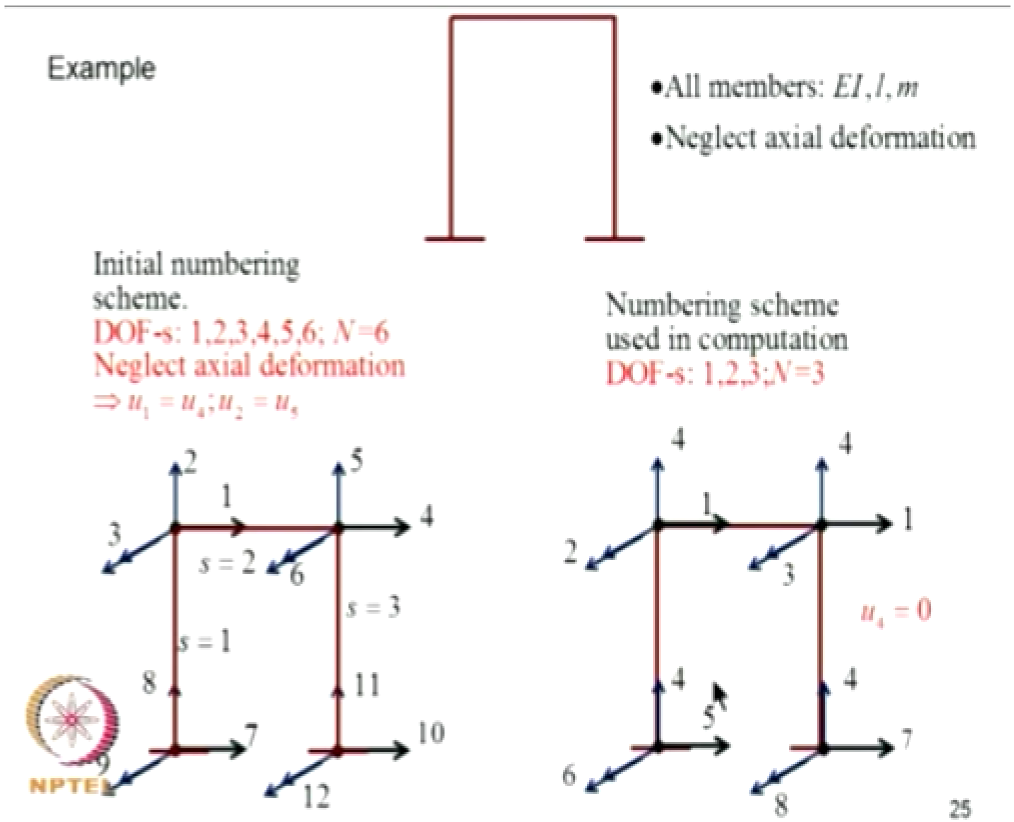
$$M = \frac{100}{420}\begin{bmatrix} 936 & -264 & -52 \\ -264 & 288 & -24 \\ -52 & -24 & 32 \end{bmatrix}$$

$$\omega = \{488.27 \quad 2087.86 \quad 7704.54\}\ \text{rad/s}$$

$$\Phi = \begin{bmatrix} 0.0562 & 0.0338 & 0.0625 \\ -0.0286 & 0.1105 & 0.1134 \\ -0.0288 & -0.1241 & 0.4168 \end{bmatrix}$$

24

This is an exercise problem a slight modification of the previous problem, we have now a propped cantilever beam carrying a supported additionally through a spring here and we can approximate this as a 3 degree freedom model, the degrees of freedom being 1 2, and 1 2 and 3 so corresponding to these numbers I have given some partial answers, the mass matrix is given and the 3 natural frequency that we obtain is given here and this is a modal matrix, so again you could verify if these are correct.



We will now consider a single bay one story portal frame, for purpose of illustration I will take that all members have same properties, flexural rigidity EI, length L, and mass per unit length M, we will assume that the axial rigidity of these members is very high therefore we can neglect axial deformation, so we need to know model this frame keeping in view this you know simplifying assumption. So now to start with we can take this as a structure made up of 3 element, S = 1, 2 and 3 and we can number the nodes expecting that the degrees of freedom at these 2 points are the unknown degrees of freedom for the structure, so I will name them as 1, 2, 3, 4, 5, 6, and then the reactions 7, 8, 9, 10, 11, 12, so since this end is clamped it automatically means that the displacement U7, U8, U9, U10, U11, U12 are 0, and the degrees of freedom are 1, 2, 3, 4, 5, 6, but now we are also, we are also given the additional information that the members are actually rigid so we can ignore axial deformation, so that would mean we have additional information now that U1 which is translation at this point is same as U4, because there is no axial deformation.

And similarly U2 which is same as U8 which is 0, and U5 which is same as U11 they are all 0s, okay, so now keeping in view these details we can renumber the nodes now, so I have read numbered here what are the degrees of freedom the horizontal translation here 1 which is the both these nodes move by the same amount because this member is actually rigid so 1 and the rotation here which is 2, the rotation here which is 3, so all other this translation and this translation are the same so I have given the same number 4, this is also same number 4 and 5, 6, 7, 8, the degrees of freedom where the displacements are specified to be 0. So now with this node numbering scheme we could proceed and find the reduced structural stiffness and mass matrices and hence an estimate of the 3 natural frequencies,

$s = 1$

$$K = \frac{EI}{l^3}\begin{array}{cc} \begin{matrix} \;5 & \;\;6 & \;\;1 & \;\;2 \end{matrix} & \\ \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} & \begin{matrix} 5 \\ 6 \\ 1 \\ 2 \end{matrix} \end{array} \qquad M = \frac{ml}{420}\begin{array}{cc} \begin{matrix} \;5 & \;\;6 & \;\;1 & \;\;2 \end{matrix} & \\ \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} & \begin{matrix} 5 \\ 6 \\ 1 \\ 2 \end{matrix} \end{array}$$

$s = 2$

$$K = \frac{EI}{l^3}\begin{array}{cc} \begin{matrix} \;4 & \;\;2 & \;\;4 & \;\;3 \end{matrix} & \\ \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} & \begin{matrix} 4 \\ 2 \\ 4 \\ 3 \end{matrix} \end{array} \qquad M = \frac{ml}{420}\begin{array}{cc} \begin{matrix} \;4 & \;\;2 & \;\;4 & \;\;3 \end{matrix} & \\ \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} & \begin{matrix} 4 \\ 2 \\ 4 \\ 3 \end{matrix} \end{array}$$

$$= \frac{EI}{l^3}\begin{array}{cc} \begin{matrix} \;1 & \;\;3 & \;\;7 & \;\;8 \end{matrix} & \\ \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} & \begin{matrix} 1 \\ 3 \\ 7 \\ 8 \end{matrix} \end{array} \qquad M = \frac{ml}{420}\begin{array}{cc} \begin{matrix} \;1 & \;\;3 & \;\;7 & \;\;8 \end{matrix} & \\ \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} & \begin{matrix} 1 \\ 3 \\ 7 \\ 8 \end{matrix} \end{array}$$

so again for the first element I have the stiffness matrix and mass matrix this data repeats what changes between S = 1, 2 & 3 are the node numbering schemes, for the first element it is 5, 6, 1, 2, second element it is 4, 2, 4, 3 and so on and so forth.

Structural matrices after imposing BCS

$$K = \frac{EI}{l^3}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 24 & 6l & 6l \\ 6l & 8l^2 & 2l^2 \\ 6l & 2l^2 & 8l^2 \end{bmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \qquad M = \frac{ml}{420}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 312 & 22l & 22l \\ 22l & 8l^2 & -3l^2 \\ 22l & -3l^2 & 8l^2 \end{bmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}$$

Governing equation of motion

$$\frac{ml}{420}\begin{bmatrix} 312 & 22l & 22l \\ 22l & 8l^2 & -3l^2 \\ 22l & -3l^2 & 8l^2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 24 & 6l & 6l \\ 6l & 8l^2 & 2l^2 \\ 6l & 2l^2 & 8l^2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = 0$$

Eigenvalue problem

$$\lambda = \frac{ml^4\omega^2}{420EI} \Rightarrow \begin{bmatrix} 24 & 6l & 6l \\ 6l & 8l^2 & 2l^2 \\ 6l & 2l^2 & 8l^2 \end{bmatrix}\{\phi\} = \lambda\begin{bmatrix} 312 & 22l & 22l \\ 22l & 8l^2 & -3l^2 \\ 22l & -3l^2 & 8l^2 \end{bmatrix}\{\phi\}$$

Eigenvalues and Natural frequencies

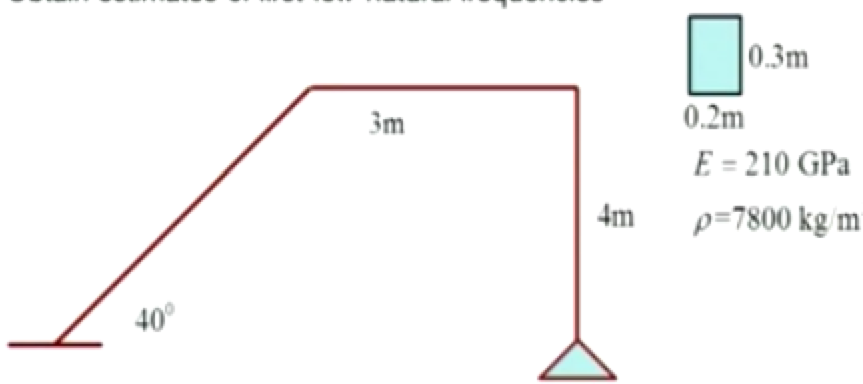
$$\Rightarrow \lambda = 0.0628, 0.5455, 4.5183$$

$$\Rightarrow \omega = 0.5706\alpha, 1.6818\alpha, 4.8402\alpha \text{ rad/s}; \ \alpha = \sqrt{\frac{EI}{m}}$$

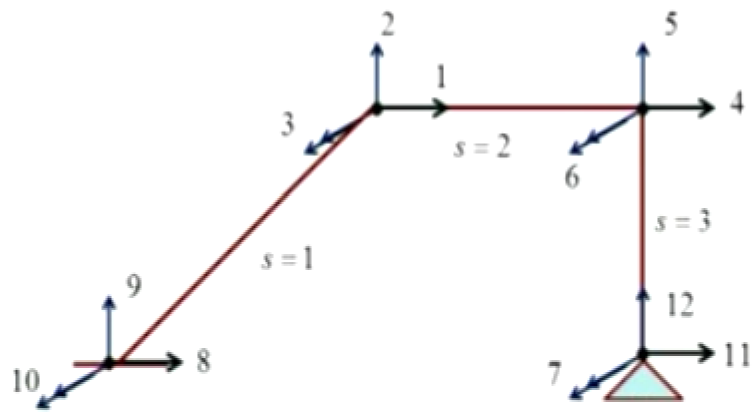
27

So now we need to assemble the reduced structural matrices after imposing the boundary conditions 1, 2, 3 are the degrees of freedom, so I have this as the structure stiffness matrix similarly this as the mass structure mass matrix, so equipped with this I can right now the governing equation of motion for the structure, this is a mass matrix, this is stiffness matrix acceleration vector, displacement vector, again by using the notation ML4 into omega square by 420EI as lambda I can derive the eigenvalue problem that we need to solve to estimate the natural frequencies and mode shapes, so if you do that I get the 3 natural frequencies to be this the eigenvalues in this case are this, and associated with this, I have the three natural frequencies expressed in terms of this factor alpha which is square root EI / M, so this completes the solution to the given problem.

Example
Formulate the governing equation of motion
Obtain estimates of first few natural frequencies



Now we can introduce some variations now, we can have now 3 member frame, one of the member is inclined like this so that the local coordinate system for this will be different from the global coordinate system, so we need to know tackle the additional issue of transforming the structural matrices to the global coordinate system. So what we do is again we model the



structure using 3 element, S = 1, 2, 3 and this is a numbering scheme that we adopted so to derive the transformation matrices we can number the nodes and we can proceed in this direction and determine the angles and the numbering as given here, as is appropriate for the local numbering scheme for the element, so I will not provide all the details, this is an exercise that you need to do, you will find it difficult to handle this with pen and paper, so it is advisable to write simple computer programs to understand how this matrix manipulations work, so I am giving some partial answers so that you could check whether my solution is similar to what you may be getting, so this is some information for the first matrix, first element, this is a transformation matrix, and this is the A1 matrix for the first element, so if you assemble the all the element level structural matrices and impose the boundary conditions we get the mass and

$$M = 10^3 \times \begin{bmatrix} 1.4846 & -0.0546 & 0.6102 & 0.2340 & 0 & 0 & 0 \\ -0.0546 & 1.5574 & -0.5066 & 0 & 0.1805 & -0.1304 & 0 \\ 0.6102 & -0.5066 & 1.1944 & 0 & 0.1304 & -0.0903 & 0 \\ 0.2340 & 0 & 0 & 1.1633 & -0.0000 & 0.3922 & -0.2318 \\ 0 & 0.1805 & 0.1304 & -0.0000 & 1.1455 & -0.2206 & 0.0000 \\ 0 & -0.1304 & -0.0903 & 0.3922 & -0.2206 & 0.4056 & -0.2139 \\ 0 & 0 & 0 & -0.2318 & 0.0000 & -0.2139 & 0.2853 \end{bmatrix}$$

$$K = 10^9 \times \begin{bmatrix} 5.3901 & 0.9947 & 0.0094 & -4.2000 & 0 & 0 & 0 \\ 0.9947 & 0.8814 & 0.0518 & 0 & -0.0420 & 0.0630 & 0 \\ 0.0094 & 0.0518 & 0.1867 & 0 & -0.0630 & 0.0630 & 0 \\ -4.2000 & 0 & 0 & 4.2177 & 0.0000 & 0.0354 & 0.0354 \\ 0 & -0.0420 & -0.0630 & 0.0000 & 3.1920 & -0.0630 & -0.0000 \\ 0 & 0.0630 & 0.0630 & 0.0354 & -0.0630 & 0.2205 & 0.0472 \\ 0 & 0 & 0 & 0.0354 & -0.0000 & 0.0472 & 0.0945 \end{bmatrix}$$

NPTEL

$\omega = (\ 87.6 \quad 332.1 \quad 585.9 \quad 941.9 \quad 1418.6 \quad 1932.1 \quad 3565.9)$ rad/s

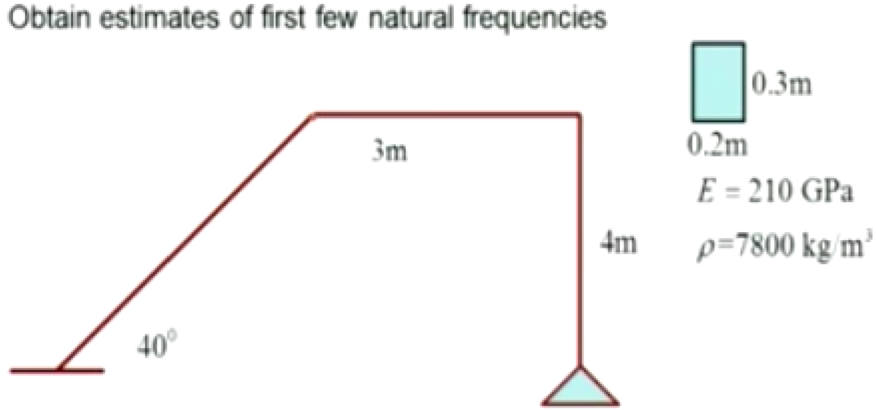
31

stiffness matrix to be this, let me see the degrees of freedom here will be, the translations here the two translation and rotation here, the two translation rotation here, and one rotation, so it is

Example

Formulate the governing equation of motion

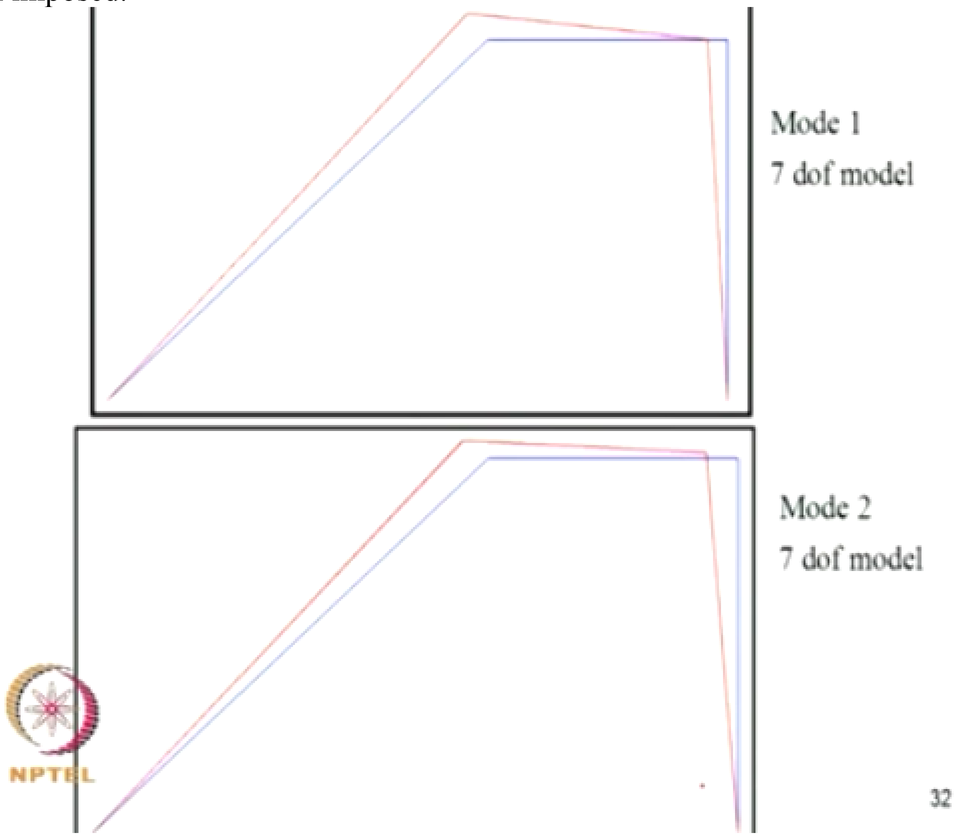
Obtain estimates of first few natural frequencies



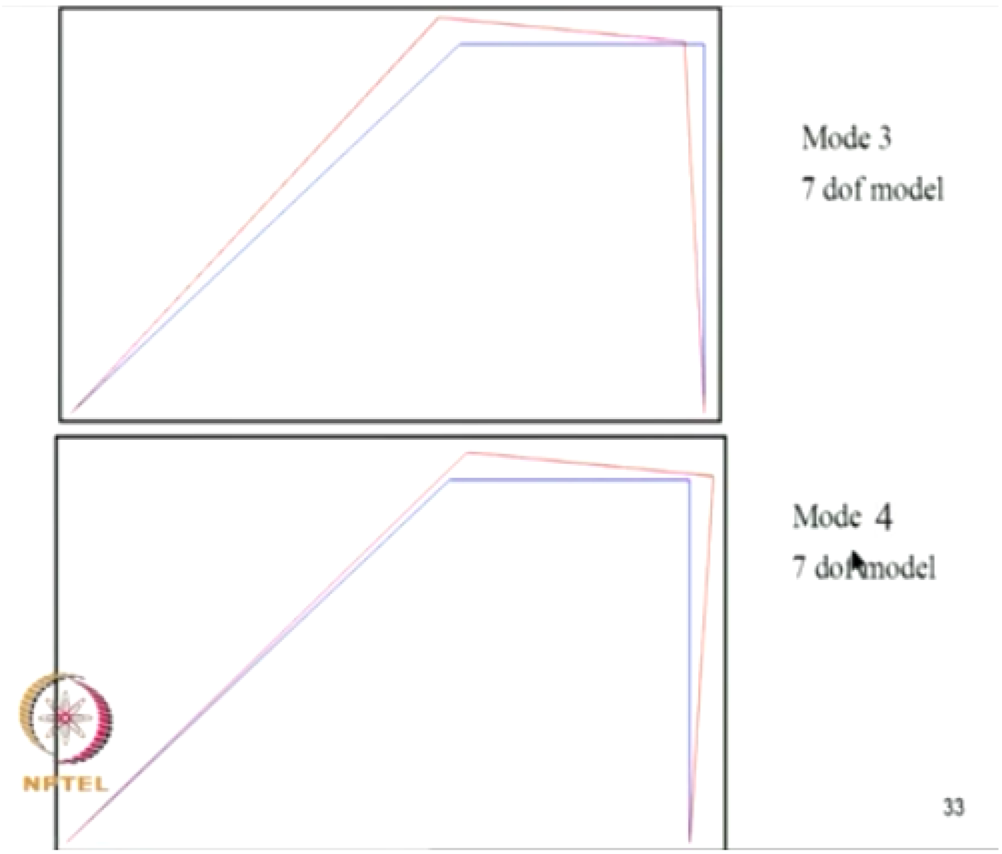
NPTEL

28

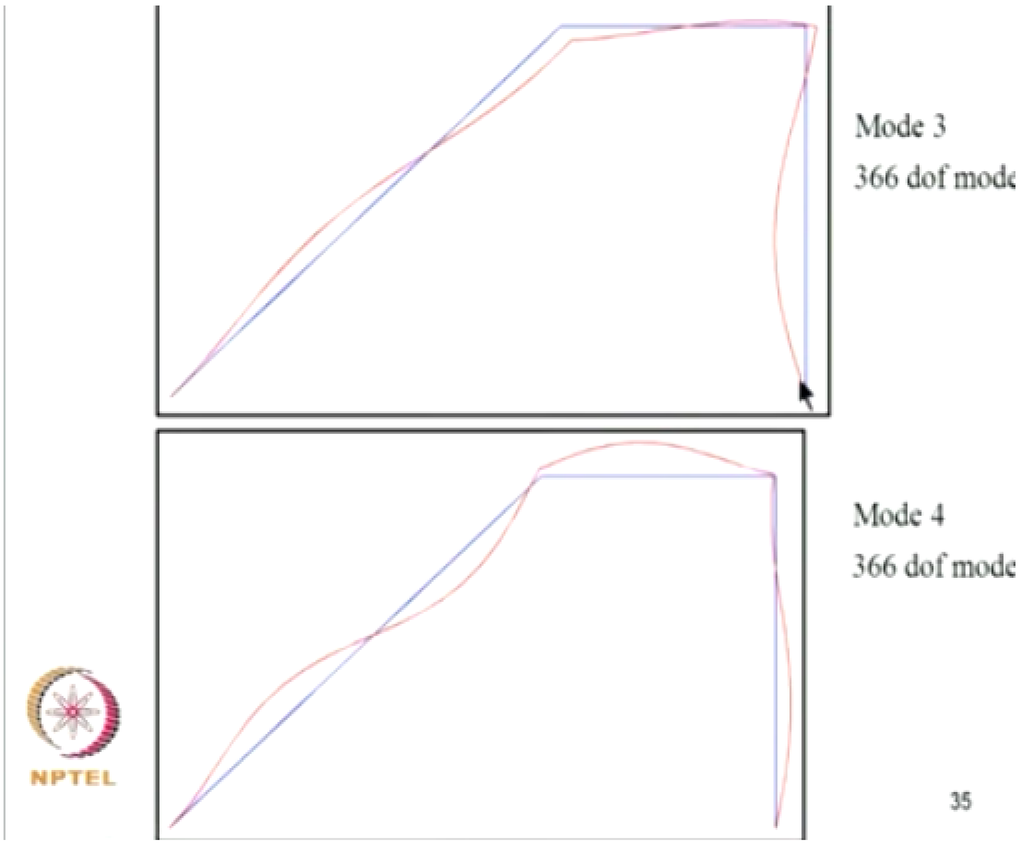
7 degree of freedom system, so I am getting the 7 by 7 mass matrix and stiffness matrix, so these are the reduced structure mass matrix and stiffness matrix after the boundary conditions have been imposed.



Now I can do the eigenvalue analysis and I get the estimates of first 7, the 7 natural frequency is available from this model as shown here, we could go ahead and construct the mode shapes as well, for example with the 7 degree of freedom model that we have the first mode shape looks like, the blue line here indicates the un-deformed configuration and the red one is a deformed configuration when the structure is vibrating in its first normal mode, this is the same result for the second normal mode, but you should understand that this is a very crude model so many of

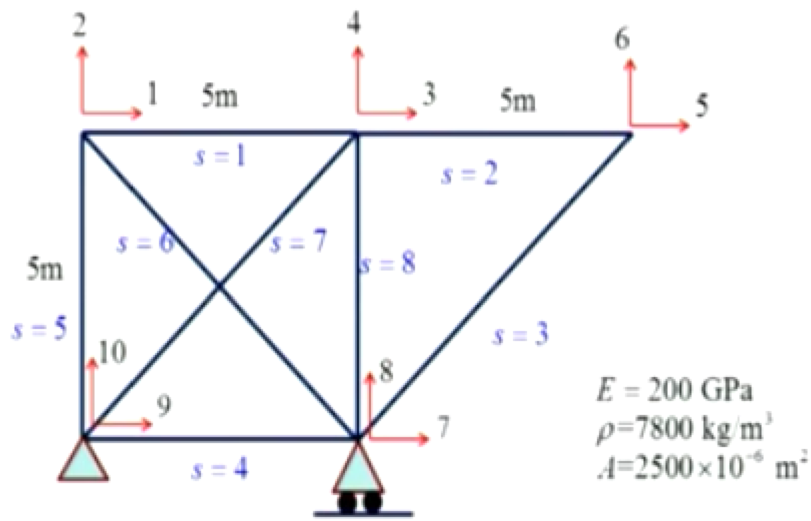


the details of these mode shapes did not be captured well here, so to illustrate that I developed a much larger, this is similar result for mode 3 and more 4 for sake of completeness I have shown here, what I have done is I've also model this structure using a more elaborate model, using say in the model that I used, I used 366 degrees of freedom, so that we can see whether the information on mode shape that we are getting is reasonable or not, it turns out that the first mode shape in a more refined model appears like this, the second one has much greater details, so this if you go back and see we are unable to capture those details in this crude model, so this is a second mode, this is a third mode, which has lot more details, this is the fourth mode so



these details will not be able to capture with the discretization scheme that we have used, so the point that is being made is if you are interested in reasonably acceptable characterization of mode shapes you should use a more refined mesh, with a crude mesh that we have used to illustrate here will not be able to get acceptable results on mode shapes.

Pin jointed truss



$u_8, u_9, u_{10} = 0$

DOF-s: $u_i, i = 1, 2, \cdots, 7$

NPTEL

Another example this is on a simple truss structure which is statically indeterminate, so this system has 8 elements as numbered here, so 1, 2, 3, 4, 5, 6, 7 and 8 and the 3 displacements at the supports U8, U9, U10 are given to be 0, so U8 is 0 and U9 and U10 are 0, so this is a truss structure so each node will have 2 degrees of freedom, so the degrees of freedom are 7 degrees of freedom 1, 2, 3, 4, 5, 6 and 7, which is the translation here. So these are some material properties and geometric properties that will be needing in the numerical work.

Now for the discretization scheme that I have shown here you can form now the mass and structure stiffness matrices for each of these elements, for each of these elements you could also form the transformation matrix the A matrix equipped with that you can assemble and get the global mass and stiffness matrix, then impose these boundary condition and partition the global mass and stiffness matrices you will get the reduced structure mass and stiffness matrices after imposing the boundary conditions and that would help you to find the natural frequencies and mode shapes, that is the problem.

$$K = \frac{AE}{l}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \qquad M = \frac{ml}{6}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{array}\right] \end{array}\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

NPTEL

So this is an element of a truss element, there are two nodes and these are the degrees of freedom, this is the stiffness matrix, this is the mass matrix and again I am giving some partial results I am not giving all the results, so these are stiffness matrices in the global coordinate

$$K_1 = K_2 = \begin{bmatrix} 100000000 & 0 & -100000000 & 0 \\ 0 & 0 & 0 & 0 \\ -100000000 & 0 & 100000000 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_3 = 10^7 \times \begin{bmatrix} 3.5355 & 3.5355 & -3.5355 & -3.5355 \\ 3.5355 & 3.5355 & -3.5355 & -3.5355 \\ -3.5355 & -3.5355 & 3.5355 & 3.5355 \\ -3.5355 & -3.5355 & 3.5355 & 3.5355 \end{bmatrix}$$

$$K_4 = 10^8 \times \begin{bmatrix} 1.0000 & 0.0000 & -1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -1.0000 & 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & -0.0000 & 0.0000 \end{bmatrix}$$

$$K_6 = 10^7 \times \begin{bmatrix} 3.5355 & -3.5355 & -3.5355 & 3.5355 \\ -3.5355 & 3.5355 & 3.5355 & -3.5355 \\ -3.5355 & 3.5355 & 3.5355 & -3.5355 \\ 3.5355 & -3.5355 & -3.5355 & 3.5355 \end{bmatrix}$$

$$K_7 = 10^7 \times \begin{bmatrix} 3.5355 & 3.5355 & -3.5355 & -3.5355 \\ 3.5355 & 3.5355 & -3.5355 & -3.5355 \\ -3.5355 & -3.5355 & 3.5355 & 3.5355 \\ -3.5355 & -3.5355 & 3.5355 & 3.5355 \end{bmatrix}$$

38

system and this needs to be assembled and if you assemble I get this reduced structure, stiffness

$$K = 10^8 \times \begin{bmatrix} 1.3536 & -0.7071 & -1.0000 & 0 & 0 & 0 & -0.3536 \\ -0.7071 & 2.4142 & 0 & 0 & 0 & 0 & 0.7071 \\ -1.0000 & 0 & 2.3536 & 0.3536 & -1.0000 & 0 & 0.0000 \\ 0 & 0 & 0.3536 & 1.3536 & 0 & 0 & 0.0000 \\ 0 & 0 & -1.0000 & 0 & 1.3536 & 0.3536 & -0.3536 \\ 0 & 0 & 0 & 0 & 0.3536 & 0.3536 & -0.3536 \\ -0.3536 & 0.7071 & 0.0000 & 0.0000 & -0.3536 & -0.3536 & 1.7071 \end{bmatrix}$$

$$M = \begin{bmatrix} 110.9619 & 0 & 16.2500 & 0 & 0 & 0 & 22.9810 \\ 0 & 248.8478 & 0 & 16.2500 & 0 & 0 & 0 \\ 16.2500 & 0 & 143.4619 & 0 & 16.2500 & 0 & 16.2500 \\ 0 & 16.2500 & 0 & 143.4619 & 0 & 16.2500 & 0 \\ 0 & 0 & 16.2500 & 0 & 78.4619 & 0 & 22.9810 \\ 0 & 0 & 0 & 16.2500 & 0 & 78.4619 & 0 \\ 22.9810 & 0 & 16.2500 & 0 & 22.9810 & 0 & 156.9239 \end{bmatrix}$$

$$\omega = (\,265.6 \quad 520.0 \quad 774.6 \quad 984.4 \quad 1261.0 \quad 1310.6 \quad 1978.3)\ \text{rad/s}$$

39

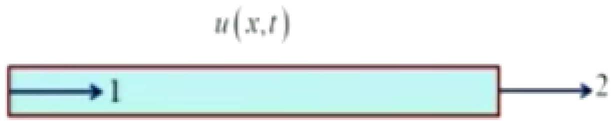
matrix and mass matrix, so if I perform an eigenvalue analysis on this I will get these numbers as the estimates of the first few natural frequency. So this is again an exercise for you to fill up the details.

Example

$$AEu'' = m\ddot{u} + c\dot{u};$$
$$u(0,t) = 0;\ AEu'(l,t) = P\exp(i\Omega t)$$
$$u(x,t) = \phi(x)\exp(i\Omega t)$$
$$\Rightarrow \phi'' + \frac{m\Omega^2 - i\Omega c}{AE}\phi = 0$$
$$\Rightarrow \phi'' + \lambda^2\phi = 0 \text{ with } \lambda^2 = \frac{m\Omega^2 - i\Omega c}{AE}$$
$$\phi(0) = 0;\ AE\phi'(l) = P$$
$$\Rightarrow \lim_{t\to\infty} u(x,t) = \frac{P\sin\lambda x}{AE\lambda\cos\lambda l}\exp(i\Omega t)\ \&\ AEu'(x,t) = \frac{P\cos\lambda x}{\cos\lambda l}\exp(i\Omega t)$$

Now we will consider few forced vibration problems now, a simple one you consider a rod actually vibrating rod which is subjected to a harmonic load at the tip as shown here so if you right now the exact partial differential equation governing this system we have this equation AE U double prime is equal to MU double dot + U dot, now the boundary conditions are at X = 0, U is 0 and at X = L there is an applied force which is AEU prime (l,t) is PE raise I omega T, now since the system is linear and it is being driven harmonically we know that such, the response of such systems would also be harmonic at the driving frequency at frequency Omega that means we are looking for solution in steady state, that means T tending to infinity this will be the form of the solution.

Now the unknown is Phi(x) so I substitute this into this partial differential equation and I get this ordinary differential equation for Phi, so this is a complex number and if I denote this by lambda square I get a second order differential equation Phi double prime + lambda square Phi = 0, now this is an eigenvalue problem and it is not an eigenvalue problem, it's a force vibration problem, so it has inhomogeneous boundary conditions, so phi(0) is 0 that is obtained here U(0,t) must be 0 therefore Phi(0) and E raise to omega T must be 0, so E raise to I omega T cannot be 0 for all T, therefore Phi(0) is 0, similarly I get AE phi prime of L is P, now using these boundary conditions I can evaluate the two integration constant that will result by solving this equation and we get this as the final solution, this is the displacement field and this is the force field. This is an exact solution, so this exact solution can serve as a benchmark to validate any approximate solution that you will develop using FEM, so let's for simplicity we will model this as, this structure as a single element with 2 degrees of freedom and U1 is given to be 0, and U2(t) is the only degree of freedom, so in this approximation we get a single degree of



$$\frac{ml}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}0\\ \ddot{u}_2\end{Bmatrix}+\frac{AE}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ u_2\end{Bmatrix}=\begin{Bmatrix}R(t)\\ P\exp(i\Omega t)\end{Bmatrix}$$

$$\Rightarrow \frac{ml}{3}\ddot{u}_2+c\dot{u}_2+\frac{AE}{l}u_2=P\exp(i\Omega t)$$

$$\Rightarrow \ddot{u}_2+2\eta\omega\dot{u}_2+\omega^2 u_2=\frac{3P}{ml}\exp(i\Omega t)\ \text{with}\ \omega^2=\frac{3AE}{ml^2}$$

$$\lim_{t\to\infty}u_2(t)=\frac{\frac{3P}{ml}}{(\omega^2-\Omega^2)+i2\eta\omega\Omega}\exp(i\Omega t)$$

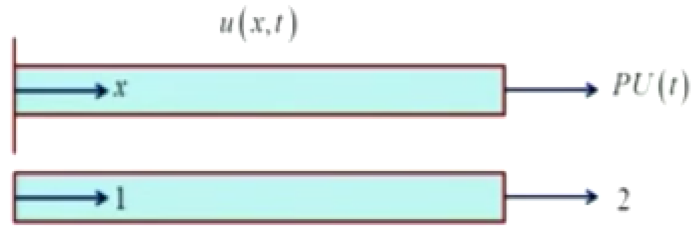
$$\lim_{t\to\infty}u(x,t)=\left(1-\frac{x}{l}\right)\frac{\frac{3P}{ml}}{(\omega^2-\Omega^2)+i2\eta\omega\Omega}\exp(i\Omega t)$$

41

freedom model damping has been introduced in ad hoc manner at the level of generalized degree of freedom, and this is the governing equation.

So now writing this in the standard form I get U2 double dot + 2 ETA omega U2 dot + omega square U2 is this forcing function, so where omega square is 3AE / ML square, so square root of this is the estimate of the natural frequency as per this model. Now as T tends to infinity we can construct this solution and we can get this as the solution, so as T tends to infinity U is given by according to this model is given by this.

Now if you want to examine the accuracy of this solution you need to compare this expression with this expression, I'll leave that as an exercise you can plot this function and superpose on this function for different values of X you will be able to see how good is this solution, so here I want to point out that the integration in time has been done exactly there is no discretization in time, therefore there is no further approximation in obtaining the solution beyond what has been done in spatial discretization.



$$\frac{ml}{3}\ddot{u}_2 + c\dot{u}_2 + \frac{AE}{l}u_2 = PU(t)$$

$$\Rightarrow \ddot{u}_2 + 2\eta\omega\dot{u}_2 + \omega^2 u_2 = \frac{3P}{ml}U(t) \text{ with } \omega^2 = \frac{3AE}{ml^2}$$

$$u_2(t) = \exp(-\eta\omega t)\left[A\cos\omega_d t + B\sin\omega_d t\right] + \frac{3P}{ml\omega^2}$$

$$u_2(0) = 0, \dot{u}_2(0) = 0 \Rightarrow A = -\frac{3P}{ml\omega^2}, B = \frac{\eta\omega}{\omega_d}A$$

$$\lim_{t\to\infty} u_2(t) \to \frac{3P}{ml\omega^2} = \frac{3P}{ml}\frac{ml^2}{3AE} = \frac{Pl}{AE} \Rightarrow \lim_{t\to\infty} u(x,t) = \left(1-\frac{x}{l}\right)\frac{Pl}{AE}$$

42

Now a similar problem where I apply instead of a harmonic load a step function, I suddenly applied load PU(t) and this will be the equation of motion using the same discretization scheme instead of PE raise to I omega T, I will be having PU(t), now again I can solve this exactly and I will get U2(t) in this form, so as limit T tends to infinity we can see that the steady-state solution becomes 3P / ML omega square and if we now substitute for omega square I get PL / AE, so you can see that in the steady state at T = 0, we are suddenly applying this load as T tends to infinity the problem becomes a problem of a statically loaded system, and the tip deflection is PL/AE, so in this approximation I am getting the exact solution, so why we are getting exact solution? This is because the trial function that we are using it so happens that it matches with the exact solution, that’s why we are getting exactly solution.

Example

Analyse the following situations

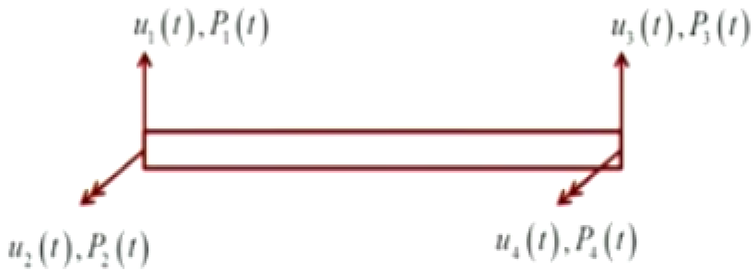
- **An udl is suddenly applied**
- **An udl is suddenly removed**
- **Supports are subjected to differential harmonic support displacements**
- **A concentrated load traverses the beam with a constant velocity**
- **Right support is subjected to a harmonic moment**
- **An unit impulsive force is applied at *x=a*.**
- **An unit harmonic load is applied at *x=a***

NPTEL

43

Now what I will do now is I will consider a simply supported beam as we have seen where in the FE model we are approximating this as a 2 degree of freedom system, so what we will do is we will consider a series of examples as relevant to this, first I will consider that this beam is carrying a UDL and that UDL is suddenly applied at T = 0, okay, so what is the dynamics of the beam under this suddenly applied load. The second case is somewhat similar but the situation is reversed now, we start with an assumption that the beam is carrying a UDL and that UDL is suddenly removed, so how does the beam vibrate? Next supports are subjected to differential harmonic displacement, so this supports vibrate harmonically something like an earthquake induced vibration for sake of discussion we will assume that the support displacement here and here are different that is the kind of situation that you expect in say for example piping in a nuclear reactor or a long span bridge where there will be special variability in the support motions.

The next one is let us assume that a concentrated load traverses the beam with a constant velocity, so this is a problem typically encountered in bridge engineering so in static analysis we do influence lines and we don't account for dynamical behavior of the beam, here we can see if we include the inertial effects of the beam how does the response, how can we can analyze the response. The next is right support is subjected to a harmonic moment, so there is inhomogeneity in the boundary conditions unlike in support displacement case here we are applying a force, then a couple of other situations like an unit impulsive force is applied at X = A, somewhere here I take a hammer and hit the beam, a impulsive load at a single point on the beam. Finally and unit harmonic load is applied at X = A, for all these cases what we will do now is we will give the exact formulation of the problem using partial differential equation model and then use finite element model with a single element that leads to a 2 degree of freedom approximation and see what we get so the discretization that I will be using for all



Structural matrices

$$M = \frac{ml}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}; K = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

NPTEL

44

these cases will be this, the two nodes 4 degrees of freedom, this is the mass matrix, this is a stiffness matrix, that is these are the structural matrices.

Case-1 An udl is suddenly applied

$$EIv^{iv} + m\ddot{v} = qU(t)$$

$$v(0,t) = 0; v(l,t) = 0$$

$$EIv''(0,t) = 0; EIv''(l,t) = 0;$$

$$v(x,0) = 0; \dot{v}(x,0) = 0 \text{ [Assumption]}$$

NPTEL

45

Now we will start with the first case and UDL is suddenly applied, so the governing equation assuming Euler-Bernoulli beam theory is EI V4 + MU double dot is Q U(t), where U(t) is the step function, so at T = 0, this beam carries a load Q is applied at T = 0, now the boundary conditions are at X = 0, displacement is 0, X = L displacement is 0 and bending moments are 0 at X = 0 and here, we assume that the system starts from rest before the application of this load the beam is at rest, so if we solve this equation we get the exact solution.

Case-1 An udl is suddenly applied

$$\frac{ml}{420}\begin{bmatrix}156 & 22l & 54 & -13l\\ 22l & 4l^2 & 13l & -3l^2\\ 54 & 13l & 156 & -22l\\ -13l & -3l^2 & -22l & 4l^2\end{bmatrix}\begin{Bmatrix}0\\ \ddot{u}_2\\ 0\\ \ddot{u}_4\end{Bmatrix}+\frac{EI}{l^3}\begin{bmatrix}12 & 6l & -12 & 6l\\ 6l & 4l^2 & -6l & 2l^2\\ -12 & -6l & 12 & -6l\\ 6l & 2l^2 & -6l & 4l^2\end{bmatrix}\begin{Bmatrix}0\\ u_2\\ 0\\ u_4\end{Bmatrix}=\{F(t)\}$$

$$\{F(t)\}=\begin{Bmatrix}R_1\\ 0\\ R_2\\ 0\end{Bmatrix}+\begin{Bmatrix}\int_0^L q\left(1-3\frac{x^2}{l^2}+2\frac{x^3}{l^3}\right)dx\\ \int_0^L q\left(x-2\frac{x^2}{l}+\frac{x^3}{l^2}\right)dx\\ \int_0^L q\left(3\frac{x^2}{l^2}-2\frac{x^3}{l^3}\right)dx\\ \int_0^L q\left(-\frac{x^2}{l}+\frac{x^3}{l^2}\right)dx\end{Bmatrix}U(t)$$

$$\frac{ml}{420}\begin{bmatrix}4l^2 & -3l^2\\ -3l^2 & 4l^2\end{bmatrix}\begin{Bmatrix}\ddot{u}_2\\ \ddot{u}_4\end{Bmatrix}+\frac{EI}{l^3}\begin{bmatrix}4l^2 & 2l^2\\ 2l^2 & 4l^2\end{bmatrix}\begin{Bmatrix}u_2\\ u_4\end{Bmatrix}=\begin{Bmatrix}F_2(t)\\ F_4(t)\end{Bmatrix}$$

NPTEL

46

Now in the FE model how do we tackle this problem, so we have this equation which is the structure equation of motion before imposing the boundary conditions. Now U1 and U3 are 0 because they are support displacements and U2 and U4 are the slopes which are not safe, so these are the degrees of freedom, these are 0, and F(t) has reactions at R1 and R2 which are the shear forces at the supports plus the contribution from the loads that are acting on the structure so these are the equivalent nodal forces that we have to tackle so for the unknown displacements U2 and U4 if you pick these equations, the second row and the fourth row I get this equation. And where F2 and F 4 are now 0 plus this, this is F2 and F4 is this, so these are polynomials you can quickly integrate and use it here, so this is multiplied by a step function, so we have to solve a 2 degree of freedom system subjected to step inputs, so that is something that we know how to deal with.

Case-2 An udl is suddenly removed

$$EIv^{iv} + m\ddot{v} = 0$$

$$v(0,t) = 0; v(l,t) = 0$$

$$EIv''(0,t) = 0; EIv''(l,t) = 0;$$

$$v(x,0) = v_0(x); \dot{v}(x,0) = 0[\text{Assumption}]$$

$$EIv_0^{iv} = q$$

$$v_0(0) = 0; v_0(l) = 0$$

$$EIv_0''(0) = 0; EIv_0''(l) = 0$$

NPTEL

47

Now how about the situation where the UDL is suddenly removed, the beams carries a UDL at T = 0 it is removed, so the boundary conditions remain the same, we have displacement at but the 2 boundaries are 0, translations are 0, and bending moments are 0, now the initial condition is the way this information can be incorporated that means at T = 0, the beam deflects under this UDL, so T = 0 the beam would be in this deflected profile and if this load is removed then the system will start vibrating from this initial condition, this deflected profile is V naught(x) and we assume that initially the velocity is here, this is the assumption, this is the actual data coming from the system, what is V naught(x), V naught(x) is the response of the system to the static load, so this satisfies this equilibrium, so you can solve this under these boundary conditions to get V naught.

Case-2 An udl is suddenly removed

$$v(x,0) = -\frac{qLx^3}{12EI} + \frac{qL^3x}{24EI} + \frac{qx^4}{24EI} \approx \sum_{i=1}^{4} u_i(0)\phi_i(x)$$

$$\Rightarrow v(x,0) = u_2(0)\phi_2(x) + u_4(0)\phi_4(x)$$

$$\Rightarrow v'(x,0) = u_2(0)\phi_2'(x) + u_4(0)\phi_4'(x)$$

Recall $\phi_2'(0) = 1, \phi_2'(l) = 0, \phi_4'(0) = 0, \phi_4'(l) = 1$

$$\left.\begin{aligned} &\Rightarrow v'(0,0) = u_2(0)\phi_2'(0) + u_4(0)\phi_4'(0) = u_2(0) \\ &\& v'(l,0) = u_2(0)\phi_2'(l) + u_4(0)\phi_4'(l) = u_4(0) \end{aligned}\right\} (*)$$

$$\Rightarrow v'(x,0) = u_2(0)\phi_2'(x) + u_4(0)\phi_4'(x)$$

NPTEL

48

Now you can show that V naught can be obtained as this, now the problem is how do you get the initial data on initial conditions for U2 and U4 which are the rotations at the supports, so at T = 0, this must be approximated through this equation. Now if you write it out since U1 and U3 are 0 I get V(x,0) U2(0) 52(x) + U4(0) 54(x) similarly V prime(x,0) is given by this. Now if you recall Phi 2 prime (0) is 1, Phi 2 prime(l) is 0, Phi 4 prime(0) is 0, Phi 2 prime (l) is 1, using these conditions we will be able to assign values to U2 and U4, so we get the, if we take now U4(0) to be V prime(l,0) and U2(0) to be V prime (0,0) will be able to satisfy this equation.

$$\frac{ml}{420}\begin{bmatrix}156 & 22l & 54 & -13l\\ 22l & 4l^2 & 13l & -3l^2\\ 54 & 13l & 156 & -22l\\ -13l & -3l^2 & -22l & 4l^2\end{bmatrix}\begin{Bmatrix}0\\ \ddot{u}_2\\ 0\\ \ddot{u}_4\end{Bmatrix}+\frac{EI}{l^3}\begin{bmatrix}12 & 6l & -12 & 6l\\ 6l & 4l^2 & -6l & 2l^2\\ -12 & -6l & 12 & -6l\\ 6l & 2l^2 & -6l & 4l^2\end{bmatrix}\begin{Bmatrix}0\\ u_2\\ 0\\ u_4\end{Bmatrix}=\begin{Bmatrix}R_1\\ 0\\ R_2\\ 0\end{Bmatrix}$$

$$\Rightarrow \frac{ml}{420}\begin{bmatrix}4l^2 & -3l^2\\ -3l^2 & 4l^2\end{bmatrix}\begin{Bmatrix}\ddot{u}_2\\ \ddot{u}_4\end{Bmatrix}+\frac{EI}{l^3}\begin{bmatrix}4l^2 & 2l^2\\ 2l^2 & 4l^2\end{bmatrix}\begin{Bmatrix}u_2\\ u_4\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}$$

$$u_2(0)=v'(0,0);\dot{u}_2(0)=0$$

$$u_4(0)=v'(l,0);\dot{u}_4(0)=0$$

$v'(x,0)$ needs to be specified

49

So using that now I am ready to launch the solution, so this is the equation before imposing boundary conditions this is the equation corresponding to the second and the fourth row I get this now this pair of equations need to be solved under these initial conditions, so the data on initial deformation is contained in these 2 numbers, so what is important to note here is to obtain solution through this method at T = 0 we need data on V prime(x,0), this is typically not needed when you write the equation into as partial differential equation, so to implement this method given the nature of the trial function that we are using this information would be needed okay.

Case-3 Supports are subjected to differential harmonic support displacements

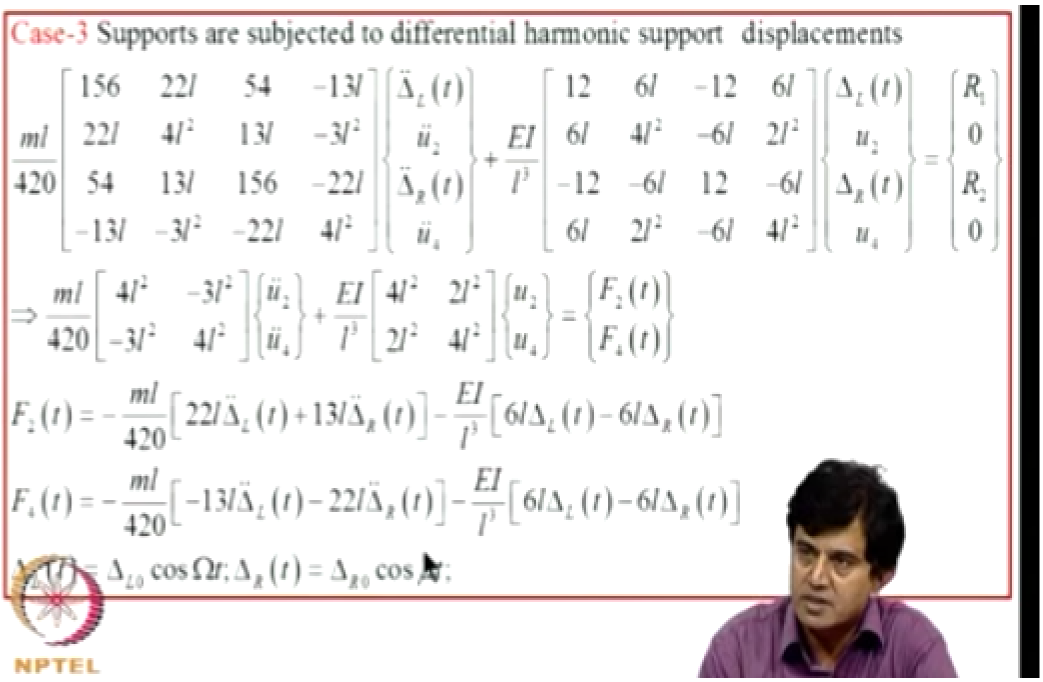
$$EIv^{iv}+m\ddot{v}=0$$

$$v(0,t)=\Delta_L(t);v(l,t)=\Delta_R(t)$$

$$EIv''(0,t)=0;EIv''(l,t)=0;$$

$$v(x,0)=0;\dot{v}(x,0)=0\,[\text{Assumption}]$$

Now the other cases, case 3 supports are subjected to differential harmonic support displacement, so the governing equation will be this, at X = 0, I have Delta L(t), Delta L(t) is the support displacement at the left end, V(l,t) is Delta R(t) the bending moments are 0 and we assume that at T = 0 system starts from rest, so this is the form problem formulation at the partial differential equation level using Euler Bernoulli beam theory.



Now if we now come to the FE discretization, now I have U1 and U3 to be specified, U1 and U3 are specified to be Delta L(t) and Delta R(t), earlier they were specified to be 0 now they are specified to be prescribed time histories, so associated with that I will have Delta L double dot Delta R double dot in the acceleration vector. So now there will be reactions at the two supports the shear forces and there are no external forces therefore these are 0.

Now if we now pick the equation for the unknown displacement that is the second row and the fourth row I get this equation, okay. Now what are these F2 and F4, F2 and F4 are the terms that are associated with this Delta L double dot, Delta R double dot, Delta L and Delta R, how do they come about, suppose if I look at the second row, when I write the equation it will be 22L into Delta L double dot + 4L square U2 double dot + 13L delta R double dot and – 3L square U4 double dot, these are the terms that I will get in the first row associated with the acceleration. So those terms are present here and the terms involving Delta L double dot and Delta R double dot are taken to the right side, so they are appearing here.

Similarly here when I write the second row equation I will have 6L Delta L (T) + 4L square U2 - 6L delta R + 2L square U4, so the terms involving delta L and delta R are taken to the right hand side and they appear as equivalent nodal forces, so similarly I have equation for U4, so you need to now solve these two, this 2 by 2 equation and get U2 and U4 and this solution could be a simple if as data, as per the data the two support motions are harmonic so we can take left hand excitation as some delta L naught cos omega T delta R(t) is Delta R naught cos mute, so you can get in finite element approximation to the system behavior.

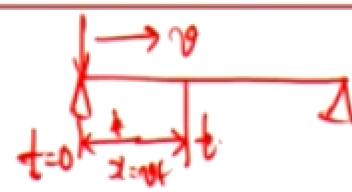
Case-4 A concentrated load P traverses the beam with a constant velocity v_0

$$EIv^{iv} + m\ddot{v} = \begin{cases} P\delta(x - v_0 t) \text{ for } 0 \le t \le \dfrac{l}{v_0} \\ 0 \text{ for for } t > \dfrac{l}{v_0} \end{cases}$$

$$v(0,t) = 0; v(l,t) = 0$$

$$EIv''(0,t) = 0; EIv''(l,t) = 0;$$

$$v(x,0) = 0; \dot{v}(x,0) = 0 \text{[Assumption]}$$



52

Now the problem of a concentrated load traveling on the bridge with velocity V, so at T = 0 suppose if we assume that the way this load enters the span and after time T the position will be, X will be VT, this is at time T, so if you do that we can right now EIY4 + MV double dot as P into direct delta of X - V naught T, this is a V naught is the velocity of the vehicle so till the time the load is on the beam this will be the forcing function, once the load leaves the beam the right hand side would be 0, so this is the forcing function. Now the boundary conditions continue to be what it was namely the displacements are, translations are 0 and bending moments are zero, and we assume again we can assume that the system starts from rest, so how does the finite element solution look like, Case-4 A concentrated load P traverses the beam with a constant velocity v_0

$$\frac{ml}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}\begin{Bmatrix} 0 \\ \ddot{u}_2 \\ 0 \\ \ddot{u}_4 \end{Bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ 0 \\ u_4 \end{Bmatrix} = \{F(t)\}$$

For $0 \le t \le \dfrac{l}{v_0}$

$$\{F(t)\} = \begin{Bmatrix} R_1 \\ 0 \\ R_2 \\ 0 \end{Bmatrix} + \begin{Bmatrix} \int_0^L P\delta(x - v_0 t)\left(1 - 3\frac{x^2}{l^2} + 2\frac{x^3}{l^3}\right)dx \\ \int_0^L P\delta(x - v_0 t)\left(x - 2\frac{x^2}{l} + \frac{x^3}{l^2}\right)dx \\ \int_0^L P\delta(x - v_0 t)\left(3\frac{x^2}{l^2} - 2\frac{x^3}{l^3}\right)dx \\ \int_0^L P\delta(x - v_0 t)\left(-\frac{x^2}{l} + \frac{x^3}{l^2}\right)dx \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ R_2 \\ 0 \end{Bmatrix} + P\begin{Bmatrix} \left(1 - 3\frac{(v_0 t)^2}{l^2} + 2\frac{(v_0 t)^3}{l^3}\right) \\ \left((v_0 t) - 2\frac{(v_0 t)^2}{l} + \frac{(v_0 t)^3}{l^2}\right) \\ \left(3\frac{(v_0 t)^2}{l^2} - 2\frac{(v_0 t)^3}{l^3}\right) \\ \left(-\frac{(v_0 t)^2}{l} + \frac{(v_0 t)^3}{l^2}\right) \end{Bmatrix}$$

NPTEL

53

so we have to now characterize the nodal forces, so this equation remains the same, these two quantities which are U1 double dot and U3 double dot are 0, because U1 and U3 are 0. Now on the right hand side F(t) there are reactions R1 R2 plus the contribution from the moving load, so to find the equivalent nodal forces we use the derivation that we have developed so we get the nodal forces to be as shown here. For example if you look at the nodal force associated with U2 this is P into direct Delta function of X - V naught T into Phi 2(x).

Now as we know the when we integrate this type of integrals X has to be replaced by V naught of T, so if I do that the forcing function is this, the P is sitting outside here, okay, so these are the equivalent nodal forces that we get even a load moves on the beam. So now if we now

$$\frac{ml}{420}\begin{bmatrix} 4l^2 & -3l^2 \\ -3l^2 & 4l^2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_2 \\ \ddot{u}_4 \end{Bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 4l^2 & 2l^2 \\ 2l^2 & 4l^2 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_2(t) \\ F_4(t) \end{Bmatrix}; 0 \le t \le \frac{l}{v_0}$$

$$= 0;\ t > \frac{l}{v_0}$$

$$F_2(t) = P\left\{(v_0 t) - 2\frac{(v_0 t)^2}{l} + \frac{(v_0 t)^3}{l^2}\right\}$$

$$F_4(t) = P\left\{-\frac{(v_0 t)^2}{l} + \frac{(v_0 t)^3}{l^2}\right\}$$

NPTEL

partition the equation and look at only those equations for unknown displacements we get this pair of equation for U2 and U4 with corresponding forcing functions and this is the forcing function for T between 0 and L / V naught and once the load leaves the bridge this is the, this quantity becomes 0 and as I already said system starts from rest, so initial conditions are 0 we can solve this problem, and one moment I find U2 and U4 I can construct my displacement field within the element using the interpolation function that we have used.

Case-5 Right support is subjected to a harmonic moment

$$EIv^{iv} + m\ddot{v} = 0$$

$$v(0,t) = 0; v(l,t) = 0$$

$$EIv''(0,t) = 0; EIv''(l,t) = \bar{M}_0 \cos\Omega t;$$

$$v(x,0) = 0; \dot{v}(x,0) = 0 \text{[Assumption]}$$

Case-5 Right support is subjected to a harmonic moment

$$\frac{ml}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}\begin{Bmatrix} 0 \\ \ddot{u}_2 \\ 0 \\ \ddot{u}_4 \end{Bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ 0 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ R_2 \\ M_0(t) \end{Bmatrix}$$

$$\frac{ml}{420}\begin{bmatrix} 4l^2 & -3l^2 \\ -3l^2 & 4l^2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_2 \\ \ddot{u}_4 \end{Bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 4l^2 & 2l^2 \\ 2l^2 & 4l^2 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ M_0(t) \end{Bmatrix}$$

NPTEL

$$M_0(t) = \bar{M}_0 \cos\Omega t$$

So now the game is clear what to do, so now if we now consider the case of a beam carrying a time dependent moment M(m) naught M(t) then the governing equation is EI 4 + MU double dot is 0, the translations are 0 the bending moment at the left hand is 0, but on the right hand side I have a time varying function. Now again we can assume that system is at rest at T = 0, so now if you write the equation where will this M naught applied moment appear? It appears here, okay, because it is an equivalent nodal force that appears here so the governing equation that we need to solve is this with this M naught (t) the forcing function here. If you want to tackle this problem using PDE we need to make a transformation where we, that is known as Midlands transformation here the right hand side is homogeneous but boundary condition is time dependent, so by making the transformation we introduce a new dependent variable and the equation, for the equation governing the new dependent variable the boundary condition, the right hand side will be inhomogeneous, the boundary conditions will be homogenous and therefore we can use normal mode approximations and get the solution, that is an approach that is exact and that follows from this partial differential equation, but this is different, there is no such transformation that is immediately evident here.

Case-6 An unit impulsive force is applied at $x = a$

$$EIv^{iv} + m\ddot{v} = \delta(x-a)\delta(t)$$

$$v(0,t) = 0; v(l,t) = 0$$

$$EIv''(0,t) = 0; EIv''(l,t) = 0;$$

$$v(x,0) = 0; \dot{v}(x,0) = 0 \left[\text{Assumption}\right]$$

NPTEL

56

Now the case of an impulsive force applied at X = A the forcing function is a product of 2 direct delta functions in space it is at X = A, in time it is at T = 0 therefore this is direct delta of X - A into direct delta (t) and the boundary conditions and initial conditions we can use the same, the description remains the same. Now therefore the what changes here, the forcing

Case-6 An unit impulsive force is applied at $x = a$

$$\frac{ml}{420}\begin{bmatrix}156 & 22l & 54 & -13l\\ 22l & 4l^2 & 13l & -3l^2\\ 54 & 13l & 156 & -22l\\ -13l & -3l^2 & -22l & 4l^2\end{bmatrix}\begin{Bmatrix}0\\ \ddot{u}_2\\ 0\\ \ddot{u}_4\end{Bmatrix}+\frac{EI}{l^3}\begin{bmatrix}12 & 6l & -12 & 6l\\ 6l & 4l^2 & -6l & 2l^2\\ -12 & -6l & 12 & -6l\\ 6l & 2l^2 & -6l & 4l^2\end{bmatrix}\begin{Bmatrix}0\\ u_2\\ 0\\ u_4\end{Bmatrix}=\{F(t)\}$$

$$\{F(t)\}=\begin{Bmatrix}R_1\\ 0\\ R_2\\ 0\end{Bmatrix}+\begin{Bmatrix}\int_0^L \delta(x-a)\left(1-3\frac{x^2}{l^2}+2\frac{x^3}{l^3}\right)dx\\ \int_0^L \delta(x-a)\left(x-2\frac{x^2}{l}+\frac{x^3}{l^2}\right)dx\\ \int_0^L \delta(x-a)\left(3\frac{x^2}{l^2}-2\frac{x^3}{l^3}\right)dx\\ \int_0^L \delta(x-a)\left(-\frac{x^2}{l}+\frac{x^3}{l^2}\right)dx\end{Bmatrix}\delta(t)$$

NPTEL

function has to now take into account the fact that we are applying a concentrated impulsive force, so these are the equivalent nodal forces and if you carry out this integration you have to replace wherever there is X by A in the second row and fourth row and we get these as the equivalent nodal forces, and this is the structure you know MX double dot + KX is equal to F(t) okay, now the forcing functions are impulsive, so that we need to handle at the level of ordinary differential equation.

Case-7 An unit harmonic load is applied at $x = a$

$$EIv^{iv}+m\ddot{v}=\delta(x-a)\cos\Omega t$$

$$v(0,t)=0; v(l,t)=0$$

$$EIv''(0,t)=0; EIv''(l,t)=0;$$

$$v(x,0)=0; \dot{v}(x,0)=0\,[\text{Assumption}]$$

NPTEL

So the final case where we are applying an harmonic concentrated harmonic load so the load is applied at X = A, and the right 4 forcing function is this, the other things remain unchanged and this only changes the equivalent nodal forces so instead of a direct delta function here I will get

Case-7 An unit harmonic load is applied at $x = a$

$$\frac{ml}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}\begin{Bmatrix} 0 \\ \ddot{u}_2 \\ 0 \\ \ddot{u}_4 \end{Bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ 0 \\ u_4 \end{Bmatrix} = \{F(t)\}$$

$$\{F(t)\} = \begin{Bmatrix} R_1 \\ 0 \\ R_2 \\ 0 \end{Bmatrix} + \begin{Bmatrix} \int_0^L \delta(x-a)\left(1 - 3\frac{x^2}{l^2} + 2\frac{x^3}{l^3}\right)dx \\ \int_0^L \delta(x-a)\left(x - 2\frac{x^2}{l} + \frac{x^3}{l^2}\right)dx \\ \int_0^L \delta(x-a)\left(3\frac{x^2}{l^2} - 2\frac{x^3}{l^3}\right)dx \\ \int_0^L \delta(x-a)\left(-\frac{x^2}{l} + \frac{x^3}{l^2}\right)dx \end{Bmatrix}\cos\Omega t$$

NPTEL

60

the harmonic function, these terms remain the same and if we now write the equation for unknown U2 and U4 the forcing functions are harmonic with these amplitudes originating from the shape functions Phi 2 and Phi 4, so this is how we tackle this problem.

$$\Rightarrow \frac{ml}{420}\begin{bmatrix} 4l^2 & -3l^2 \\ -3l^2 & 4l^2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_2 \\ \ddot{u}_4 \end{Bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 4l^2 & 2l^2 \\ 2l^2 & 4l^2 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_2(t) \\ F_4(t) \end{Bmatrix}$$

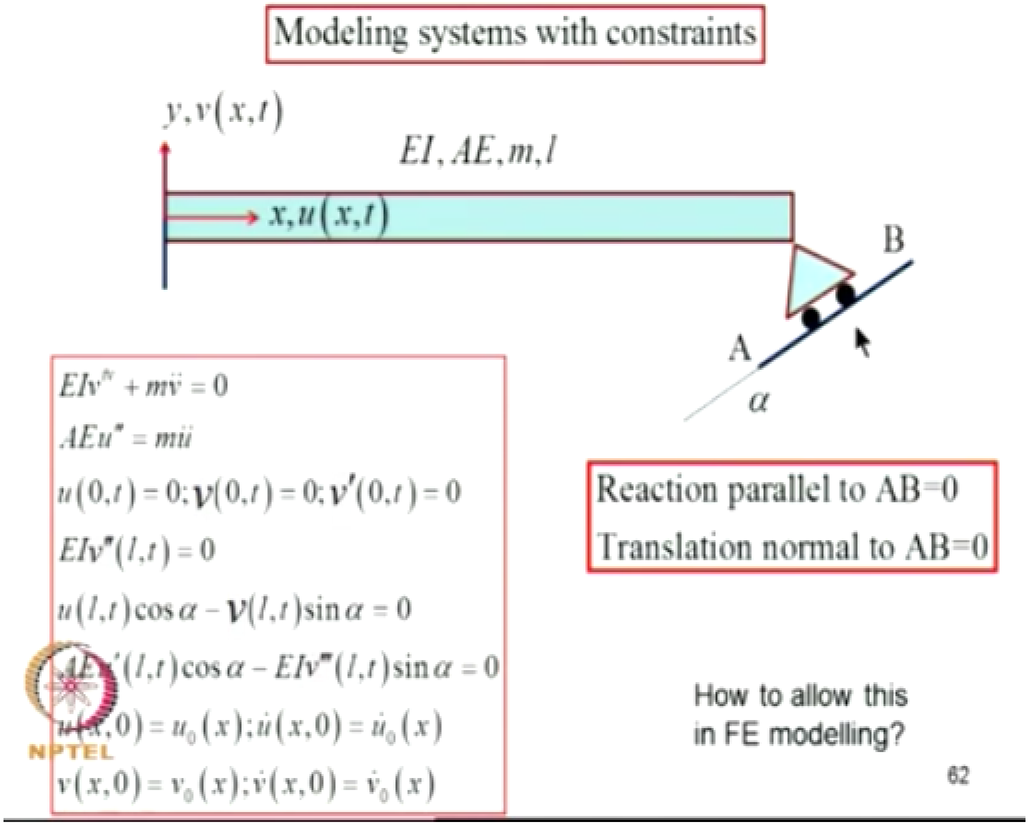
$$F_2(t) = \left(a - 2\frac{a^2}{l} + \frac{a^3}{l^2}\right)\cos\Omega t$$

$$F_4(t) = \left(-\frac{a^2}{l} + \frac{a^3}{l^2}\right)\cos\Omega t$$

NPTEL

61

Now we will conclude this lecture at this stage, there are few more complicating issues that we need to handle this is something to do with constraint equations for example if we have a



system which is supported on an inclined roller, how do you make a model? So you have to introduce certain constraint equations to treat this type of problems, then the other issues that we need to handle or how to compute stresses and how to introduce damping models, this I have not discussed in detail in this course, so we will address some of these issues in the forthcoming lectures. So this lecture concludes here.

Programme Assistance

Guruprakash P
Dipali K Salokhe

Technical Supervision
B K A N Singh
Gururaj Kadloor

Indian Institute of Science
Bangalore