Indian Institute of Science Bangalore

National Programme on Technology

Enhanced Learning

Finite element method for structural dynamic and stability analyses

Lecture -40

Closure

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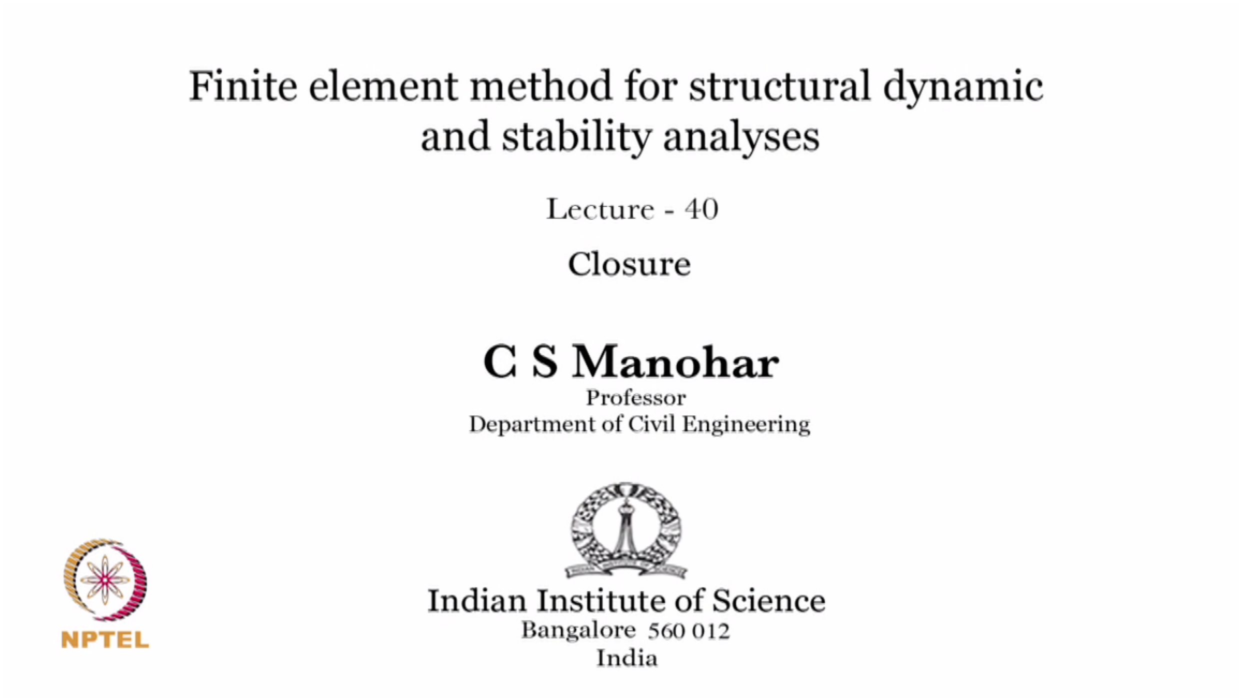
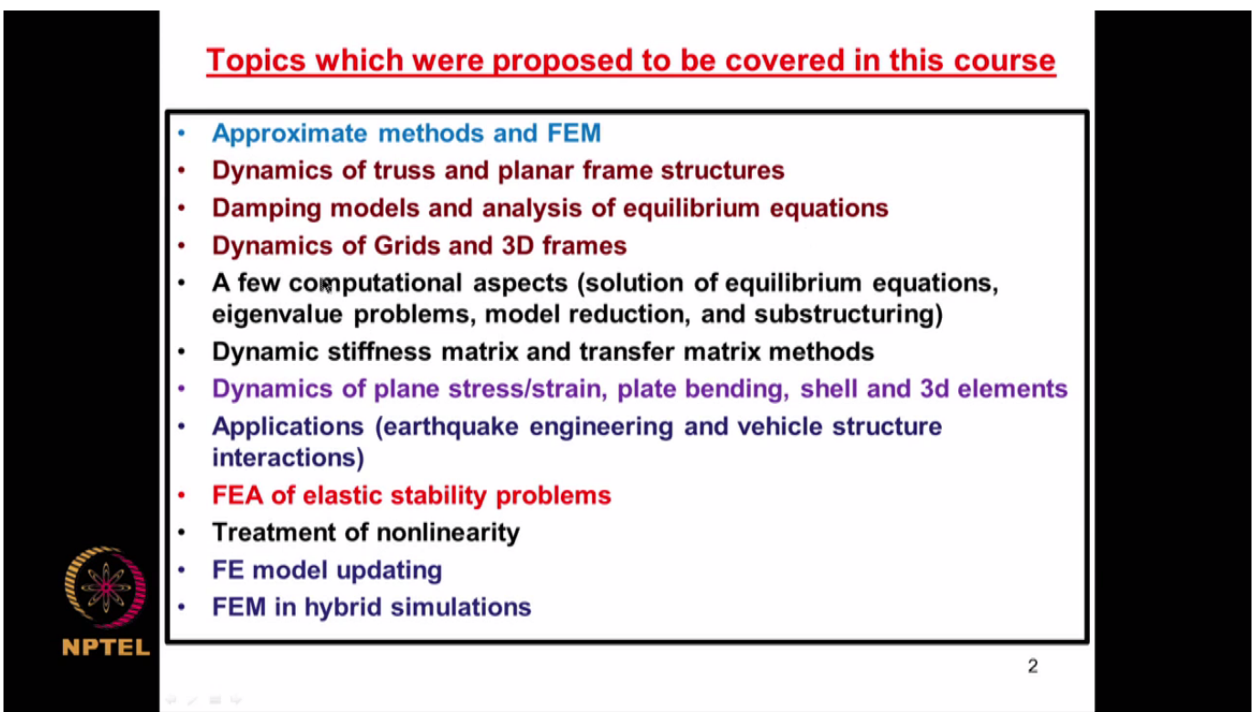


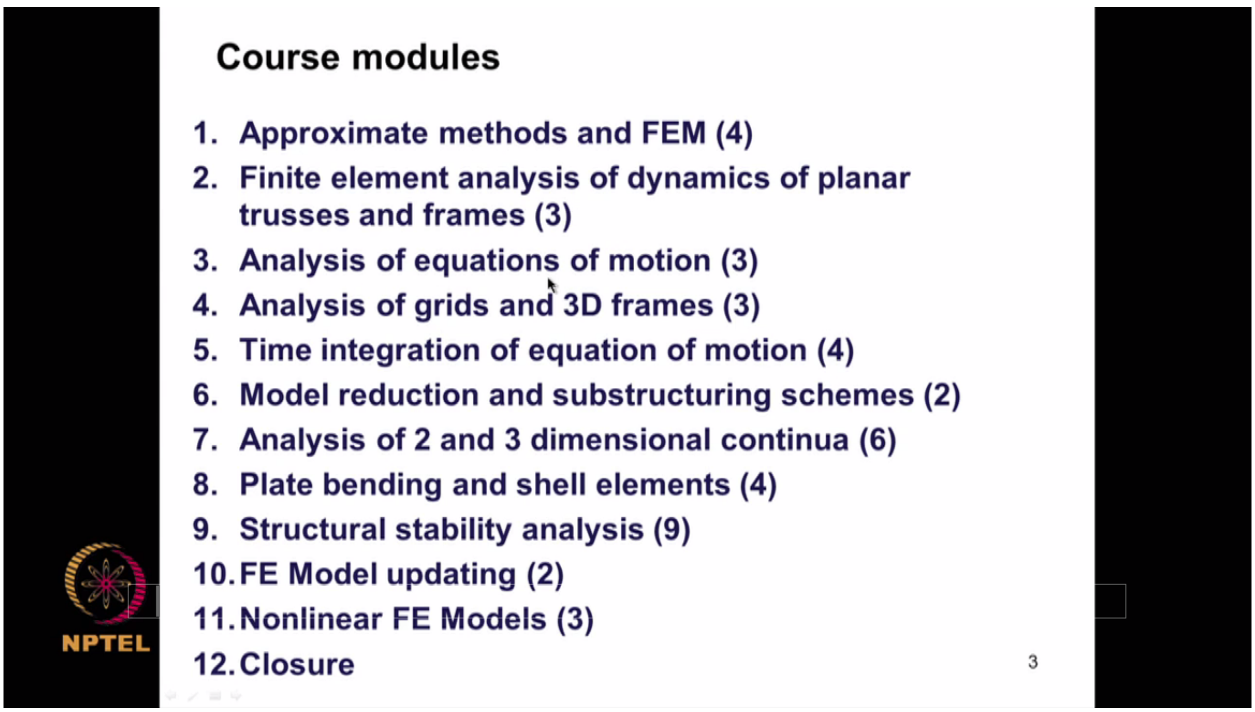
This is the last lecture in this course. So what we will do is we will quickly recall what we did in this course and briefly discuss a couple of directions in which we can move forward.



So when we started this course these were the topics which were proposed to be covered. The approximate methods and finite element method, dynamics of truss and planar frames, damping models and analysis of equilibrium equations, dynamics of grids and 3D frames, and some computational aspects like solution or equilibrium equations. Again value problems, model reduction and substructuring. Then review of dynamic stiffness matrix and transfer matrix methods. Then dynamics of plane stress/strain, plate bending shell and 3D elements.

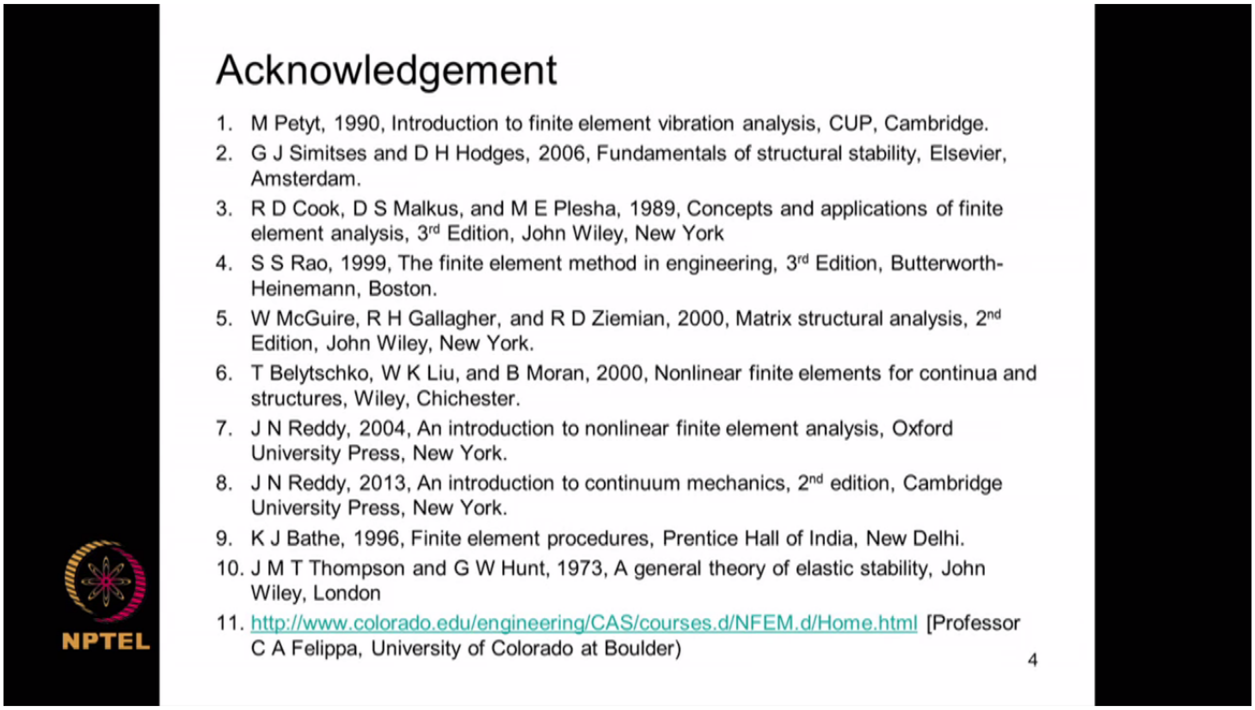
Then applications to earthquake engineering and vehicle structure interactions. Then application of finite element analysis for elastic stability problems. Some issues about treatment of nonlinearity and questions on finite element model updating, and how finite element method is used in hybrid simulations.

So accepting the last topic we have broadly covered all these issues.

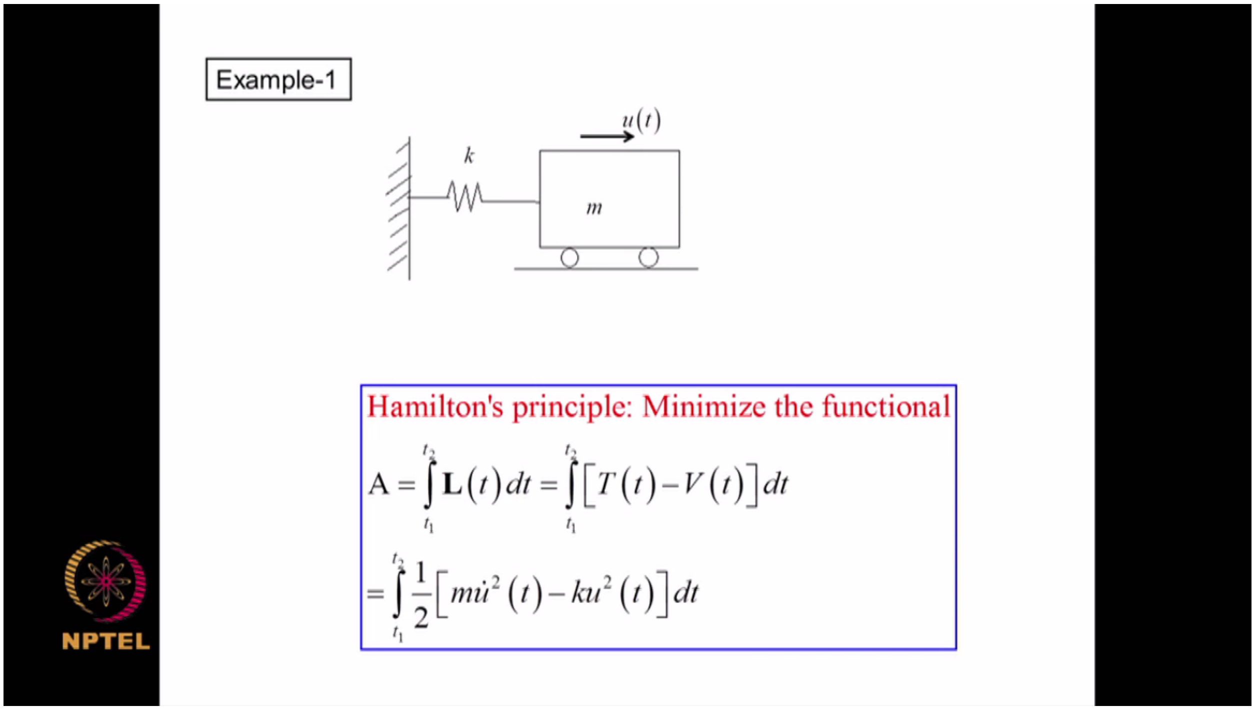


So the course has been divided into basically eleven modules. In the first module we started with approximate methods and finite element method. Then this was followed by analysis of planar, trusses and frames. Then we spend time on integration of equations of motion. Then analysis of grids and three-dimensional frames. Then time integration of equation of motion.

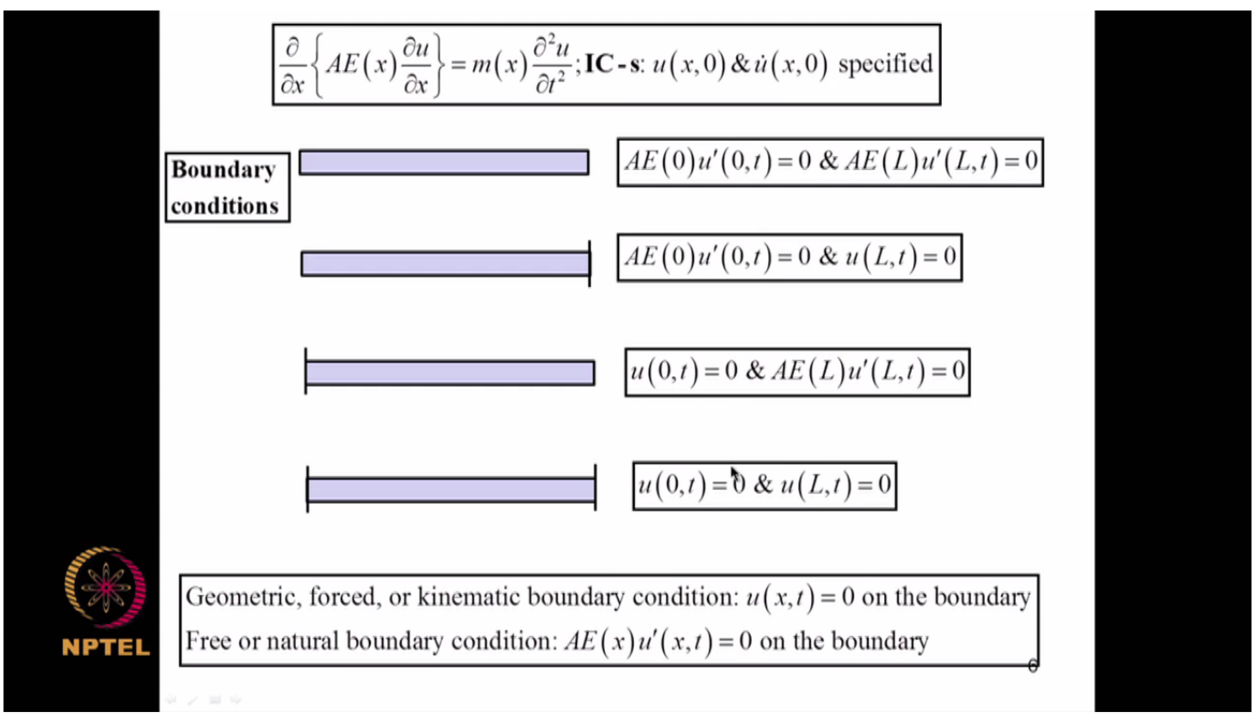
In the third module we considered again normal mode representations. Here we look at a time integration. Then questions on model reduction and substructuring were addressed in model module six. Then analysis of two and three dimensional continuum. We spent some time discussing plain stress element, plate bending elements and facet shell elements. So that covered the model seven and eight and in module nine we covered quite a bit of ground on structural stability analysis. Then we briefly consider two topics that is finite element model updating and how to deal with nonlinearity in finite element models.



So I would like to acknowledge at this stage authors of these books whose work I found quite useful in preparing this lectures. So in many places I have – my lectures were fashioned after some of the coverage in some of this text books.



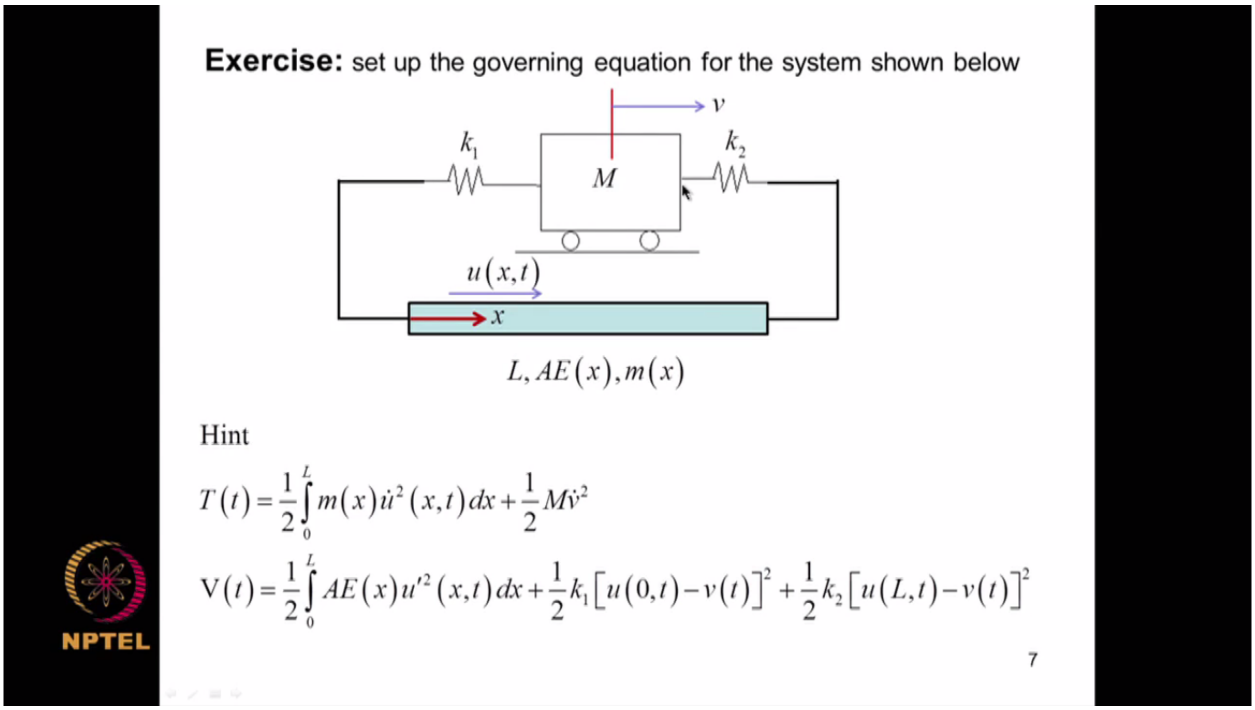
So we begin by considering application of Hamilton's principle to simple systems like mass spring dashpot system and we derived the equation of motion by applying Hamilton's principle.



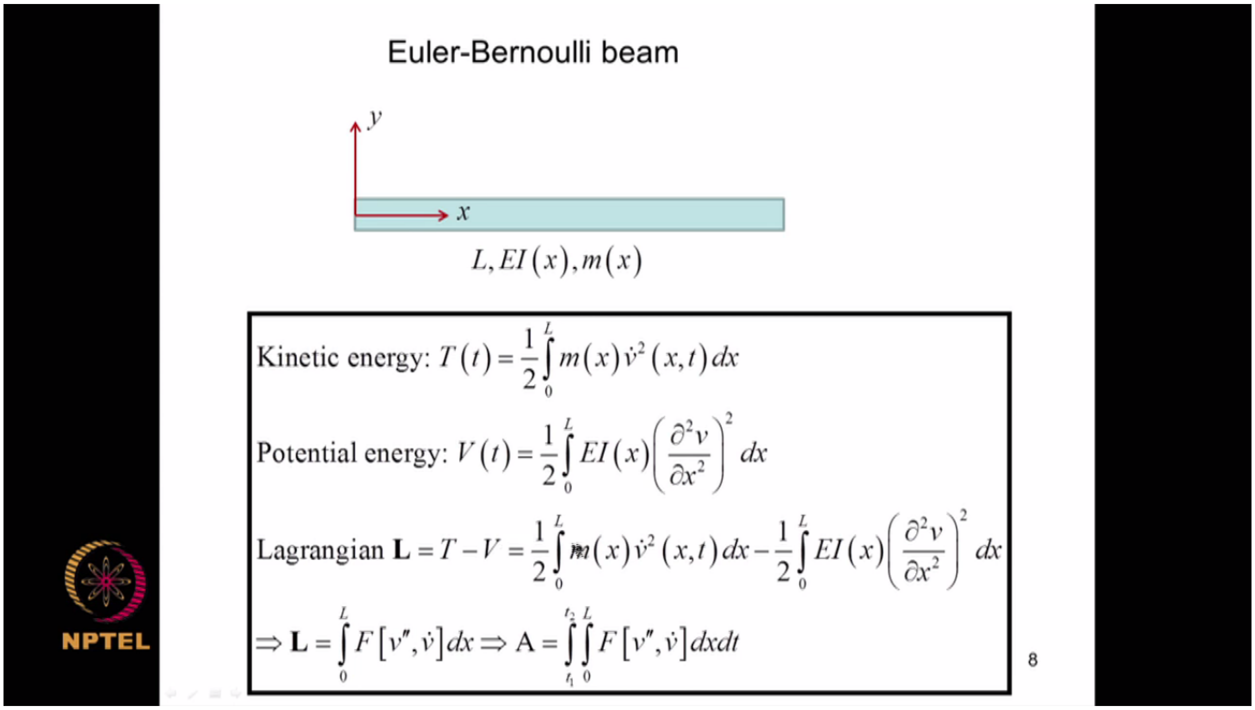
And then generalize this to continuous systems.

So this is actually vibrating rod and we saw that application of Hamilton's principle enables us to derive the equilibrium equation and also a set of valid boundary conditions.

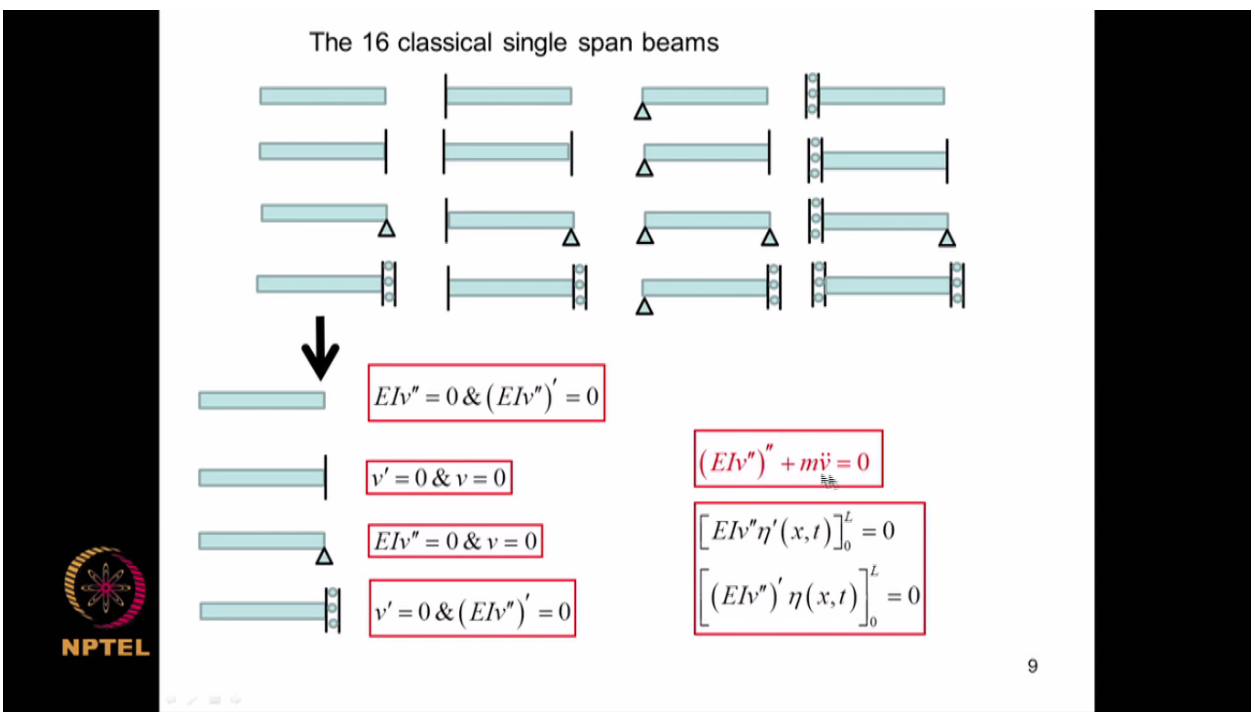
Now the boundary conditions themselves were classified as geometric forced or kinematic conditions and then natural boundary conditions.



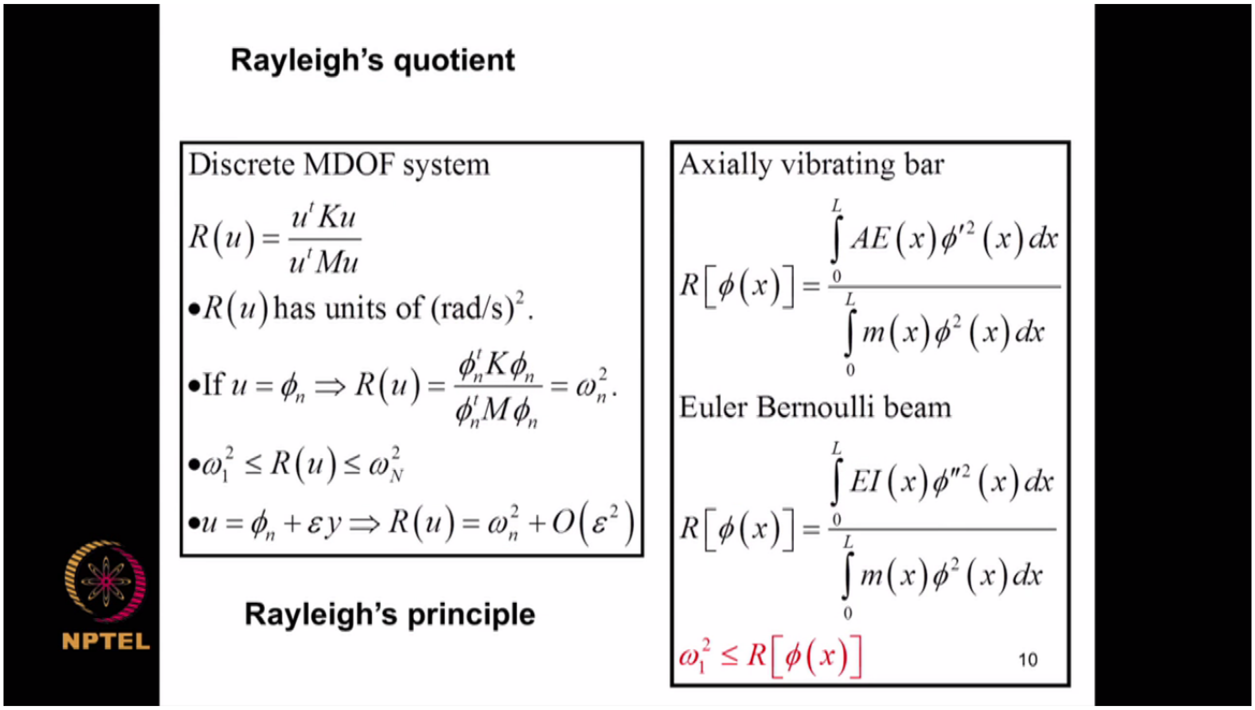
We covered several examples where there was combination of continuum elements and discrete elements used Hamilton's principle and derived the equations.



We then moved on to theory of Euler-Bernoulli beam. Again by using the Hamiltons principle we derived the governing equation and the relevant boundary conditions.

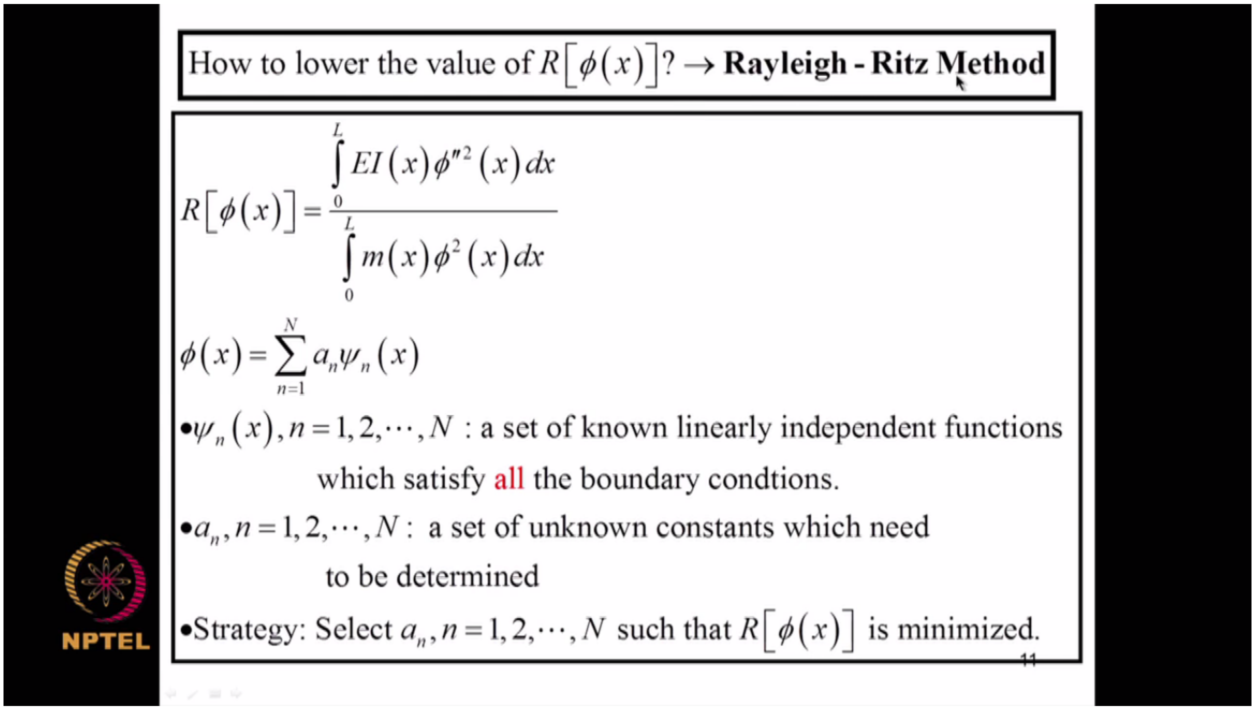


And we saw that the boundary can – the field equation will be of this form in free vibration and there are 16 combinations of single span beams which emanate from the application of Hamilton's principle. So these are various boundary conditions which are appropriate for Euler-Bernoulli beam cases.



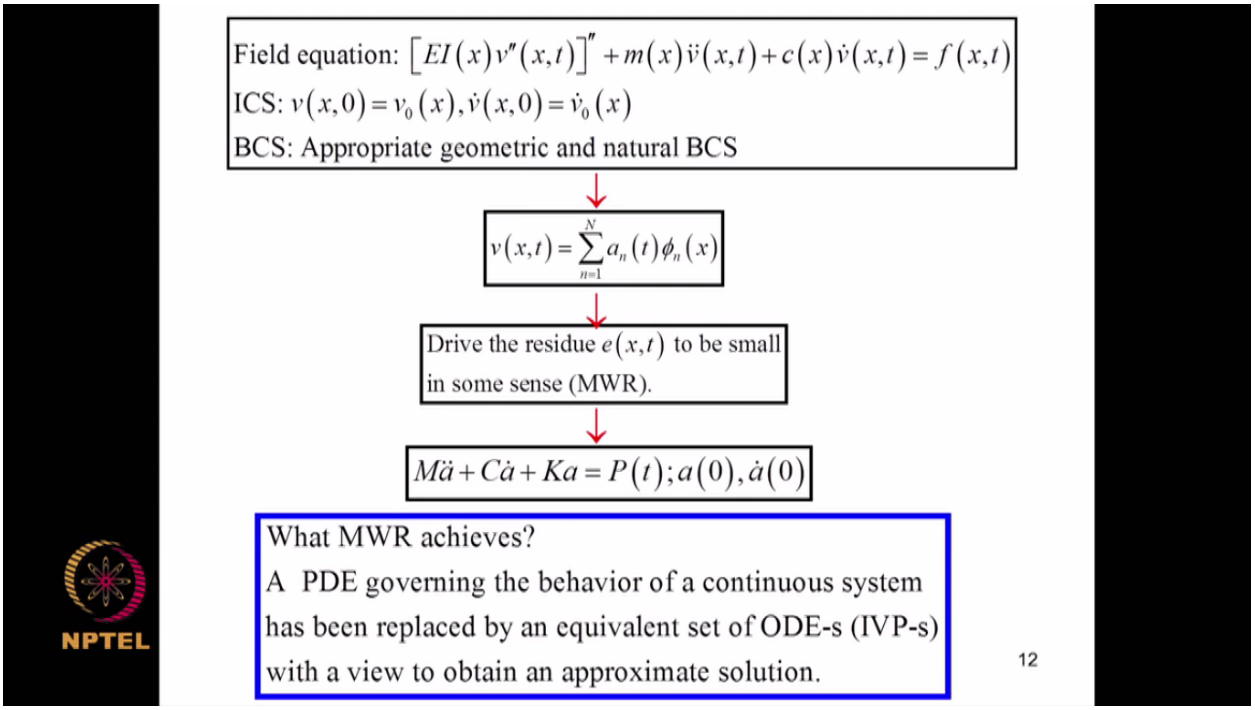
Now in the analysis of free vibration characteristics we begin by discussing about Raleigh's quotient. So we formulated the Raleigh's quotient for discrete multi degree-of-freedom system and few continuous systems and an important property of Raleigh's quotient was that it was bounded between square of first natural frequency and square of the N natural frequency in the end degree freedom system.

Whereas for continuous system there was only a bound on the lower value R was greater than Omega 1 square. Now, Raleigh's quotient you can derive without writing the equation of motion. So that's a quick way of finding natural frequencies and phi of X are the trial functions and the choice of these trial functions plays important role in application of these methods and among different choices of trial functions, the one which provides the lowest value of Raleigh's quotient provides the best estimate to the first natural frequency.



So in our endeavor to lower the value of Raleigh's quotient, we introduce that Raleigh-Ritz method where the trial function was represented in a series of orthogonal functions typically A ends where the generalized coordinates and we minimize Raleigh's quotient with respect to these A ends. So this phi of X are set of known linearly independent function which satisfy all the boundary conditions to start with.

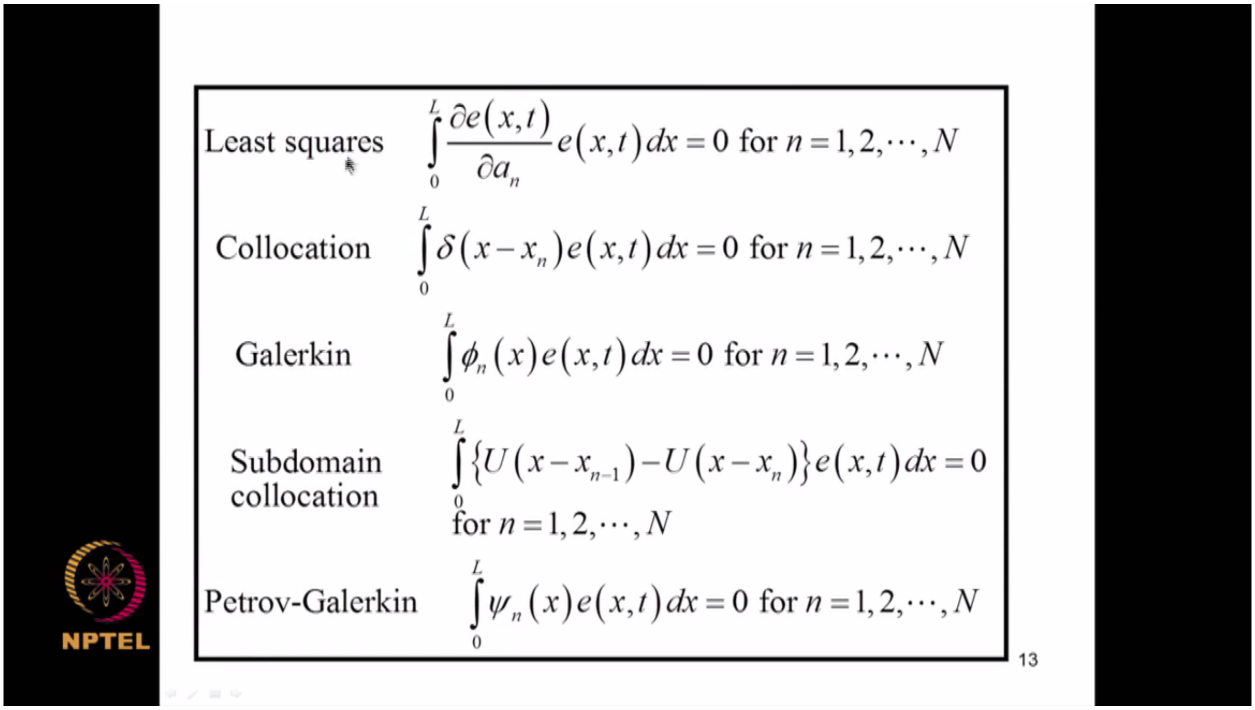
Then, we found out A ends which minimize this Raleigh's quotient and that helped us to find not only the first natural frequency but also approximations to higher natural frequencies.



Then starting from field equations we developed the so-called method of weighted residuals. So, for example, for the beam equation, this was a field equation and we started by approximating the solution or 0 to L in terms of N generalized coordinates N of T to be determined, and a set of known trial functions Phi n of X. So this phi n of X are valid over the entire domain of the beam.

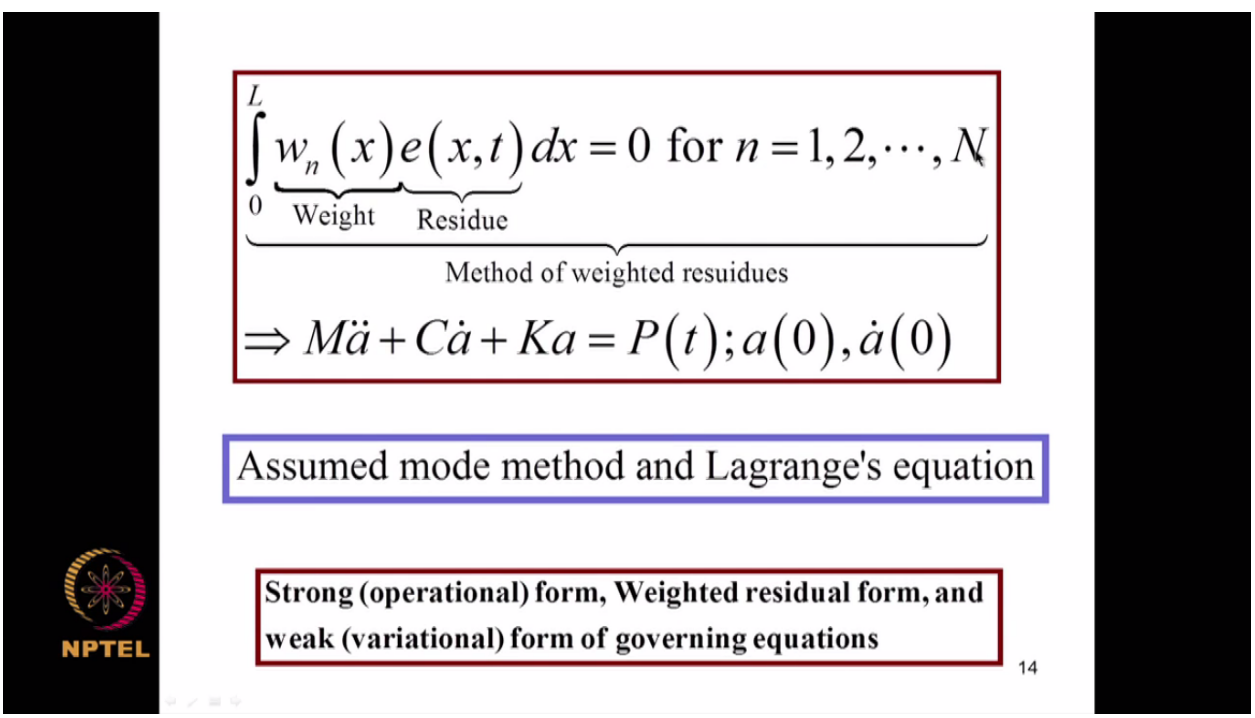
So when we substitute this into the field equation, we get an error that we call as residue and the method of weighted residual essentially minimizes this residue in some sense. So that leads to a set of discrete equations that is the space variable X has been discretized and we are left with time which is still continuous. So this is a semi discretized equation of motion for the system and this leads to the concept of mass damping and stiffness matrix and forcing vector.

So what method of weighted residual achieves is that a partial differential equation governing the behavior of a continuous system has been replaced by an equivalent set of ordinary differential equation which are initial value problems with a view to obtain approximate solutions. The notion of this method of weighted residuals was not unique and there were several possibilities.



And we talked about least square method, collocation method, and Galerkin method, sub domain collocation, Petrov–Galerkin and so on and so forth.

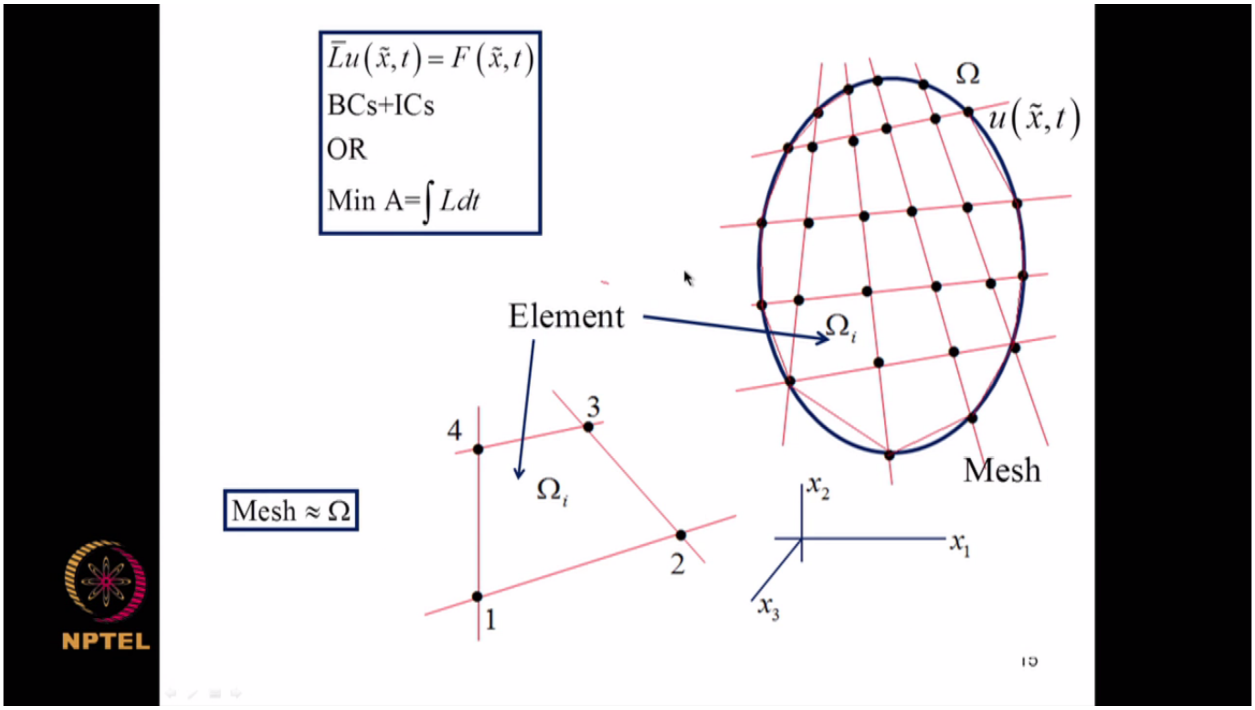
The basic idea here is the weighting function changes for these different methods. And we also applied this to a few problems and saw how it works.



So basically there was a residue and a weight function and by equating, selecting N weight functions and equating this to 0 we got N number of equations and this is a resulting equation. This is broadly the class of methods known as method of weighted residues.

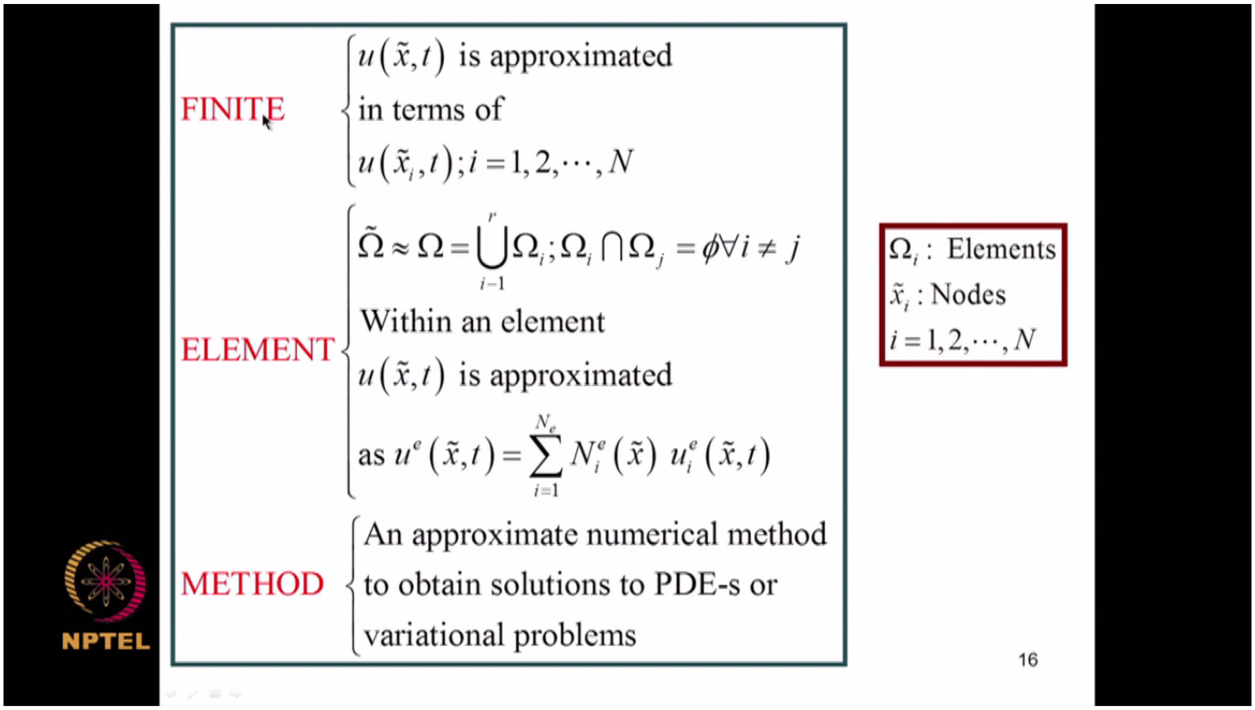
We also considered certain other additional methods and we introduced the strong form, weighted residual form and weak form of governing equations while discussing this.

Now the problem with method of weighted residuals as discussed here is that these trial functions were globally valid.



So if we have a more complicated domain to construct phi of X, which are globally valid will be difficult and also the generalized coordinates N of T that arises here do not have a direct physical meaning. Now this led to the discussion on finite element method, what we did was the domain of interest was partitioned as shown here and the field variable at these points known as nodes were taken as generalized coordinates and these sub domains which are known as elements, the field variable within an element was interpolated in terms of these nodal values using polynomials.

So each of these elements have, for example, have simple for unknown sign a interpolation function. So there is nothing really super relative about the performance of a single element. But the entire finite element procedure is such that when all this is assembled it delivers a very super relative performance in terms of analysis of very complex problems involving complexity in geometry, constitutive loss and so on and so forth.



Now the word finite element, the phrase finite element method we try to explain the word finite here originates because the field variable is approximated in terms of the value of the field variables as a set of nodes, finite number of nodes, that is where the word finite arises. The element, of course, is that we are the domain of interest is approximated by Omega tilde and this is taken as union of a set of non intersecting subdomains and each one is an element and within an element the field variable is approximated in terms of interpolation function and nodal values.

So this renders credence to the name element. The method is of course it indicates that finite element method is an approximate numerical method to obtain solutions to partial differential equations or variational problems. So this Omega I R elements X, tilde are nodes and capital N is a number of nodes in the system – in the approximated model.

Summary

Element level EOM in local coordinate system: $M_s\ddot{u}_s + C_s\dot{u}_s + K_s u_s = F_s(t)$

Element level EOM in global coordinate system:

$$\bar{M}_s\ddot{\bar{U}}_s + \bar{C}_s\dot{\bar{U}}_s + \bar{K}_s\bar{U}_s = \bar{F}_s(t)$$
$$\bar{U}_s = T_s^t u_s; \bar{M}_s = T_s^t M_s T_s; \bar{K}_s = T_s^t K_s T_s;$$
$$\bar{C}_s = T_s^t C_s T_s$$

Global EOM after assembly of structural matrices and before imposing boundary condtions:

$$\bar{M}\ddot{\bar{U}} + \bar{C}\dot{\bar{U}} + \bar{K}\bar{U} = \bar{F}(t)$$
$$\bar{M} = \sum_{s=1}^{p}[A]_s^t[\bar{M}]_s[A]_s \bar{K} = \sum_{s=1}^{p}[A]_s^t[\bar{K}]_s[A]_s$$
$$\bar{C} = \sum_{s=1}^{p}[A]_s^t[\bar{C}]_s[A]_s; \bar{F}(t) = \sum_{s=1}^{p}[A]_s\bar{F}_s(t)$$

Equations for unknown reactions:

$$M_{0I}\ddot{\bar{U}}_I + C_{0I}\dot{\bar{U}}_I + K_{0I}\bar{U}_I = \bar{F}_0(t)$$

Equations for unknown displacements:

$$M_{II}\ddot{\bar{U}}_I + C_{II}\dot{\bar{U}}_I + K_{II}\bar{U}_I = \bar{F}_I(t)$$
$$M\ddot{U} + C\dot{U} + KU = F(t)$$
$$U(0) = U_0; \dot{U}(0) = \dot{U}_0$$

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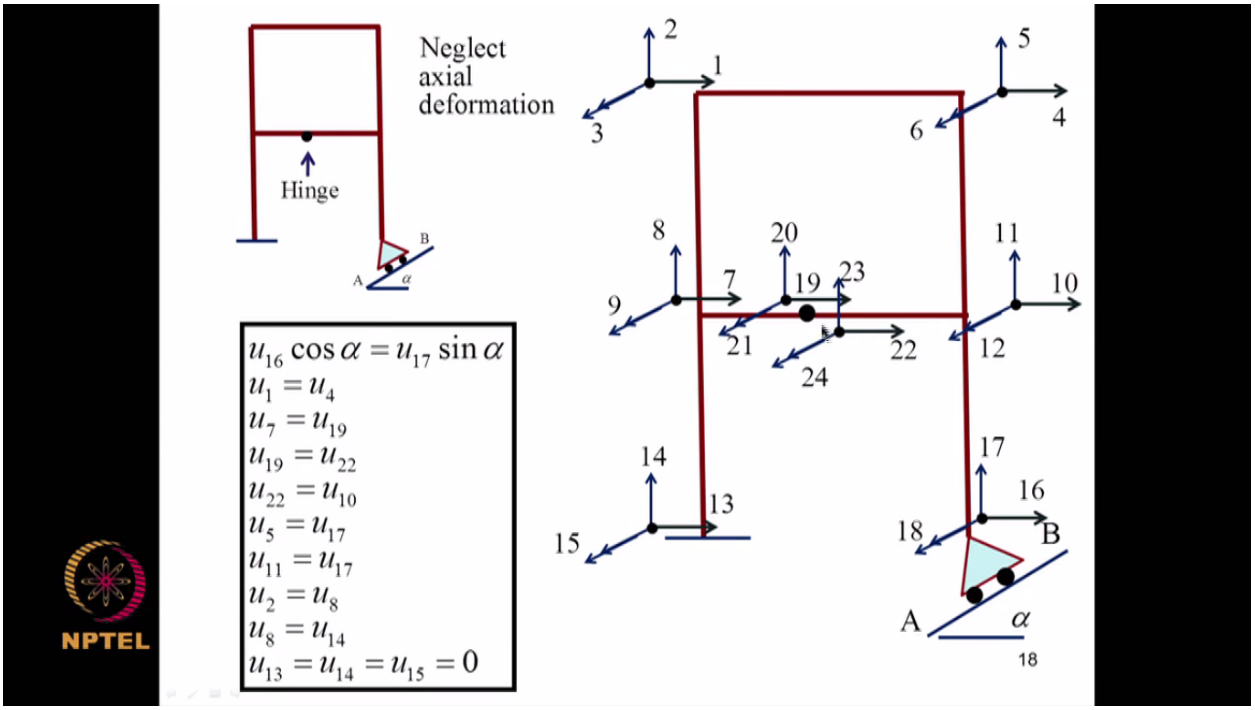
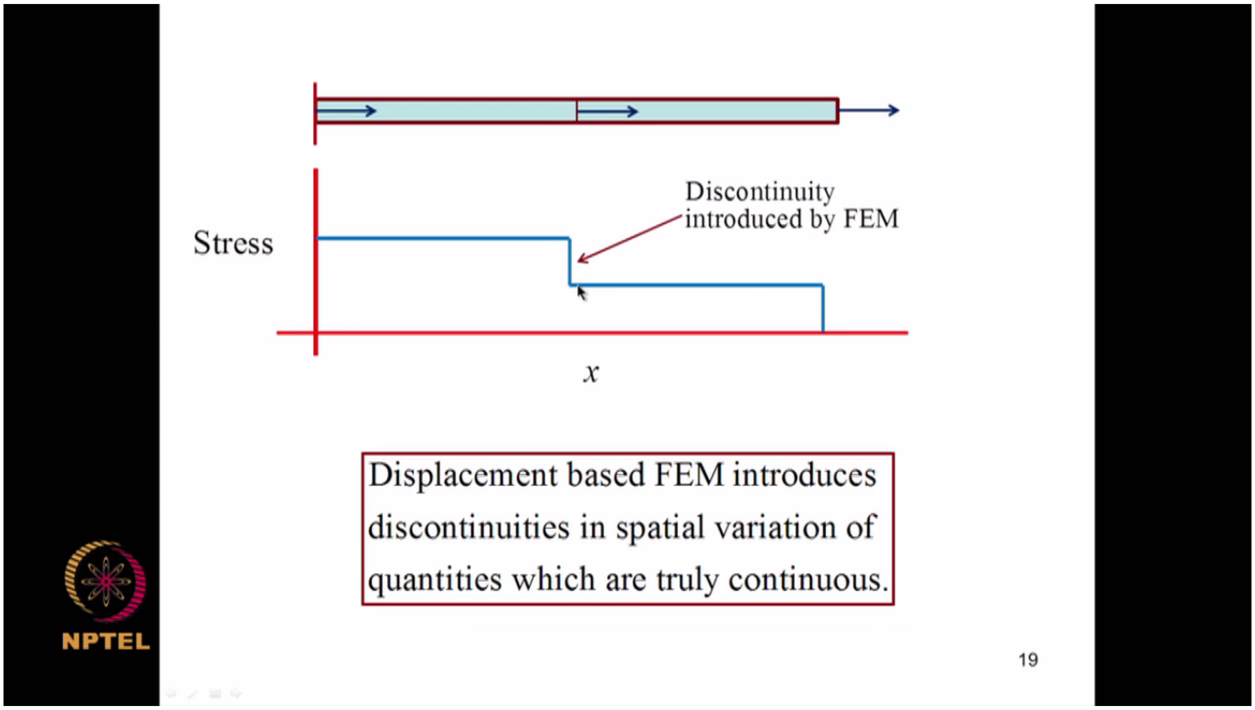
Now using that what we did was for each of these elements we derived the equation of motion that led to the notion of element level, mass damping and stiffness matrix and a forcing vector and this is formulated in the local coordinate system. Then the element level equation of motion is transformed to global coordinate systems.

We developed how displacement, velocity and acceleration and force force vectors are all vectors and they obey certain rules of transformation and based on that we can construct the transformation needed to for the mass stiffness and damping matrices in the global coordinate system.

Then to construct the structure matrix for the entire structure these matrices need to be assembled. So basically energies in different subdomains add and that forms the basis of forming the global equation of motion and at this stage we have not yet partitioned the equation into unknown displacements and unknown reactions and we are not yet impose the boundary conditions. So upon imposing the boundary conditions and partitioning the vector U accordingly we got the final equation of motion for the system for the unknown displacements as MU double dot plus U dot plus K = F of T where U of 0 U naught and u dot 0 is u naught dot which are initial displacement and velocities.

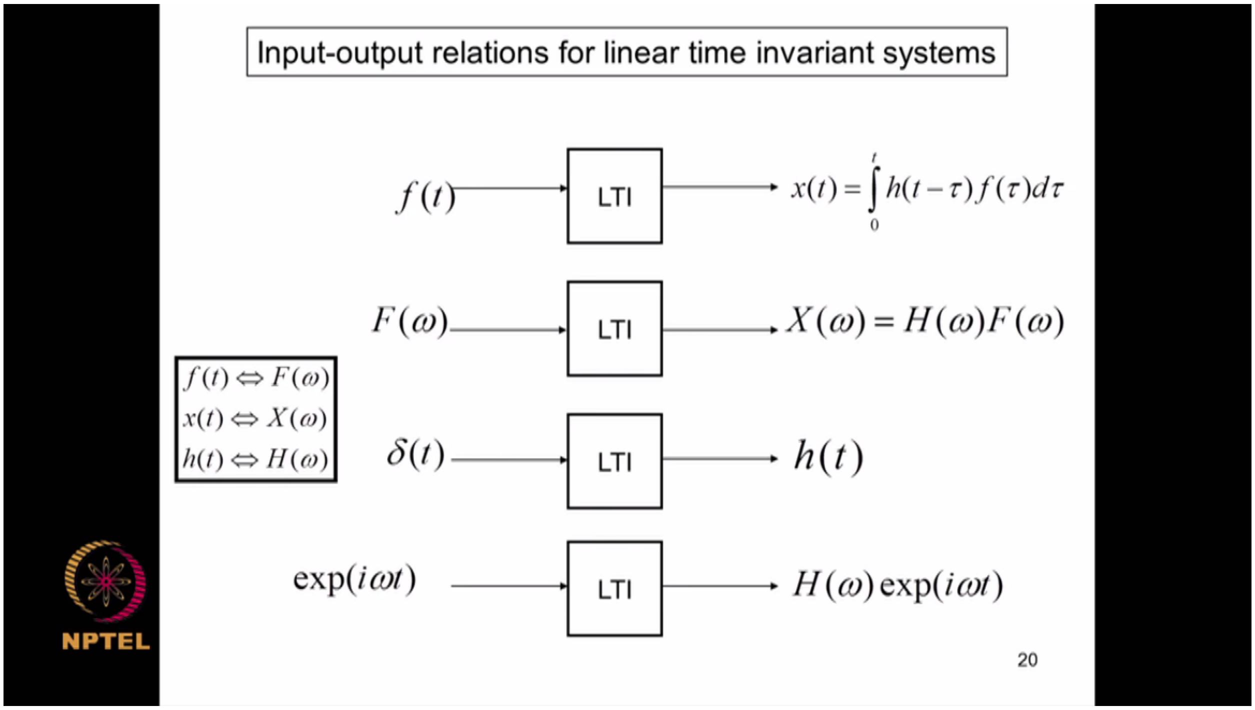


We considered various complicating issues. We started by studying planar frames and modeled elements as Euler-Bernoulli beams. We considered questions about how to deal with the hinge and a roller on an inclined support, etcetera. that is how to set up suitable constraint equations in terms of degrees of freedom to correctly capture the presence of a hinge or inclined roller and so on and so forth.

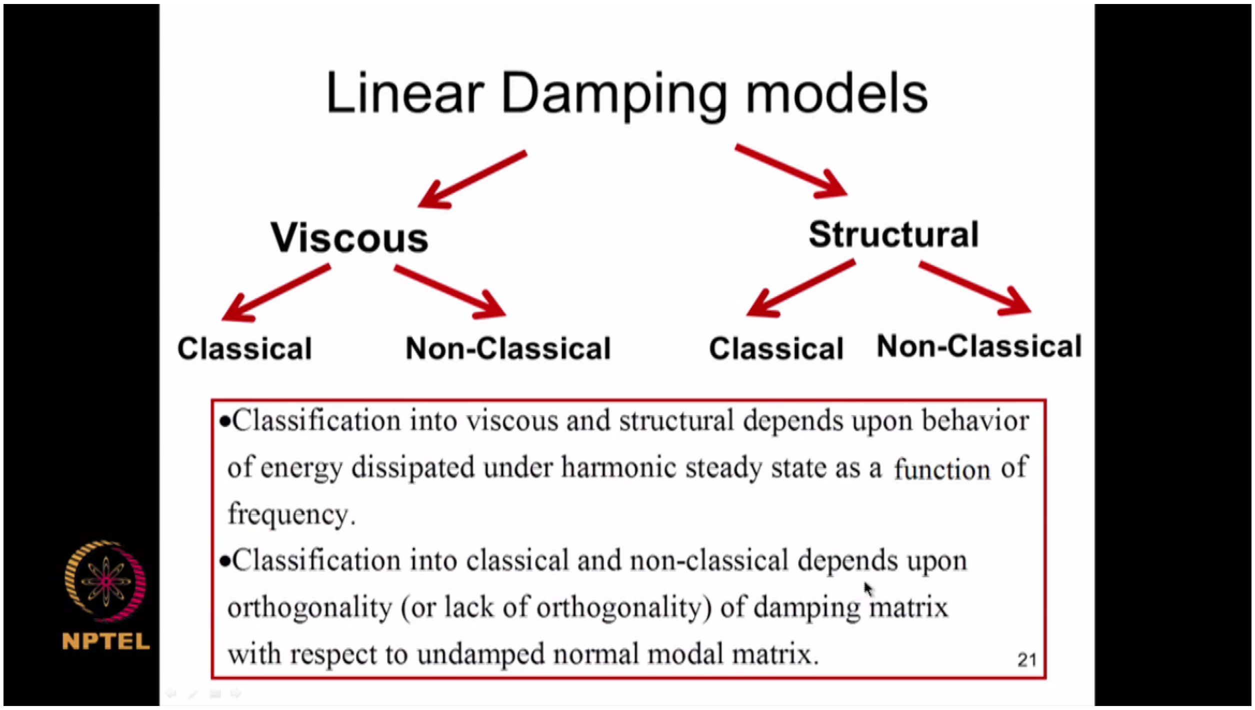


One important observation we made was that while the displacement field across the element is continuous but when it comes to evaluation of stresses, for example; in the case of an actual vibrating bar, there will be discontinuities. So the finite element method thus introduces discontinuities across element boundaries in certain quantities which are not – there is no discontinuity in an exact solution.

So this is one of the limitations of finite element method.



Next we started looking at some issues about dynamics. So we derived the input output relation in time and frequency domains. We introduced the notion of impulse response function and complex frequency response function. These quantities become matrices for multi degree freedom systems and these are the definitions for the impulse response function and the complex frequency response function.



Now in formulating – in trying to solve the governing equations of motion we started by looking for certain coordinates in which the degrees of freedom become uncoupled. And we found that there are transformations from a given coordinate system to certain natural coordinate systems in which the degrees of freedom become uncoupled. The consequence of that is a coupled set of ordinary differential equations are solved as a set of uncoupled either second order differential equation or first order differential equation depending on how we represent the equation of motion.

So we started another element in our solution strategy, modeling strategy was introduction of damping. For linear systems, the damping models were either viscous or structural and so-called classical or non-classical. So the classical damping models are those, in which the undamped normal modes uncouple the equation of motion. Whereas that is not true for non-classical damping. So we addressed question of how to uncouple equation of motion for each one of these for damping models. And we introduced in doing so several quantities like receptance, mobility, accelerants, dynamic stiffness, mechanical impedance, apparent mass, and so on and so forth.

Nomenclature for FRF		
Response Quantity (R)	$\frac{R}{F}$	$\frac{F}{R}$
Displacement	Receptance Admittance Dynamic compliance Dynamic flexibility	Dynamic stiffness
Velocity	Mobility	Mechanical impedance
Acceleration	Accelerance	Apparent mass

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Summary

FRF calculations (valid for both viscous and structural damping models

Direct calculation

(a) Viscously damped system

$$M\ddot{U} + C\dot{U} + KU = F\exp(i\omega t)$$

$$\left[\alpha(\omega)\right] = \left[-\omega^2 M + i\omega C + K\right]^{-1}$$

(b) Structurally damped system

$$M\ddot{U} + (K + iD)U = F\exp(i\omega t)$$

$$\left[\alpha(\omega)\right] = \left[-\omega^2 M + K + iD\right]^{-1}$$

Calculation based on mode superposition

(c) Viscously damped system with classical damping

$$\alpha_{rs}(\omega) = \sum_{n=1}^{N} \frac{\Phi_{rn}\Phi_{sn}}{\left(\omega_n^2 - \omega^2 + i2\eta_n\omega_n\omega\right)}$$

$$K\Phi = M\Phi\Lambda; \Phi' M\Phi = I; \Phi' K\Phi = \Lambda; \Lambda = \text{Diag}\left[\omega_i^2\right]$$

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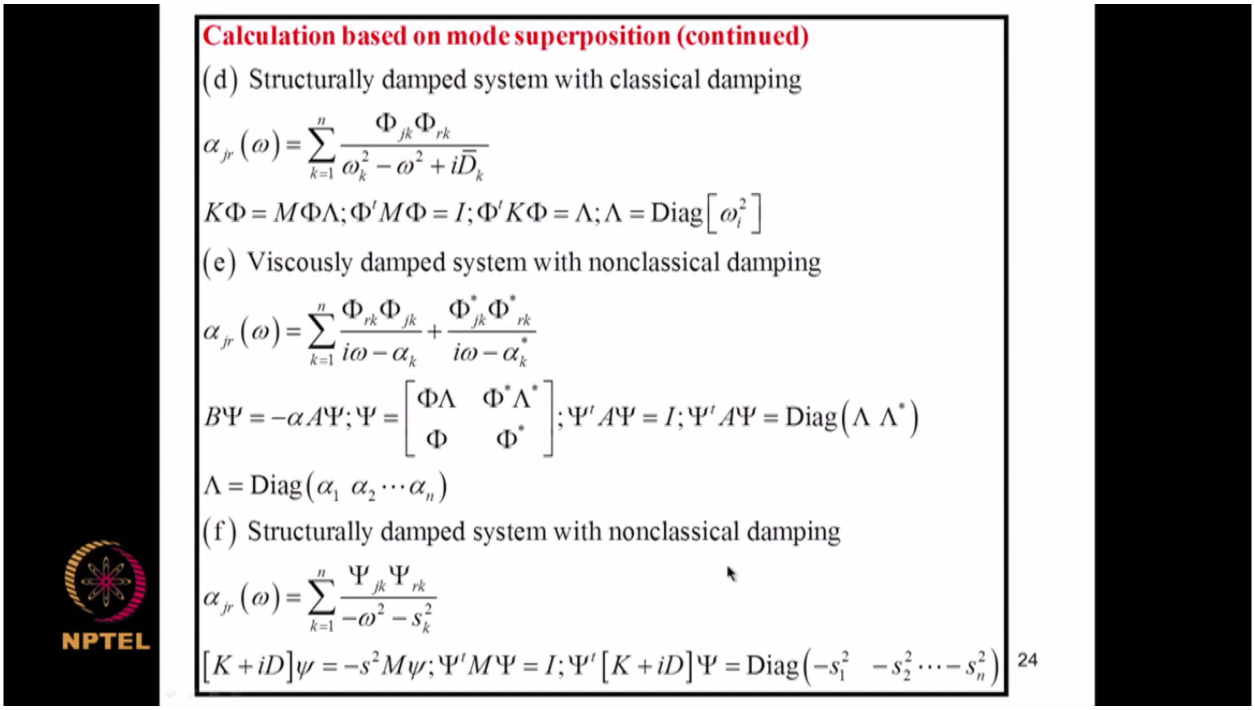
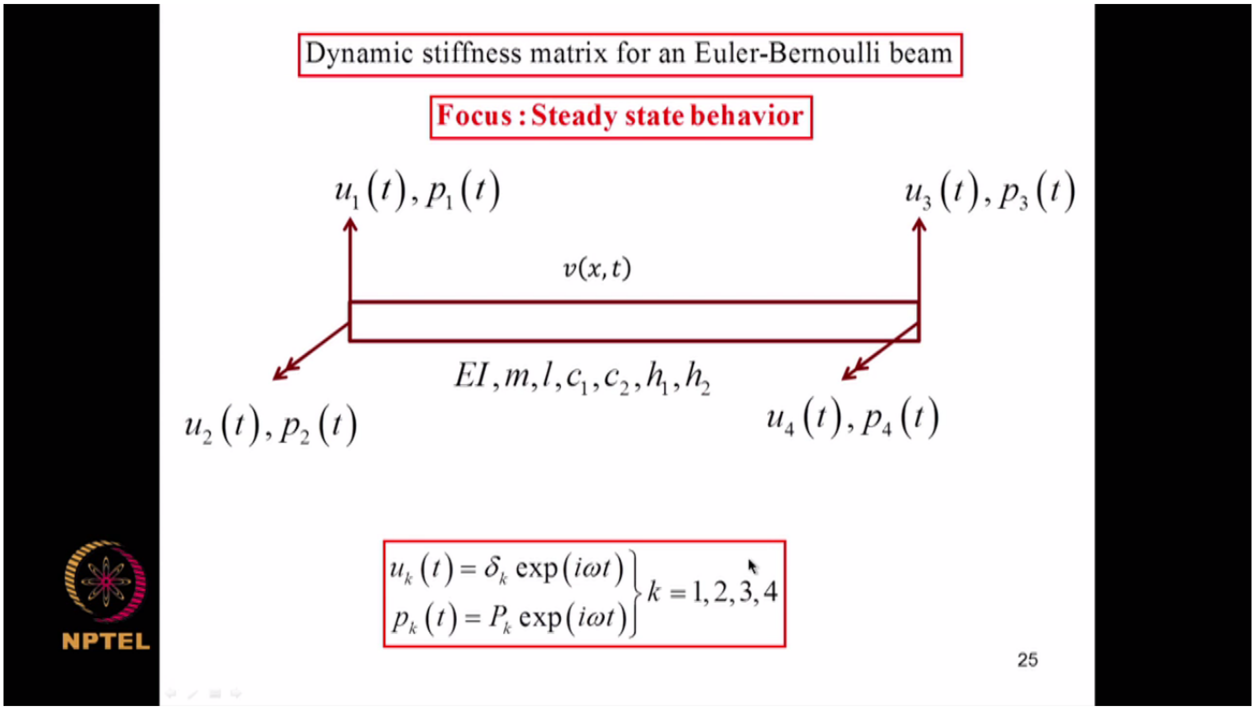
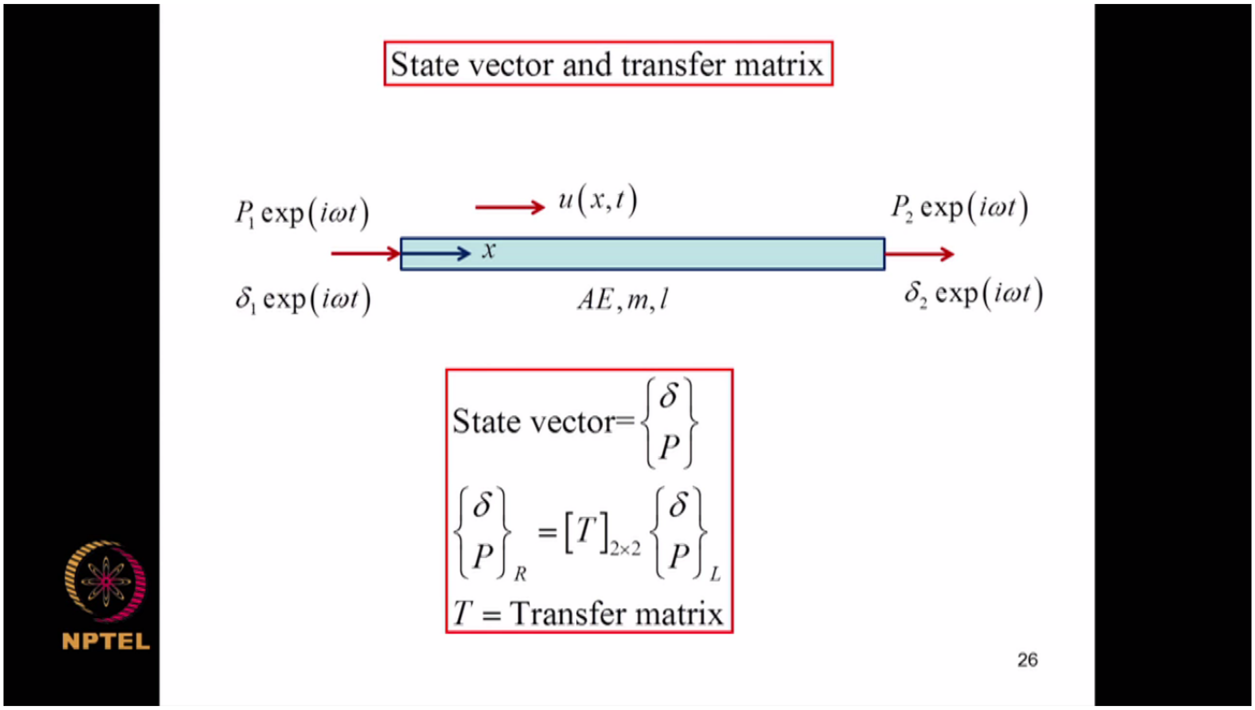
So the FRF of calculations that is the representation of system response in frequency domain we developed several methods, for example, in viscously damped system one could directly invert the dynamic stiffness matrix and get that and similar strategy can also be adopted for structurally damped systems but each one of this calculation can be carried out in terms of mode superposition method by using mode superposition method, and that requires the determination of the appropriate natural frequencies and mode shapes.

So this we did for all the four cases of damping that is structural, viscous, proportional and non-proportional damping models.



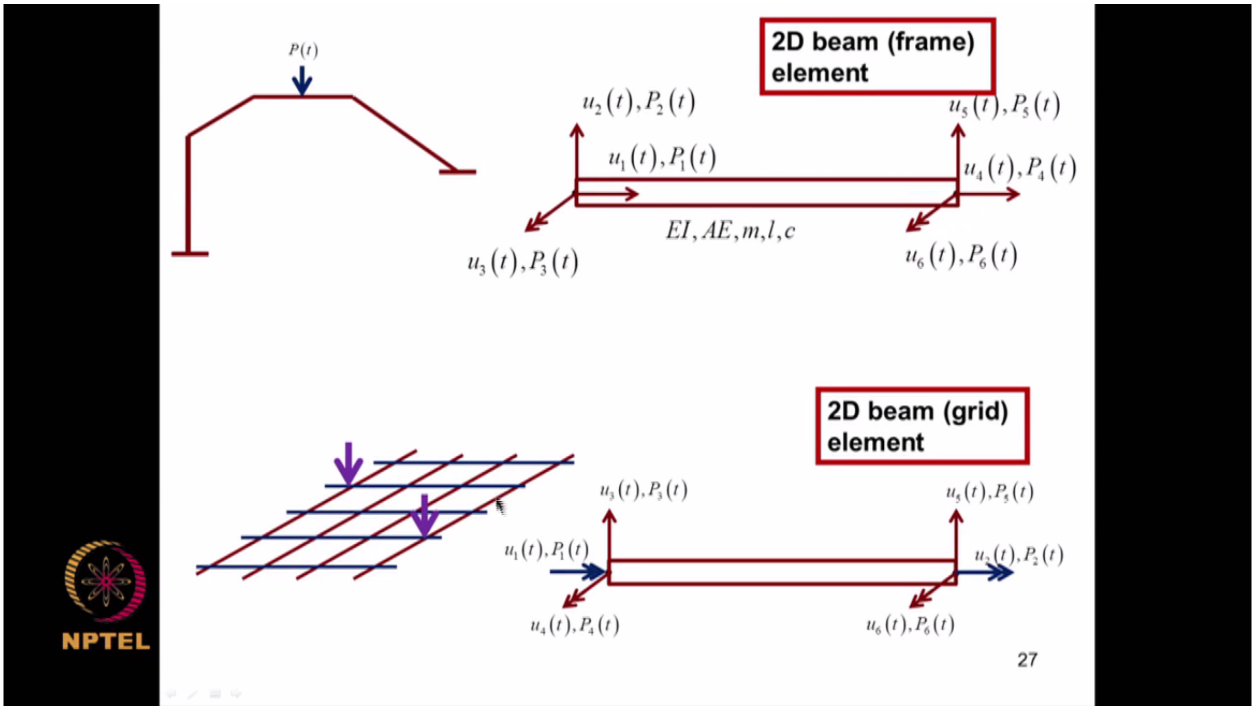


In certain class of problems in linear structural dynamics where [Indiscernible] [0:15:53] focused only on steady-state behavior under harmonic loads or loads which permit for a representation one can use what method known as dynamic stiffness matrix. Here we assume that in a beam for example; shown here this is an Euler-Bernoulli beam if the nodal displacement and forces are all harmonic at the same frequency as shown here what conditions these amplitudes Delta K, and PK should satisfy so that the governing equation is satisfied. So that leads us to the definition of the dynamic stiffness matrix that we derived for Euler-Bernoulli beam.

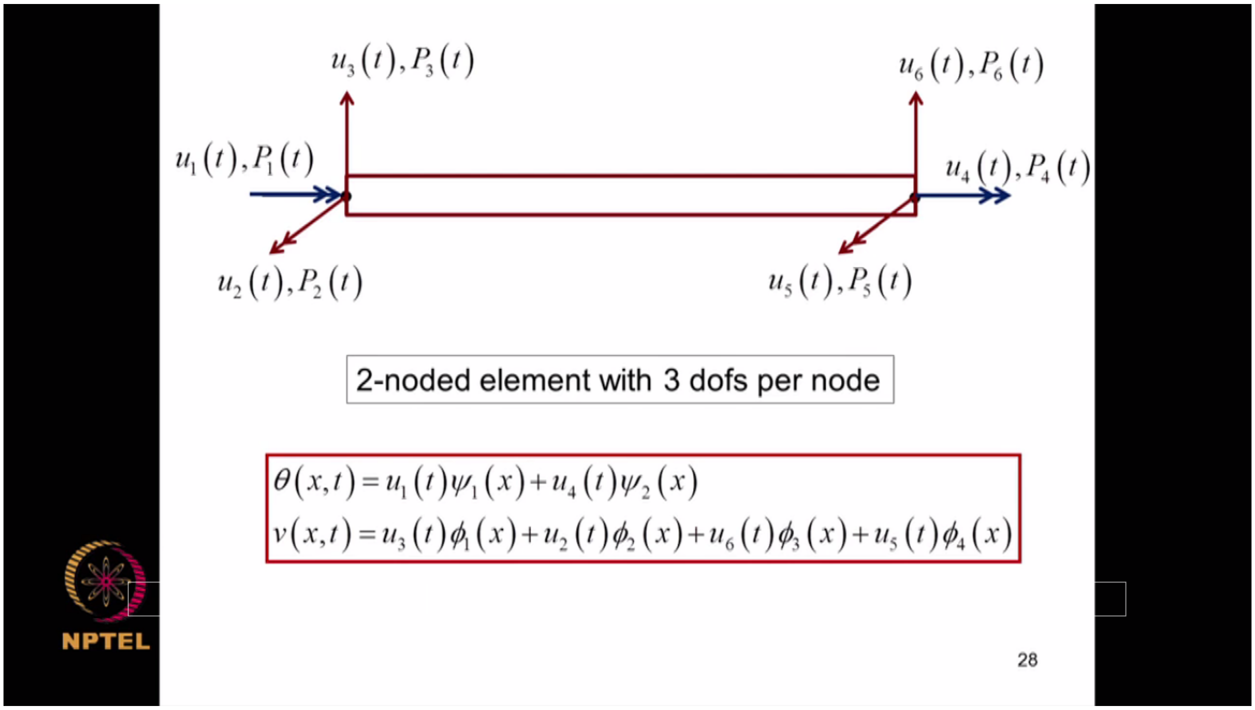


And we also considered alternative representation in terms of transfer matrices briefly and to do that we introduced for example in actually vibrating rod a state vector comprising of displacement and axial force and how it is related at this node the amplitude and displacement of – amplitude of force and displacement how they are related to amplitude have force and displacement at the other node. So this is so called transfer matrix.

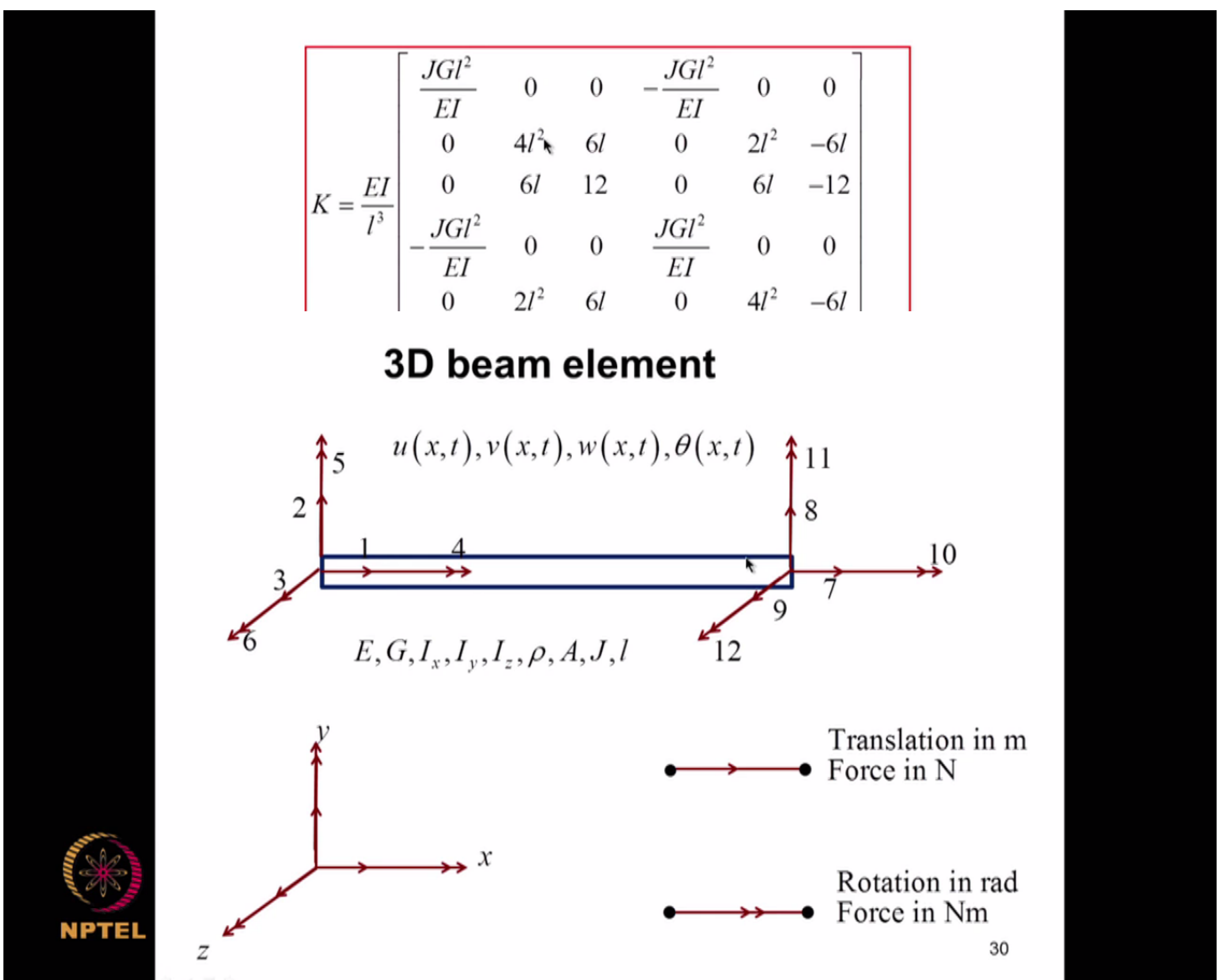
So this can also be derived for actually vibrating bar as well as beam and other more general problems.



Then we moved on to analysis of certain built up structures, a planar frame like this, a grid structure like this or in more general situations a three-dimensional frame. So to analyze a grid element which is displayed here bending in these beams cause torsion in the other beams. So we need to include effect of twisting and that we spend some time and develop the stiffness matrix associated with torsional degrees of freedom.



And we developed the relevant structural matrices that included the flexure and twisting.



This we generalize to three-dimensional beam element and here we considered U beam element typically has two nodes and six degrees of freedom per node. The degrees of freedom are translation in X direction that is the axial deformation, and rotations about Z axis and about Y axis at the two nodes. So and of course translations here. So this we developed and the way we approach the problem was we assume that cross sections have symmetric properties.

$$U = \underbrace{\frac{1}{2}\int_0^L AE\left(\frac{\partial u}{\partial x}\right)^2 dx}_{\text{Axial deformation}} + \underbrace{\frac{1}{2}\int_0^L GJ\left(\frac{\partial \theta}{\partial x}\right)^2 dx}_{\text{Twisting}}$$

$$+\underbrace{\frac{1}{2}\int_0^L EI_z\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx}_{\text{Bending@z}} + \underbrace{\frac{1}{2}\int_0^L EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx}_{\text{Bending@y}}$$

$$\left(\text{with } J = \int_A \left\{\left(-y+\frac{\partial \psi}{\partial x}\right)^2 + \left(z+\frac{\partial \psi}{\partial z}\right)^2\right\} dA\right)$$

$$T = \underbrace{\frac{1}{2}\int_0^L m\dot{u}^2 dx}_{\text{Axial deformation}} + \underbrace{\frac{1}{2}\int_0^L I_{\bar{m}}\dot{\theta}^2 dx}_{\text{Twisting}} + \underbrace{\frac{1}{2}\int_0^L m\dot{v}^2 dt}_{\text{Bending@z}} + \underbrace{\frac{1}{2}\int_0^L m\dot{w}^2 dt}_{\text{Bending@y}}$$

$$\left(\text{with } I_{\bar{m}} = \int_A \rho\left(y^2+z^2\right) dA\right)$$

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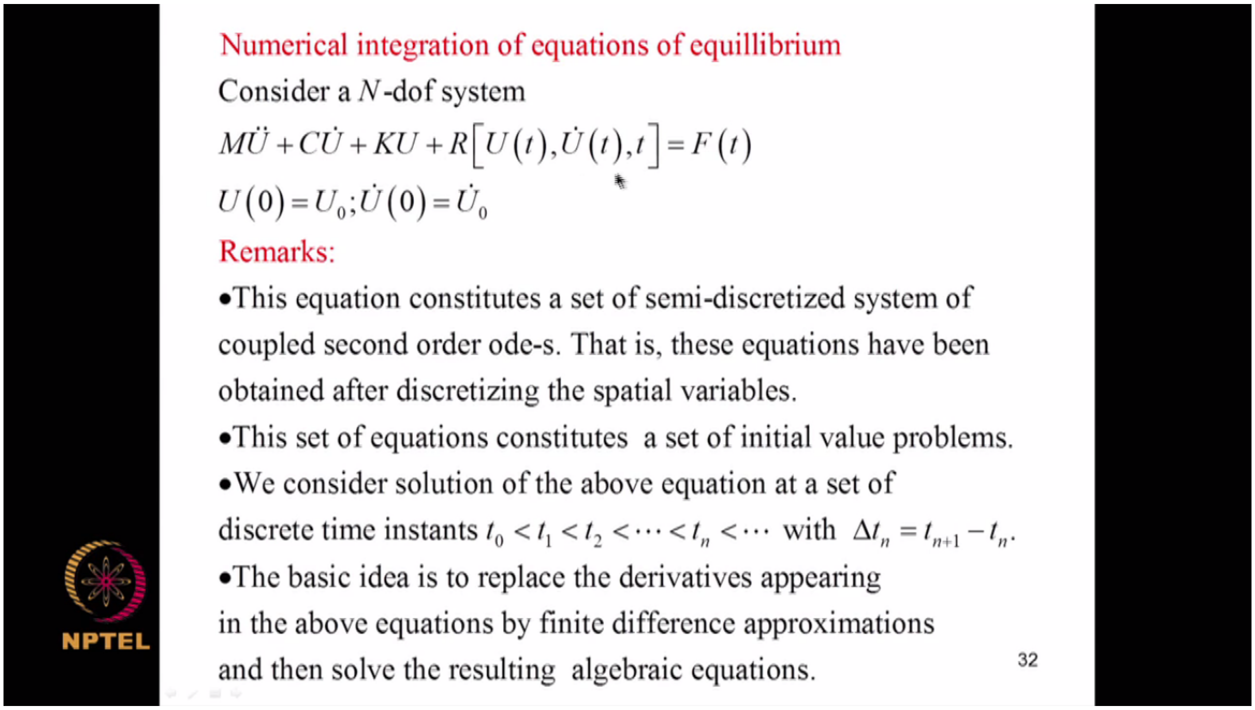
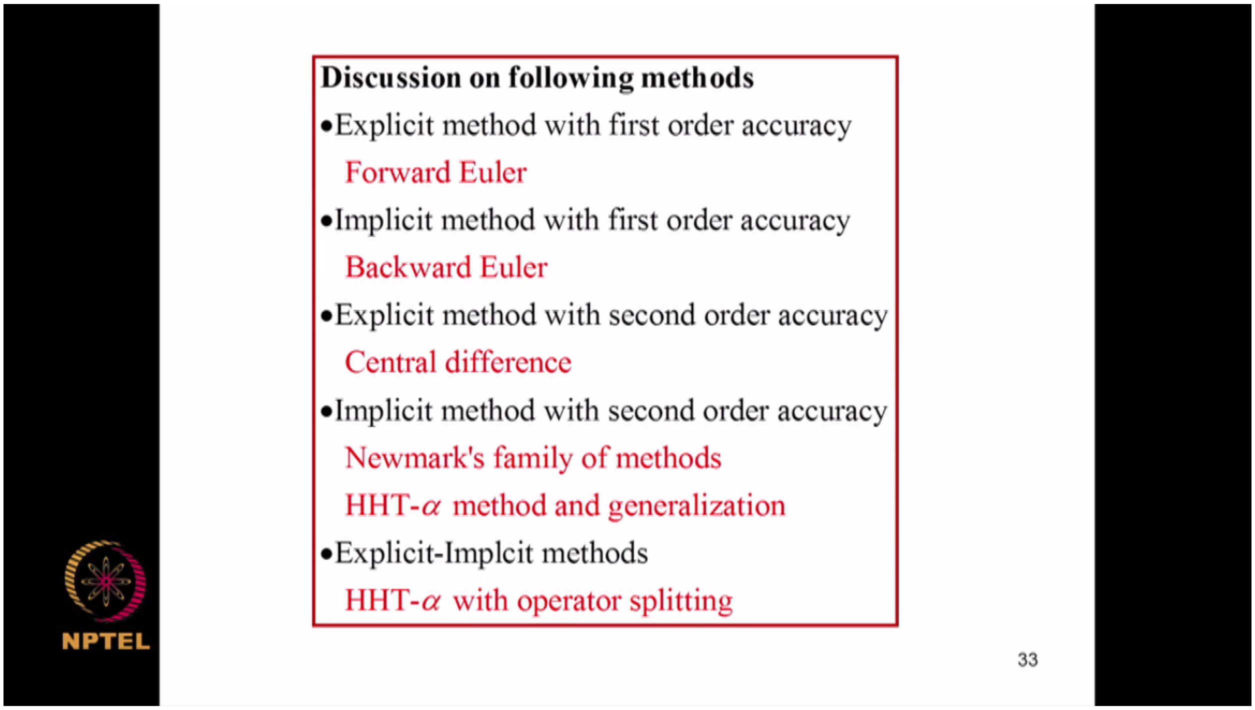
And we set up the expression for axial deformation, twisting, bending about Z and bending about Y where for twisting we use the theory of torsion of prismatic members which we developed in some detail as we went along.

The kinetic energy itself comprised of axial deformation, twisting, bending about Z and bending about Y. In later discussions, we also introduced strain energy due to shear deformation and kinetic energy due to rotary inertia. That led to the treatment of deep beams. Now for all these class of systems that is fairly broad class of system.

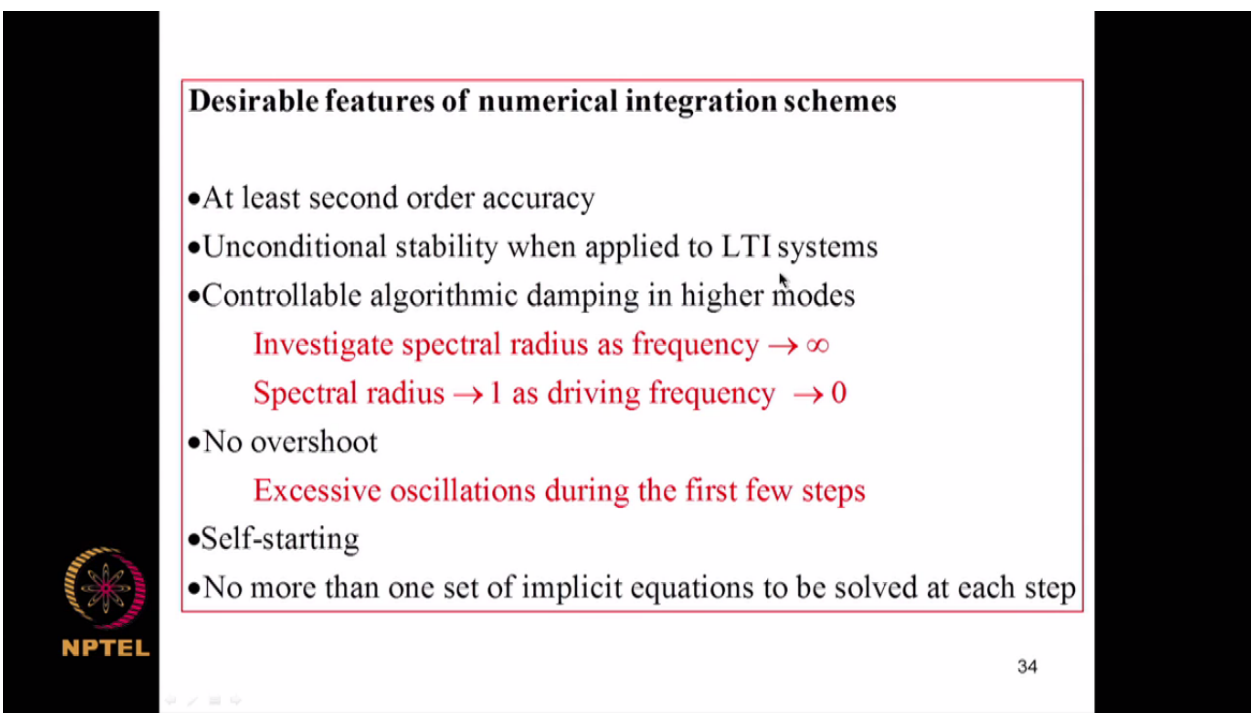


The final equation of motion always had this form. We for the purpose of discussing numerical methods for solving these equations, we introduced the nonlinear terms, with an anticipation that this we will be dealing with later in the course.

So we developed several methods for numerical integration of these equations. These equations at this stage are semi discretized that means time is still continuous. The special variables have been discretized. So the basic idea was to replace the time variation in time again by discretizing time. So we considered solution of this equation at a set of discrete time instant T naught, T1, T2, Tn with some steps as delta T N, TN plus 1 minus TN. The basic idea was to replace the derivatives appearing in these equations by suitable finite difference approximations and convert these equations into appropriate algebraic equations and treat their solution using algebraic methods.



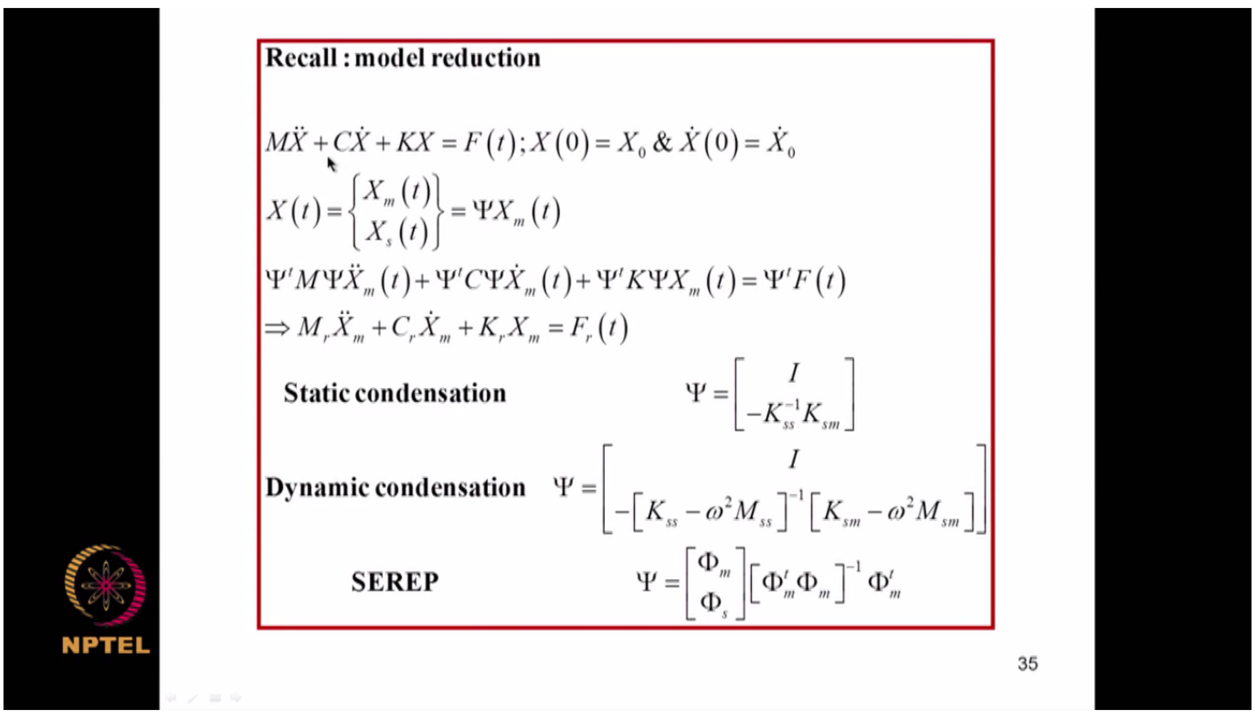
So we discussed several methods forward Euler, backward Euler, central difference and numerous family of methods and the so-called HST alpha method and the generalization and HST alpha with operator splitting. We discussed questions about explicitness of this algorithms and what is implicit and explicit algorithms so on and so forth.



We also discussed for linear time invariant systems, the question of spectral radius and how it influences choice of step size that would mean how to select step size so that the solution should be stable.

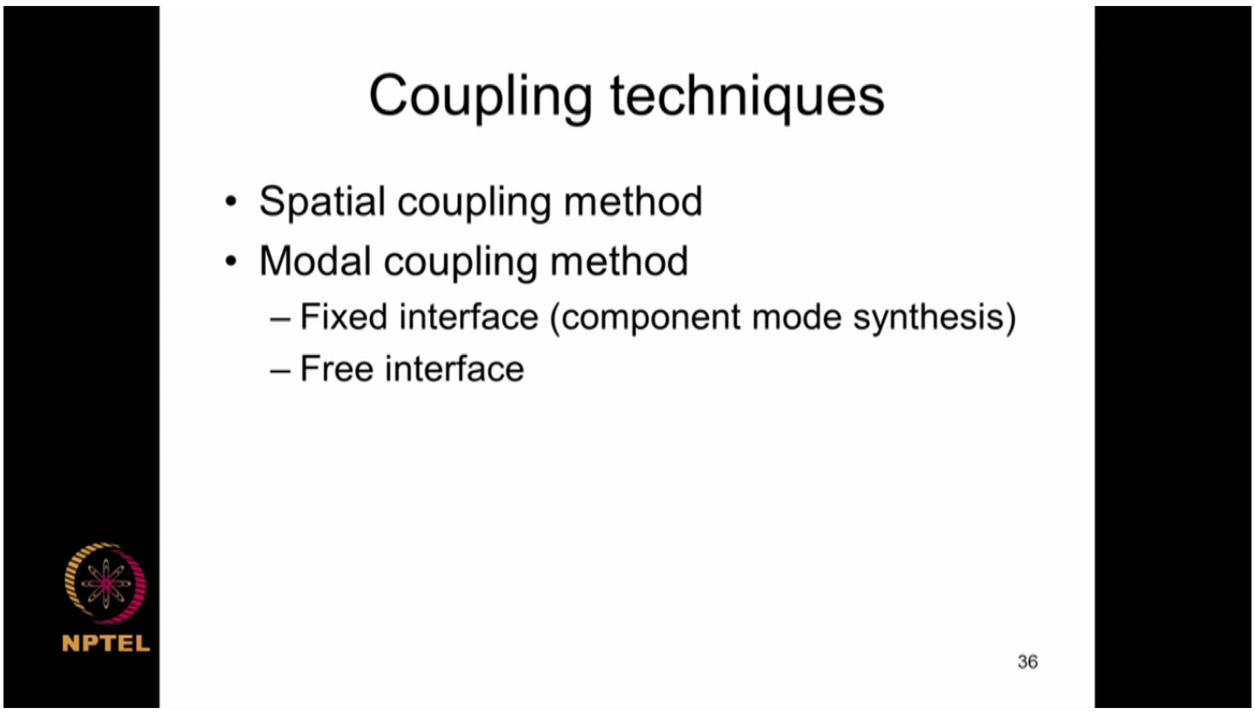
In discussing this, we identified certain desirable features for numerical integration schemes that is -- that we needed to use in dealing with large scale problems. The algorithm should have at least second-order accuracy. They should be unconditionally stable when applied to linear time invariant systems. Then there should be controllable algorithmic damping in higher modes which would require us to investigate the spectral radius as frequency becomes large and frequency becomes small. There should be no overshoot that is excessive oscillations during first few steps should not be there and the solution should be self-starting and no more than one set of implicit equations to be solved at each step.

So we illustrated some of these requirements with specific methods.



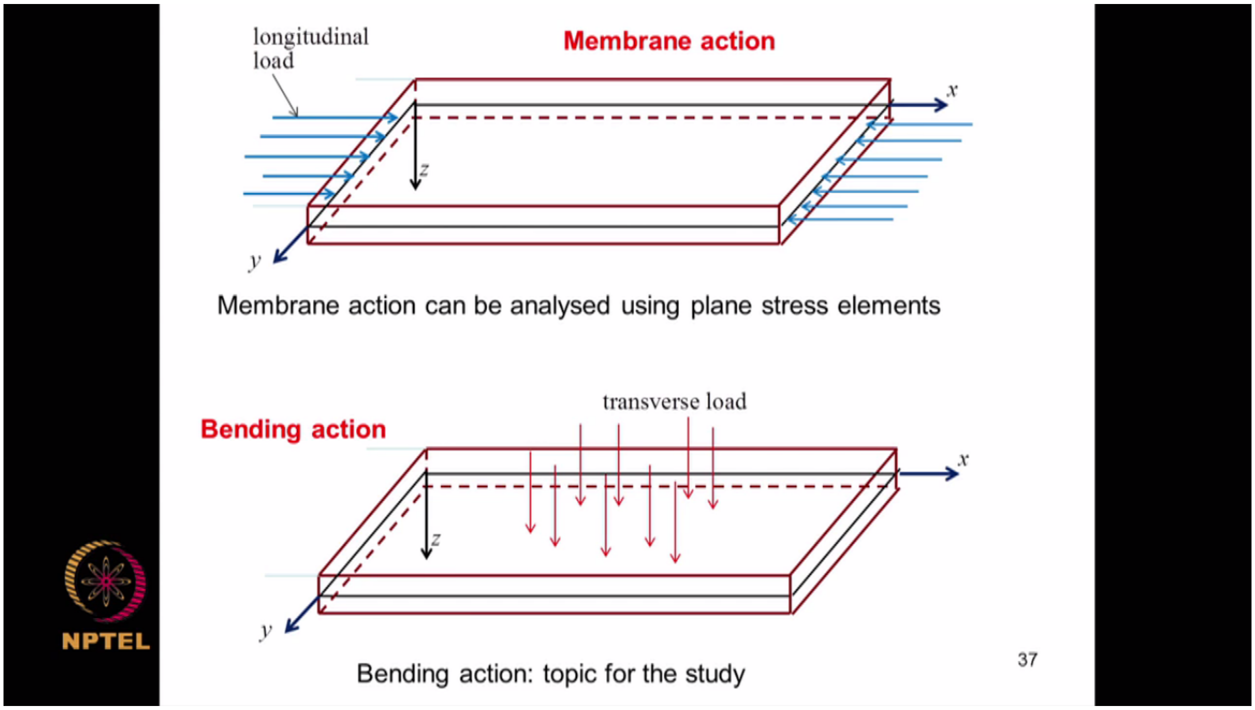
Next we considered questions two questions on dealing with large scale problems. The first question was on model reduction. That means suppose if I have a large scale finite element model such as this how can we reduce it a model with lesser degrees of freedom. This arises originally this type of questions were asked in the context of when computational resources were limited but presently those questions are probably not that relevant. But what is still relevant is when we deal with situations where we are comparing their performance of a computational model with an experimental model and we are trying to reconcile the two models, then we need to reduce the, for example, typically the size of computational model to match the degree of freedoms that are measured in an experimental work. So the experimental degrees of freedom typically tend to be small than the computational model.

So either we can reduce the size of the computational model or expand the size of an experimental model. So that requires certain computational procedures and we discussed three methods; steady condensation, dynamic condensation and system equivalent reduction and expansion process. We saw that this SEREP method is the most versatile. It preserves a set of normal modes in the reduced model whereas in a static condensation there is no such promise, not even one mode need to be correctly captured whereas in dynamic condensation it is possible to capture at least one mode correctly.

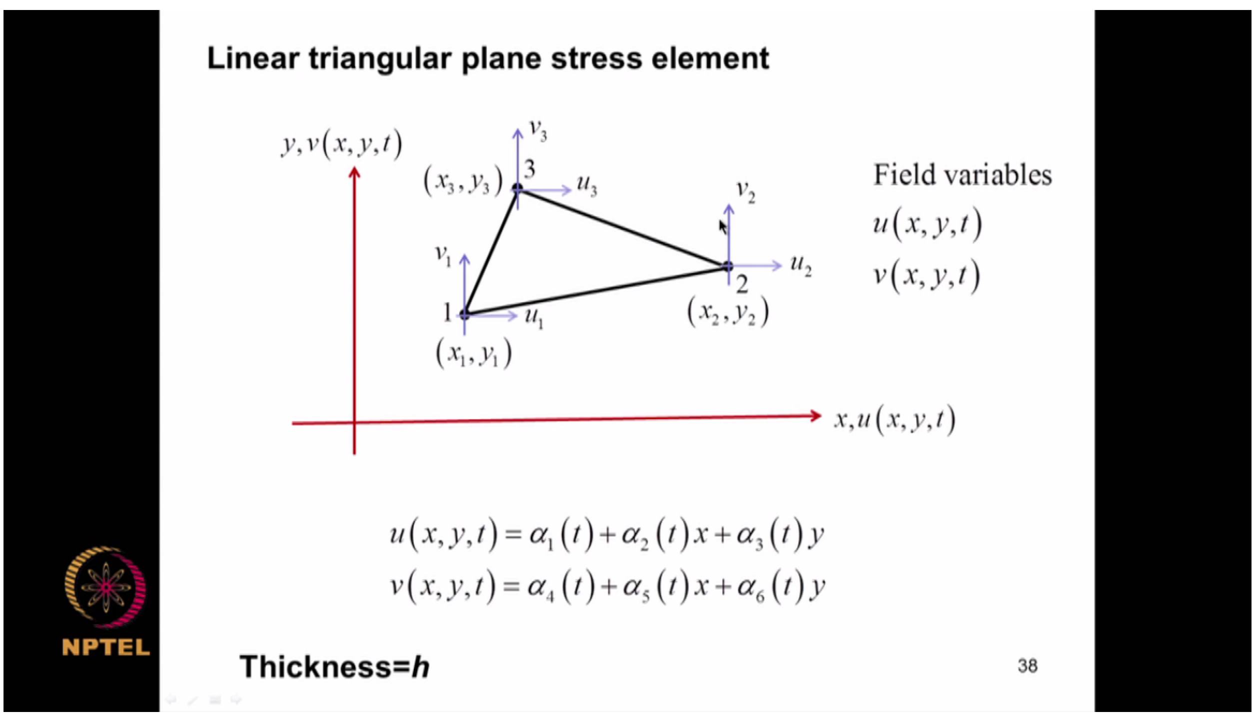


Next we consider situation where again a structure made up of different components and the question was suppose these different components, for example, as in a satellite structure if are being designed and developed by different teams how do we produce a finite element model for the combined system?

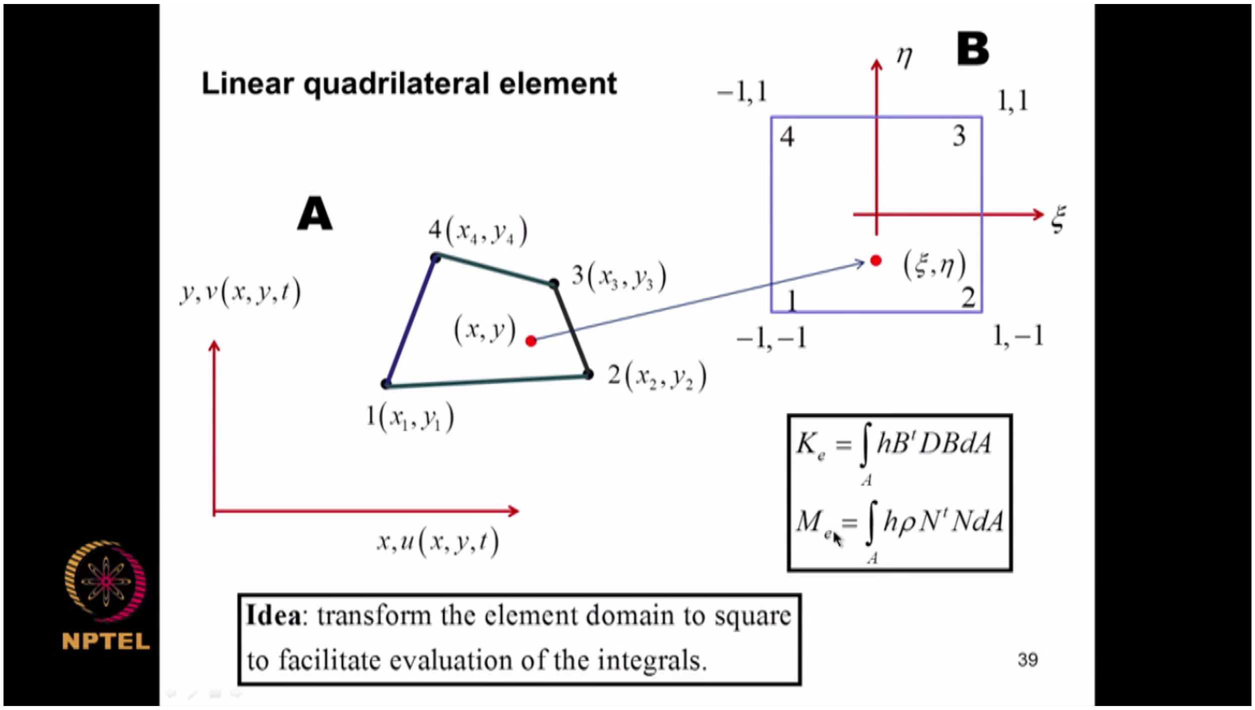
So here we discuss two class of methods that is spatial coupling method and modal coupling. That in discussing the modal coupling method we discuss the component mode synthesis which is one of the well-known coupling techniques with that is used in practice.



After the foray into certain computational aspect we return to the question of element development and we moved on to two dimensional elements. We considered the membrane action due to in plane loads and bending action due to transverse load. So we began by discussing membrane action.

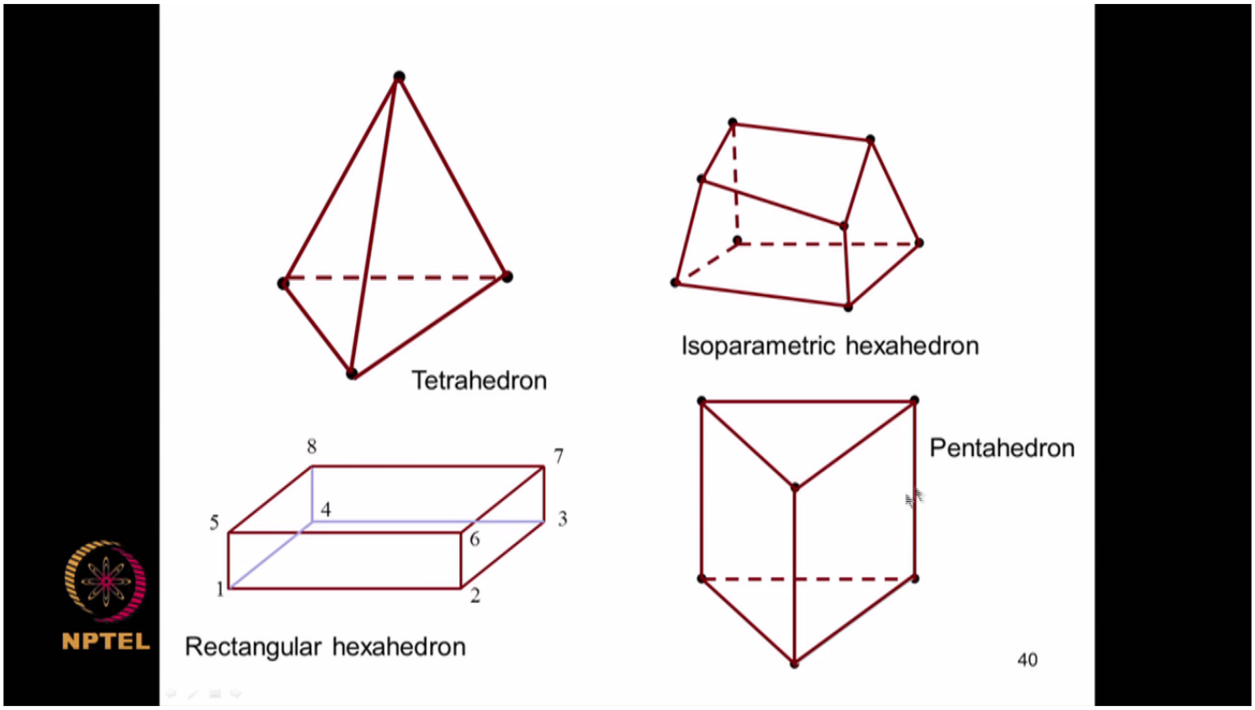


We developed linear triangular plane stress element.

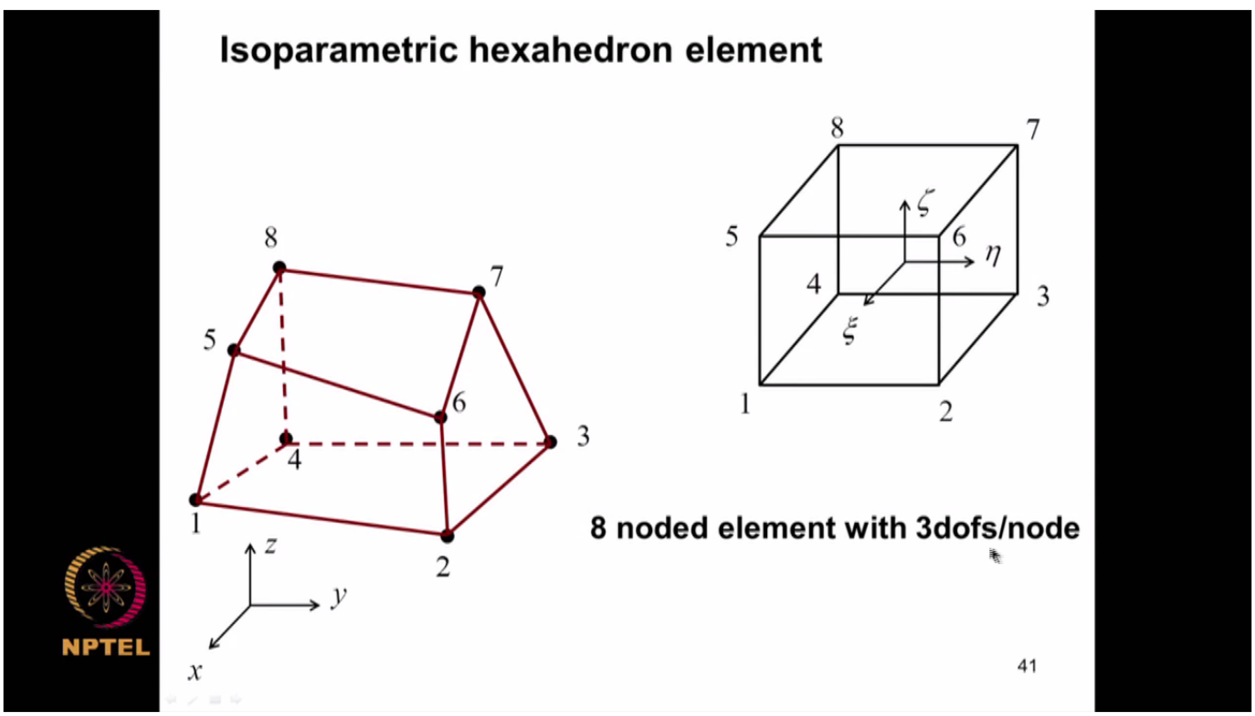


And we generalized it to linear quadrilateral element and that led to the notion of isoparametric formulation which basically was necessitated by requirement to integrate these integrals that appear in the formulation of element stiffness and mass matrices. So the idea is the geometry of this structure is mapped to a master element like this and we interpolate the coordinates in this X and Y in terms of sine theta using the same trial functions as is used to represent the field variables, displacement field variables.

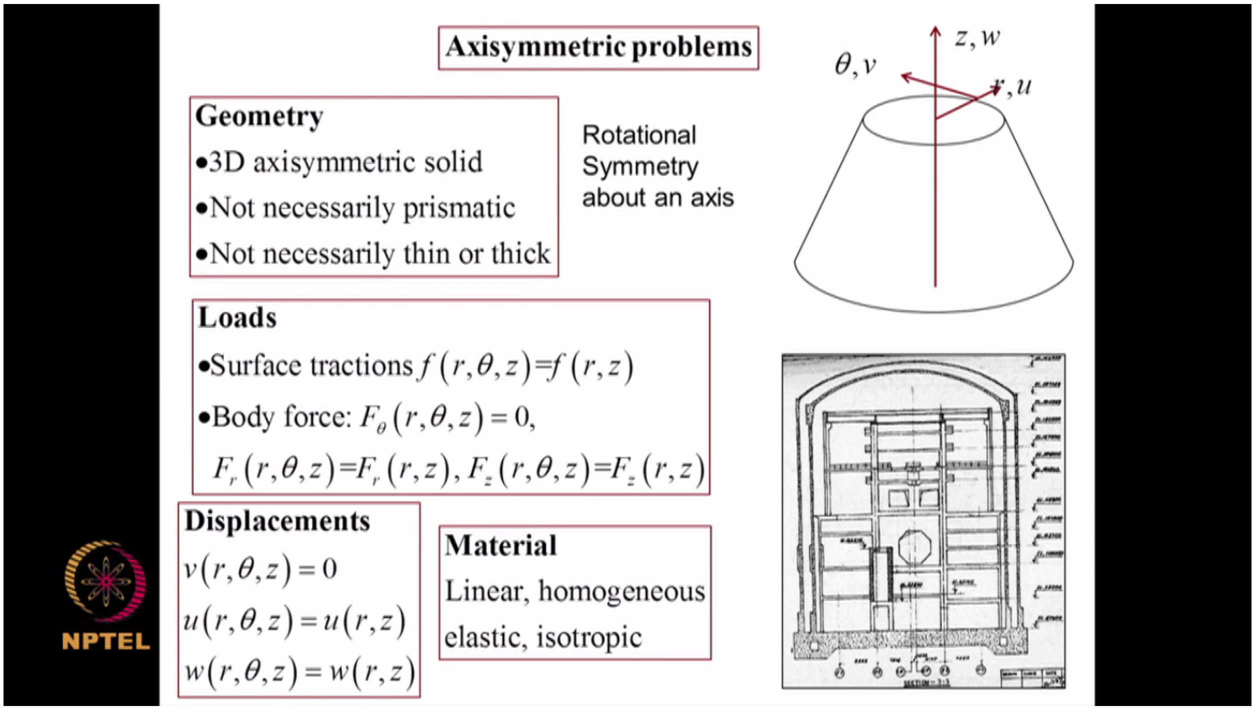
So this transformation was essentially made to facilitate evaluation of the integral leading to the determination of stiffness, mass and forcing stiffness and mass matrices and the forcing vectors.



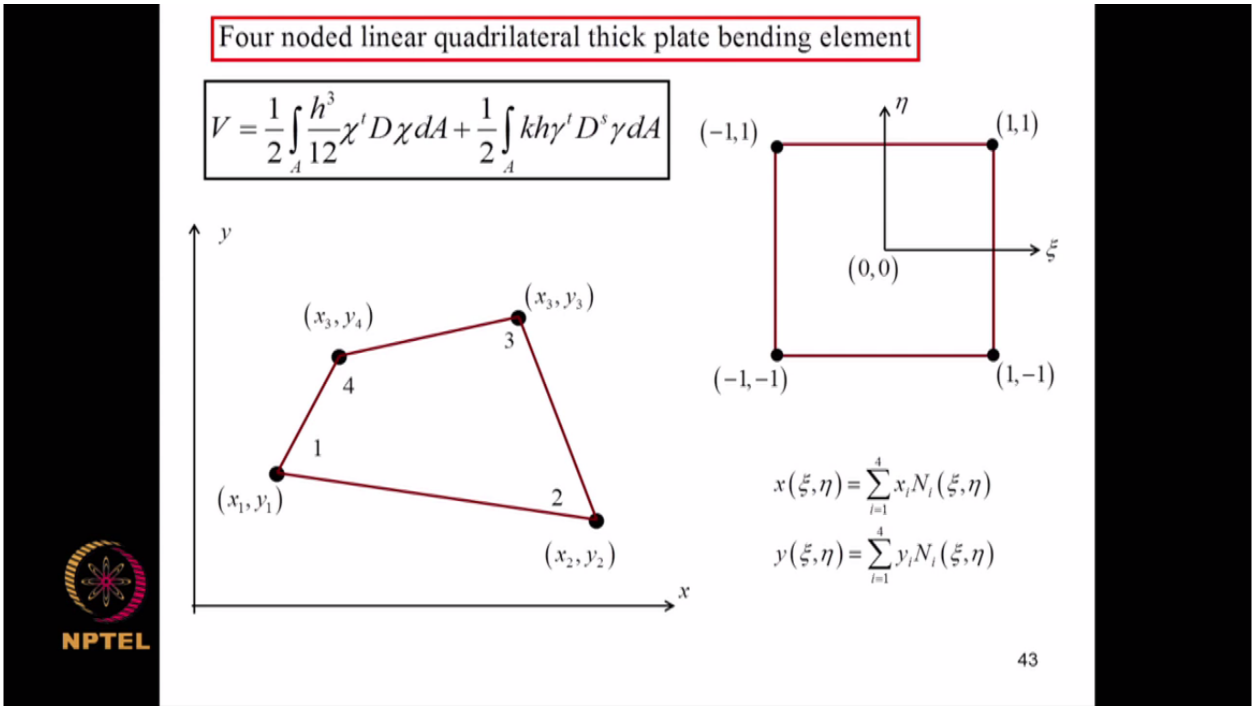
We move on to the discussion of solid elements. We consider tetrahedral element, rectangular hexahedron element, pentahedron, isoparametric hexahedron elements and we developed some of these formulations.



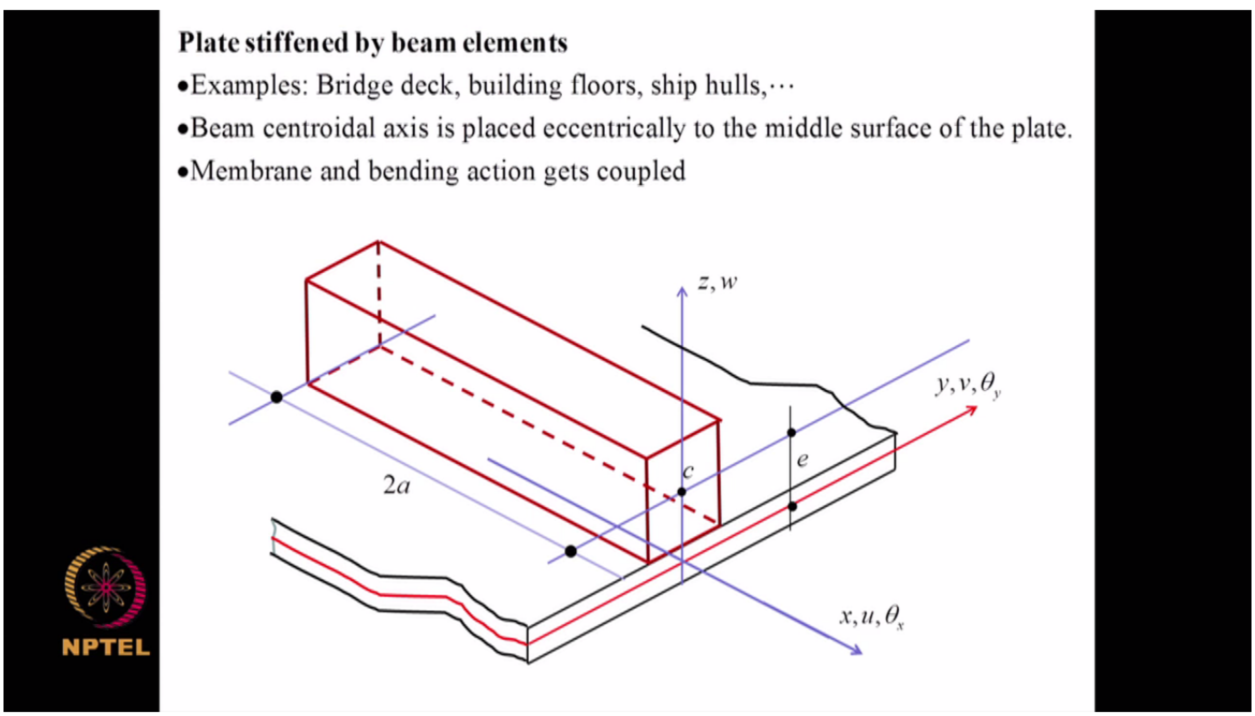
And we discussed we think 8 noded element with three degrees of freedom per node in isoparametric formulation how this can be done.



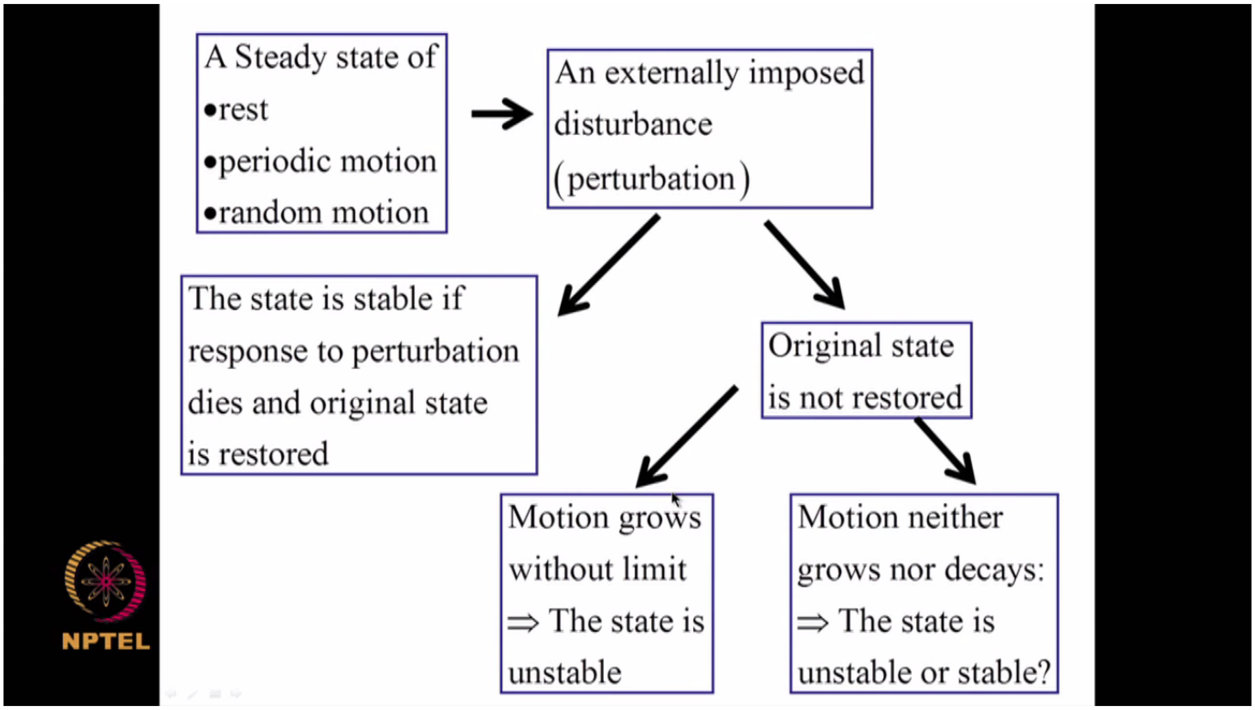
Next we considered problems where there was a axis symmetry in the geometry that means geometry was 3D axis symmetric solid. It is not -- the object is not necessarily prismatic and it is not necessarily thin or thick as in the case of plane stress plane strain models. There were certain restriction that were placed on surfaced traction and body forces and that under certain assumptions on these variations this type of problems can also be reduced to two-dimensional problems and we formulated one such element.



Next we considered plate bending problems and we developed several plate bending element. We saw the question of how certain element formulations lead to non-conforming, nonconformity and how to overcome that. We discussed several of those strategies.

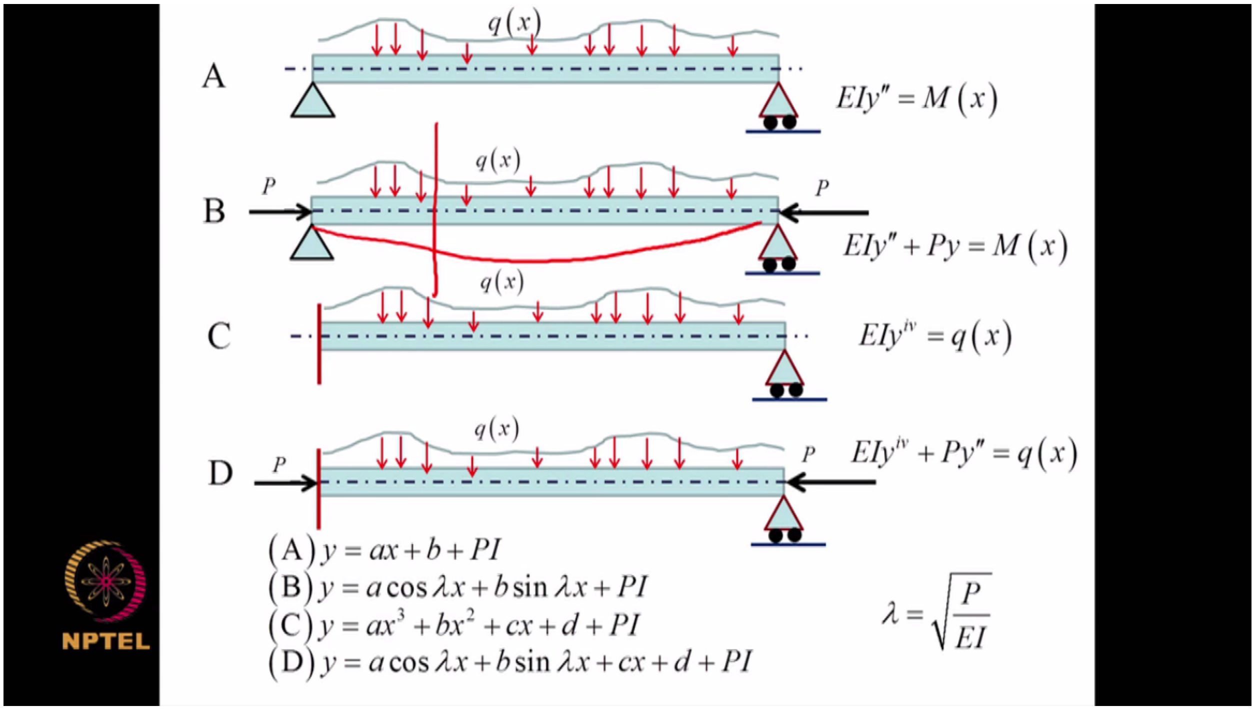


Then we also discussed how to deal with plates that are stiffened by beam elements. So we developed elements for dealing with this type of situations.



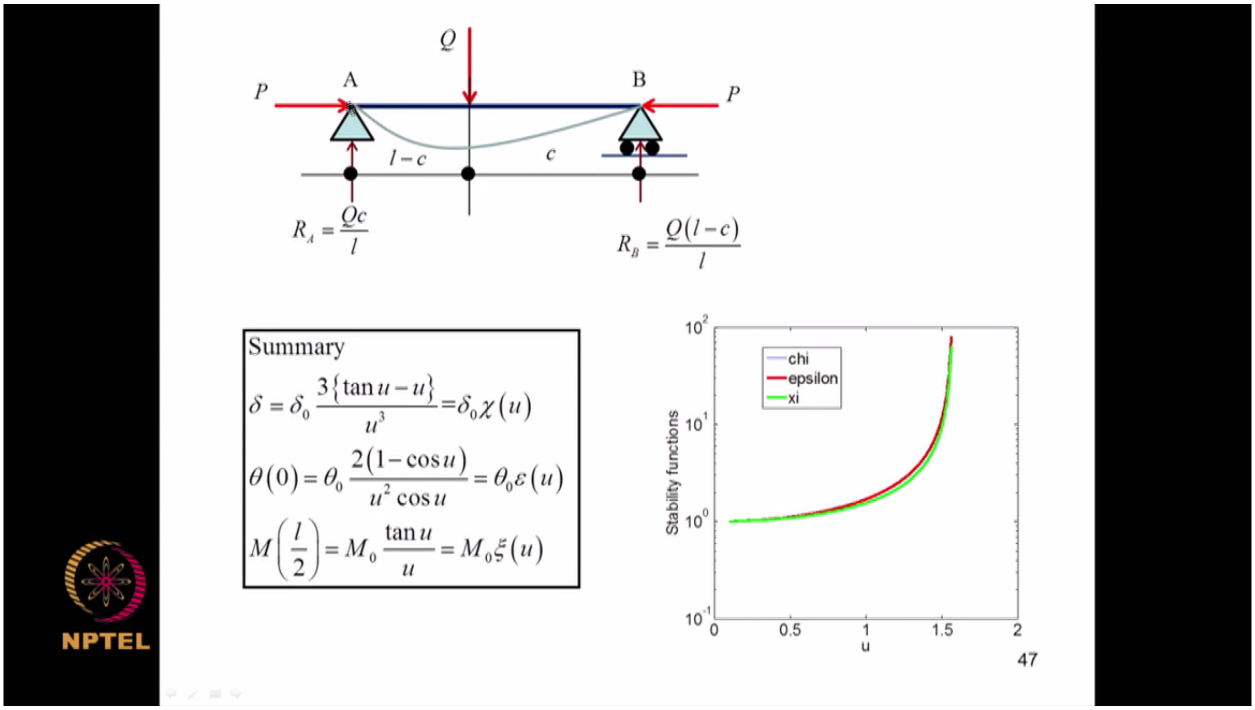
So at the end of these lectures we were ready with stiffness and mass matrices and damping matrices and forcing vector for a wide class of problems and next we moved on to questions about stability analysis. The question of stability was addressed with respect to either a steady state of rest or periodic motion or random motion. We can address in this form. We considered state of rest and periodic motion. We didn't consider random motion in these lectures. So the question we asked was what is the influence of an externally imposed disturbance on these states of rest and periodic motion. If as a consequence of this disturbance if the response dies there disturbance dies off then we say that the state is stable.

Then if the original state is not restored the two possibilities is motion grows without limit and the state is unstable or if motion neither grows nor decays then we reach a stage where we will not be able to resolve using the first order approximation whether the state is stable or unstable.



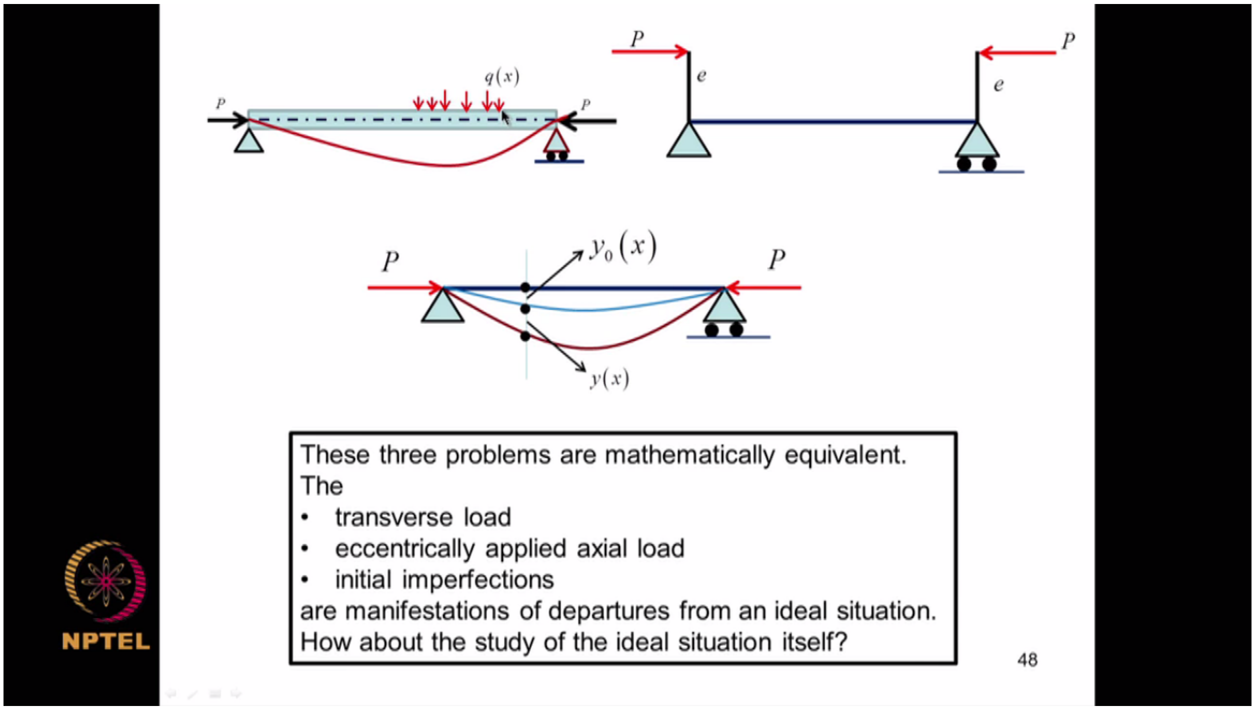
This type of questions we begin by addressing in context of beam columns and for beams carrying axial loads, we notice that there is a dramatic change in the nature of the solution. For example, if we consider this problem where there is a single span beam carrying a transverse load Q of X and axial load P while writing the expression for bending moment at this cross-section, we were able to write the bending moment due to Q of X but when it came to a question of writing the expression for bending moment due to P, we use the deform configuration of the beam.

So this was our first test of issue of nonlinearities where to find bending moment M of X we use undeformed geometry whereas to find the contribution from axial load we use the deformed geometry. So that played a crucial role and we saw that the presence of an axial load plays a crucial - produces dramatic effect and some basic notions like principle of superposition become – they become the first casualty in treatment of axial loads.



So we recalled some results on beam columns and for example, this single span beam carrying a load Q and axial load P we derived the expression for mid span displacement and rotations at the supports and maximum bending moment when load was applied symmetrically and we showed that these responses typically had a structure where Delta naught, theta naught and M naught were responses with P equal to 0, when the axial loads were absent whatever was the response they got magnified or modified by certain functions known as stability functions.

So we introduced Chiphi of U, epsilon U, XI of U and this graph shows typical plots of the stability functions and the most interesting aspect of these functions is that at certain values of load parameter, axial load parameter, these modifications become unbounded thereby indicating that at these values of axial loads the structure would not be stable.

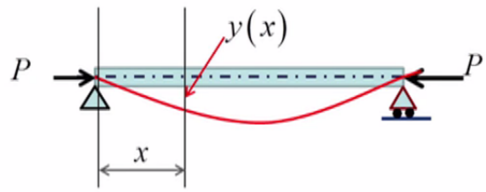


So we considered -- we interpreted presence of a transverse load or our inability to apply axial loads in a truly perfect manner or presence of an initial imperfection as manifestations of certain imperfections in the system. That means these three problems we showed that they are all mathematically equivalent and we interpreted them as manifestations of departures from an ideal situation.

Now how about the study of an ideal situation itself?

What should be P such that a adjoining equillibrium position becomes possible?

Assume: an adjoining equillibrium position is indeed possible.



$$EI\frac{d^2y}{dx^2}+Py=0; y(0)=0; y(L)=0$$

This is an eigenvalue problem.

For what values of P does this equation admit a nontrivial solution?

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That led to the notion of an eigenvalue problem to determine the value of axial load P at which the a neighboring equilibrium position becomes possible. So this led to the notion of Euler buckling loads and we developed that theory.

Stability of equillibrium points

$\dot{x}=f(x,y); \dot{y}=g(x,y)$

Equillibrium points

$f(x^*,y^*)=0$

$g(x^*,y^*)=0$

$x(t)=x^*+\eta(t); y(t)=y^*+\xi(t)$

$$\begin{Bmatrix}\dot{\eta}(t)\\ \dot{\xi}(t)\end{Bmatrix}=\begin{bmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{bmatrix}\begin{Bmatrix}\eta(t)\\ \xi(t)\end{Bmatrix}\Rightarrow\begin{Bmatrix}\eta(t)\\ \xi(t)\end{Bmatrix}=\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix}\exp(st)\Rightarrow\begin{bmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{bmatrix}\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix}=s\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix}$$

$s=a\pm ib$

If $a>0$, $\lim_{t\to\infty}\begin{Bmatrix}\eta(t)\\ \xi(t)\end{Bmatrix}\to\infty\Rightarrow$ the fixed point (x^*,y^*) is unstable

If $a<0$, $\lim_{t\to\infty}\begin{Bmatrix}\eta(t)\\ \xi(t)\end{Bmatrix}\to 0\Rightarrow$ the fixed point (x^*,y^*) is stable

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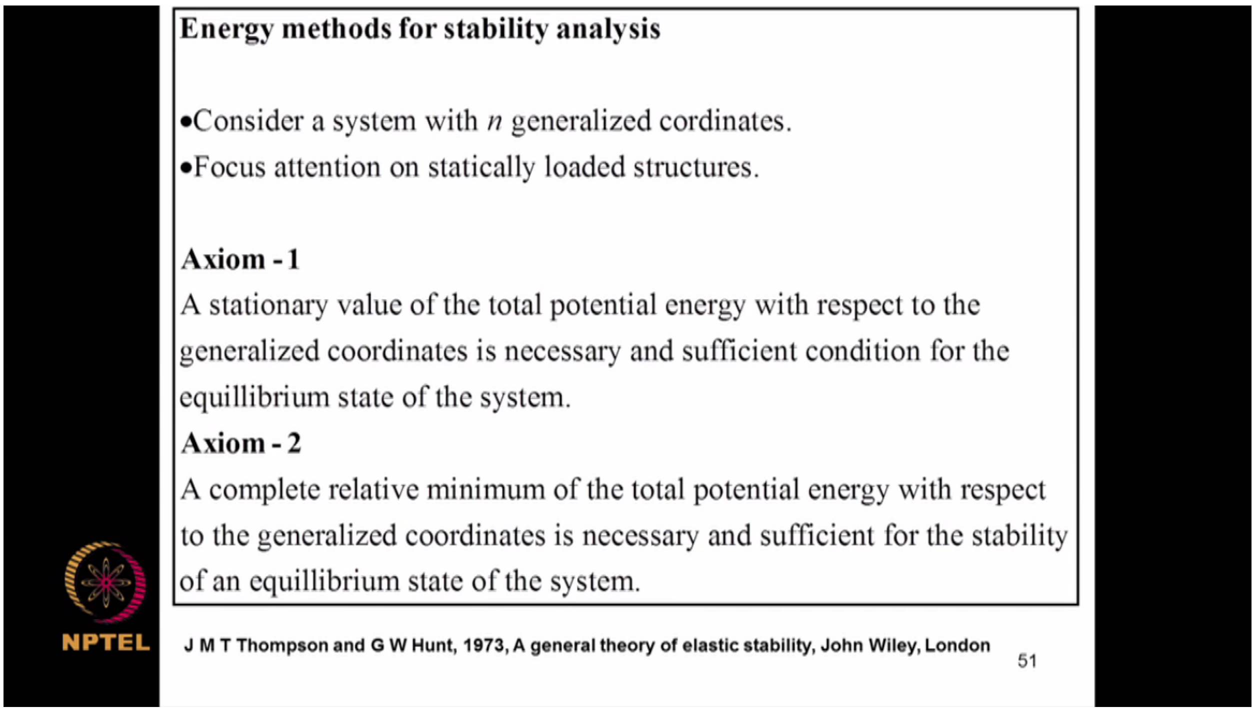
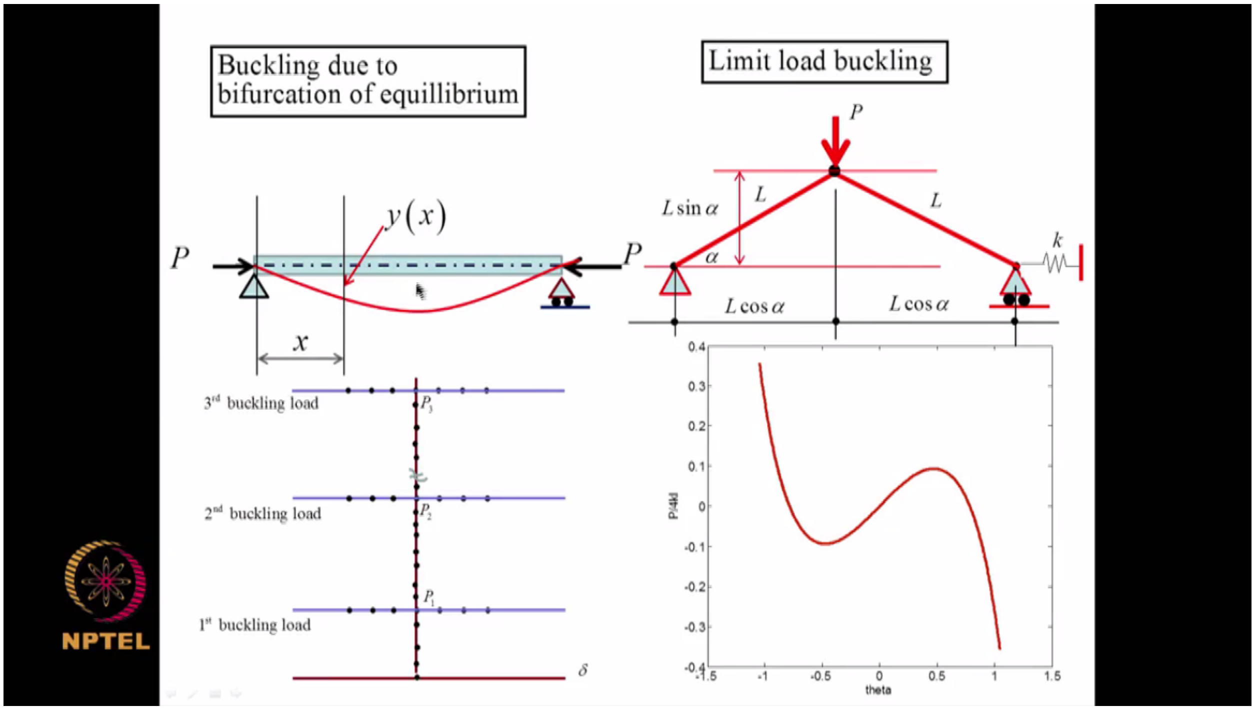
Then we briefly consider a question of stability of dynamical systems because we also asked the question on equilibrium of systems in motion. So we consider differential equations of this form and we define the notion of equilibrium points where X dot and Y dot are zero and we investigated the influence of small perturbations on these equilibrium positions and we found that the nature of the equilibrium points also known as fixed point depend on Eigenvalues values of this gradient matrix when it is evaluated around these fixed points.



Next we initiated two axioms for analyzing stability of more general class of problems and these axioms led to the energy methods for stability analysis. The first axiom stated that stationary value of the total potential energy with respect to generalized coordinates is necessary and sufficient condition for the equilibrium of the system. So the first axiom provided the condition for equilibrium. The second axiom helped us to establish whether that equilibrium position is stable or not. A complete relative minimum of the total potential energy with respect to generalized coordinate is necessary and sufficient for the stability of an equilibrium state of the system.



Now based on this we analyzed the several problems. We considered special specifically two problems one is buckling of Euler-Bernoulli beam and we derived this load deflection diagram of P versus this transverse displacement Delta and we traced the the load deflection paths lose their stability at certain points and the bifurcate. That is one aspect of it. In the structure such as this which are -- shell structures can be thought of as something being similar to this type of structure. Suppose there are two rigid links supported as shown here and there is a load P and we start loading this structure that is at theta equal to zero, I mean P equal to zero we have initially some displacement and as we go on increasing this, the loading path – the load and displacement increased simultaneously and they reach a critical value here at which the structure loses stability and moves to a faraway equilibrium position.

So this type of behavior is known as limit load buckling that means when the structure loses stability the equilibrium position is far away from the position at which it lost its stability. Whereas, here when P approaches P critical, a neighboring equilibrium position becomes possible.

$$U_\sigma = \int_V \left[\sigma_{xx0} \left\{ \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial x} \right)^2 \right\} + \cdots + \sigma_{zx0} \frac{1}{2} \left\{ \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z} \right\} \right] dV$$

$$\delta = \Xi u$$

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = [N]\{u\}_e \Rightarrow \delta = \Xi [N]\{u\}_e = [G]\{u\}_e$$

$$U_\sigma = \frac{1}{2}\int_V \delta' \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} \delta dV \text{ with } s = \begin{bmatrix} \sigma_{xx0} & \sigma_{xy0} & \sigma_{xz0} \\ \sigma_{xy0} & \sigma_{yy0} & \sigma_{yz0} \\ \sigma_{xz0} & \sigma_{yz0} & \sigma_{zz0} \end{bmatrix}$$

$$U_\sigma = \frac{1}{2}\int_V \{u\}_e^t [G]^t \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} [G]\{u\}_e dV = \{u\}_e^t K_\sigma \{u\}_e$$

$$\text{with } K_\sigma = \frac{1}{2}\int_V [G]^t \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} [G] dV = \text{Geometric stiffness matrix}$$

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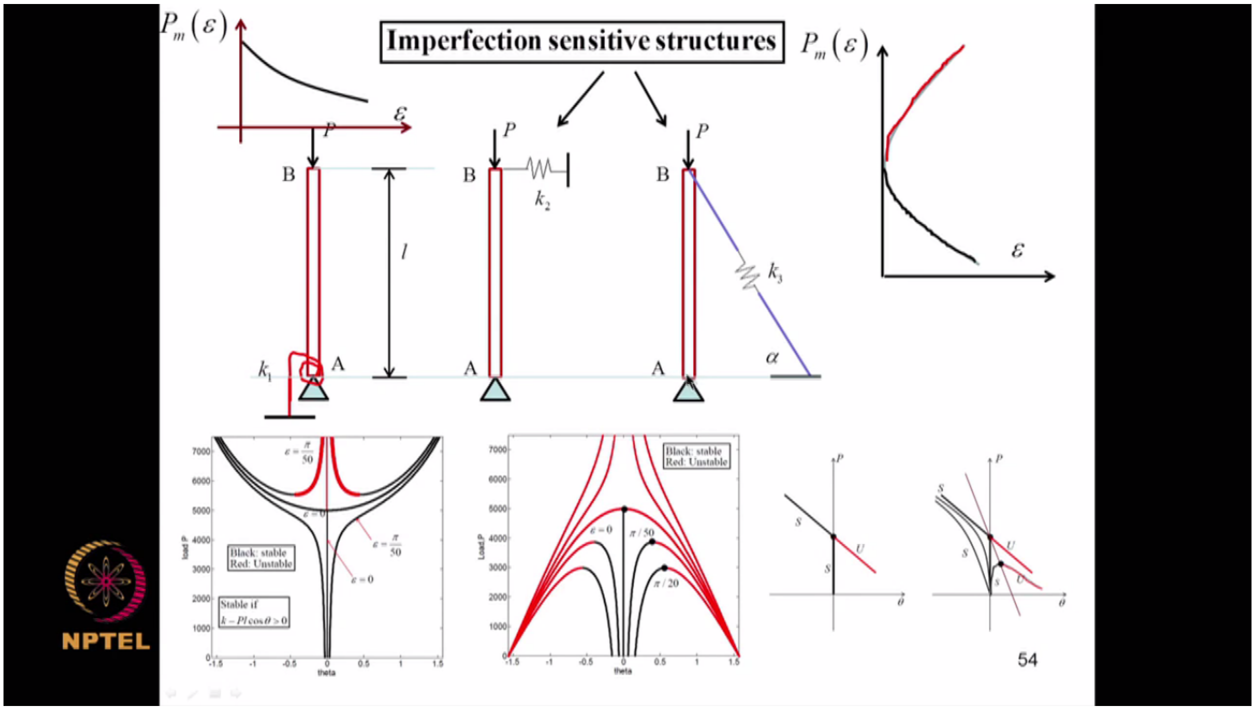
Now to formulate the problems of stability using finite element method we developed what is known as geometric stiffness matrix for different elements and a general theory for that required us to introduce the nonlinear relationship between displacements and strains. So a pre-stress that exists does work on the subsequent deformations and that helps us to formulate the geometric stiffness matrix and we showed that the an eigenvalue analysis of the elastic stiffness matrix and the geometric stiffness matrix that mean KX equal to K Sigma, lambda K Sigma X, helped us to determine the loads at which the structure would lose stability.

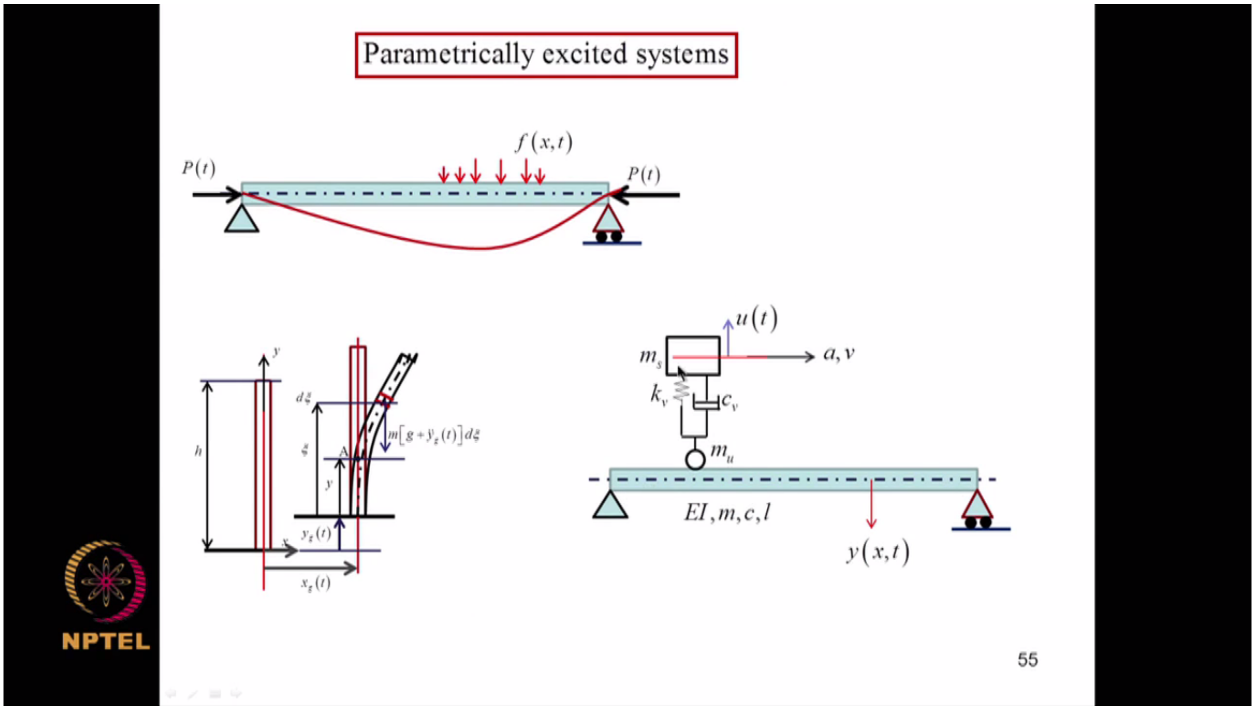


Now another important question we addressed was the interaction between nonlinearity and imperfections. So to illustrate that we considered three archetypal problems. Here in these three problems AB is a rigid bar which is identical in all these three cases but supported in three different ways. Here it is supported through a spring here. Here it is supported through a spring here, whereas, here it is supported through an inclined spring.

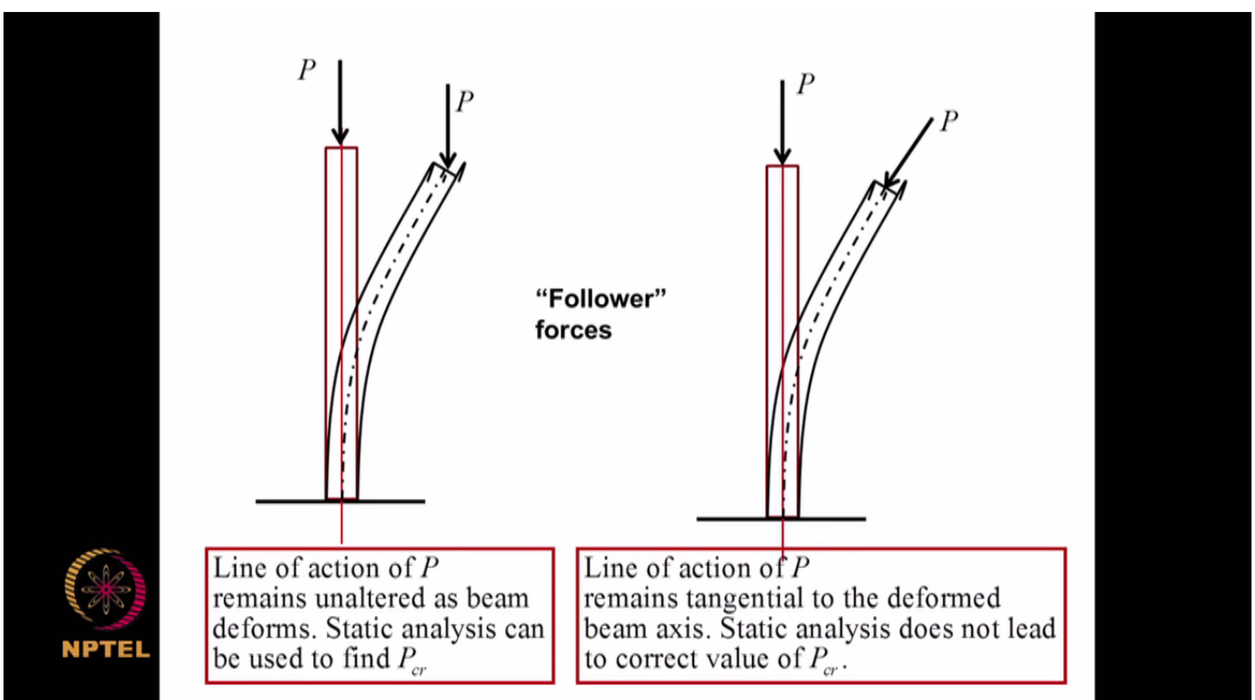
Conceptually we did a thought experiment in which we designed this values of K we selected K1, K2, K3 so that all these three systems had the same value of critical load. But we plotted the load deflection diagram for each one of this and for this case we found that the load deflection path rises along a stable path and it continues to evolve along a stable path without encountering a unstable path.

Now we also introduced in each of these cases slight imperfections and investigated the influence of these imperfections. In the second case, the system with a slight imperfection the load rises, load deflection path rises on a stable path and it encounters an unstable path and the structure loses stability. So here the influence of imperfection is to lower the load carrying capacity of the structure and this kept -- this in the final case similar behavior is observed and again stable path culminates in a unstable - it encounters an unstable trajectory.

So in an experimental work what happens is although these three systems are designed to have the same critical value, it is observed that the critical load evaluated for the third system will be less than what is evaluated for the second system and this will be less than what is evaluated for this system. So these two systems are known as imperfection sensitive structures where the critical load carrying capacity depends on imperfection and plates and shells display this type of behavior.

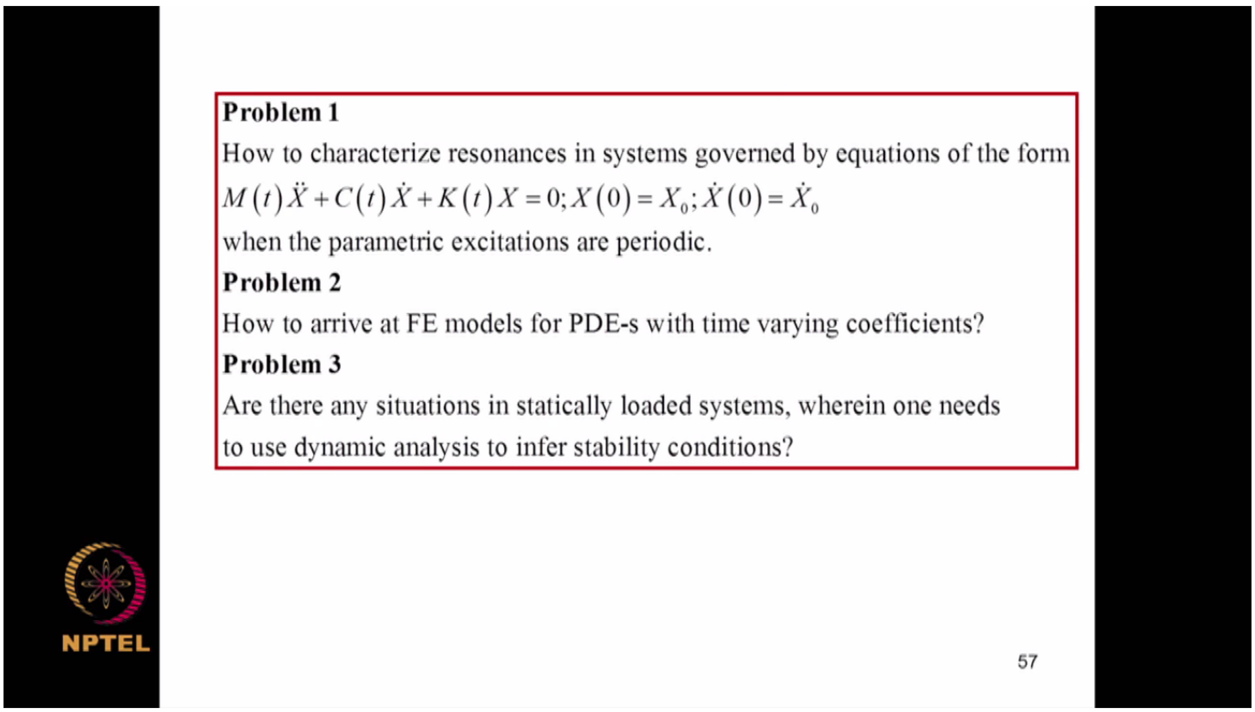


We considered subsequently another class of problems where the action loads were time varying. We consider two situations where there is a tall stack under biaxial earthquake ground motion and also a bridge structure which is traversed by a vehicle. So we formulated equation for these two systems and showed that the coefficients in the governing partial differential equations are time dependent. So such systems are known as parametrically excited systems.

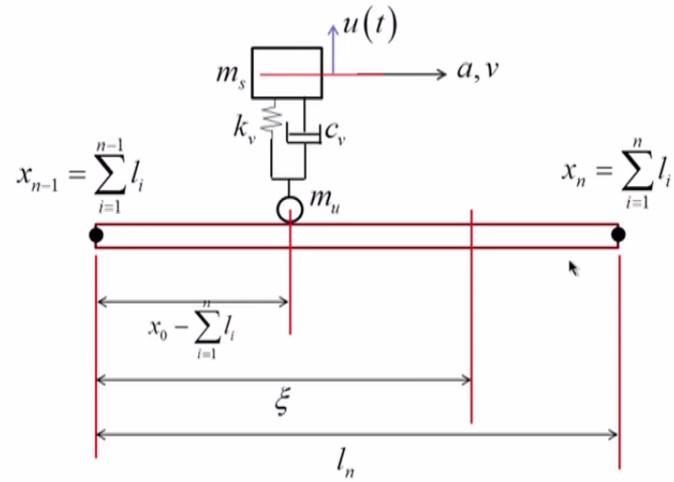


We also considered another class of problems wherein the loads were not time-dependent but the direction of the load in one case for example; if the load P is direction does not change during process of deformation, a static analysis would tell us how this structure behaves. But if the load were to be such that it remains tangential to the deformed axis, a static analysis reveals that this structure is always stable but a dynamic analysis shows that there is a finite value for P beyond which the structure becomes unstable which is consistent with what we anticipate.

So these two cases parametrically excited system and the so called follower force models indicated situations under which a dynamic analysis needs to be done to infer stability of the system.



Now in the in this context we consider three problems. First one was how to characterize resonances in systems governed by equations of this form where the mass damping and stiffness matrices are time-dependent. And especially when these time variations are periodic. Next how to arrive at finite element models for partial differential equations with time-varying coefficients? Subsequently are there – we also consider the question if there are any situations in statically loaded systems wherein one needs to use dynamic analysis to infer stability conditions. So while answering the first question we developed the theory of flow case coefficients and we developed a procedure on how to evaluate the flow K matrix and we showed that eigenvalues of the flow case matrix enable us to answer these questions.



$$\xi = x - \sum_{i=1}^{n-1} l_i \Rightarrow 0 < \xi < 1$$

$$\xi_0 = x_0 - \sum_{i=1}^{n-1} l_i = \text{position of the wheel in local coordinate system}$$

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So to answer the second type of deal with second type of problems we considered beam carrying a single – moving single degree freedom system and we formulated the governing equation and developed a finite element model based on weak formulation.

Element level equation of motion

$$\text{Let } d_n = \begin{Bmatrix} \alpha(t) \\ u(t) \end{Bmatrix} \; \& \; F_n = \begin{Bmatrix} -V_{n-1} \\ M_{n-1} \\ V_n \\ -M_n \end{Bmatrix}$$

$$\begin{bmatrix} M + \tilde{M}(t) & 0 \\ 0 & m_s \end{bmatrix} \begin{Bmatrix} \ddot{\alpha}(t) \\ \ddot{u}(t) \end{Bmatrix} +$$

$$\begin{bmatrix} C + \tilde{C}(t) & I\left[t_{n-1} < t < t_n\right]\Phi^t(\xi_0)c_v \\ -I\left[t_{n-1} < t < t_n\right]\overline{\psi}_1(\xi_0) & c_v \end{bmatrix} \begin{Bmatrix} \dot{\alpha}(t) \\ \dot{u}(t) \end{Bmatrix}$$

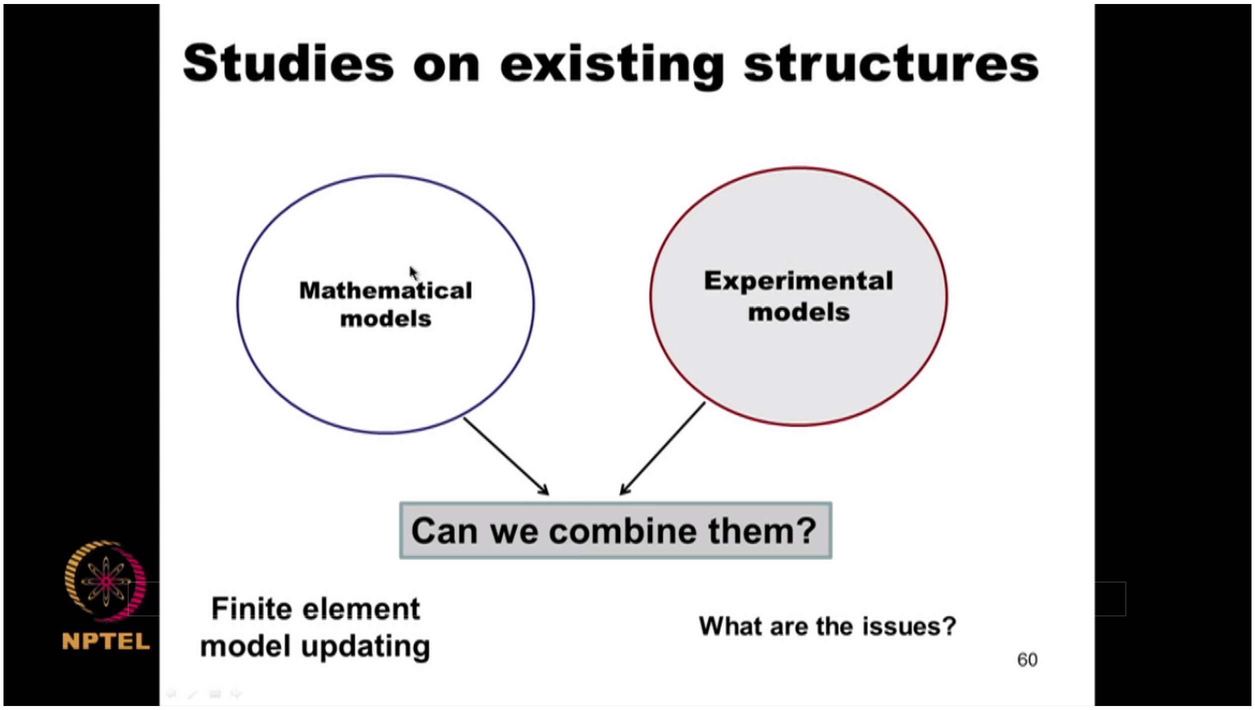
$$+ \begin{bmatrix} K + \tilde{K}(t) & -I\left[t_{n-1} < t < t_n\right]\Phi^t(\xi_0)k_v \\ -I\left[t_{n-1} < t < t_n\right]\overline{\psi}_1(\xi_0) & k_v \end{bmatrix} \begin{Bmatrix} \alpha(t) \\ u(t) \end{Bmatrix}$$

$$= \begin{Bmatrix} I\left[t_{n-1} < t < t_n\right]\Phi^t(\xi_0)(m_u + m_s)g \\ 0 \end{Bmatrix} + \begin{Bmatrix} -F_n \\ 0 \end{Bmatrix}$$

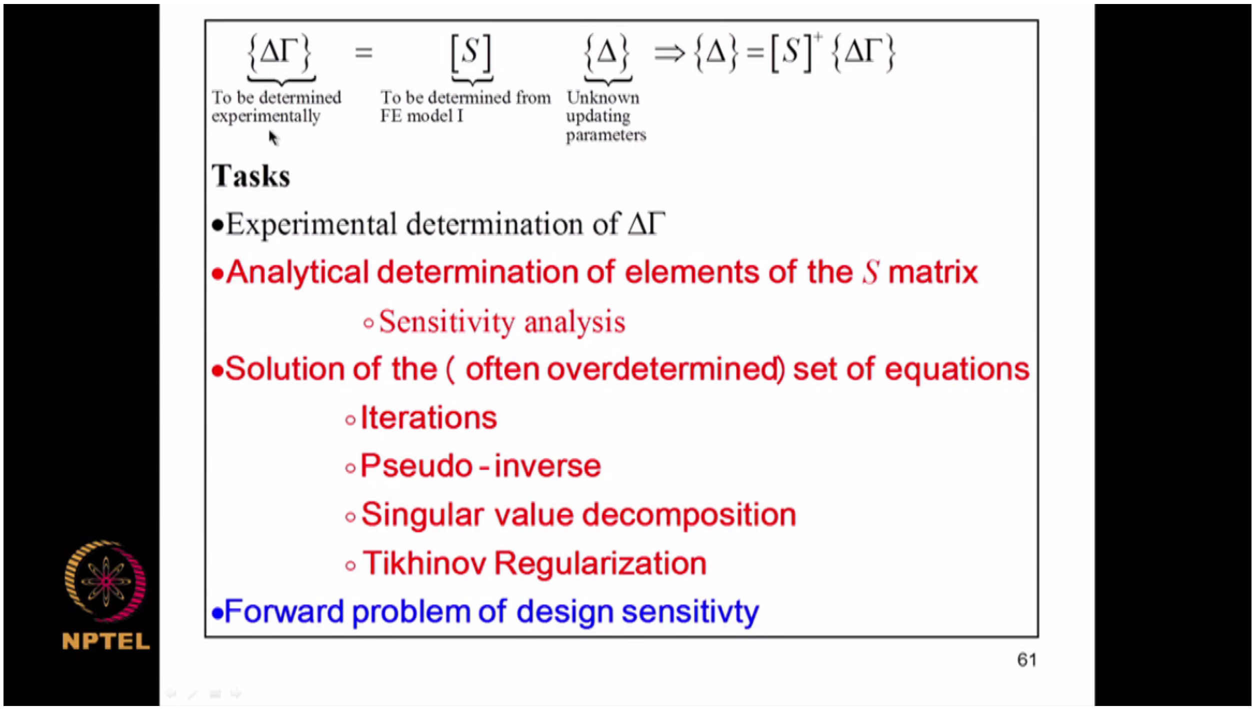
Further steps: assembly, imposition of BCs

And showed that the element mass matrix, damping matrix and stiffness matrices are unsymmetric and time-dependent for this class of problems. The interaction between the vehicle and the structure induces this unusual features.

So this type of problems need to be handled only in time. They do not have natural coordinates. The concept of natural frequency mode shapes are no longer valid.

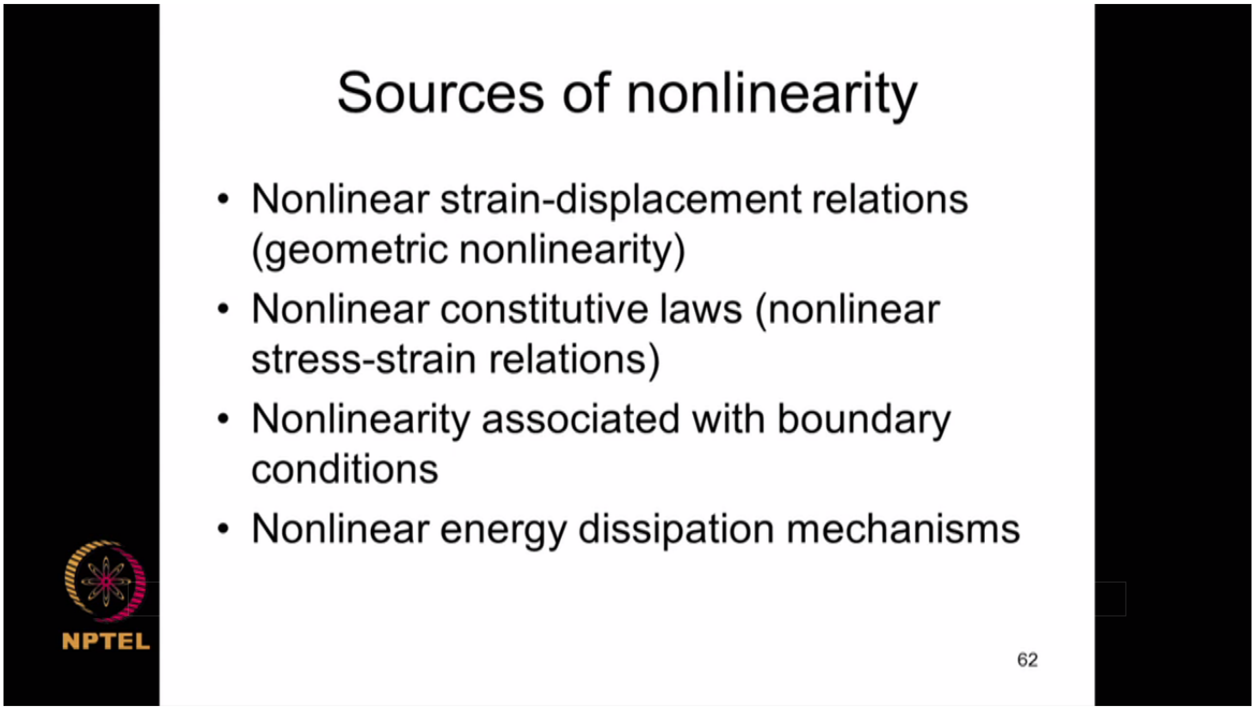


We briefly then considered questions on the role of finite element modeling in dealing with existing structures. So before a structure comes into existence we have only mathematical models to deal with that type of problems. That is typically what we do when we design a structure which is yet to come into existence. Moment the structure comes into existence of course the mathematical modeling technique still remain valid but also additionally we have experimental tools becoming available to us and thus we can measure the performance of the structure under either diagnostic loads or operational loads. So the prediction in these situations from an experimental model and a mathematical model often do not agree and the question is how do we update the mathematical model to reconcile these two predictions. So this leads to the topic of finite element model updating and we briefly reviewed the issues related to this question and developed specifically one approach that was based on so called inverse sensitivity analysis.

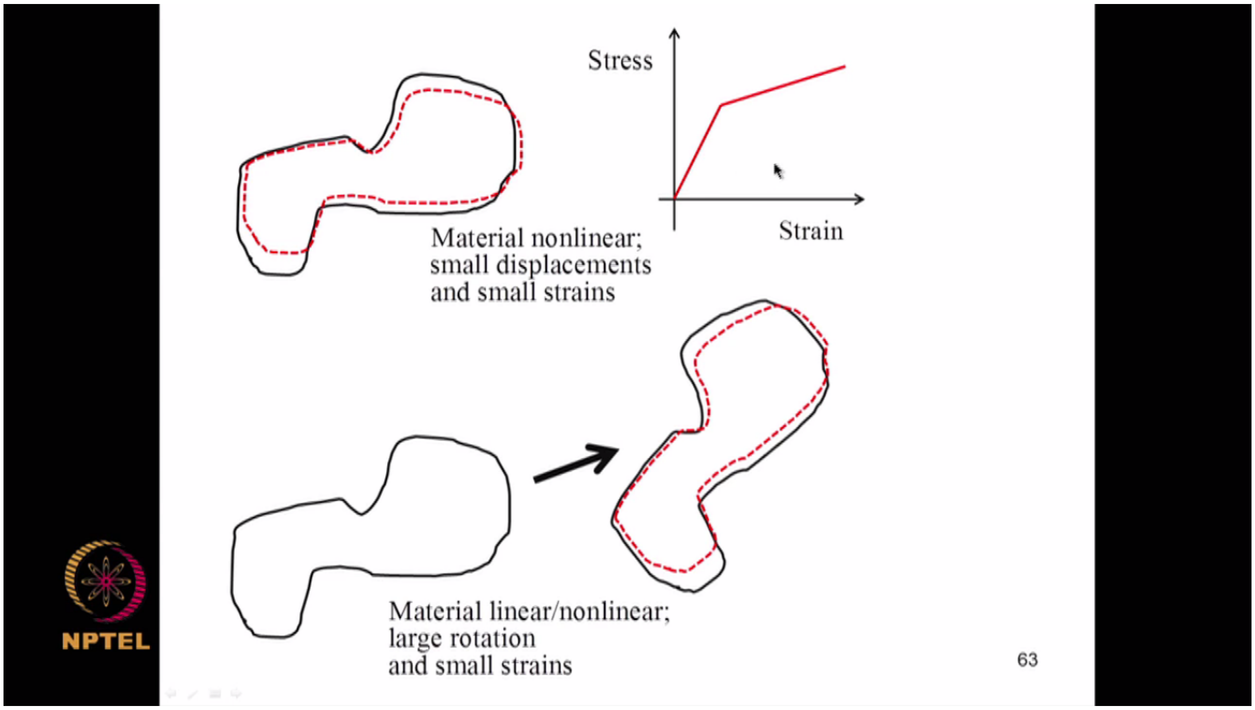


So we derived the updating equations in terms of changes to be made to system parameters based on observed changes in certain system responses and these we showed is connected through a matrix known as sensitivity matrix.

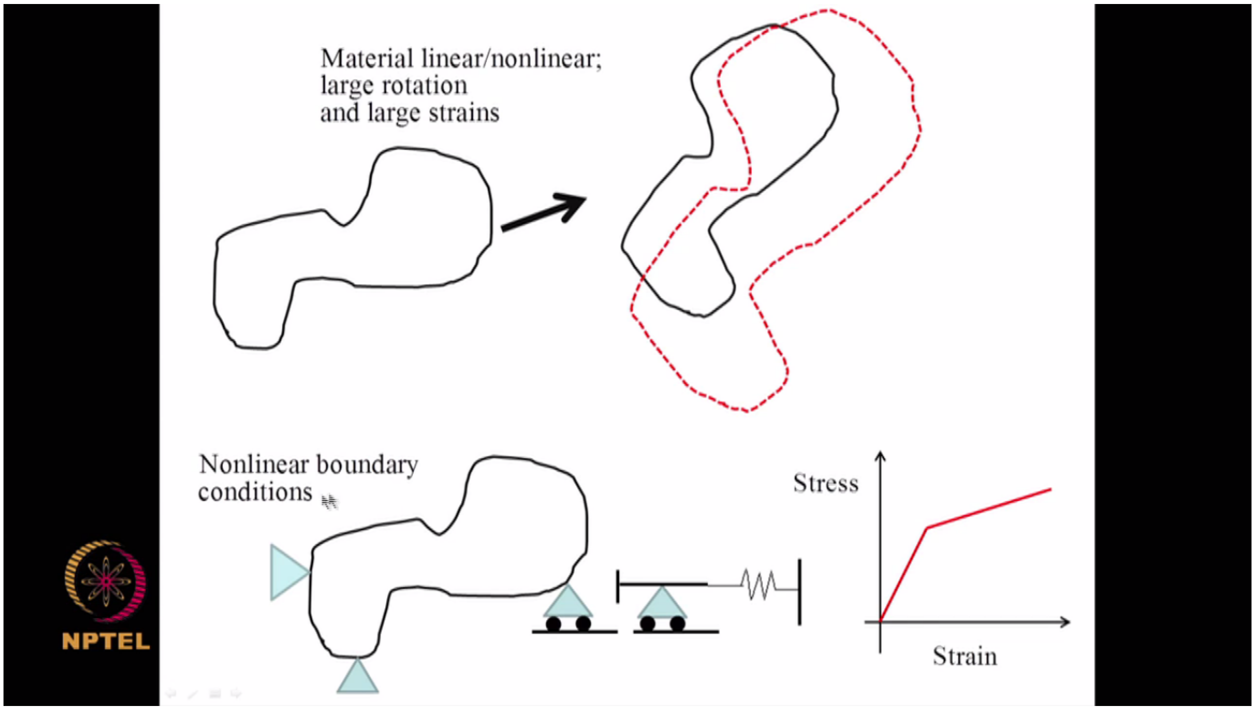
So we carry – we considered several issues here how to formulate this S matrix, and how to – this the resulting equations will be often a set of or determined equations that require special techniques to solve them. We discussed pseudo-inverse method, singular value decomposition and Tikhonov regularization approaches for dealing with this class of problems.



In the final part of our course we started talking about how to deal with nonlinearity. So the sources of nonlinearity we identified were either related to nonlinear strain displacement relations in which case we said the nonlinearity is geometric nonlinearity or the relationship between stress and strain could be nonlinear. Then we call this a nonlinear constitutive. I mean here we had nonlinear constitute laws and this type of nonlinearity was called material nonlinearity. Then nonlinearity associated with boundary conditions as in contact problems or free plays and so on and so forth induces a different kind of nonlinearity. The energy dissipation mechanisms also bring in newer forms of nonlinearity. It could be friction. It could be impacting free place etc.

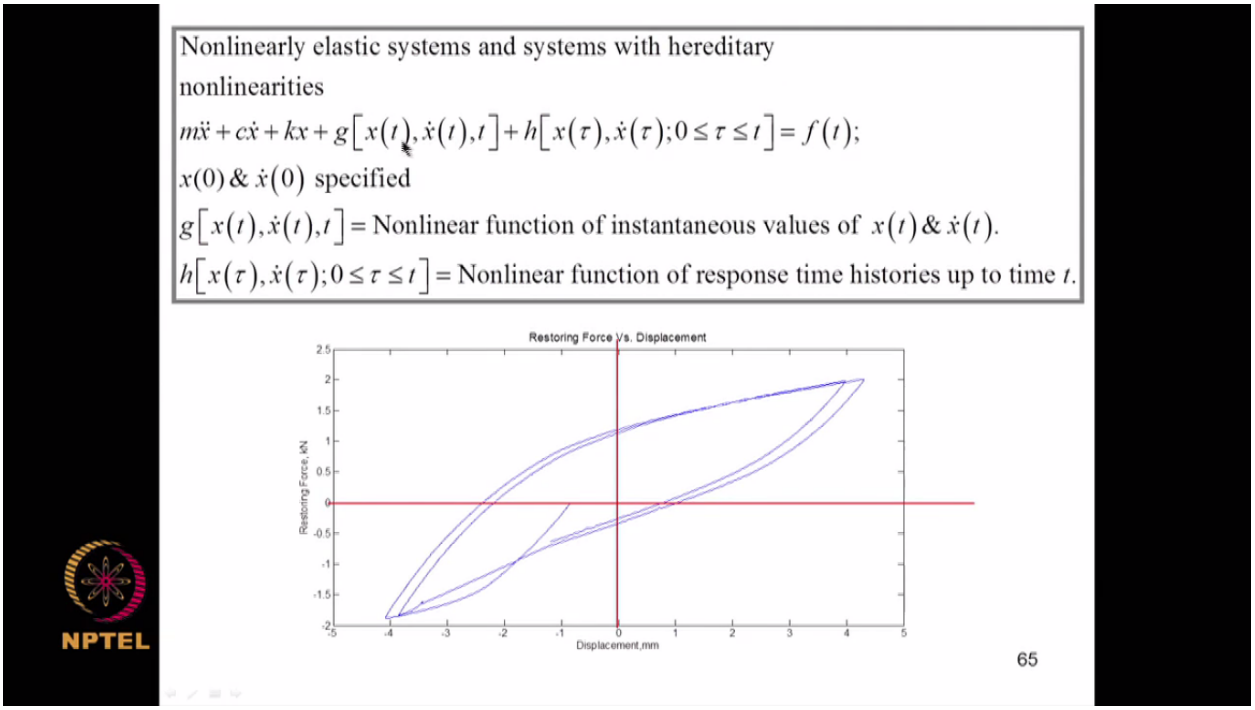


Now while formulating this we can conceptualize different frameworks for example the displacement and rotation of a structure could be small but the stress-strain relationship could be nonlinear. So this is small deformation but materially nonlinear. Geometrically linear; materially nonlinear. Here the material could be linear or nonlinear but there are large rotations and small strains. So one of the questions that needs to be carefully addressed in dealing with nonlinear problems is treatment of rotations.

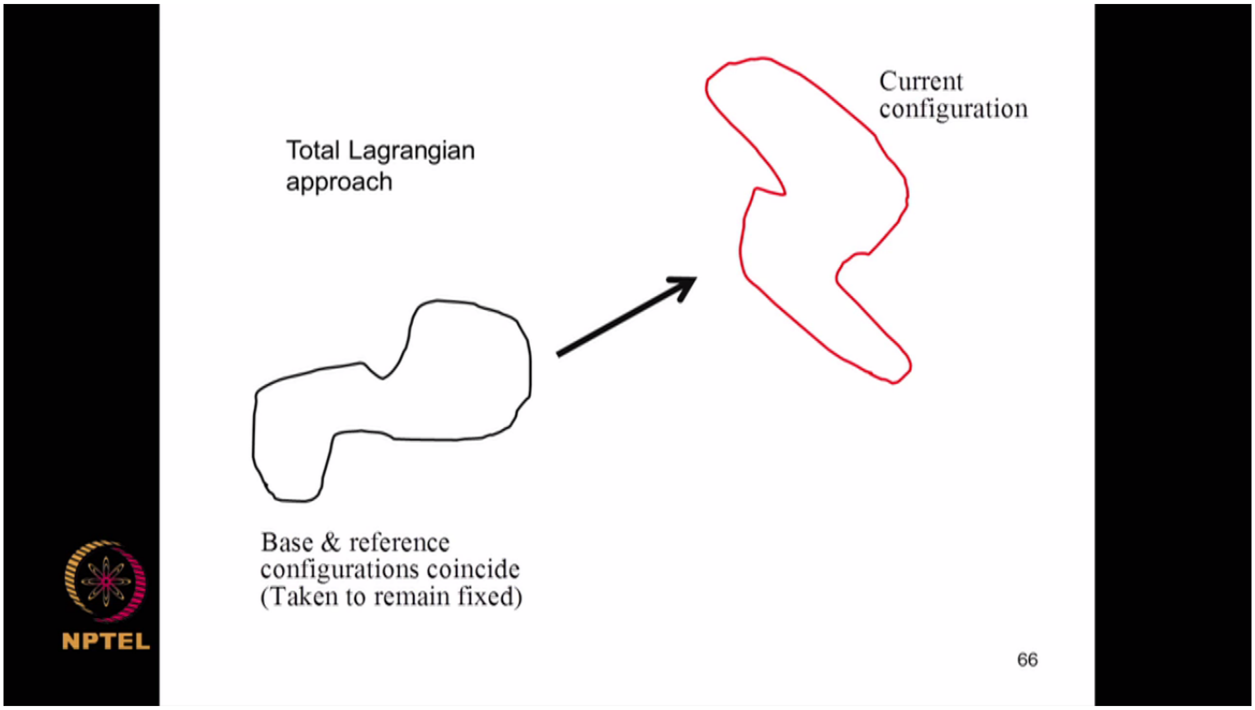


So in a more general class of problems, of course, material could be linear or nonlinear. There are large rotations and large strains. These are the most general class of problems which are most difficult to deal with.

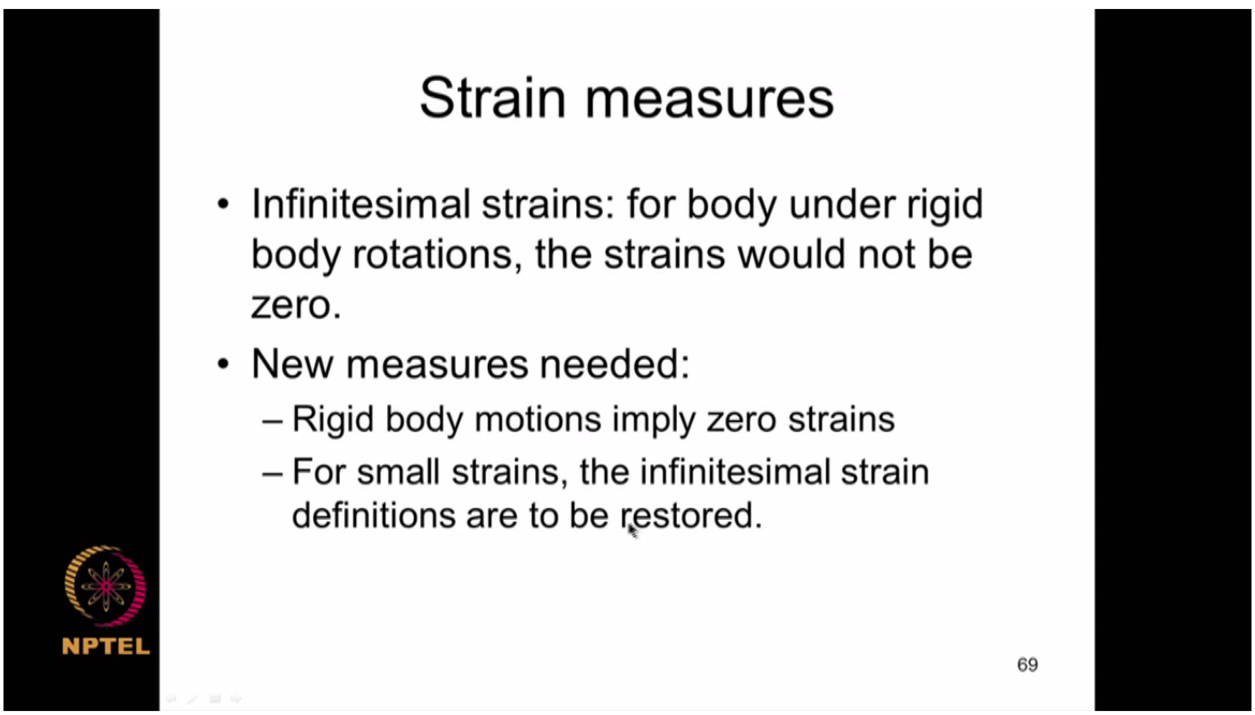
A simple illustration of a nonlinear problem with nonlinear boundary condition is shown here. Suppose this support there is a free player, this gap and as the structure deforms if this gap is negotiated then this spring stiffness will come into action and the stress strain typical stress strain plot will have this kind of feature. There the loading and unloading path will trace each other. This is a geometric nonlinearity whereas, if you are dealing with material nonlinearity the loading and unloading parts will be different. Upon removal of the load there will be a permanent set.



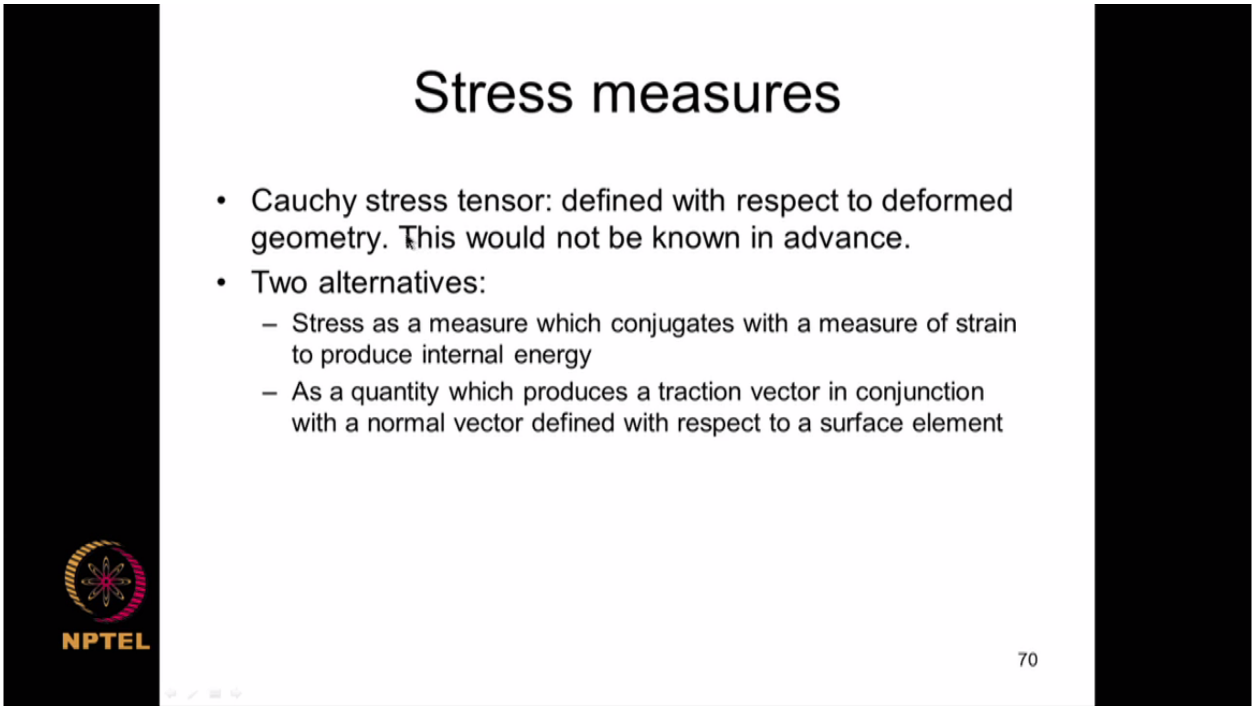
So in dynamical systems we saw that the governing equation will be of the form mx double dot plus cx dot plus kx and a nonlinear function of instantaneous values of displacement and velocity and this is the geometric nonlinearity or elastically nonlinear system behavior, whereas, this one the second term the force of resistance depends on entire time history of the response up to the current time instant. So this is hereditary or memory dependent nonlinearity typically arising due to material nonlinearity. So this is a more general class of problems.



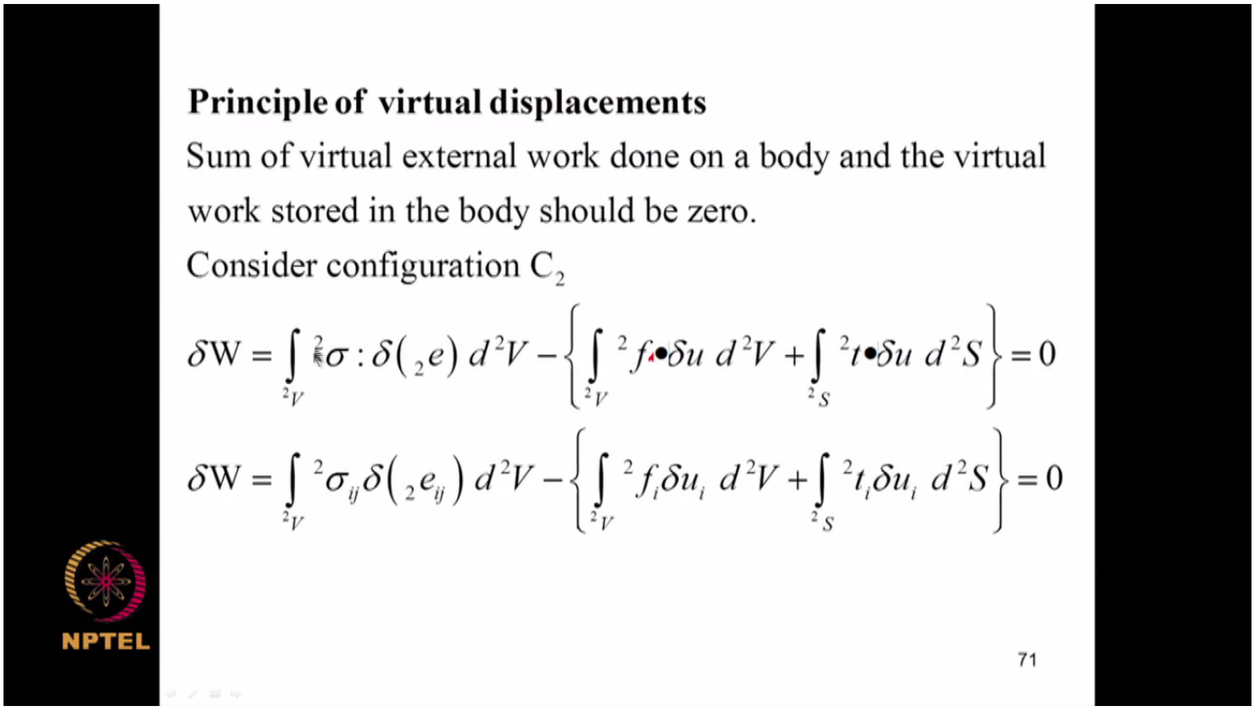
In formulating the finite element formulations for nonlinear problems we found that we needed to introduce certain newer measures for strains and stresses.



For example, if you use infinite decimal strains for a body under rigid body rotations the strains would not be zero, whereas, we know that under rigid body rotation the strain should be zero. So a newer measures of strains are needed which satisfy two requirements namely that rigid body motions imply zero strains and for small strains the infinitesimal strains definitions are restored.



Now similar issues about stresses definition of stress measure also were considered. The Cauchy stress tensor which deals with deformed configuration, the force field and area in a deformed configuration was difficult to use in an analysis simply because we would not know the deform geometry when we developed this solution. So we developed a two alternative definition for stresses that is Piola-Kirchhoff, first Piola-Kirchhoff and second Piola-Kirchhoff stress tensors. So to develop the different measures of stress, we interpreted stress as a measure which conjugates with a measure of strength to produce internal energy or as a quantity which produces a traction vector in conjunction with a normal vector defined with respect to a surface element.



Principle of virtual displacements was used to formulate the problems and the problem was if use Cauchy stress tensor and Eulerian strain tensor which are defined – both are defined with respect to deform geometry we can set up the expression for the virtual work. But the problem is as I already said the deform geometry would not be known. So the volume over which we need to evaluate these integrals would not be known in advance.

So what we do is therefore introduce suitable strain and stress measures that helps us to evaluate these integrals over known configurations. So that is how we develop nonlinear. In nonlinear problems we recognize that if a structure is carrying a particular load P we cannot apply the load entire load in stroke. So we divide the load 0 to P into small increments and we trace the solution as the load is incremented by small amounts. During an increment of a load we can linearize certain system behavior and that helped us to formulate the complete finite element procedure to deal with these problems. So kinematically we considered three types of approaches; the so-called Total Lagrangian approach in which the base and reference configurations coincide and this problem is solved with respect to the configuration in the reference with respect to the reference configuration. Whereas, in Updated Lagrangian approach the reference configuration was updated as the loads were incremented and the increment between the reference configuration and current configurations were taken to be small. But the reference configuration itself was updated at every step. This we didn't discuss although I briefly mentioned co-rotational formulation where we first form the co-rotated configuration that is the base configuration undergoes ritz body motions. This is exaggerated. This won't be so large, need not be so large. Actually CG scan would coincide here.

The base configuration is used as a reference to measure rotations whereas, co-rotated configuration is used as reference to measure current stress of stress and strains.

Updated Lagrangian approach

All quantities reckoned with respect to the latest known configuration (C_1).

$$\int_{^2V} {}^2\sigma_{ij}\,\delta\left({}_2e_{ij}\right)\, d^2V = \int_{^1V} {}^2_1S_{ij}\,\delta\left({}^2_1\varepsilon_{ij}\right)\, d^1V$$

$$\int_{^2V} {}^2f_i\,\delta u_i\; d^2V = \int_{^1V} {}^2_1f_i\,\delta u_i\; d^1V$$

$$\int_{^2S} {}^2t_i\,\delta u_i\; d^2S = \int_{^1S} {}^2_1t_i\,\delta u_i\; d^1S$$

${}^2_1\varepsilon_{ij}$ = updated Green-Lagrange strain tensor

${}^2_1f_i$ = body force refered to in C_1.

${}^2_1t_i$ = surface traction refered to in C_1.

$$\int_{^1V} {}^2_1S_{ij}\,\delta\left({}^2_1\varepsilon_{ij}\right)\, d^1V - \delta\left({}^2_1R\right) = 0$$

$$\delta\left({}^2_1R\right) = \int_{^1V} {}^2_1f_i\,\delta u_i\; d^1V + \int_{^1S} {}^2_1t_i\,\delta u_i\; d^1S$$

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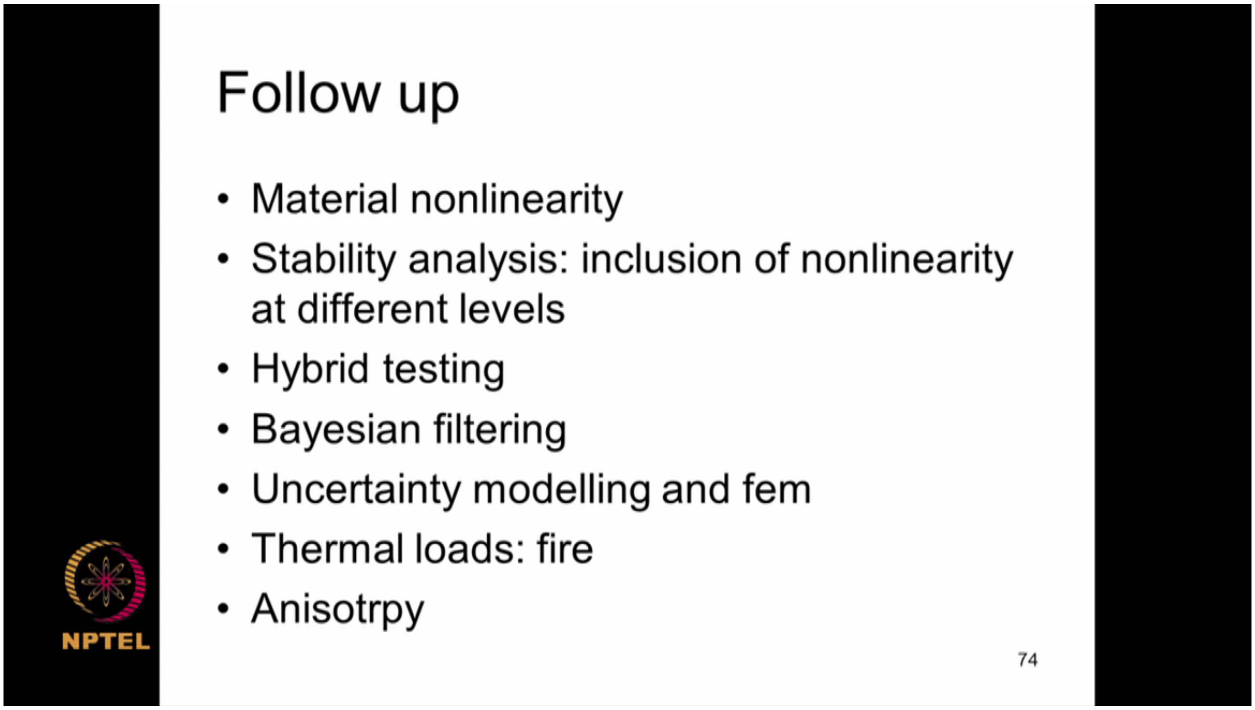
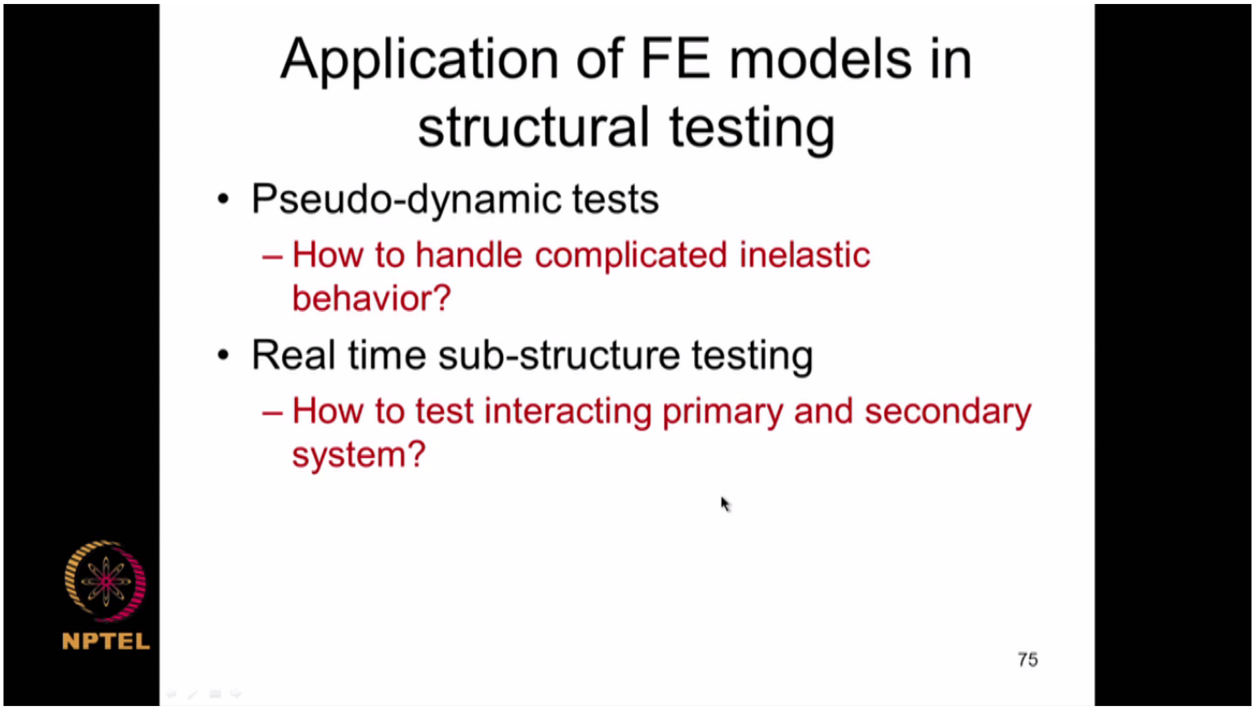
Now we develop the Total Lagrangian approach and Updated Lagrangian approach based on which we formulated the virtual work – these are the virtual work principles and we formulated the structural matrices.

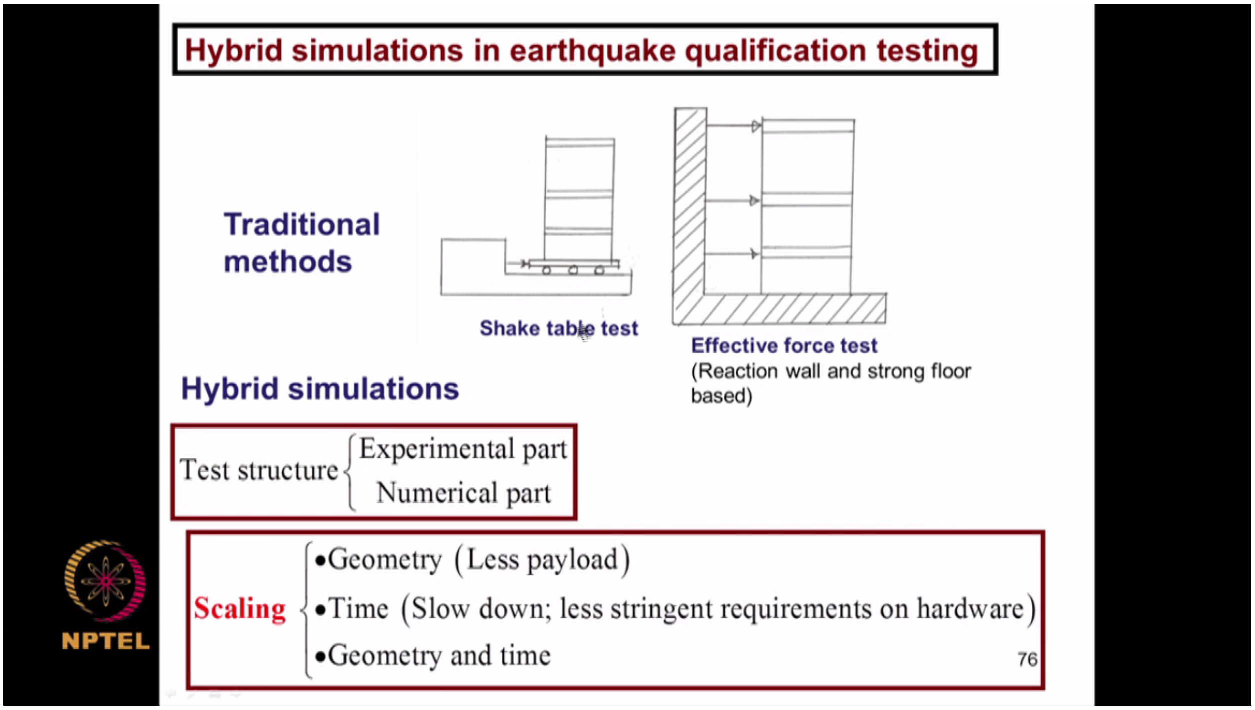


So this is a kind of a gist of what we try to achieve during the course and towards the end of the last lecture I also briefly mentioned topics that could be followed up as a based on material covered in this course. What was not covered in any I mean we didn't pay any attention was questions about material only. So we should study, you should study the subject of plasticity and pay more attention to formulation of constitutive laws to be able to do this.

Similarly in the questions over stability analysis we again didn't consider material nonlinear. Now there are other topics about which I have briefly talked about. One is hybrid testing among other Bayesian filtering etc. I would like to spend few minutes explaining what is hybrid testing and what are the issues about uncertainty modeling and finite element method.

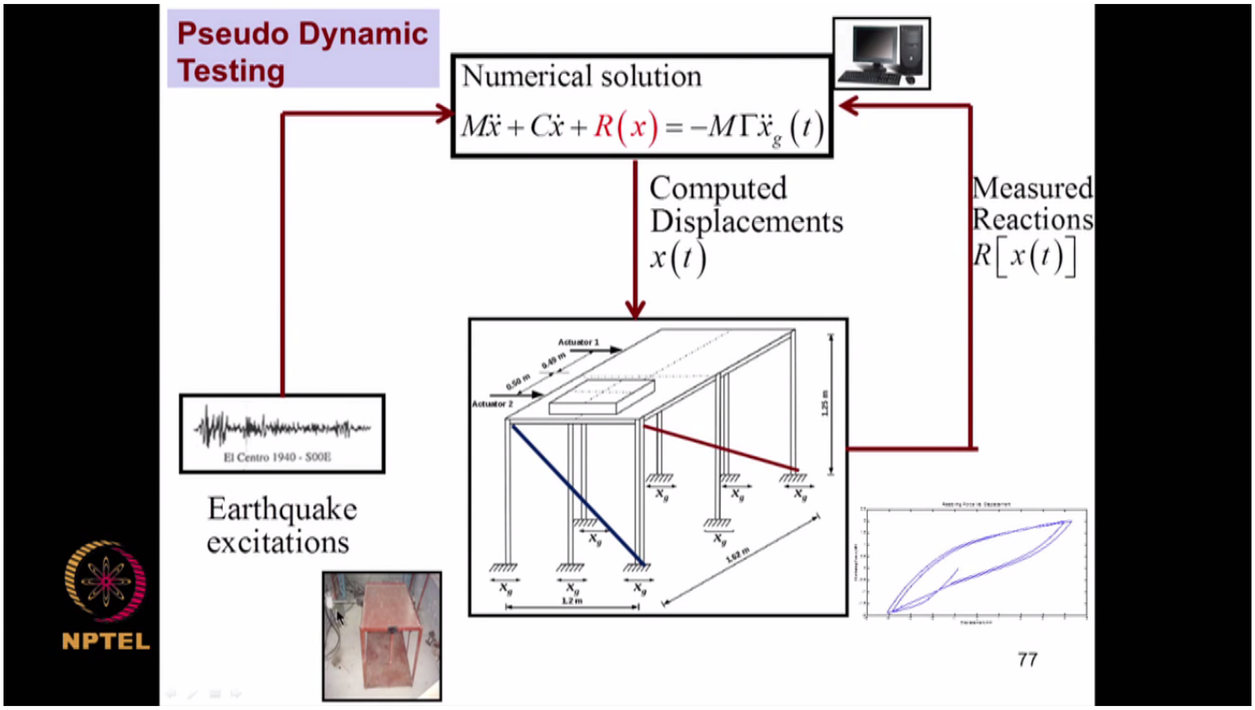


Now how – what is the role that finite element models play in structural testing? Now there are different tests – [Indiscernible] [0:49:07] testing strategies available in vibration engineering. One is what is pseudo-dynamic tests that typically helps us to handle questions about complicated in elastic behavior under dynamic loads using basically static methods of experimental investigations. Then there is another one known as real-time sub-structure testing. It deals with treatment of interacting primary and secondary systems in an vibrating environment.



So these techniques are being developed in the field of earthquake engineering. So traditionally in earthquake engineering we have either a shake table on which we mount the structure to be tested or we have a reaction wall based system where there are several hydraulic actuators which apply dynamic loads on models like this. The shake table testing, the load time history that is earthquake load time history is applied in real time. The length of duration of the applied acceleration is equal to the length of the observed earthquake signal. The problem with this approach is limitations on payload capacity of shake tables.

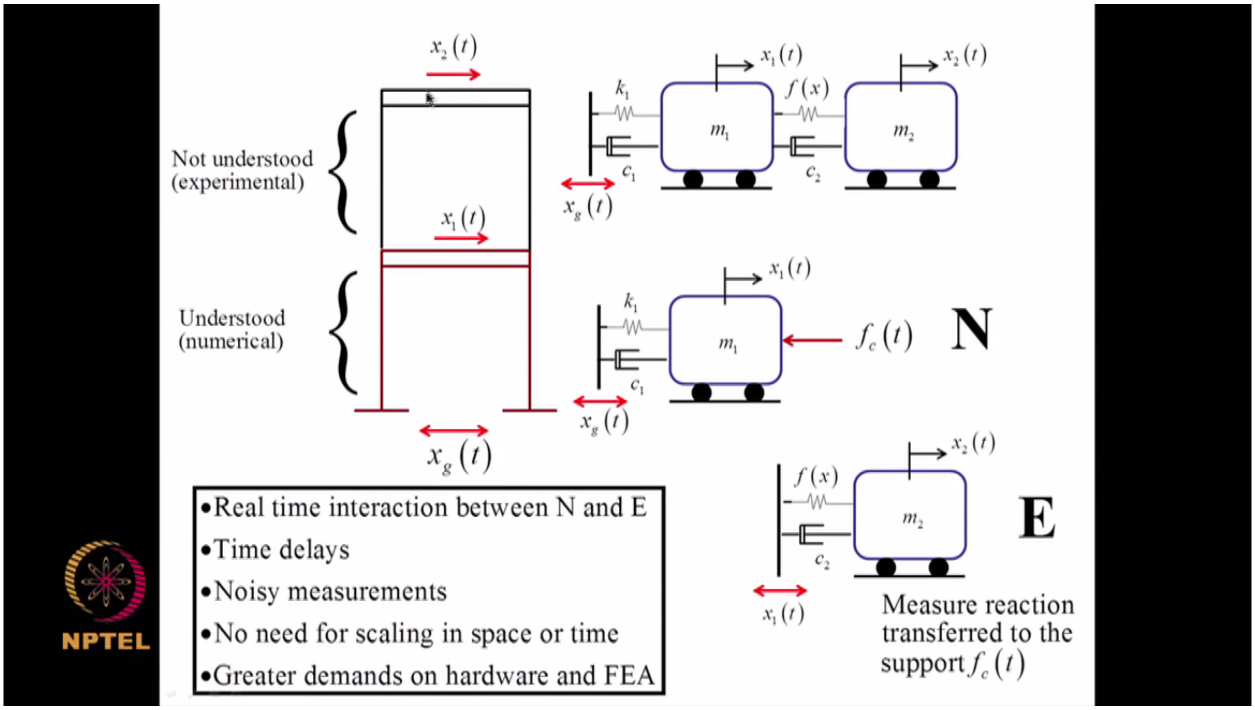
The best shake table that is currently available may not be -- you may not be able to test say a building which is taller than say five-story building or steady interaction between soil and structure or fluid and structure and issues like that. So there is a need to geometrically scale the test structures to be able to use shake tables. On the other hand, in effective force testing, this limitation is overcome to some extent but still the dimensions of the structure to be tested is governed by the dimension of the reaction wall system. Now in hybrid simulations what we do is we divide the test structure into an experimental part and then numerical part and we try to avoid scaling either of time or of geometry to the extent that is possible.



So in a typical pseudo dynamic testing suppose this is the test structure and these arrows represents actuators and this is schematically the actual hardware is shown here. This is our hydraulic actuator which is under computer control. Now we start by modeling this system as mx double dot plus cx dot plus r of x is equal to the applied ground motion. We treat that r of x which captures the inelastic behavior of the system as unknown. On the other hand, we assume that the inertial properties and damping properties of the system can be modeled computationally. So part of the structure namely inertia and damping properties are modeled computationally and part of the structural model which captures stiffness properties is measured experimental. The way the experiment proceeds is we start integrating this equation possibly in a linearized model for the stiffness and integrate from say 0 to Delta T. That tells us how the building has displaced x of t with a vector of different nodal displacements and we apply those actual displacements on the structure. These actuators are under displacement control. And as a consequence of applying those displacements, there will be reactions that are set up in the actuators which we measure using load cell and that helps us to determine the stiffness. That value of stiffness is put into the equation of motion and the load is incremented from Delta T to 2 Delta T and at every time the actuator displacement is determined by solving this equation and the stiffness is determined by actual measurement.

Now these displacements were applied statically. So that is why this is called pseudo-dynamic testing. So in this approach the time is slowed down. A 30 second earthquake event can be expanded for two hours or three hour depending on the test duration that you are willing to – time that you are willing to expend on the testing. Now the question here from the perspective of this course is we have to integrate this equation of motion. We saw that even when stiffness was completely specified the question on how these errors grow was not very easy to answer. There were several questions on how to select the schemes of integration and so on and so forth.

Now in this pseudo-dynamic testing what happens is the errors due to using finite step size and adopting certain integration schemes is compounded by errors due to experimental errors in making measurements. So when we say that we have to apply a displacement of say 2mm we may not be able to apply exactly that displacement. There will be an error and when we measure the force transferred there will be again an error due to errors in measuring through the load cells. So all those errors also propagate. So the major challenge in this approach is, of course, how to combine finite element models, part of finite element model with an experimental protocol and also how to deal with newer sources of errors in establishing this integration schemes.



The another testing procedure is a real-time substructuring that can conceptually be explained like this. Suppose we have a two-story frame. We assume that part of this structure – the behavior of part of the structure is understood well and a computational model is adequate for that. The remaining part has to be experimentally tested. Suppose we make a finite element model or a simple dynamic model this will be a two degree of freedom system say. Now in this analysis this part of the structure is treated as a numerical model and this part of the structure is treated as an experimental model. N is numerical E is experiment. So this resides on a finite element platform in our test protocol and this is the hardware. Only part of the structure is mounted on a reaction wall or a shake table. Now we again start integrating the numerical model and while doing this we need to know what is the reaction transferred by the experimental component. So there will be an iteration. So we start integrating. We move from say zero to Delta T and find out what is the displacement. We apply this displacement maybe a shake table in real time now. It is not pseudo time. I mean it is not static and we measure the reaction transferred to the supports on our shake table or reaction wall and that is transferred to the numerical model.

So the integration of equation of motion and dynamic testing of the experimental component take place hand in hand in real time. So a 30-second event is analyzed in 30 seconds that means the speed of integration and speed of testing must coincide. So this again leads to several complicating questions and there are questions of time delays, noisy measurements and issues like that which we need to implement while taking care of.

So these hybrid testing methods are modern developments in earthquake engineering and there actually the finite element modeling need to shake hands with experimental tools and there are lots of newer challenges that one has to face. So this is a brief introduction about hybrid simulations. As you can take off from what we discussed in this course and probably be able to address some of these issues.

So at this point we will close the discussion on this this lecture as well as this course.

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