Indian Institute of Science Bangalore

National Programme on Technology

Enhanced Learning

Copyright

All rights reserved. No part of this work may be reproduced, stored or transmitted in any form or by any means, electronic or mechanical, including downloading, recording, photocopying or by using any information storage and retrieval system without prior permission in writing from the copyright owner.

Provided that the above condition of obtaining prior permission from the copyright owner for reproduction, storage or transmission of this work in any form or by any means, shall not apply for placing this information in the concerned Institute's library, departments, hostel or any other place suitable for academic purposes in any electronic form purely on non-commercial basis.

Any commercial use of this content in any form is forbidden.

Finite element method for structural dynamic and stability analyses

Lecture -39

Total and Updated Lagrangian formulations

CS Manohar

Professor

Department of Civil Engineering

Indian Institute of Science

Bangalore 560012

India

Finite element method for structural dynamic and stability analyses

Lecture - 39

Total and updated Lagrangian formulations

C S Manohar
Professor
Department of Civil Engineering

Indian Institute of Science
Bangalore 560 012
India

We have been discussing problems of non-linear structural mechanics and we are trying to formulate finite element models for that. With that in mind we have reviewed the basic principles of continuum mechanics. So in a linear analysis displacements and strains are small.

Linear analysis

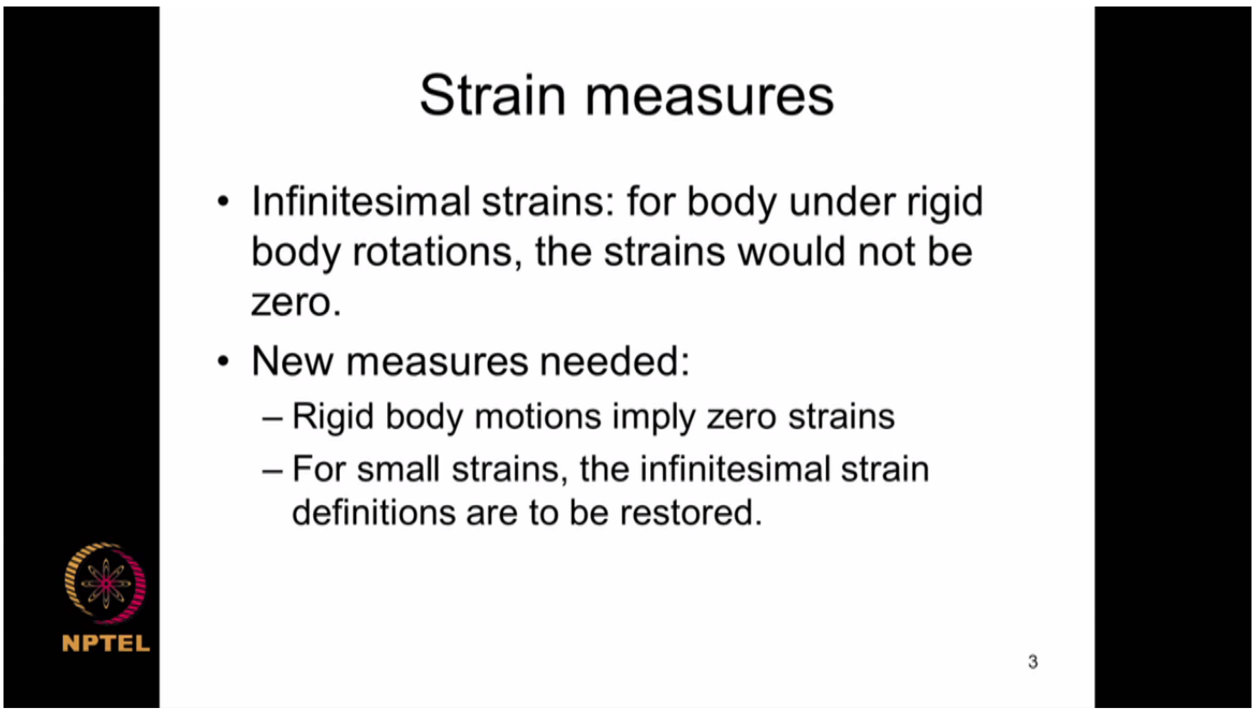
- Displacements and strains are small
- Stress-strain relations are linear
- Equillibrium equations are derived based on undeformed geometry
- Principle of superposition holds

Nonlinear analysis

- Displacement and strains need not be small; special efforts needed to characterize rotations
- The geometry of the object, stress-strain relations, and BCs could change during the process of deformation.
- The definitions of stress and strain and the formulation of the governing equations need to take into account these changes.
- Necessiates introduction of newer measures of stress and strain
- Principle of superposition does not hold

2

Stress strain relations are linear and equilibrium equations are derived based on undeform geometry and principle of superposition holds. So these are some of the facts which make linear analysis very simple. Now whereas in a nonlinear analysis displacement and strains need not be small. Special efforts need to be expanded to characterize rotations. Then the geometry of the object, the stress strain relations, the boundary conditions could change during process of deformation and that has to be taken into account. The definitions of stress and strain and the formulation of the governing equations also need to take into account the changes in the geometry of the object and constitutive laws and boundary conditions. This necessity introduction of newer measures of stress and strain. We need to have properly understood definitions of stress and strain. Then principle of superposition which is the bread and butter of linear analysis no longer holds. So if there are multiple loads attacked on the structure you cannot perform separate analysis and try to superpose. For example in earthquake response analysis, the self-weight effect and the response due to earthquake loads cannot be analyzed separately. You need to handle them together.



Now we introduced few strain measures. The infinitesimal strains we showed that for body and the rigid body rotation the strains would not be 0. So that is we cannot do the infinitesimal definitions of the strain. Now new measures are needed so the criteria for establishing these measures we looked at two such requirements that rigid body motions implies 0 strains and then for small strains the infinitesimal strain definitions are restored.

Green - Lagrange strain measure

$$dx = FdX$$

$$ds^2 - dS^2 = dx^t dx - dX^t dX = dX^t F^t F dX - dX^t dX = dX^t \left(F^t F - I \right) dX$$

$$= dX^t 2EdX$$

$$E = \frac{1}{2}\left(F^t F - I \right) = \text{Green-Lagrange strain measure}$$

$$MF = \frac{1}{2}\left[\left(\frac{ds}{dS} \right)^2 - 1 \right] = E_{\alpha\beta} N_\alpha N_\beta \ \& \ \Gamma_{AB} = \cos\theta + 2E_{\alpha\beta} M_\alpha N_\beta$$

Almansi - Hamel (Eulerian) strain

$$dX = F^{-1} dx$$

$$ds^2 - dS^2 = dx^t dx - dX^t dX = dx^t dx - dx^t F^{-t} F^{-1} dx$$

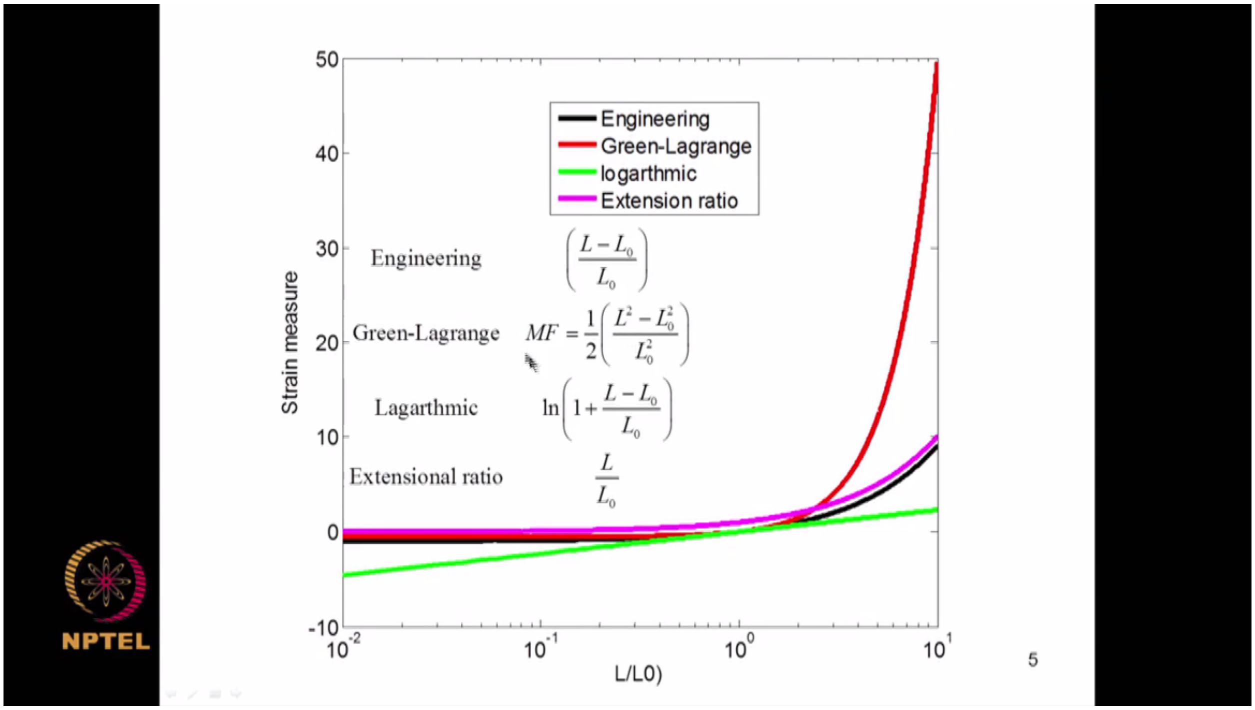
$$= dx^t \left(I - F^{-t} F^{-1} \right) dx = 2dx^t e dx$$

$$e = \frac{1}{2}\left(I - F^{-t} F^{-1} \right)$$

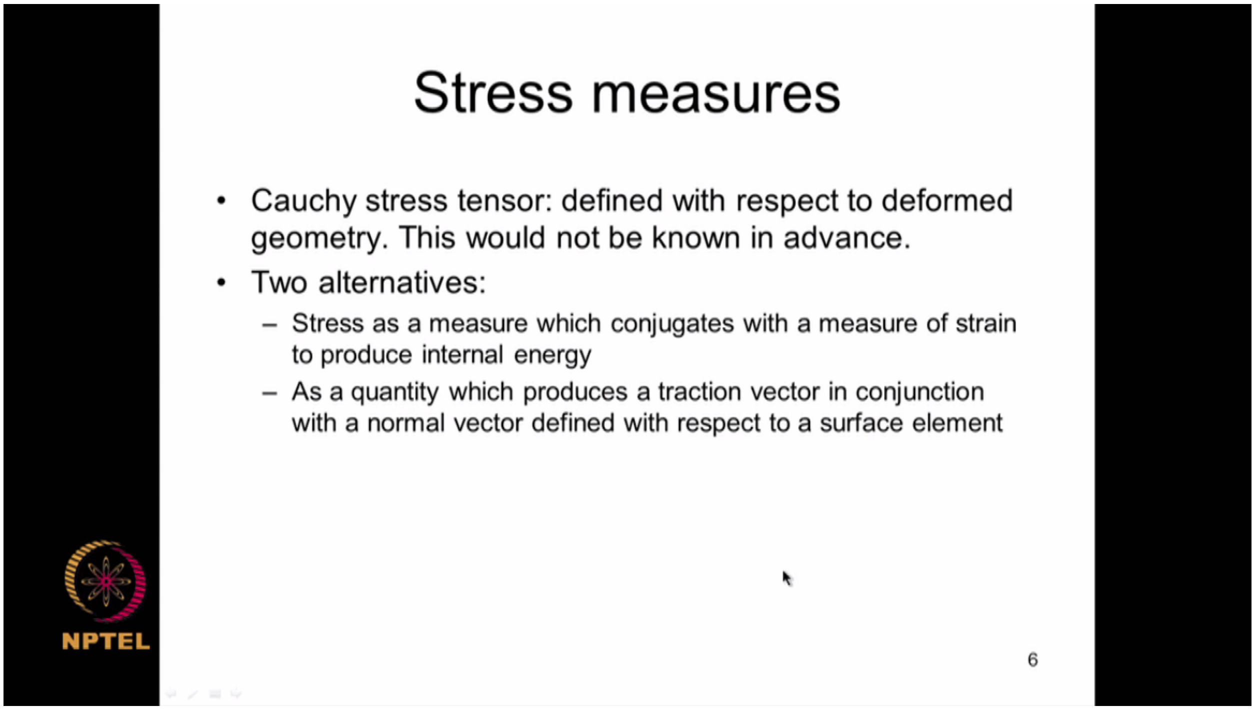
4

That helped us to introduce the Green-Lagrange strain measure the dx is FdX that is the capital dX is the line element in the undeformed geometry upon deformation it become dx and F is the matrix that relates them and this is a deformation matrix and in terms of deformation matrix we consider the change in square of the lengths of the line segments in deformed and undeformed positions and we obtained this quantity. And this quantity E which is half of F transpose F minus I is the Green-Lagrange strain measure. And this helped us to – this is associated with F quantity known as magnification factor as defined here and in terms of the Green-Lagrange strain measure, this magnification factor is given by this and this is a measure of shearing strain.

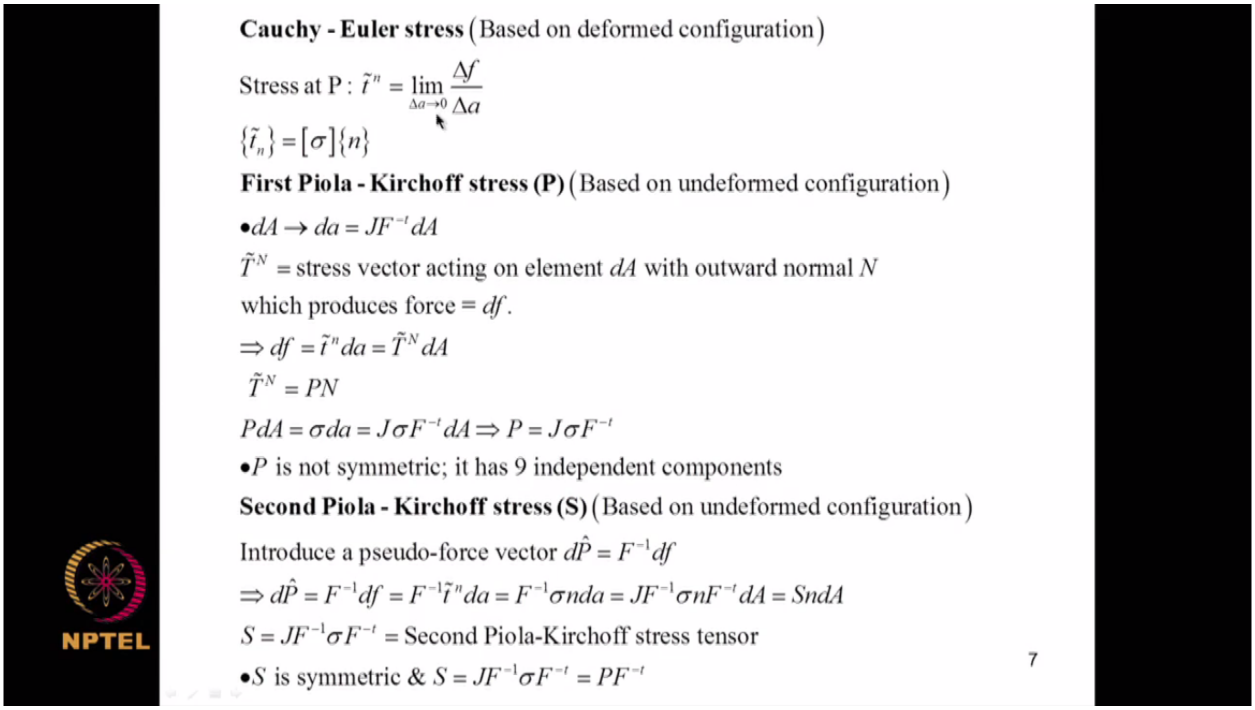
So if a line element with direction cosines N1, N2, N3 deforms and it still it undergoes a change in length and this magnification factor is expressed in terms of elements of Green-Lagrange tensors. So if the direction -- if the line segment is aligned with X-axis the magnification factor will be E11. If it is aligned along Y-axis it will be E22 and so on and so. Similarly the shear strain measures if one of the line segment is alone the line to X-axis and other one aligned to Y-axis, then gamma AB gives the shearing strain which is epsilon XX and so on and so forth. So we also introduced – this definition was with respect to the undeformed geometry. With respect to the deform geometry we get another strain measure known as Eulerian strain or Almansi—Hamel strain and this is as shown here.



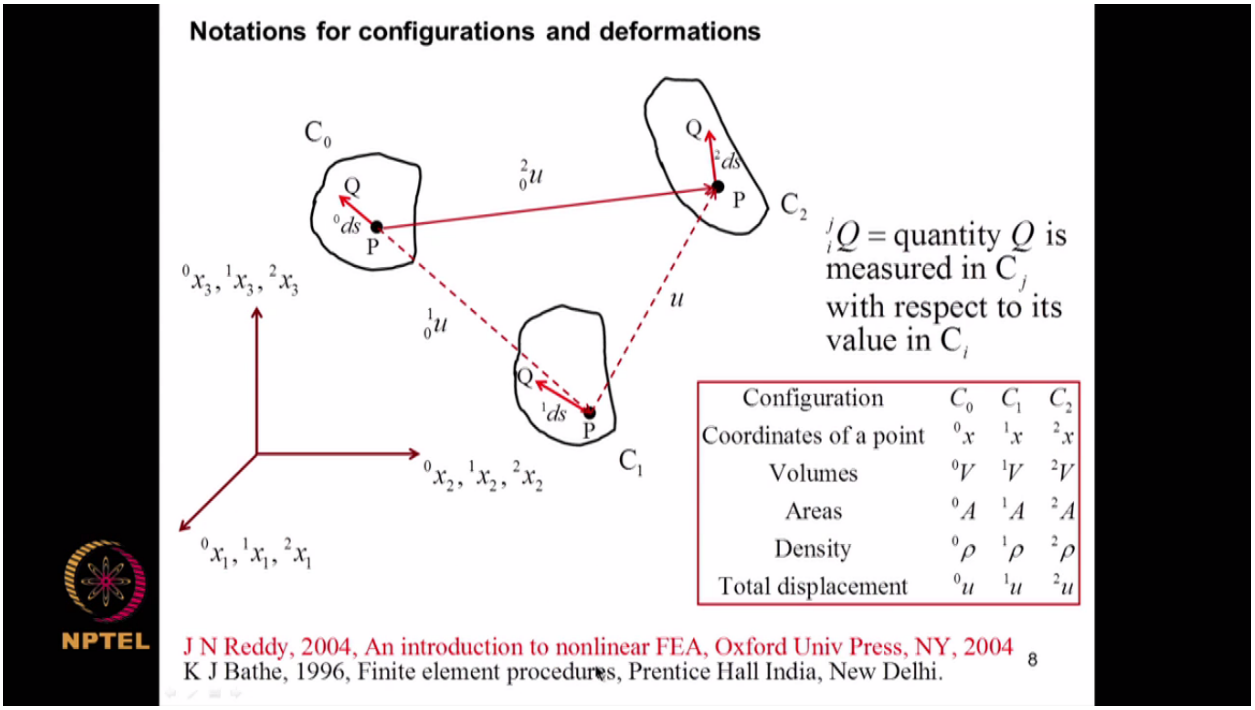
Now there are – in literature there are different definitions of measures of strain. So an engineering strain is different as change in length by the original length. Green-Lagrange the magnification factor L square minus L naught square by L naught square. Logarithmic measure is given by this and so called extensional ratio is given by this. So this I have plotted on X-axis L by L naught has been plotted and on Y-axis these different strain measures are plotted and as you can see that they show different behavior especially for large strains and when we are reporting results this has to be borne in mind.



The stress measures the commonly used stress measure is a Cauchy stress which is defined with respect to the deformed geometry. So we define this with respect to the internal forces in the deformed object with respect to area defined with respect to deform geometry and that looks to be the most natural way of defining stress. But when we apply this definition to problems of nonlinear analysis we will not be knowing the deform geometry in advance. So that this utility of Cauchy stress tensor thus becomes difficult and this tensor becomes it becomes difficult to use this tensor in nonlinear analysis. So this has necessitated alternative definitions. There are two approaches to develop measures of stress one stresses a measure which conjugates with a measure of strain to produce internal energy. That is one way of looking at it. The other one is as a quantity which produces a traction vector in conjunction with a normal vector defined with respect to a surface element. So based on this different measures have been introduced.



The Cauchy-Euler stress based on deformed configuration is we have discussed. It is given by this. This is a traction vector on a line element with outward normal n and this is expressed in terms of stress components defined with respect to certain cardinal coordinate systems. The first Piola-Kirchoff stress is based on undeformed configuration. We know how the surface element transform through this relation and we define a stress vector acting on element da with outward normal n which produces force df. df is this forces in the deform geometry and we introduce a quantity capital P fashion in the same way as the Cauchy stress is defined and this capital P is known as the first Piola-Kirchoff stress and we have seen that this is not symmetric. It has nine independent components. The second Piola-Kirchoff stress eliminates this awkwardness in the stress matrix. It produces a symmetric matrix. It is based on undeformed configuration. We introduce a pseudo force vector dp hat which is F inverse df. This again this transformation is fashioned after the transformation of dX equal to FdX for line elements. So based on this we introduce the second Piola-Kirchoff stress tensor and we also – we saw that this is symmetric and we also establish how these three stress measures are related to each other.



Now to develop the finite element formulations we need to have certain clarity on notations for configurations and deformations. So we start with configuration C naught and we consider a point P and a line segment PQ of length dsnaught. So we introduced a special notation system. We use left superscripts and subscripts. So Qji is to be interpreted like this; quantity Q is measured in configuration cj with respect to its value in ci. It is how we should interpret that. So here we introduce the coordinate system. The left superscript corresponds to the configurations and the right subscripts responds to the components of the quantity that we are considering. So this is x1, x2, x3 in configuration 012,012 and 0. Now configuration is denoted by C0, C1, C2. Coordinates of a point X left superscript 0 and so on and so, 1x, 2x this. Volumes 0V,1V, 2V. Area 0A, 1A, 2A. Similarly density. Total displacement 0U, 1U, 2U. So this is configuration C0 this is some intermediate configuration C1 and this is the current configuration. So this U is the incremental displacement from C1 to C2. Okay. Now the discussion that I am going to present and the notational system that we are going to use is based on the material available in two books. One is by Professor J N Reddy An introduction to nonlinear finite element analysis and the other one is by KJ Bathe Finite element procedures. Now the material that I am going to use in this lecture is largely based on the coverage given in the book by Professor J N Reddy. The idea here in this lecture is not to develop the complete procedure for nonlinear finite element analysis but instead to give a flavor of how to – what are the issues that one must be aware of in formulating such problems. The detailed formulation of finite elements and their application in specific problems etcetera. are not covered in this lecture. So the objective is to simply introduce you to the basic issues that arise in dealing with non-linearity and some preliminary indication on how one could proceed with producing a finite element model. Okay. So there are a lot more details that I will not be able to touch upon but hopefully this gives you necessary motivation to study this topic further.

Deformation: A particle in

C_0 at ${}^0x = \left({}^0x_1 \quad {}^0x_2 \quad {}^0x_3\right)$

moves to

C_1 at ${}^1x = \left({}^1x_1 \quad {}^1x_2 \quad {}^1x_3\right)$

and to

C_2 at ${}^2x = \left({}^2x_1 \quad {}^2x_2 \quad {}^2x_3\right)$

Motion from C_0 to C_1: ${}^1_0u = {}^1x - {}^0x$

$\Rightarrow {}^1_0u_i = {}^1x_i - {}^0x_i ; i = 1,2,3$

Motion from C_1 to C_2: ${}^2_1u = {}^2x - {}^1x$

$\Rightarrow {}^2_1u_i = {}^2x_i - {}^1x_i ; i = 1,2,3$

Conservation of mass : $\int_{{}^2V} {}^2\rho\, d\,{}^2V = \int_{{}^1V} {}^1\rho\, d\,{}^1V = \int_{{}^0V} {}^0\rho\, d\,{}^0V$

${}^2x_i \rightarrow {}^0x_i \Rightarrow \int_{{}^2V} {}^2\rho\, d\,{}^2V = \int_{{}^0V} {}^2\rho\, {}^2_0J\, d\,{}^0V$

$$\int_{{}^2V} {}^2\rho\, d\,{}^2V = \int_{{}^0V} {}^2\rho\, {}^2_0J\, d\,{}^0V \quad \text{with } {}^2_0J = \left|{}^2_0F\right| = \begin{vmatrix} \frac{\partial\,{}^2x_1}{\partial\,{}^0x_1} & \frac{\partial\,{}^2x_1}{\partial\,{}^0x_2} & \frac{\partial\,{}^2x_1}{\partial\,{}^0x_3} \\ \frac{\partial\,{}^2x_2}{\partial\,{}^0x_1} & \frac{\partial\,{}^2x_2}{\partial\,{}^0x_2} & \frac{\partial\,{}^2x_2}{\partial\,{}^0x_3} \\ \frac{\partial\,{}^2x_3}{\partial\,{}^0x_1} & \frac{\partial\,{}^2x_3}{\partial\,{}^0x_2} & \frac{\partial\,{}^2x_3}{\partial\,{}^0x_3} \end{vmatrix}$$

$\Rightarrow {}^0\rho = {}^2\rho\, {}^2_0J \quad \& \quad {}^1\rho = {}^2\rho\, {}^2_1J$

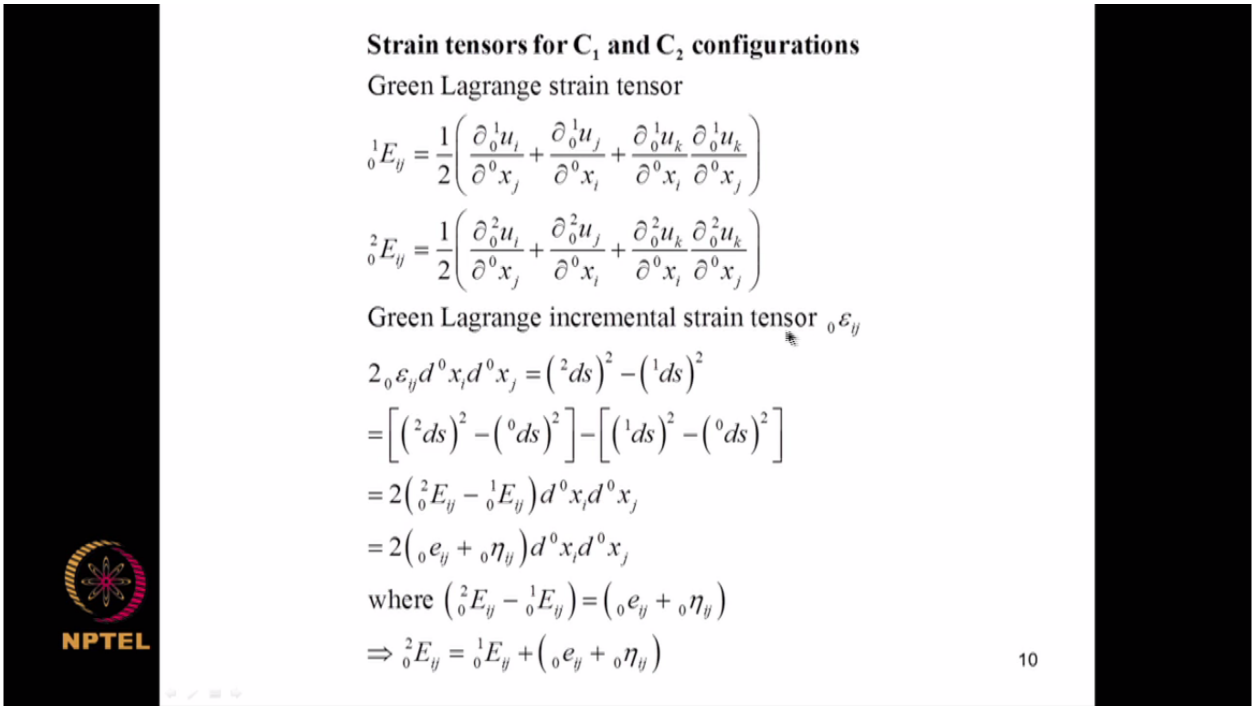
9

NPTEL

Now a particle in C0 the position coordinate is x1 x2 x3 with superscript 0. It moves to C1 at and it occupies a point x1 x2 x3 configuration one. And at C2 it is at x1 x2 x3 configuration two. So the motion from C naught to C1 the displacement is given by this U from configuration 0 to 1 is given by. So the components are given as shown here. For motion from C1 to C2 U configuration 1 to 2 is given by this x2 minus x1. Similarly components are this.

Now the law of conservation of mass demands that the mass of the object in different configuration should be the same. So if I have now in second configuration integral over 2V of 2 row must be equal to integral of one row over d1V and that must be equal to this volume is 1V and this is 0V that you should notice and this is in the undeformed or reference configuration. Now you must be careful in interpreting this quantity. It is not d square V. It is d of 2V because we are using left superscripts and subscripts it should not be confused with powers for the preceding quantity. Hopefully that confusion if you carefully understand the flow of logic you will not get into.

Now by substituting for X2i in terms of X0i we can show that the law of conservation of mass leads to this requirement between configuration 0 and 2. So this – that would mean this 02 Jacobian is given by this determinant and we have this relation between densities at various configurations. Okay. If this is satisfied the law of conservation of mass is obeyed.



Now let us look at strain tensors for C1 and C2 configurations. The Green-Lagrange strain tensor we have derived this for configuration from 0 to 1 is given by this. From 0 to 2 it is given by this. If you look at incremental strain tensor that is defined like this. It is change in square length of the line element in configuration to 21. Okay. So we are – this is an increment – incremental deformation from C1 to C2. The idea is that as we apply the surface traction and body forces for each increment of the force the body occupies different configurations and we will while solving the problem we will divide the applied loads into small increments and the increment from says C1 to C2 is affected by a small increment in the load. We can linearize the relations when moving from C1 to C2. That is the idea. Okay. So that you should keep in mind. Now this incremental Green-Lagrange strain tensor is given by this and we can define this with respect to the ds in the original configuration I am adding and subtracting that and this enables me to write this in terms of the Green-Lagrange tensor from configuration 0 to 2 and 0 to 1, and this by rearranging the terms I defined these quantities eij ETA ij. These are linear parts of the incremental strain tensor and the nonlinear parts respectively. So this is given by this.

$$2\,{}_0\varepsilon_{ij}\, d^0x_i d^0x_j = 2\left({}^2_0E_{ij} - {}^1_0E_{ij}\right) d^0x_i d^0x_j$$

$$= 2\left({}_0e_{ij} + {}_0\eta_{ij}\right) d^0x_i d^0x_j$$

$${}_0e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial^0 x_j} + \frac{\partial u_j}{\partial^0 x_i} + \frac{\partial\, {}^1_0u_k}{\partial^0 x_i}\frac{\partial u_k}{\partial^0 x_j} + \frac{\partial u_k}{\partial^0 x_i}\frac{\partial\, {}^1_0u_k}{\partial^0 x_j}\right) \text{ (linear in increment } u_i\text{)}$$

$${}_0\eta_{ij} = \frac{1}{2}\frac{\partial u_k}{\partial^0 x_i}\frac{\partial u_k}{\partial^0 x_j} \text{ (nonlinear in increment } u_i\text{)}$$

NPTEL

11

So if you use the definitions of Green-Lagrange tensor capital E you will be able to show that the linear increment in ui, the strain due to linear increment in ui is given by this and this is nonlinear increment in ui. So these are incremental Green-Lagrange tensors. This will be needing in our formulation.

Updated Green Lagrange strain tensor

$${}^2_0E_{ij} = \frac{1}{2}\left(\frac{\partial\, {}^2_0u_i}{\partial^0 x_j} + \frac{\partial\, {}^2_0u_j}{\partial^0 x_i} + \frac{\partial\, {}^2_0u_k}{\partial^0 x_i}\frac{\partial\, {}^2_0u_k}{\partial^0 x_j}\right) \text{is helpful in total Lagrangian formulations.}$$

To facilitate updated Lagrangian formulation we introduce

$$2\left({}^2_1\varepsilon_{ij}\right) d^1x_i d^1x_j = \left({}^2ds\right)^2 - \left({}^1ds\right)^2$$

${}^2_1\varepsilon_{ij}$ = updated Green-Lagrange strain tensor.

Using the following results

$$d^2x_i = {}^2_1F_{ij}\, d^1x_j = \frac{\partial^2 x_i}{\partial^1 x_j} d^1x_j \Rightarrow \left({}^2ds\right)^2 = \frac{\partial^2 x_k}{\partial^1 x_i}\frac{\partial^2 x_k}{\partial^1 x_j} d^1x_i d^1x_j,$$

$$u_i = {}^2x_i - {}^1x_i \;\&\; \frac{\partial^2 x_k}{\partial^1 x_i} = \frac{\partial u_k}{\partial^1 x_i} + \delta_{ki}$$

we get $${}^2_1\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial^1 x_j} + \frac{\partial u_j}{\partial^1 x_i} + \frac{\partial u_k}{\partial^1 x_i}\frac{\partial u_k}{\partial^1 x_j}\right) = {}_1e_{ij} + {}_1\eta_{ij}$$

with $${}_1e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial^1 x_j} + \frac{\partial u_j}{\partial^1 x_i}\right) \;\&\; {}_1\eta_{ij} = \frac{1}{2}\left(\frac{\partial u_k}{\partial^1 x_i}\frac{\partial u_k}{\partial^1 x_j}\right)$$

NPTEL

12

Then the updated Green-Lagrange tensor you see that the Green-Lagrange tensor from configuration 0 to 2 is helpful in total Lagrangian formulations where we refer when formulate the problem with respect to quantities in the undeformed configuration. To facilitate updated Lagrangian formulation we introduce another quantity. This is 2 epsilon ij, see this is epsilon ij with respect to 0 but here for the increment from one to two I introduce this definition and again using the following results again using the deformation matrix for movement deformation from one to two we get all this and we get the increments, the linear part and the nonlinear part as shown here.

Euler strain tensor

Consider that the body has reached configuration C_1 from C_0 in several increments. The deformation from C_0 to C_1 could be large.

We now wish to move to configuration C_2. The increment from C_1 to C_2 is taken to be small and we could refer to strains with respect to C_2.

$$2\,{}^1_1\varepsilon_{ij}\, d\,{}^1x_i\, d\,{}^1x_j = \left({}^1ds\right)^2 - \left({}^0ds\right)^2$$

$$2\,{}^2_2\varepsilon_{ij}\, d\,{}^2x_i\, d\,{}^2x_j = \left({}^2ds\right)^2 - \left({}^1ds\right)^2 = \left(\delta_{ij} - \frac{\partial\,{}^1x_k}{\partial\,{}^2x_i}\frac{\partial\,{}^1x_k}{\partial\,{}^2x_j}\right) d\,{}^2x_i\, d\,{}^2x_j$$

$$\Rightarrow {}^2_2\varepsilon_{ij} = \frac{1}{2}\left(\delta_{ij} - \frac{\partial\,{}^1x_k}{\partial\,{}^2x_i}\frac{\partial\,{}^1x_k}{\partial\,{}^2x_j}\right) = \text{Euler strain tensor}$$

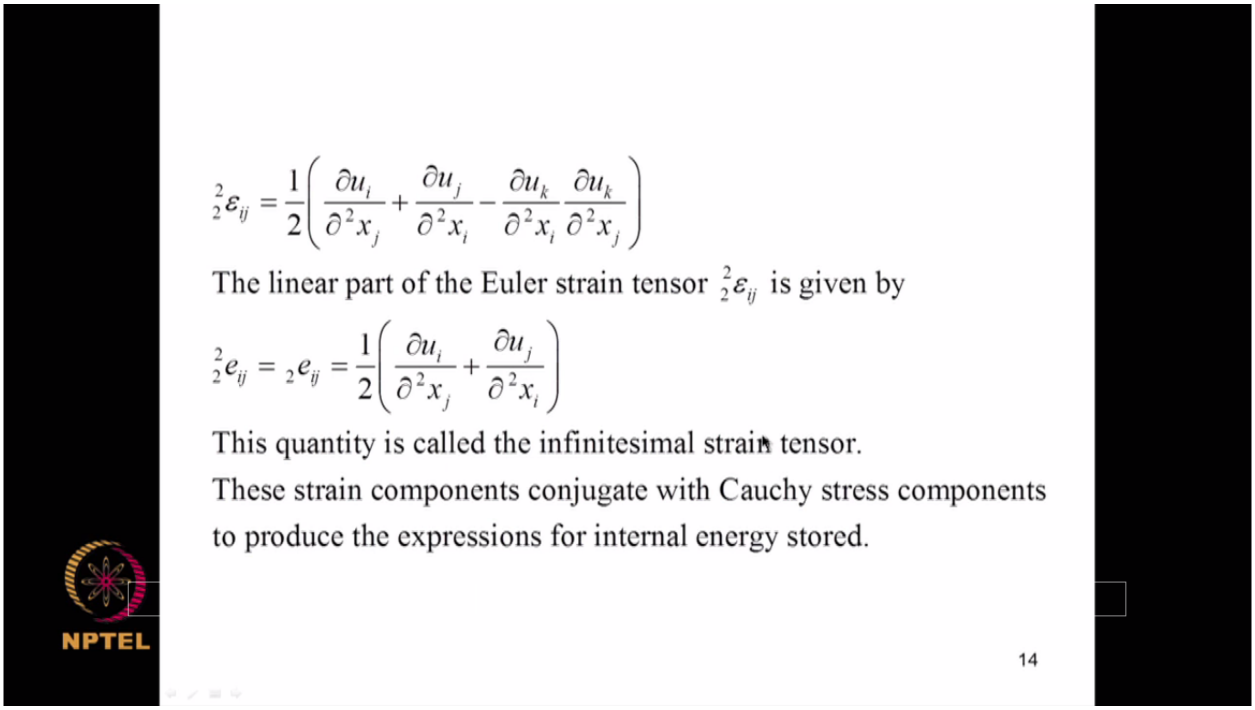
We have $u_k = {}^2x_k - {}^1x_k \Rightarrow \dfrac{\partial\,{}^1x_k}{\partial\,{}^2x_j} = \dfrac{\partial}{\partial\,{}^2x_j}\left({}^2x_k - u_k\right) = \delta_{jk} - \dfrac{\partial u_k}{\partial\,{}^2x_j}$

$${}^2_2\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial\,{}^2x_j} + \frac{\partial u_j}{\partial\,{}^2x_i} - \frac{\partial u_k}{\partial\,{}^2x_i}\frac{\partial u_k}{\partial\,{}^2x_j}\right)$$

13

Now the Almansi-Hamel strain tensor or the Euler strain tensor we can consider. If you consider that the body has reached configuration C1 from C0 in several increments. The deformation from C0 to C1 could be large but we now wish to move to configuration C2. The increment from C1 to C2 is taken to be small and we could refer to strains with respect to C2. So that means we are looking at current configuration as in the definition Euler strain but we are – movement is from C1 to C2 and not from C0 to C2. So with that certain simplifications become possible and we will be able to introduce what are known as Euler strains. This is epsilon 22ij.

So the same configuration both current and reference configurations are the same. Therefore this Euler strain tensor. Now this can be again expressed by using definitions of displacement vector we get this.



And the linear part of this is given by this and this we omit now the two subscripts and simply write it as 2Eij and this quantity is called the infinitesimal strain tensor. So it is Eulerian strain tensor but the linear incremental part. The strain components these strain components conjugate with Cauchy stress components to produce the expression for internal energy stone. It is clear that Eulerian strains are defined with respect to deformed configuration and Cauchy stress is defined with respect to deform configuration. Therefore, one can expect that they will conjugate to produce the internal work done. Internal energy stored.

Stress tensors

Cauchy stress: internal force reckoned in the deformed configuration and area reckoned in the deformed configuration.

Configuration C_1: ${}^1\sigma_{ij} = {}^1_1\sigma_{ij}$

Configuration C_2: ${}^2\sigma_{ij} = {}^2_2\sigma_{ij}$

Second Piola - Kirchoff stress tensor: force in C_2 transformed to C_0 and area reckoned in $C_0 \Rightarrow$

$$\left({}^0\hat{n}\bullet S\right) d\,{}^0A = d\,{}^0f = {}^2_0F^{-1} d\,{}^2f = \left[\frac{\partial^2 x}{\partial^0 x}\right]^{-1} d\,{}^2f$$

${}^0\hat{n}$ = unit normal to $d\,{}^0A$ in C_0.

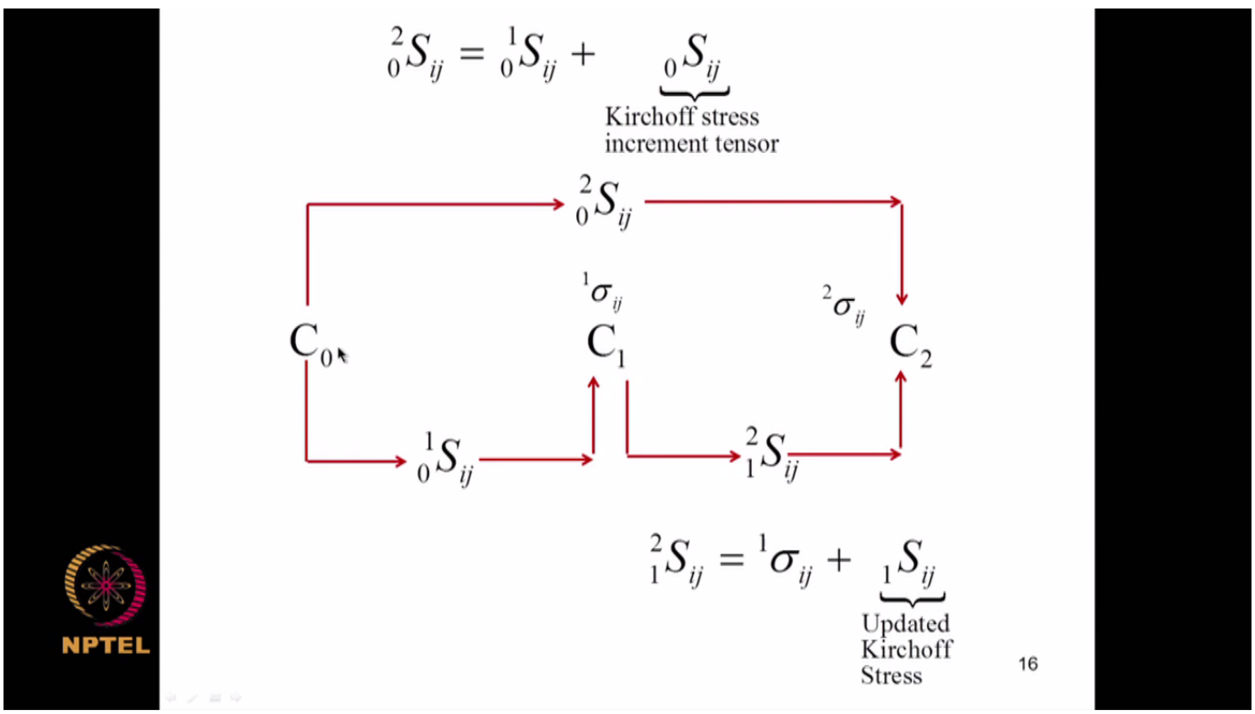
Updated Kirchoff stress tensor (useful in updated Lagrangian formulations)

Consider the point $P\left({}^1x_1, {}^1x_2, {}^1x_3\right)$ in C_1.

Cauchy stress tensor at P in C_1 is denoted by ${}^1\sigma_{ij}$

15

We have Cauchy stress. These are internal force reckoned in the deformed configuration and area reckoned in the deformed configuration. So the configuration C1 this is 1 Sigma ij which is nothing but 11 Sigma ij. Configuration C2 it is 2 Sigma ij which is nothing but 22 Sigma ij.



Now we can look at this diagram. We have three configurations C naught, C1, C2. In C1 and C2 we have the Cauchy stress tensor this. This is in the deformed configuration C1 and C2. Now the Green-Lagrange know the second Piola-Kirchoff stress tensor between C0 to C2 is this. So this can be written in terms of 01 plus an increment that means 0 to 1 there is one Piola-Kirchoff stress and there is an increment. So this is Kirchhoff stress increment tensor. On the other hand, I can move from C naught to C1 that is this, this is a second Piola-Kirchoff stress from 0 to 1 and similarly I get Sij from 1 to 2. This Sij 1 to 2 I can write in terms of the Cauchy stress plus an increment. This increment is with respect to the second Piola-Kirchoff stress whereas this increment is due to is based on the Cauchy stress and this is known as updated Kirchhoff stress. So we have Kirchhoff stress increment tensor and updated Kirchhoff stress. So that becomes important in our discussions. In the second Piola-Kirchoff stress tensor force in C2 is transformed to C0 and area is reckoned in C naught. So this is the relation we get between the Piola-Kirchoff stress and the displacement gradient and the normal.

Then updated Kirchhoff stress tensor it is useful in updated Lagrangian formulation. Consider the point in P x1 x2 x3 in C1 the Cauchy stress tensor at P in C1 is this.

$$ {}_1^2S_{ij} = {}^1\sigma_{ij} + \underbrace{{}_1S_{ij}}_{\text{Updated Kirchoff Stress}} $$

$$ {}_0^2S_{ij} = {}_0^1S_{ij} + \underbrace{{}_0S_{ij}}_{\text{Kirchoff stress increment tensor}} $$

$$ {}^2\sigma_{ij} = \left({}_0^2J\right)^{-1}\left(\frac{\partial\,{}^2x_i}{\partial\,{}^0x_m}\right){}_0^2S_{mn}\left(\frac{\partial\,{}^2x_i}{\partial\,{}^0x_n}\right) $$

$$ {}_0^2S_{ij} = \left({}_0^2J\right)\left(\frac{\partial\,{}^0x_i}{\partial\,{}^2x_m}\right){}^2\sigma_{mn}\left(\frac{\partial\,{}^0x_j}{\partial\,{}^2x_n}\right) $$

Since ${}^0\rho = {}^2\rho\,{}_0^2J$, we also get

$$ {}^2\sigma_{ij} = \frac{{}^2\rho}{{}^0\rho}\left(\frac{\partial\,{}^2x_i}{\partial\,{}^0x_m}\right)\left(\frac{\partial\,{}^2x_i}{\partial\,{}^0x_n}\right){}_0^2S_{mn} $$

$$ {}_0^2S_{ij} = \frac{{}^0\rho}{{}^2\rho}^{2}\left(\frac{\partial\,{}^0x_i}{\partial\,{}^2x_m}\right)\left(\frac{\partial\,{}^0x_j}{\partial\,{}^2x_n}\right){}^2\sigma_{mn} $$

Relation between Cauchy stress in C_2 and updated Kirchoff stress

$$ {}_1^2S_{ij} = \frac{{}^1\rho}{{}^2\rho}\left(\frac{\partial\,{}^1x_i}{\partial\,{}^2x_p}\right)\left(\frac{\partial\,{}^1x_j}{\partial\,{}^2x_q}\right){}^2\sigma_{pq} $$

$$ {}^2\sigma_{ij} = \frac{{}^2\rho}{{}^1\rho}\left(\frac{\partial\,{}^2x_i}{\partial\,{}^1x_p}\right)\left(\frac{\partial\,{}^2x_j}{\partial\,{}^1x_q}\right){}_1^2S_{pq} $$

PK-2 stress in different configurations

$$ {}_0^2S_{ij} = \frac{{}^0\rho}{{}^1\rho}\left(\frac{\partial\,{}^0x_i}{\partial\,{}^1x_p}\right)\left(\frac{\partial\,{}^0x_j}{\partial\,{}^1x_q}\right){}_1^2S_{pq} $$

$$ {}_0^1S_{ij} = \frac{{}^0\rho}{{}^1\rho}\left(\frac{\partial\,{}^0x_i}{\partial\,{}^1x_p}\right)\left(\frac{\partial\,{}^0x_j}{\partial\,{}^1x_q}\right){}^1\sigma_{pq} $$

Relations between incremental stresses

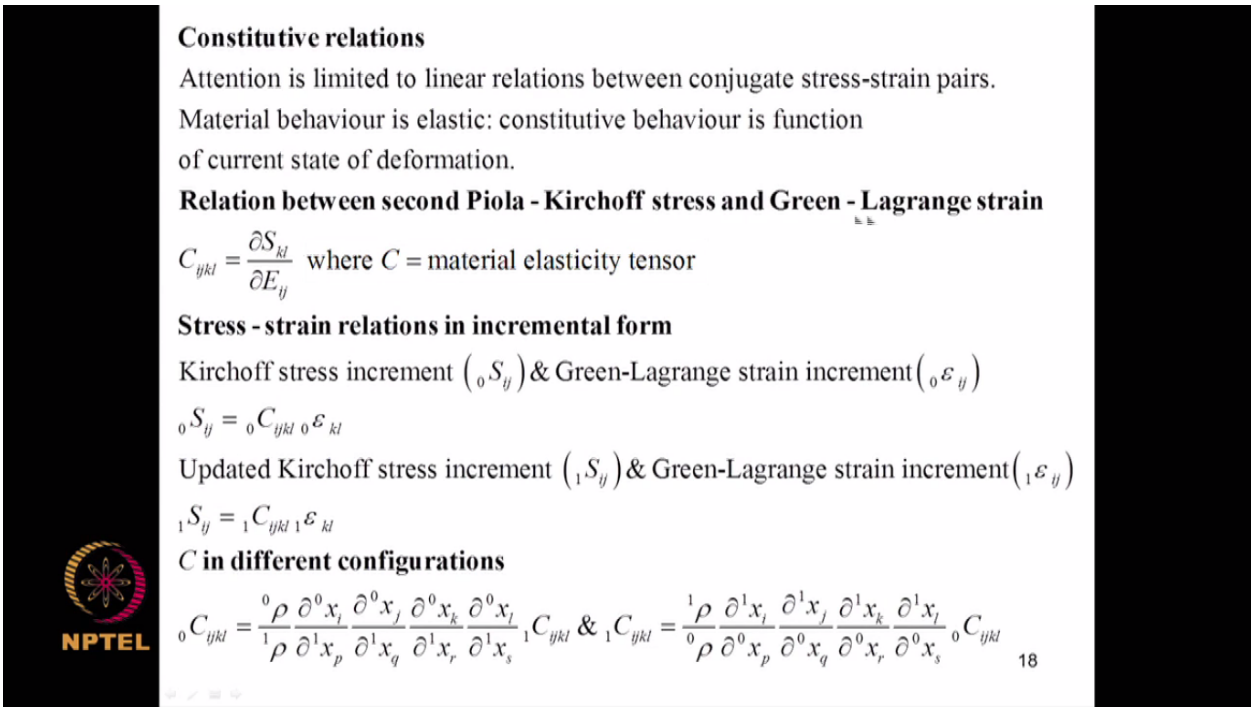
$$ {}_0S_{ij} = \frac{{}^0\rho}{{}^1\rho}\left(\frac{\partial\,{}^0x_i}{\partial\,{}^1x_p}\right)\left(\frac{\partial\,{}^0x_j}{\partial\,{}^1x_q}\right){}_1S_{pq} $$

$$ {}_1S_{ij} = \frac{{}^1\rho}{{}^0\rho}\left(\frac{\partial\,{}^1x_i}{\partial\,{}^0x_p}\right)\left(\frac{\partial\,{}^1x_j}{\partial\,{}^0x_q}\right){}_0S_{pq} $$

That is what I am trying to define and here we have summarized. So this is the Piola-Kirchoff from 1 to 2 in terms of Cauchy stress plus an increment. This is Piola-Kirchoff from 0 to 2 in terms of the Piola-Kirchoff at the configuration 1 plus an increment. So this is the Cauchy stress in the second configuration and we know these relations. I am simply putting it in the indicial notation. So this is the relationship between Cauchy stress and the second Piola-Kirchoff stress and this is the relationship between the second Piola-Kirchoff stress and the Cauchy stress.

Now since we conservation of mass we have seen that this relation must hold good we can also express these relations. We can instead of writing the Jacobian I can write the ratios of masses and I get this relation. This is same as this except that Jacobian is written in terms of ratios of masses. Now what is relationship between Cauchy stress in C2 and updated Kirchhoff stress. So that means I want now 2 Sigma ij in terms of updated Kirchhoff stress. So that if we consider now the second Piola-Kirchoff from 1 to 2 this is given by this as you have seen and this is the relationship between the Cauchy stress in the second configuration with the S from 1 to 2.

Now this PK-2 stress in different configurations we can write by using this relation explicitly just for reference I have given. Then relationship between incremental stresses. So Sij0 and S1pq I mean this is how it is related. So this and this how they are related is what I am explaining. I mean these are fairly simple if you observe I mean sit with a pen and paper you can easily understand but all these relations would be needed as we proceed.



Now for the discussion in this lecture we are limiting our attention to linear relations between conjugate stress-strain pairs. Material behavior is elastic that means constitutive behavior is function of current state of deformation and the loading unloading path will be identical. Upon removal of the load the original configuration will be restored. Then relationship between Piola-Kirchoff stress and Green-Lagrange strain is defined through this quantity Cijkl where C is the material elasticity tensor. The stress strain relations in an incremental form we can write in terms of this. This is the Kirchhoff stress increment tensor and epsilon kl this is the incremental strain and this is a C matrix. Similarly updated Kirchhoff stress increment and Green-Lagrange strain increment can be related through this relation. Now C in different configurations can be written in this form and this is C in first configuration related to zeroth configuration. This is the other way, the relation other way.

Principle of virtual displacements

Sum of virtual external work done on a body and the virtual work stored in the body should be zero.

Consider configuration C_2

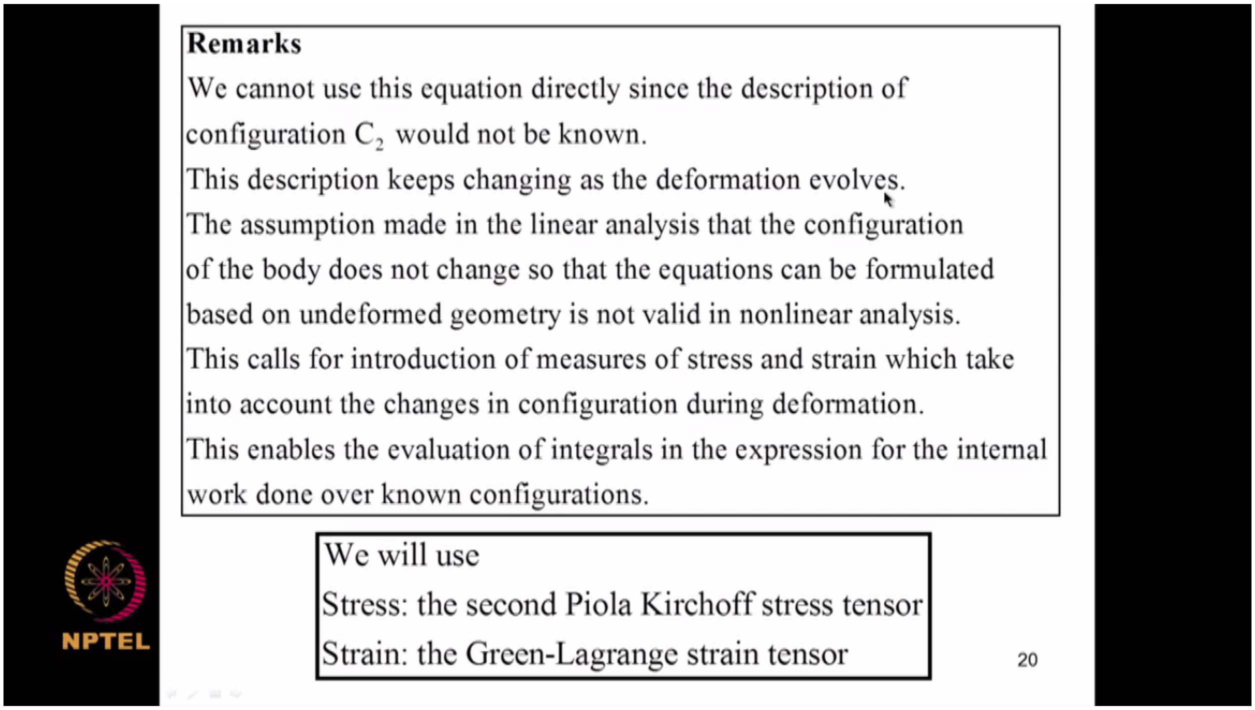
$$\delta W = \int_{^2V} {}^2\sigma : \delta\left({}_2e\right)\, d\,^2V - \left\{ \int_{^2V} {}^2f \bullet \delta u\ d\,^2V + \int_{^2S} {}^2t \bullet \delta u\ d\,^2S \right\} = 0$$

$$\delta W = \int_{^2V} {}^2\sigma_{ij}\, \delta\left({}_2e_{ij}\right)\, d\,^2V - \left\{ \int_{^2V} {}^2f_i\, \delta u_i\ d\,^2V + \int_{^2S} {}^2t_i\, \delta u_i\ d\,^2S \right\} = 0$$

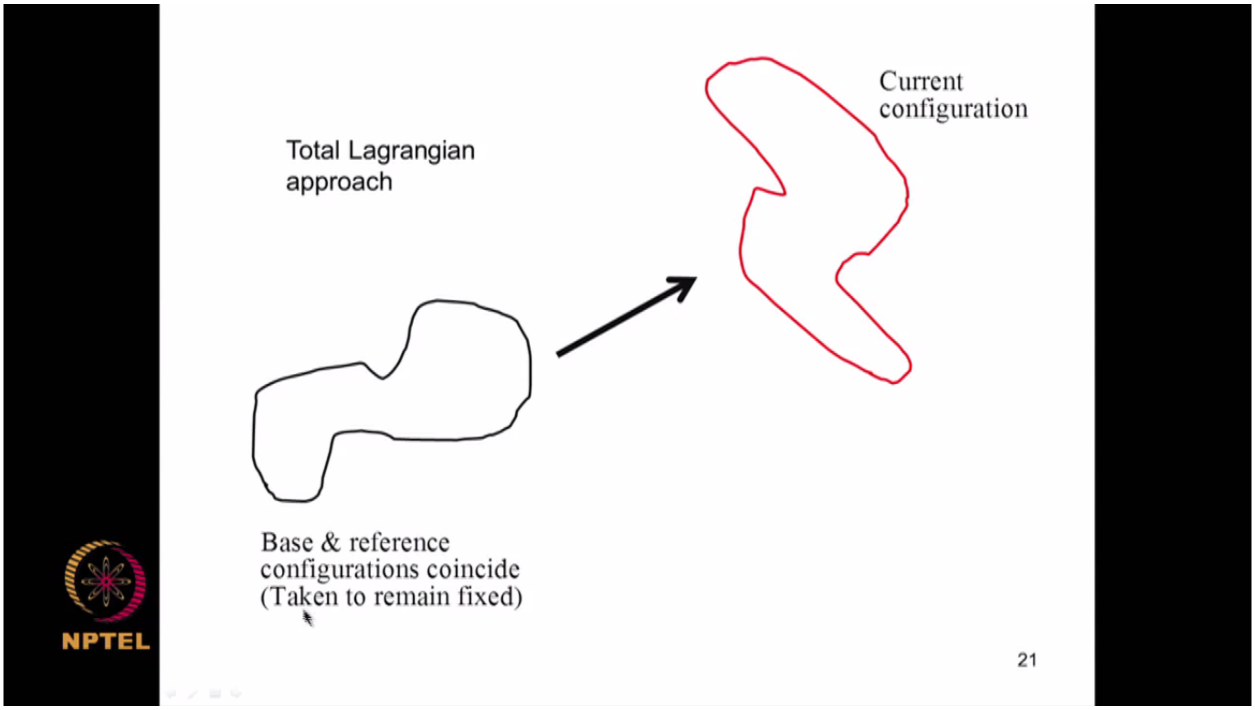
NPTEL

19

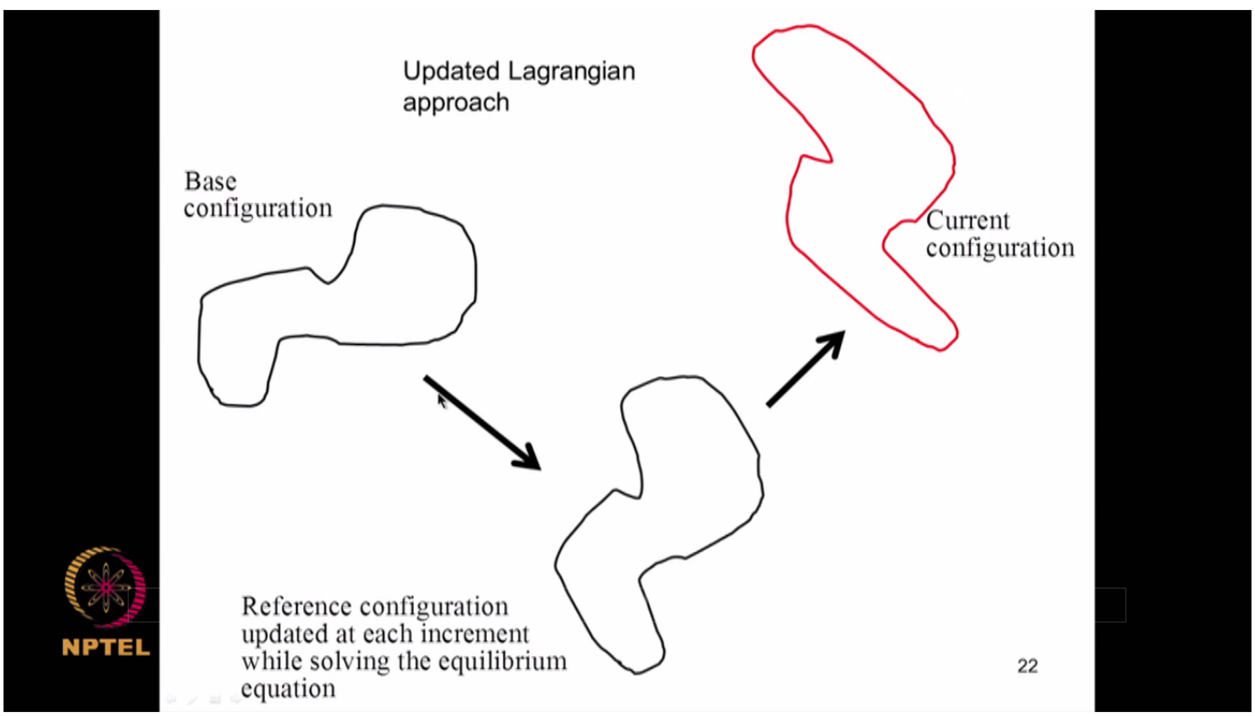
Now the formulation of finite element in this case will be based on principle of virtual displacements. What it says is somehow virtual external work done on a body and the virtual work stored in the body should be zero. So if you consider configuration C2 the virtual work is the Cauchy stress into the Euler strain virtual Euler strain integrated over the volume in the second configuration, and this is the contribution due to virtual – from virtual displacement from the surfaced. This body forces. This is surface traction. Now this in the indicial notation it is given in this form.



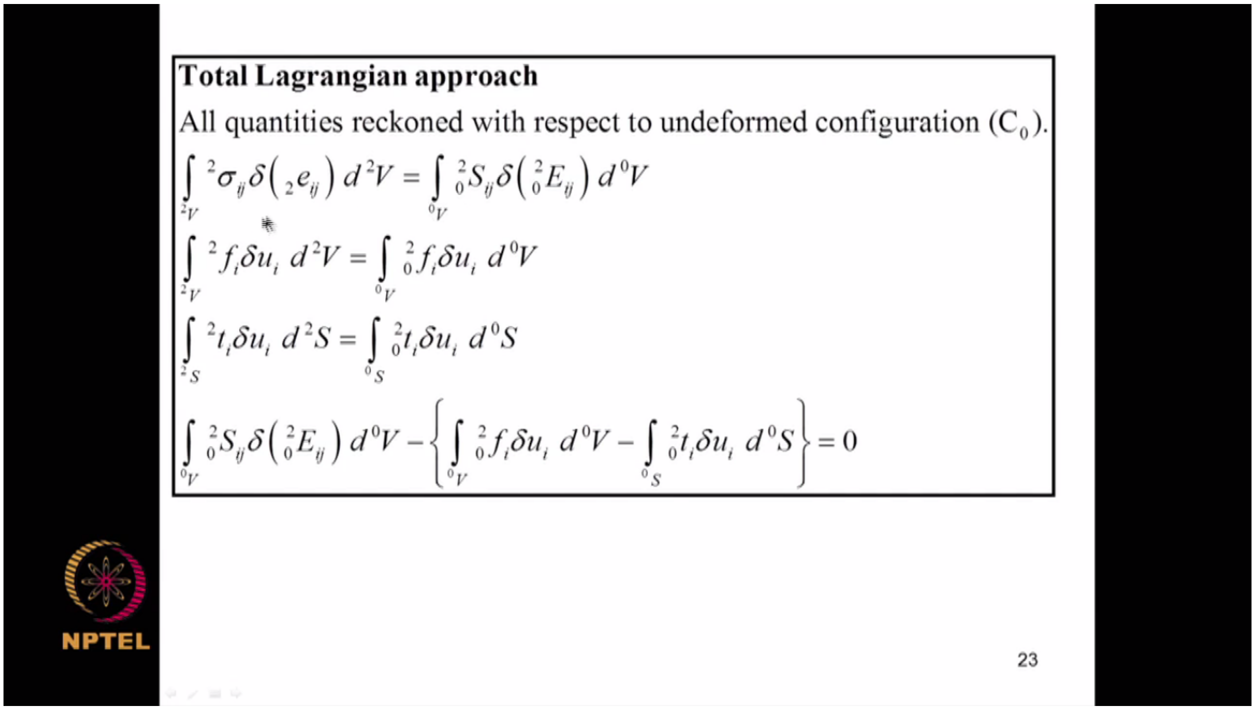
Now we cannot use this equation directly. See if you see here we have used the Cauchy stress and the Eulerian strain. That means both of them correspond to the deformed configuration. So it looks nice but as I already said the definition of the configuration C2 would not be known so we cannot use this equation directly. So the configuration keeps changing as the deformation evolves but in a linear analysis you can quickly recall or the assumption made in the linear analysis that the configuration of the body does not change so that the equations can be formulated based on undeformed geometry is not valid in nonlinear analysis. This calls for introduction of measures of stress and strain which take into account changes in configuration during deformation. This point I have been making couple of times already. I have made up this point couple of times already. This enables the evaluation of integrals in the expression for the internal work done or known configuration. By introducing the alternative measures of stress and strain what we will be doing is will be able to evaluate these integrals with respect to volumes in known configurations. So that is the another way of expressing the need for alternative definitions of stress and strain. So in our lecture we will use stress as the second Piola-Kirchoff stress tensor and strain is a Green-Lagrange strain tensor. They are the conjugate in the sense that they produce the internal work done.



This I have mentioned in the previous one of the previous lecture in the total Lagrangian formulation the base and reference configuration coincide and it is taken to remain fixed and this is a current configuration but all equations are formulated with respect to the reference configuration.



Whereas in updated Lagrangian approach the reference configuration is updated with each increment in the load and we analyze the incremental displacement with respect to a reference state that keeps evolving as the loads are incremented.



So let us take a look at the total Lagrangian approach. So here all quantities are reckoned with respect to undeformed configuration. So the virtual work statement leads to these terms. These are the terms that are present in the virtual work statement. If you see here we have to analyze each one of them now. So this is the Cauchy stress into the virtual Eulerian strain and now this is second Piola-Kirchoff stress and the virtual Green-Lagrange strain. And this is – similarly these are identities that they should be equal is an identity. So nothing is gained or lost okay. So this is the terms involving body forces. This is terms involving surface traction. And this is the virtual work – statement of the virtual work principle in terms of second Piola-Kirchoff stress and Green-Lagrange strain measure and all volumes are now in the first configuration. This is 0V and this is surface in the zeroth configuration.

$$\int_{{}^0V} {}^2_0S_{ij}\,\delta\left({}^2_0E_{ij}\right) d\,{}^0V - \left\{ \int_{{}^0V} {}^2_0f_i\,\delta u_i\; d\,{}^0V - \int_{{}^0S} {}^2_0t_i\,\delta u_i\; d\,{}^0S \right\} = 0$$

Recall: $${}^2_0S_{ij} = {}^1_0S_{ij} + \underbrace{{}_0S_{ij}}_{\text{Kirchoff stress increment tensor}}$$

Consider $\delta\left({}^2_0E_{ij}\right)$

Recall: $${}^2_0E_{ij} = {}^1_0E_{ij} + \left({}_0e_{ij} + {}_0\eta_{ij}\right)$$

$$\Rightarrow \delta\left({}^2_0E_{ij}\right) = \delta\left({}^1_0E_{ij}\right) + \delta\left({}_0e_{ij}\right) + \delta\left({}_0\eta_{ij}\right)$$

$$\Rightarrow \delta\left({}^2_0E_{ij}\right) = \delta\left({}_0e_{ij}\right) + \delta\left({}_0\eta_{ij}\right)$$

$\left\{\delta\left({}^1_0E_{ij}\right) = 0 \because {}^1_0E_{ij} \text{ does not depend on unknown displacements}\right\}$

NPTEL

24

Now our idea is to now work with this and develop a weak form based on which we can develop a finite element model. Now as I already said we are going to take an incremental loading approach. So as the load is incremented by a small amount we wish to deal with linearized stress strain relations and to formulate that we have to get into many details. So in this approach we will not formulate a nonlinear set of equations and then develop an independent solution strategy. The solution strategy is embedded into the development of the finite element model itself. So we have this statement.

Now we start working with the second Piola-Kirchoff stress from 0 to 2 is expressed in terms of PK-2 from 0 to 1 and the Kirchhoff stress increment. Now so we will consider each one separately and then we will look at these quantities in terms of increments and certain reference values. So this 0 to 2 Green-Lagrange strain tensor is written in terms of 0 to 1 plus these incremental quantities and therefore the virtual strains you use the Delta operator and you will get this. Now Delta of Eij01 is 0 because this does not depend on the unknown displacements. The unknown displacements are the increments. Therefore, this will be 0 and I get this.

Consider $\delta\left({}_0e_{ij}\right)$

Recall: $${}_0e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial {}^0x_j} + \frac{\partial u_j}{\partial {}^0x_i} + \frac{\partial {}_0^1u_k}{\partial {}^0x_i}\frac{\partial u_k}{\partial {}^0x_j} + \frac{\partial u_k}{\partial {}^0x_i}\frac{\partial {}_0^1u_k}{\partial {}^0x_j}\right)$$

$$\delta\left({}_0e_{ij}\right) = \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial {}^0x_j} + \frac{\partial \delta u_j}{\partial {}^0x_i} + \frac{\partial {}_0^1u_k}{\partial {}^0x_i}\frac{\partial \delta u_k}{\partial {}^0x_j} + \frac{\partial \delta u_k}{\partial {}^0x_i}\frac{\partial {}_0^1u_k}{\partial {}^0x_j}\right)$$

Consider $\delta\left({}_0\eta_{ij}\right)$

Recall: $${}_0\eta_{ij} = \frac{1}{2}\frac{\partial u_k}{\partial {}^0x_i}\frac{\partial u_k}{\partial {}^0x_j}$$

$$\Rightarrow \delta\left({}_0\eta_{ij}\right) = \frac{1}{2}\frac{\partial \delta u_k}{\partial {}^0x_i}\frac{\partial u_k}{\partial {}^0x_j} + \frac{1}{2}\frac{\partial u_k}{\partial {}^0x_i}\frac{\partial \delta u_k}{\partial {}^0x_j}$$

NPTEL

25

Now we have the expression for this quantity E0ij that is this and therefore I will be able to compute the variations and for linear component and nonlinear component. Okay. So this is matter of derivations that you have to observe.

$$\int_{{}^0V} {}_0^2S_{ij}\,\delta\left({}_0^2E_{ij}\right) d\,{}^0V - \delta\left({}_2^0R\right) = 0$$

$$\delta\left({}^0R_2\right) = \int_{{}^0V} {}_0^2f_i\,\delta u_i\; d\,{}^0V + \int_{{}^0S} {}_0^2t_i\,\delta u_i\; d\,{}^0S$$

$$\int_{{}^0V} {}_0^2S_{ij}\,\delta\left({}_0^2E_{ij}\right) d\,{}^0V = \int_{{}^0V} \left({}_0^1S_{ij} + {}_0S_{ij}\right)\delta\left({}_0\varepsilon_{ij}\right) d\,{}^0V$$

$$= \int_{{}^0V} \left\{ {}_0S_{ij}\,\delta\left({}_0\varepsilon_{ij}\right) + {}_0^1S_{ij}\left[\delta\left({}_0e_{ij}\right) + \delta\left({}_0\eta_{ij}\right)\right]\right\} d\,{}^0V$$

Consider the term $\int_{{}^0V} {}_0^1S_{ij}\,\delta\left({}_0e_{ij}\right) d\,{}^0V$

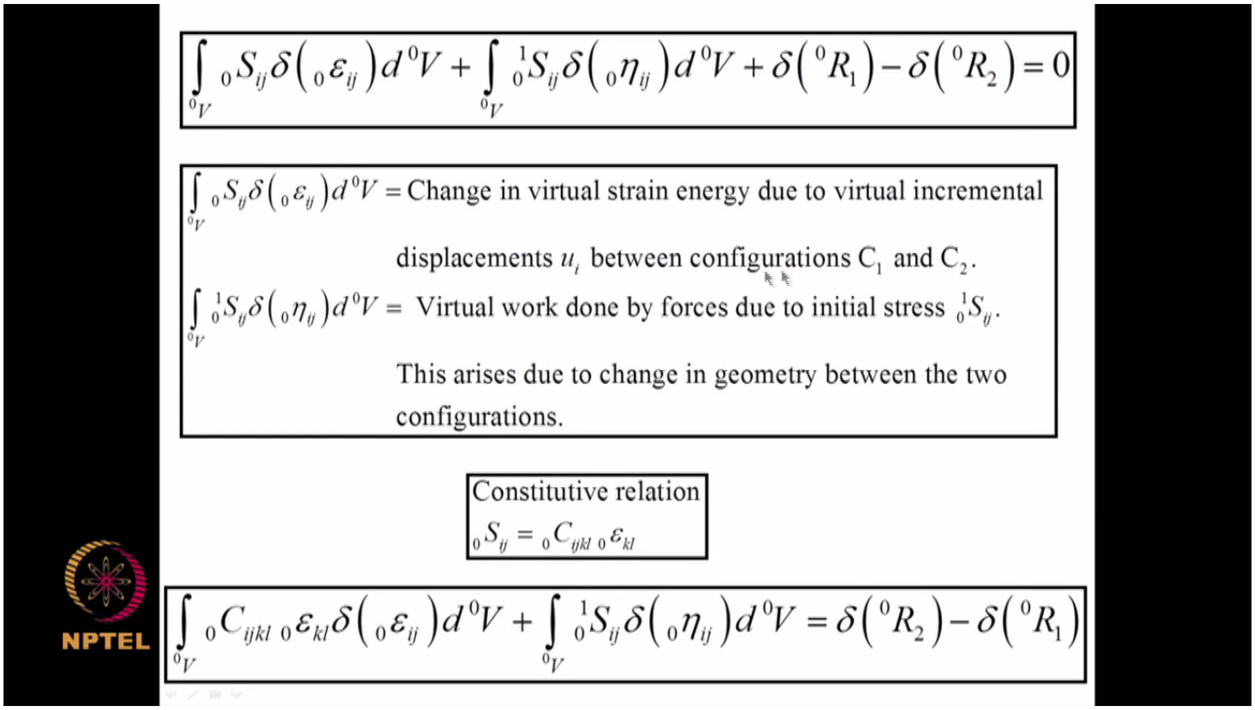
This represents the virtual internal energy stored in the body in configuration C_1. By applying virtual work principle to body in configuration C_1, one gets

$$\delta\left({}^0R_1\right) = \int_{{}^0V} {}_0^1S_{ij}\,\delta\left({}_0e_{ij}\right) d\,{}^0V = \int_{{}^0V} {}_0^1f_i\,\delta u_i\; d\,{}^0V + \int_{{}^0S} {}_0^1t_i\,\delta u_i\; d\,{}^0S$$

NPTEL

26

And similarly this -- the contribution from body force and surface traction is denoted as Delta of 0 r 2 and that is given by this. Now let us consider this part and write the 02PK in terms of this and we will expand this and this virtual strain also is written in terms of the increment and as we multiply we get one of the terms as Sij01 Delta of 0 Eij $ naught V. This quantity represents a virtual internal energies stored in the body in configuration C1. Now by applying virtual work principle to body in configuration C1 we can show that this quantity is given by the body force and surface traction contributions in configuration one.



Equipped with that we have now the statement that we will need for formulating the finite element model.

Now this is this expression we get. Now we can again summarize what these terms are. The first term is change in virtual strain energy due to virtual incremental displacement UI between configuration C1 and C2. The next term is the virtual work done by forces due to initial stress 01Sij. This arises due to change in geometry between the two configuration. Now we need to introduce the constitutive relations. So the constitutive relation we use in the incremental form as shown here and that leads to the equation that forms the basis for developing the finite element model.

$$\int_{{}^0V} {}_0S_{ij}\,\delta\left({}_0\varepsilon_{ij}\right)d\,{}^0V + \int_{{}^0V} {}_0^1S_{ij}\,\delta\left({}_0\eta_{ij}\right)d\,{}^0V + \delta\left({}^0R_1\right) - \delta\left({}^0R_2\right) = 0$$

Assuming the displacements u_i to be small

(reasonable assumption for small load increments)

$${}_0^1S_{ij} \approx {}_0C_{ijkl}\,{}_0e_{kl}$$

$$\delta\left({}_0\varepsilon_{ij}\right) \approx \delta\left({}_0e_{ij}\right)$$

$\Rightarrow$

$$\int_{{}^0V} {}_0C_{ijkl}\,{}_0e_{kl}\,\delta\left({}_0e_{ij}\right)d\,{}^0V + \int_{{}^0V} {}_0^1S_{ij}\,\delta\left({}_0\eta_{ij}\right)d\,{}^0V = \delta\left({}^0R_2\right) - \delta\left({}^0R_1\right)$$

$${}_0^1S_{ij} = {}_0C_{ijkl}\,{}_0^1{}_0E_{kl}$$

28

Now before we jump into that we can simplify this further. If you assume the displacement UI to be small that means the increments are small if load increments are small, the displacement increments are also small. If we assume that then I can simplify and remove certain nonlinear terms and the constitutive law can be expressed as shown here and consequently the virtual work statement gets further simplified to this form.

Inertia forces

$$\int_{{}^2V} {}^2\rho\,{}^2\ddot{u}_i\,\delta\,{}^2u_i\,d\,{}^2V = \int_{{}^0V} {}^0\rho\,{}^2\ddot{u}_i\,\delta\,{}^2u_i\,d\,{}^0V \qquad \left(\delta\,{}^2u_i = \delta u_i\right)$$

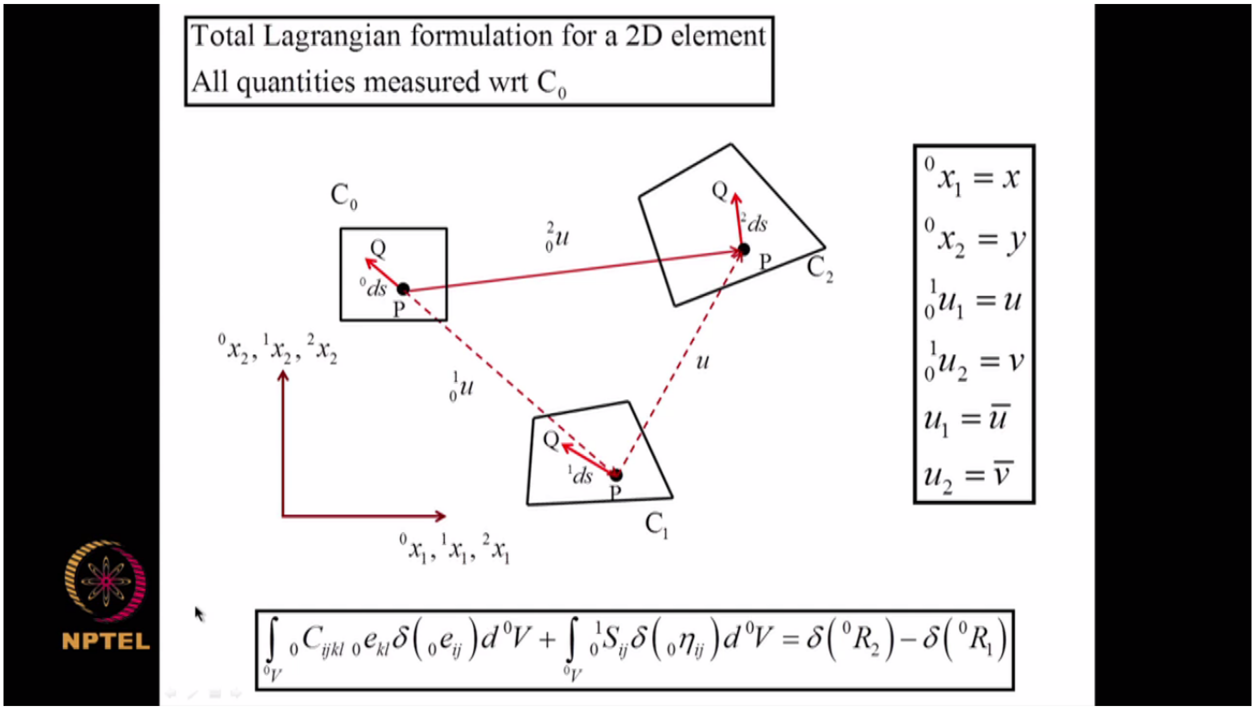
$\Rightarrow$ Equation of motion

$$\int_{{}^0V} {}^0\rho\,{}^2\ddot{u}_i\,\delta\,{}^2u_i\,d\,{}^0V + \int_{{}^0V} {}_0C_{ijrs}\,{}_0e_{rs}\,\delta\left({}_0e_{ij}\right)d\,{}^0V + \int_{{}^0V} {}_0^1S_{ij}\,\delta\left({}_0\eta_{ij}\right)d\,{}^0V$$

$$= \delta\left({}^0R_2\right) - \delta\left({}^0R_1\right)$$

29

Now I have not talked about inertia forces till now, but if inertia forces and I am not going to discuss this at least in this part of the lecture but we can quickly see how we can deal with this if such terms were to be present. So the identity that will use is the total inertial force in configurations 0 & 2 would be equal and this is the statement that is identity. Then equation of motion while formulating the principle of virtual work now we need to consider inertial forces also and we get this statement. So this is a new term that has to be handled if you are dealing with dynamics.



Now for purpose of illustration we can consider a 2D element and develop the total Lagrangian formulation. Suppose this is configuration C naught. This is configuration C1 and this is C2 and these are the coordinates of 2D element therefore, the third direction is not taken and for sake of simplifying the notation we call x1 x2 and zeroth configuration as xy and displacement 01 U1 that is this components as UV and the increments as u bar v bar. So this is a UV. This is u bar, v bar and these coordinates are XY. So this is the weak form that we will have to use to formulate that.

Weak form

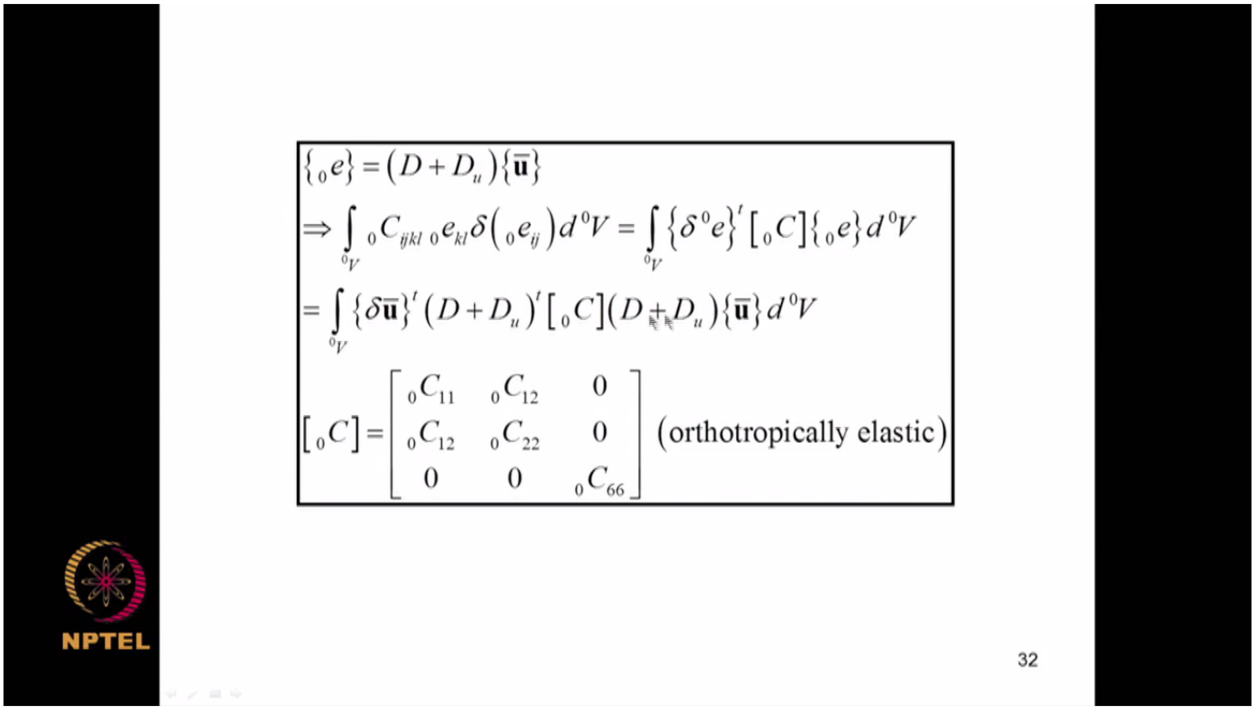
$$\int_{^0V} {}_0C_{ijkl}\,{}_0e_{kl}\,\delta\left({}_0e_{ij}\right)d\,{}^0V + \int_{^0V} {}_0^1S_{ij}\,\delta\left({}_0\eta_{ij}\right)d\,{}^0V = \delta\left({}^0R_2\right) - \delta\left({}^0R_1\right)$$

$$\left\{{}_0e\right\} = \begin{Bmatrix} {}_0e_{xx} \\ {}_0e_{yy} \\ 2\,{}_0e_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \bar{u}}{\partial x} \\ \dfrac{\partial \bar{v}}{\partial y} \\ \dfrac{\partial \bar{u}}{\partial y} + \dfrac{\partial \bar{v}}{\partial x} \end{Bmatrix} + \begin{Bmatrix} \dfrac{\partial u}{\partial x}\dfrac{\partial \bar{u}}{\partial x} + \dfrac{\partial v}{\partial x}\dfrac{\partial \bar{v}}{\partial x} \\ \dfrac{\partial u}{\partial y}\dfrac{\partial \bar{u}}{\partial y} + \dfrac{\partial v}{\partial y}\dfrac{\partial \bar{v}}{\partial y} \\ \dfrac{\partial u}{\partial x}\dfrac{\partial \bar{u}}{\partial y} + \dfrac{\partial v}{\partial x}\dfrac{\partial \bar{v}}{\partial y} + \dfrac{\partial u}{\partial y}\dfrac{\partial \bar{u}}{\partial x} + \dfrac{\partial v}{\partial y}\dfrac{\partial \bar{v}}{\partial x} \end{Bmatrix}$$

$$= \left(\begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} + \begin{bmatrix} \dfrac{\partial u}{\partial x}\dfrac{\partial}{\partial x} & \dfrac{\partial v}{\partial x}\dfrac{\partial}{\partial x} \\ \dfrac{\partial u}{\partial y}\dfrac{\partial}{\partial y} & \dfrac{\partial v}{\partial y}\dfrac{\partial}{\partial y} \\ \dfrac{\partial u}{\partial y}\dfrac{\partial}{\partial x} + \dfrac{\partial u}{\partial x}\dfrac{\partial}{\partial y} & \dfrac{\partial v}{\partial y}\dfrac{\partial}{\partial x} + \dfrac{\partial v}{\partial x}\dfrac{\partial}{\partial y} \end{bmatrix} \right) \begin{Bmatrix} \bar{u} \\ \bar{v} \end{Bmatrix} = \left(D + D_u\right)\left\{\bar{\mathbf{u}}\right\}$$

31

Now what follows is we have to now convert all the expressions to the expressions in terms of incremental displacements and the incremental displacement need to be interpolated in terms of nodal values and once that interpolation form is established we return to this equation and get the the finite element model. That is the story. So we have 0E as this and in terms of incremental displacements it is given by a linear part and a nonlinear part. This is a still linear in u bar so we get this equation. This is in the matrix form. We introduce D and DU and this is u bar.



So this is a relationship between strain and the displacements. And now if I substitute that into the terms in the virtual work statement we can simplify the first term in terms of this representation and we get this term simplifies to this where this matrix is the elastic C matrix that is less stress and strain. We are assuming in this formulation that it is a an orthotopically elastic material.

$$\int_{{}^0V} {}_0^1S_{ij}\,\delta\left({}_0\eta_{ij}\right)d\,{}^0V = \int_{{}^0V} \delta\left({}_0\eta\right)^t\,{}_0^1S\,d\,{}^0V$$

$$\left({}_0\eta\right) = \begin{Bmatrix} {}_0\eta_{xx} \\ {}_0\eta_{yy} \\ 2\,{}_0\eta_{xy} \end{Bmatrix} = \frac{1}{2}\begin{Bmatrix} \dfrac{\partial\bar{u}}{\partial x}\dfrac{\partial\bar{u}}{\partial x} + \dfrac{\partial\bar{v}}{\partial x}\dfrac{\partial\bar{v}}{\partial x} \\ \dfrac{\partial\bar{u}}{\partial y}\dfrac{\partial\bar{u}}{\partial y} + \dfrac{\partial\bar{v}}{\partial y}\dfrac{\partial\bar{v}}{\partial y} \\ 2\left(\dfrac{\partial\bar{u}}{\partial x}\dfrac{\partial\bar{u}}{\partial y} + \dfrac{\partial\bar{v}}{\partial x}\dfrac{\partial\bar{v}}{\partial y}\right) \end{Bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} \dfrac{\partial\bar{u}}{\partial x}\dfrac{\partial}{\partial x} & \dfrac{\partial\bar{v}}{\partial x}\dfrac{\partial}{\partial x} \\ \dfrac{\partial\bar{u}}{\partial y}\dfrac{\partial}{\partial y} & \dfrac{\partial\bar{v}}{\partial y}\dfrac{\partial}{\partial y} \\ \dfrac{\partial\bar{u}}{\partial y}\dfrac{\partial}{\partial x} + \dfrac{\partial\bar{u}}{\partial x}\dfrac{\partial}{\partial y} & \dfrac{\partial\bar{v}}{\partial y}\dfrac{\partial}{\partial x} + \dfrac{\partial\bar{v}}{\partial x}\dfrac{\partial}{\partial y} \end{bmatrix}\begin{Bmatrix} \bar{u} \\ \bar{v} \end{Bmatrix} = \frac{1}{2}\left[D_{\bar{u}}\right]\left\{\bar{\mathbf{u}}\right\}$$

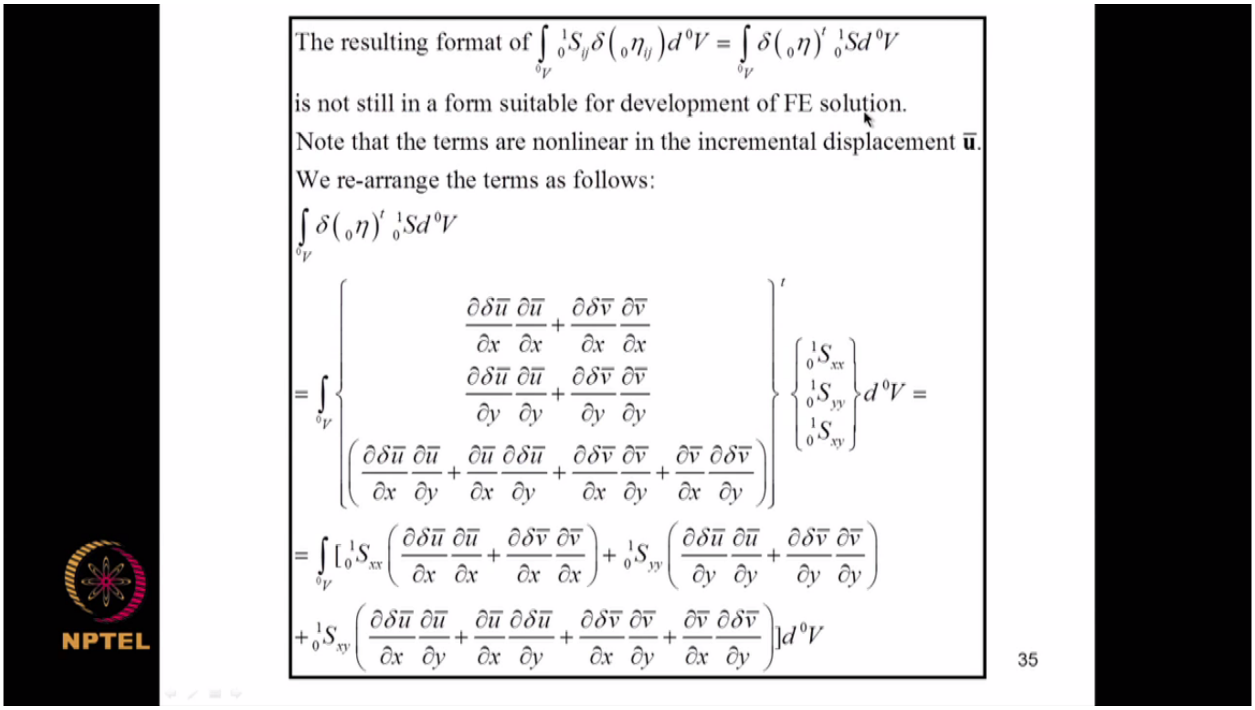
Now the other terms from eta increments eta that again if we look at eta this is non-linear terms. So we write in this form u bar, v bar is D of u bar because you can see her u bar. The U bars are here. This is non-linear and the variation is obtained in this form.

$$\left(\delta\,{}_0\eta\right) = \begin{Bmatrix} \delta\,{}_0\eta_{xx} \\ \delta\,{}_0\eta_{yy} \\ 2\delta\,{}_0\eta_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial\delta\bar{u}}{\partial x}\dfrac{\partial\bar{u}}{\partial x} + \dfrac{\partial\delta\bar{v}}{\partial x}\dfrac{\partial\bar{v}}{\partial x} \\ \dfrac{\partial\delta\bar{u}}{\partial y}\dfrac{\partial\bar{u}}{\partial y} + \dfrac{\partial\delta\bar{v}}{\partial y}\dfrac{\partial\bar{v}}{\partial y} \\ \left(\dfrac{\partial\delta\bar{u}}{\partial x}\dfrac{\partial\bar{u}}{\partial y} + \dfrac{\partial\bar{u}}{\partial x}\dfrac{\partial\delta\bar{u}}{\partial y} + \dfrac{\partial\delta\bar{v}}{\partial x}\dfrac{\partial\bar{v}}{\partial y} + \dfrac{\partial\bar{v}}{\partial x}\dfrac{\partial\delta\bar{v}}{\partial y}\right) \end{Bmatrix} = \left[D_{\bar{u}}\right]\left\{\delta\bar{\mathbf{u}}\right\} = \left[D_{\delta\bar{u}}\right]\left\{\bar{\mathbf{u}}\right\}$$

$$\left\{{}_0^1S\right\} = \begin{Bmatrix} {}_0^1S_{xx} \\ {}_0^1S_{yy} \\ {}_0^1S_{xy} \end{Bmatrix} = \begin{bmatrix} {}_0C_{11} & {}_0C_{12} & 0 \\ {}_0C_{12} & {}_0C_{22} & 0 \\ 0 & 0 & {}_0C_{66} \end{bmatrix}\begin{Bmatrix} {}_0^1E_{xx} \\ {}_0^1E_{yy} \\ 2\,{}_0^1E_{xy} \end{Bmatrix}$$

$$\left\{{}_0^1E\right\} = \begin{Bmatrix} {}_0^1E_{xx} \\ {}_0^1E_{yy} \\ 2\,{}_0^1E_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} + \dfrac{1}{2}\left[\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial v}{\partial x}\right)^2\right] \\ \dfrac{\partial v}{\partial y} + \dfrac{1}{2}\left[\left(\dfrac{\partial u}{\partial y}\right)^2 + \left(\dfrac{\partial v}{\partial y}\right)^2\right] \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} + \left(\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y} + \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}\right) \end{Bmatrix}$$

So this is written in this form where we introduce certain notations as shown here.


The resulting format of $\int_{{}^0V} {}_0^1S_{ij}\delta\left({}_0\eta_{ij}\right)d\,{}^0V = \int_{{}^0V} \delta\left({}_0\eta\right)^t\,{}_0^1S\,d\,{}^0V$

is not still in a form suitable for development of FE solution.

Note that the terms are nonlinear in the incremental displacement $\bar{\mathbf{u}}$.

We re-arrange the terms as follows:

$$\int_{{}^0V} \delta\left({}_0\eta\right)^t\,{}_0^1S\,d\,{}^0V$$

$$= \int_{{}^0V} \begin{Bmatrix} \dfrac{\partial\delta\bar{u}}{\partial x}\dfrac{\partial\bar{u}}{\partial x}+\dfrac{\partial\delta\bar{v}}{\partial x}\dfrac{\partial\bar{v}}{\partial x} \\ \dfrac{\partial\delta\bar{u}}{\partial y}\dfrac{\partial\bar{u}}{\partial y}+\dfrac{\partial\delta\bar{v}}{\partial y}\dfrac{\partial\bar{v}}{\partial y} \\ \left(\dfrac{\partial\delta\bar{u}}{\partial x}\dfrac{\partial\bar{u}}{\partial y}+\dfrac{\partial\bar{u}}{\partial x}\dfrac{\partial\delta\bar{u}}{\partial y}+\dfrac{\partial\delta\bar{v}}{\partial x}\dfrac{\partial\bar{v}}{\partial y}+\dfrac{\partial\bar{v}}{\partial x}\dfrac{\partial\delta\bar{v}}{\partial y}\right) \end{Bmatrix}^t \begin{Bmatrix} {}_0^1S_{xx} \\ {}_0^1S_{yy} \\ {}_0^1S_{xy} \end{Bmatrix} d\,{}^0V =$$

$$= \int_{{}^0V} [\,{}_0^1S_{xx}\left(\frac{\partial\delta\bar{u}}{\partial x}\frac{\partial\bar{u}}{\partial x}+\frac{\partial\delta\bar{v}}{\partial x}\frac{\partial\bar{v}}{\partial x}\right)+{}_0^1S_{yy}\left(\frac{\partial\delta\bar{u}}{\partial y}\frac{\partial\bar{u}}{\partial y}+\frac{\partial\delta\bar{v}}{\partial y}\frac{\partial\bar{v}}{\partial y}\right)$$

$$+{}_0^1S_{xy}\left(\frac{\partial\delta\bar{u}}{\partial x}\frac{\partial\bar{u}}{\partial y}+\frac{\partial\bar{u}}{\partial x}\frac{\partial\delta\bar{u}}{\partial y}+\frac{\partial\delta\bar{v}}{\partial x}\frac{\partial\bar{v}}{\partial y}+\frac{\partial\bar{v}}{\partial x}\frac{\partial\delta\bar{v}}{\partial y}\right)]\,d\,{}^0V$$

See actually the resulting format of these terms is still not in a form suitable for development FE solution. We need to reorganize these terms. So that requires certain introduction of certain vectors and we write the stress matrix in terms of a vector like this and these coefficients in this form and we multiply them out and if we do that we will be able to get the expressions in this form.

$$\int_{^0V} \delta({}_0\eta)^t\, {}^1_0S\, d^0V = \int_{^0V} \begin{Bmatrix} \dfrac{\partial\delta\bar{u}}{\partial x}\\ \dfrac{\partial\delta\bar{u}}{\partial y}\\ \dfrac{\partial\delta\bar{v}}{\partial x}\\ \dfrac{\partial\delta\bar{v}}{\partial y}\end{Bmatrix}^t \begin{bmatrix} {}^1_0S_{xx} & {}^1_0S_{xy} & 0 & 0\\ {}^1_0S_{xy} & {}^1_0S_{yy} & 0 & 0\\ 0 & 0 & {}^1_0S_{xx} & {}^1_0S_{xy}\\ 0 & 0 & {}^1_0S_{xy} & {}^1_0S_{yy}\end{bmatrix} \begin{Bmatrix} \dfrac{\partial\bar{u}}{\partial x}\\ \dfrac{\partial\bar{u}}{\partial y}\\ \dfrac{\partial\bar{v}}{\partial x}\\ \dfrac{\partial\bar{v}}{\partial y}\end{Bmatrix} d^0V$$

$$= \int_{^0V} (\delta\bar{\mathbf{u}})^t \left[\bar{D}\right]^t \left[{}^1_0S\right]\left[\bar{D}\right]\{\bar{\mathbf{u}}\}\, d^0V$$

$$\left[{}^1_0S\right] = \begin{bmatrix} {}^1_0S_{xx} & {}^1_0S_{xy} & 0 & 0\\ {}^1_0S_{xy} & {}^1_0S_{yy} & 0 & 0\\ 0 & 0 & {}^1_0S_{xx} & {}^1_0S_{xy}\\ 0 & 0 & {}^1_0S_{xy} & {}^1_0S_{yy}\end{bmatrix} \;\&\; \left[\bar{D}\right] = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0\\ \dfrac{\partial}{\partial y} & 0\\ 0 & \dfrac{\partial}{\partial x}\\ 0 & \dfrac{\partial}{\partial y}\end{bmatrix} \begin{Bmatrix}\bar{u}\\ \bar{v}\end{Bmatrix}$$

We are now ready to launch the FE formulation.

36

Some of these details you need to observe. Once you are done with this we are now ready to launch the FE formulations.

Incremental displacement: $\{\bar{\mathbf{u}}\} = \begin{Bmatrix}\bar{u}\\ \bar{v}\end{Bmatrix} = \begin{Bmatrix} \sum_{j=1}^{n} \bar{u}_j \psi_j(x)\\ \sum_{j=1}^{n} \bar{v}_j \psi_j(x)\end{Bmatrix} = [\Psi]\{\bar{\Delta}\}$

Total displacement: $\{\mathbf{u}\} = \begin{Bmatrix}u\\ v\end{Bmatrix} = \begin{Bmatrix} \sum_{j=1}^{n} u_j \psi_j(x)\\ \sum_{j=1}^{n} v_j \psi_j(x)\end{Bmatrix} = [\Psi]\{\Delta\}$

$$[\Psi] = \begin{bmatrix} \psi_1 & 0 & \psi_2 & 0 & \cdots & \psi_n & 0\\ 0 & \psi_1 & 0 & \psi_2 & \cdots & 0 & \psi_n\end{bmatrix}$$

$$\{\bar{\Delta}\}^t = \{\bar{u}_1 \;\; \bar{v}_1 \;\; \bar{u}_2 \;\; \bar{v}_2 \;\; \cdots \;\; \bar{u}_n \;\; \bar{v}_n\}$$

$$\{\Delta\}^t = \{u_1 \;\; v_1 \;\; u_2 \;\; v_2 \;\; \cdots \;\; u_n \;\; v_n\}$$

37

So what we do is the incremental displacement u bar, v bar is interpolated in terms of n nodal values. Sij are the interpolation functions and this is written like this and the total displacement is also represented in the same form which is this. So I have Si capital Si matrix and Delta bar vector and Delta vector as shown here.

$$\int_{{}^0V} \{\delta\,{}^0e\}^t [{}_0C]\{{}_0e\}\, d\,{}^0V$$

$$= \int_{{}^0V} \{\delta\bar{\mathbf{u}}\}^t (D + D_u)^t [{}_0C](D + D_u)\{\bar{\mathbf{u}}\}\, d\,{}^0V$$

$$= \int_{{}^0V} \{\delta\bar{\Delta}\}^t [\Psi]^t (D + D_u)^t [{}_0C](D + D_u)[\Psi]\{\bar{\Delta}\}\, d\,{}^0V$$

$$= \int_{{}^0V} \{\delta\bar{\Delta}\}^t [B_L]^t [{}_0C][B_L]\{\bar{\Delta}\}\, d\,{}^0V \text{ with } [B_L] = (D + D_u)[\Psi].$$

$$\int_{{}^0V} \delta({}_0\eta)^t\, {}_0^1S\, d\,{}^0V = \int_{{}^0V} (\delta\bar{\mathbf{u}})^t [\bar{D}]^t [{}_0^1S][\bar{D}]\{\bar{\mathbf{u}}\}\, d\,{}^0V$$

$$= \int_{{}^0V} \{\delta\bar{\Delta}\}^t [\Psi]^t [\bar{D}]^t [{}_0^1S][\bar{D}][\Psi]\{\bar{\Delta}\}\, d\,{}^0V$$

$$= \int_{{}^0V} \{\delta\bar{\Delta}\}^t [B_{NL}]^t [{}_0^1S][B_{NL}]\{\bar{\Delta}\}\, d\,{}^0V \text{ with } [B_{NL}] = [\bar{D}][\Psi]$$

38

And then we get into the principle of virtual work statement and we consider each term and submit these substitutions and then assemble all of them and obtain the final finite element equation. So we start with this. So we have now already derived the terms this Delta E naught in terms of u bar is here and that we are using here and here I am now making the substitution for u bar in terms of the assume shape functions. That is this and we introduce a matrix BL which is D plus DU into Si. Similarly the other terms we simplify and we get BNL and so on and so forth.

$$\delta\left({}_0^1R\right)=\int_{{}^0V} {}_0^1S_{ij}\,\delta\left({}^0e_{ij}\right)d\,{}^0V=\int_{{}^0V}\left\{\delta\left({}^0e\right)\right\}^t\left\{{}_0^1S\right\}d\,{}^0V$$

$$=\int_{{}^0V}\left\{\delta\bar{\Delta}\right\}^t\left[B_L\right]^t\left\{{}_0^1S\right\}d\,{}^0V$$

$$\delta\left({}_0^2R\right)=\int_{{}^0V} {}_0^2f_i\,\delta u_i\;d\,{}^0V+\int_{{}^0S} {}_0^2t_i\,\delta u_i\;d\,{}^0S$$

$$=\int_{{}^0V}\left\{\delta\bar{\Delta}\right\}^t\left[\Psi\right]^t\left\{{}_0^2\mathbf{f}\right\}\;d\,{}^0V+\int_{{}^0S}\left\{\delta\bar{\Delta}\right\}^t\left[\Psi\right]^t\left\{{}_0^2\mathbf{t}\right\}\delta u_i\;d\,{}^0S$$

39

And body force and surface traction terms are also simplified in the same way.

$$\int_{{}^0V} {}_0C_{ijkl}\,{}_0e_{kl}\,\delta\left({}_0e_{ij}\right)d\,{}^0V+\int_{{}^0V} {}_0^1S_{ij}\,\delta\left({}_0\eta_{ij}\right)d\,{}^0V=\delta\left({}^0R_2\right)-\delta\left({}^0R_1\right)$$

$$\Rightarrow\left(\left[K_L\right]+\left[K_{NL}\right]\right)\left\{\bar{\Delta}\right\}=\left\{{}_0^2\mathbf{F}\right\}-\left\{{}_0^1\mathbf{F}\right\}$$

$$\left[K_L\right]=\int_{{}^0V}\left[B_L\right]^t\left[{}_0C\right]\left[B_L\right]d\,{}^0V$$

$$\left[K_{NL}\right]=\int_{{}^0V}\left[B_{NL}\right]^t\left[{}_0^1S\right]\left[B_{NL}\right]d\,{}^0V$$

$$\left\{{}_0^2\mathbf{F}\right\}=\int_{{}^0V}\left[\Psi\right]^t\left\{{}_0^2\mathbf{f}\right\}\;d\,{}^0V+\int_{{}^0S}\left[\Psi\right]^t\left\{{}_0^2\mathbf{t}\right\}\delta u_i\;d\,{}^0S$$

$$\left\{{}_0^1\mathbf{F}\right\}=\int_{{}^0V}\left[B_L\right]^t\left\{{}_0^1S\right\}d\,{}^0V;\quad\left\{{}_0^2\mathbf{f}\right\}=\begin{Bmatrix}{}_0^2f_x\\{}_0^2f_y\end{Bmatrix};\qquad\left\{{}_0^2\mathbf{t}\right\}=\begin{Bmatrix}{}_0^2t_x\\{}_0^2t_y\end{Bmatrix}$$

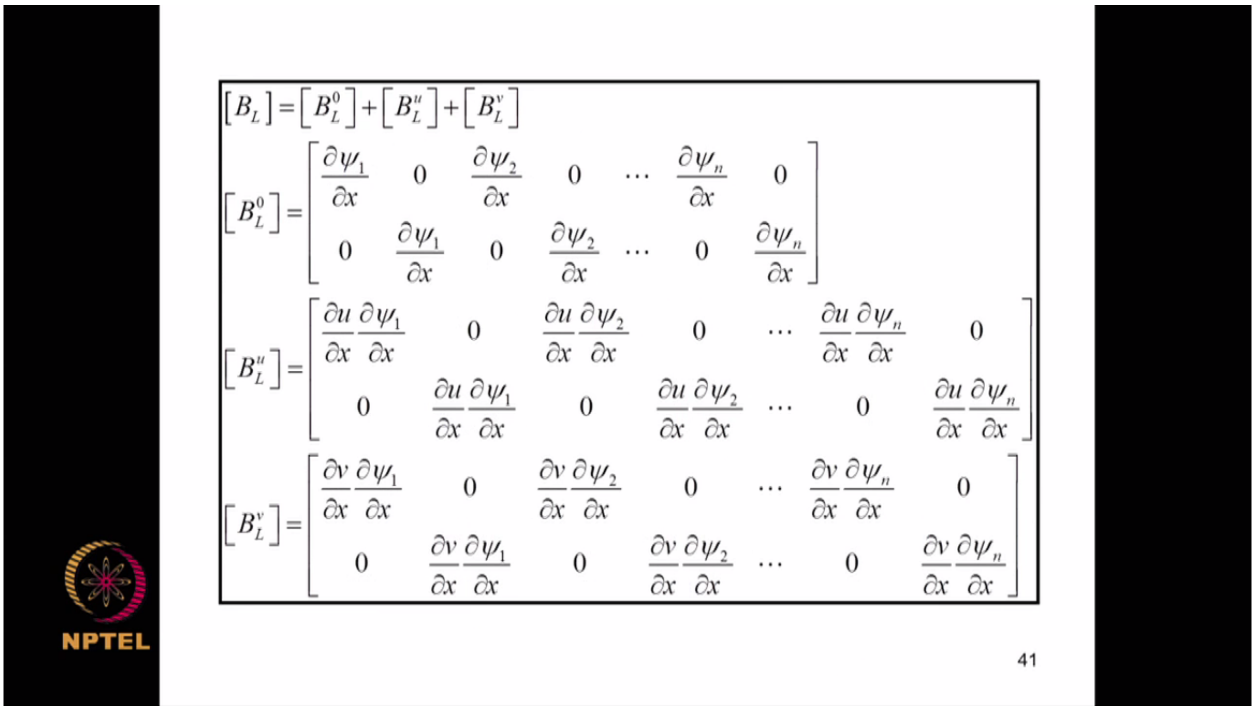
Remarks

- $K=\left[K_L\right]+\left[K_{NL}\right]=K^t\quad\left(\because\left[{}_0^1S\right]\&\left[{}_0C\right]\text{ are symmetric}\right)$
- This is an incremental formulation. The stiffness matrix here is the tangent stiffness matrix.
- For a linear analysis, $\left\{\bar{\Delta}\right\}=\left\{\Delta\right\},\left\{{}_0^1\mathbf{F}\right\}=0,\&\left[K_{NL}\right]=0$

40

And that leads to this is the weak form and this is a finite element equation.

The KL is this. The KNL is this and these are the forcing terms. You can spend some time and go through that. Some of the observation that we can make is that this K matrix is symmetric. This is because these matrices are symmetric. The material constitutive matrix is symmetric and this S01 is also symmetric. And also has already said this is an incremental formulation. So the stiffness matrix here is the tangent stiffness matrix that means we already that linearization is implied in our formulation. Okay. So – but, however, we can still note that if you were to go out do the linear analysis this Delta bar will be simply Delta and this quantity will be 0 and KNL will be 0. So this reduces to the linear stiffness matrix if we are not including nonlinear behavior.



$$[B_L] = [B_L^0] + [B_L^u] + [B_L^v]$$

$$[B_L^0] = \begin{bmatrix} \frac{\partial \psi_1}{\partial x} & 0 & \frac{\partial \psi_2}{\partial x} & 0 & \cdots & \frac{\partial \psi_n}{\partial x} & 0 \\ 0 & \frac{\partial \psi_1}{\partial x} & 0 & \frac{\partial \psi_2}{\partial x} & \cdots & 0 & \frac{\partial \psi_n}{\partial x} \end{bmatrix}$$

$$[B_L^u] = \begin{bmatrix} \frac{\partial u}{\partial x}\frac{\partial \psi_1}{\partial x} & 0 & \frac{\partial u}{\partial x}\frac{\partial \psi_2}{\partial x} & 0 & \cdots & \frac{\partial u}{\partial x}\frac{\partial \psi_n}{\partial x} & 0 \\ 0 & \frac{\partial u}{\partial x}\frac{\partial \psi_1}{\partial x} & 0 & \frac{\partial u}{\partial x}\frac{\partial \psi_2}{\partial x} & \cdots & 0 & \frac{\partial u}{\partial x}\frac{\partial \psi_n}{\partial x} \end{bmatrix}$$

$$[B_L^v] = \begin{bmatrix} \frac{\partial v}{\partial x}\frac{\partial \psi_1}{\partial x} & 0 & \frac{\partial v}{\partial x}\frac{\partial \psi_2}{\partial x} & 0 & \cdots & \frac{\partial v}{\partial x}\frac{\partial \psi_n}{\partial x} & 0 \\ 0 & \frac{\partial v}{\partial x}\frac{\partial \psi_1}{\partial x} & 0 & \frac{\partial v}{\partial x}\frac{\partial \psi_2}{\partial x} & \cdots & 0 & \frac{\partial v}{\partial x}\frac{\partial \psi_n}{\partial x} \end{bmatrix}$$

So we can go through the details of these formulations and derive the expressions for different components of these B matrices as shown here.

$$B_{NL}=\begin{bmatrix} \frac{\partial\psi_1}{\partial x} & 0 & \frac{\partial\psi_2}{\partial x} & 0 & \cdots & \frac{\partial\psi_n}{\partial x} & 0 \\ \frac{\partial\psi_1}{\partial y} & 0 & \frac{\partial\psi_2}{\partial y} & 0 & \cdots & \frac{\partial\psi_n}{\partial y} & 0 \\ 0 & \frac{\partial\psi_1}{\partial x} & 0 & \frac{\partial\psi_2}{\partial x} & \cdots & 0 & \frac{\partial\psi_n}{\partial x} \\ 0 & \frac{\partial\psi_1}{\partial y} & 0 & \frac{\partial\psi_2}{\partial y} & \cdots & 0 & \frac{\partial\psi_n}{\partial y} \end{bmatrix}$$

$$\left([K_L]+[K_{NL}]\right)\{\bar{\Delta}\}=\{{}_0^2\mathbf{F}\}-\{{}_0^1\mathbf{F}\}$$

$$\begin{bmatrix} K^{11L}+K^{11N} & K^{12L} \\ K^{12L} & K^{22L}+K^{22N} \end{bmatrix}\begin{Bmatrix} \bar{u} \\ \bar{v} \end{Bmatrix}=\begin{Bmatrix} {}_0^2F^1 \\ {}_0^2F^2 \end{Bmatrix}-\begin{Bmatrix} {}_0^1F^1 \\ {}_0^1F^2 \end{Bmatrix}$$

And we can proceed and write these equations in a way that the elements of these matrices can be evaluated. So I am not going to get into these details. I will stop the discussion on total Lagrangian with this.

Updated Lagrangian approach

All quantities reckoned with respect to the latest known configuration (C_1).

$$\int_{{}^2V} {}^2\sigma_{ij}\,\delta\left({}_2e_{ij}\right)\,d^2V=\int_{{}^1V} {}_1^2S_{ij}\,\delta\left({}_1^2\varepsilon_{ij}\right)\,d^1V$$

$$\int_{{}^2V} {}^2f_i\,\delta u_i\,d^2V=\int_{{}^1V} {}_1^2f_i\,\delta u_i\,d^1V$$

$$\int_{{}^2S} {}^2t_i\,\delta u_i\,d^2S=\int_{{}^1S} {}_1^2t_i\,\delta u_i\,d^1S$$

${}_1^2\varepsilon_{ij}$ = updated Green-Lagrange strain tensor

${}_1^2f_i$ = body force refered to in C_1.

${}_1^2t_i$ = surface traction refered to in C_1.

$$\int_{{}^1V} {}_1^2S_{ij}\,\delta\left({}_1^2\varepsilon_{ij}\right)\,d^1V-\delta\left({}_1^2R\right)=0$$

$$\delta\left({}_1^2R\right)=\int_{{}^1V} {}_1^2f_i\,\delta u_i\,d^1V+\int_{{}^1S} {}_1^2t_i\,\delta u_i\,d^1S$$

We can quickly take a look at this updated Lagrangian approach. Now here all quantities are reckoned with respect to the latest known configuration, that is C1. So the identities are now the Cauchy stress in configuration to and the Eulerian stress strains; they are related -- the increments these are related through this and this is the relation for body force. This is relation for surface traction. So here this epsilon ij 1 to 2 is updated Green-Lagrange tensor. This F12 fi is the body force refer to in C1. This is surface traction refer to in C1. So we can now rewrite the virtual work statement in this form.

We have $${}_1^2\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial {}^1x_j} + \frac{\partial u_j}{\partial {}^1x_i} + \frac{\partial u_k}{\partial {}^1x_i}\frac{\partial u_k}{\partial {}^1x_j}\right) = {}_1e_{ij} + {}_1\eta_{ij}$$

$$\delta\left({}_1e_{ij}\right) = \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial {}^1x_j} + \frac{\partial \delta u_j}{\partial {}^1x_i}\right)$$

$$\delta\left({}_1\eta_{ij}\right) = \frac{1}{2}\left(\frac{\partial \delta u_k}{\partial {}^1x_i}\frac{\partial u_k}{\partial {}^1x_j} + \frac{\partial u_k}{\partial {}^1x_i}\frac{\partial \delta u_k}{\partial {}^1x_j}\right)$$

Consider $$\int_{{}^1V} {}_1^2S_{ij}\,\delta\left({}_1^2\varepsilon_{ij}\right) d\,{}^1V - \delta\left({}_1^2R\right) = 0$$

Recall $${}_1^2S_{ij} = {}^1\sigma_{ij} + \underbrace{{}_1S_{ij}}_{\text{Updated Kirchoff Stress}}$$

$$\int_{{}^1V} {}_1^2S_{ij}\,\delta\left({}_1^2\varepsilon_{ij}\right) d\,{}^1V - \delta\left({}_1^2R\right) = \int_{{}^1V}\left({}^1\sigma_{ij} + {}_1S_{ij}\right)\delta\left({}_1^2\varepsilon_{ij}\right) d\,{}^1V - \delta\left({}_1^2R\right)$$

$$= \int_{{}^1V} {}_1S_{ij}\,\delta\left({}_1^2\varepsilon_{ij}\right) d\,{}^1V + \int_{{}^1V} {}^1\sigma_{ij}\,\delta\left({}_1e_{ij} + {}_1\eta_{ij}\right) d\,{}^1V - \delta\left({}_1^2R\right)$$

NPTEL

44

And now using the definitions of the strains we can derive the virtual strains for linear and non-linear parts and the Piola-Kirchoff stress from one to two is decomposed into Cauchy stress in configuration 1 plus updated Kirchoff stress. So that if we substitute and simplify we get the – we are aiming at getting the weak form appropriate for this model.

$$\int_{^1V} {}^2_1S_{ij}\delta\left({}^2_1\varepsilon_{ij}\right)\, d^1V - \delta\left({}^2_1R\right)$$

$$= \int_{^1V} {}_1S_{ij}\delta\left({}^2_1\varepsilon_{ij}\right)\, d^1V + \int_{^1V} {}^1\sigma_{ij}\delta\left({}_1e_{ij} + {}_1\eta_{ij}\right)\, d^1V - \delta\left({}^2_1R\right)$$

$$= \int_{^1V} {}_1S_{ij}\delta\left({}^2_1\varepsilon_{ij}\right)\, d^1V + \int_{^1V} {}^1\sigma_{ij}\delta\left({}_1\eta_{ij}\right)\, d^1V + \delta\left({}^1_1R\right) - \delta\left({}^2_1R\right) = 0$$

$$\delta\left({}^1_1R\right) = \int_{^1V} {}^1\sigma_{ij}\delta\left({}_1e_{ij}\right) d^1V$$

Considering the equillibrium of body in C_1

$$\delta\left({}^1_1R\right) = \int_{^1V} {}^1_0f_i\delta u_i\, d^1V + \int_{^1S} {}^1_0t_i\delta u_i\, d^1S$$

We have the constitutive relation ${}_1S_{ij} = {}_1C_{ijkl}\varepsilon_{kl}$

$$\Rightarrow \int_{^1V} {}_1C_{ijkl}\varepsilon_{kl}\delta\left({}^2_1\varepsilon_{ij}\right)\, d^1V + \int_{^1V} {}^1\sigma_{ij}\delta\left({}_1\eta_{ij}\right)\, d^1V = \delta\left({}^2_1R\right) - \int_{^1V} {}^1\sigma_{ij}\delta\left({}_1e_{ij}\right) d^1V$$

As we did in the total Lagrangian approach we take

$${}_1S_{ij} \approx {}_1C_{ijkl}\,{}_1e_{kl} \;\&\; \delta\left({}^2_1\varepsilon_{ij}\right) \approx \delta\left({}_1e_{ij}\right)$$

45

So this is the weak form. Now we are going to use the virtual strain expressions and if we consider the equilibrium of body in C1 this is the principle of virtual work applied to that and from this if we now use a constitutive relations, we get this equation and as we did in the total Lagrangian approach if we now linearize we get these equations.

This leads to the required weak form

$$\int_{^1V} {}_1C_{ijkl}\,{}_1e_{kl}\delta\left({}_1e_{ij}\right)\, d^1V + \int_{^1V} {}^1\sigma_{ij}\delta\left({}_1\eta_{ij}\right)\, d^1V = \delta\left({}^2_1R\right) - \int_{^1V} {}^1\sigma_{ij}\delta\left({}_1e_{ij}\right) d^1V$$

The Cauchy stress ${}^1\sigma_{ij}$ are evaluated using ${}^1\sigma_{ij} = {}^1C_{ijkl}\,{}^1_1\varepsilon_{kl}$

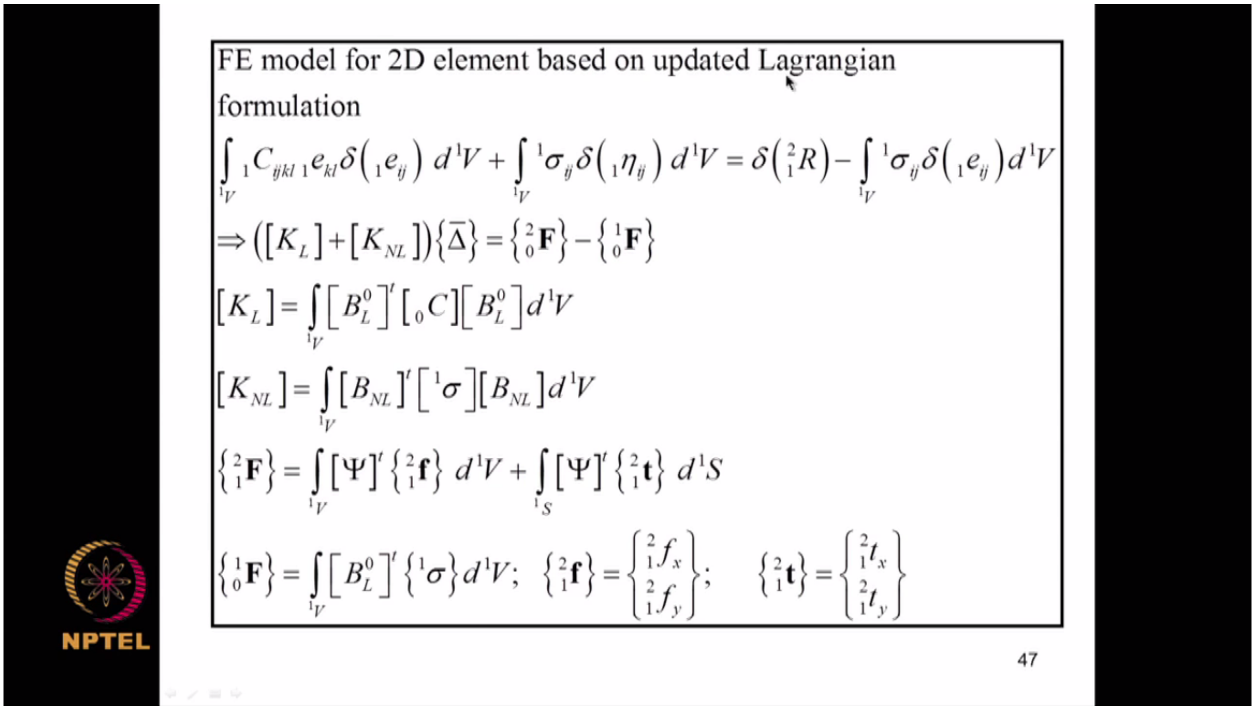
${}^1_1\varepsilon_{kl}$ = Almansi strain components

$$2\,{}^1_1\varepsilon_{ij}d^1x_id^1x_j = \left({}^1ds\right)^2 - \left({}^0ds\right)^2$$

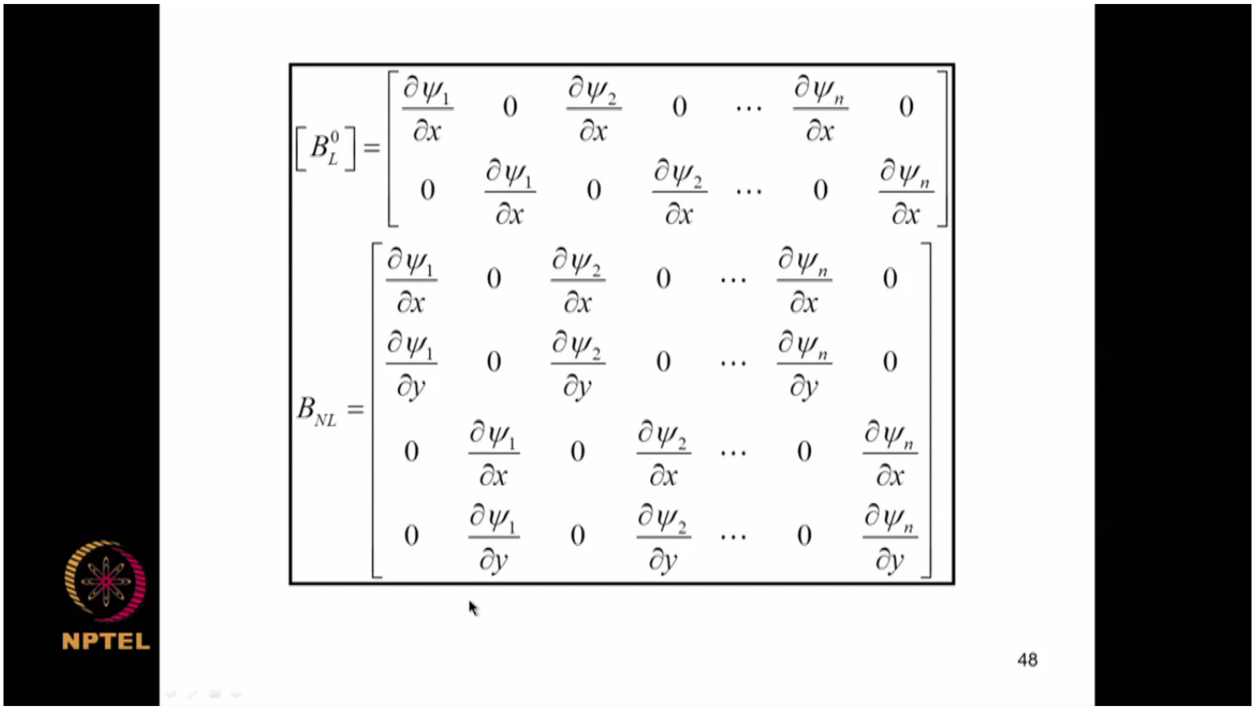
$${}^1_1\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial\,{}^0_1u_i}{\partial\,{}^1x_j} + \frac{\partial\,{}^0_1u_j}{\partial\,{}^1x_i} - \frac{\partial\,{}^0_1u_k}{\partial\,{}^1x_i}\frac{\partial\,{}^0_1u_k}{\partial\,{}^1x_j}\right)$$

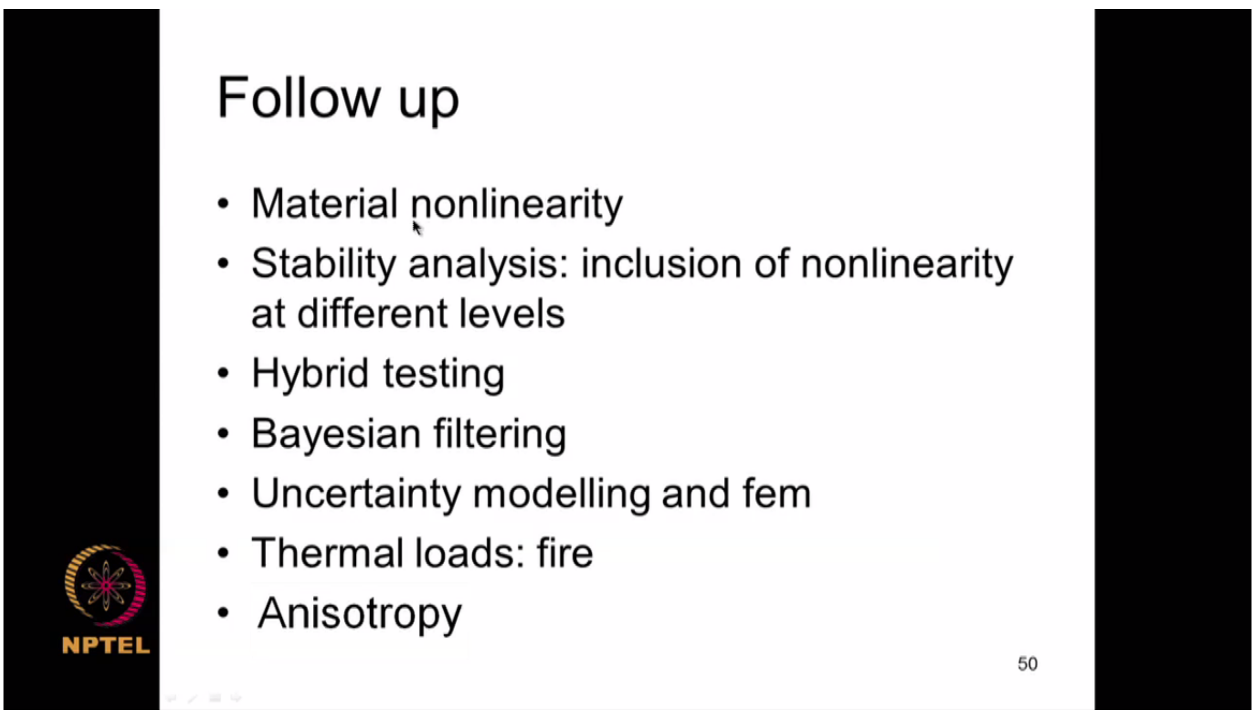
46

And this leads to the required weak form that forms a basis for application of development of an finite element model. The Cauchy stress here are evaluated using this relation where this epsilon KL 1 to 1 are the Almansi strain components or the Eulerian strain components and that is defined through these terms.



Now I have not given all the steps but we can use updated Lagrangian formulation and derive the finite element model for the 2D element based on the procedure that we use for total Lagrangian except now that you have to take care of the fact that we are dealing with all quantities in configuration one instead of 0. So I have reproduced here the basic equations and the various matrices that arise in this formulation.





As I said in the beginning of this lecture the objective of this discussion has been basically to provide a flavor of how to carry out nonlinear analysis, what issues arise as geometry of the body changes during deformation, why we need to introduce newer definitions of stresses and strains and consequently what is the role played by these newer definitions in the formulation of the FE model. Now with this we have come to the conclusion of this course. So in the next lecture what I wish to do is to quickly review what we have done so far and where we can go from what we have learned till now.

Now the material that is covered in the course from the coverage that we are already achieved in this course we can move on further to study problems of material non-linear. I have not addressed this issue in most of the discussion that we had. So to study this you need to first understand theory of plasticity and continuum mechanics has to be generalized to allow for non-linear material constitutive laws and that requires newer preparation. Now in our discussion of stability analysis again we have included the geometric non-linearity alright, but there are different layers of sophistication that is possible in carrying out stability analysis. We could deal with the modified definitions of stresses and strains and updated Lagrangian, total Lagrangian approaches and investigate the problem of stability once again and see what we did earlier without being aware of all these refinements what was that we achieved that can be reexamined.

Now in the problems of stability analysis what would happen if we consider material non-linearity? That is one issue that we have not taken into account. There is another topic what is known as hybrid testing. In recent years the sophistication in experimental hardware and computing have led to newer testing procedures in a laboratory. They are known as hybrid testing procedures. This hybrid – the word hybrid here connotes indicates the combined use of computational and experimental methods. So there are methods especially in earthquake engineering these methods are being developed. So I will be briefly touching upon in the next class. The idea here is to model that part of the structural behavior which is reasonably well understood through computational schemes and to resort to experimentation only to complement what is not clearly understood in computational modeling. And there are complicating issues that an analyst computational modeling and its analysis of a resulting model and the testing of the structure need to go hand -in-hand many times in real time so that includes -- I mean that introduces newer challenges. I also briefly talked about problems of structural health monitoring. So in structural health monitoring we make finite element models for existing structures typically and we wish to combine the predictions from the computational model with what we actually observe the observations that we make on structural performance. so there exists a need to combine the experimental observations with computational models and a framework for that as I already mentioned is so called Bayesian filtering methods. So here the finite element methods have to be combined with experimentally observed measurements and a newer entity which is essentially a mathematical model that has assimilated the observations made on existing structure need to be developed. That new entity can be used to prognosis the future behavior of the structure. One of the question that we have not discussed in this course is what happened to uncertainty. We know that loads like earthquake wind, [Indiscernible] [0:45:34] etcetera. are uncertain in nature and they are typically modeled using theory of probability and random processes. Now when structural behavior is captured through finite element methods either the loads are the constitutive relations or inertial properties etc. if they are all modeled as random processes or random variables how do we combine the finite element formulation with the probabilistic modeling of structural behavior, and how do you specify inputs, how do you compute the responses, and what are they expected response descriptors. So this is another area and one of the computational tools that is available in the literature is the so-called stochastic finite element method. So what you learned in the course can be a framework to pursue this study.

Another area where we have what we have studied could be of a application in structural engineering is analysis of structures under fire loads. So the fire loads create time varying temperature environment in which the structure need to perform and stress analysis and heat transfer analysis need to be carried out to understand the behavior of the structure. They can be carried out in uncoupled manner that is first way perform the heat transfer analysis and find out the thermal heat distribution, temperature distribution in the body and if constitutive laws are described as a function of temperature then that temperature dependence need to be accounted for in our formulation of structural behavior. Typically in problems of practical interest one need to consider both geometric and material non-linearity in problems of analyzing structures under fire loads. so that offers immense challenge to analyst in terms of being able to handle the combined problem of thermo mechanical stress analysis, treatment of uncertainties, treatment of material and geometric non-linearity, etcetera. Many structural elements display Anisotropy. So that is another feature that we need to develop that is where basically most of our studies resume the isotropic except in one or two small examples but many structural components like composites, etcetera. inherently an isotropic in nature and inclusion of these features into mathematical modeling again offers challenges and some of this need to be taken up if we have to take the subject forward.

So what I will do is in the will close this lecture at this point. In the next lecture I will briefly review the whole content of the course and again revisit some of these questions with slightly more details. So we close this lecture at this point.

Programme Assistance

Guruprakash P

Ramachandran M V

Depali K Salokhe

Technical Supervision

B K A N Singh

Gururaj Kadloor

Indian Institute of Science

Bangalore