Indian Institute of Science Bangalore

National Programme on Technology

Enhanced Learning

Finite element method for structural dynamic and stability analyses

Lecture -38

Review of measures of strain and stress; balance laws

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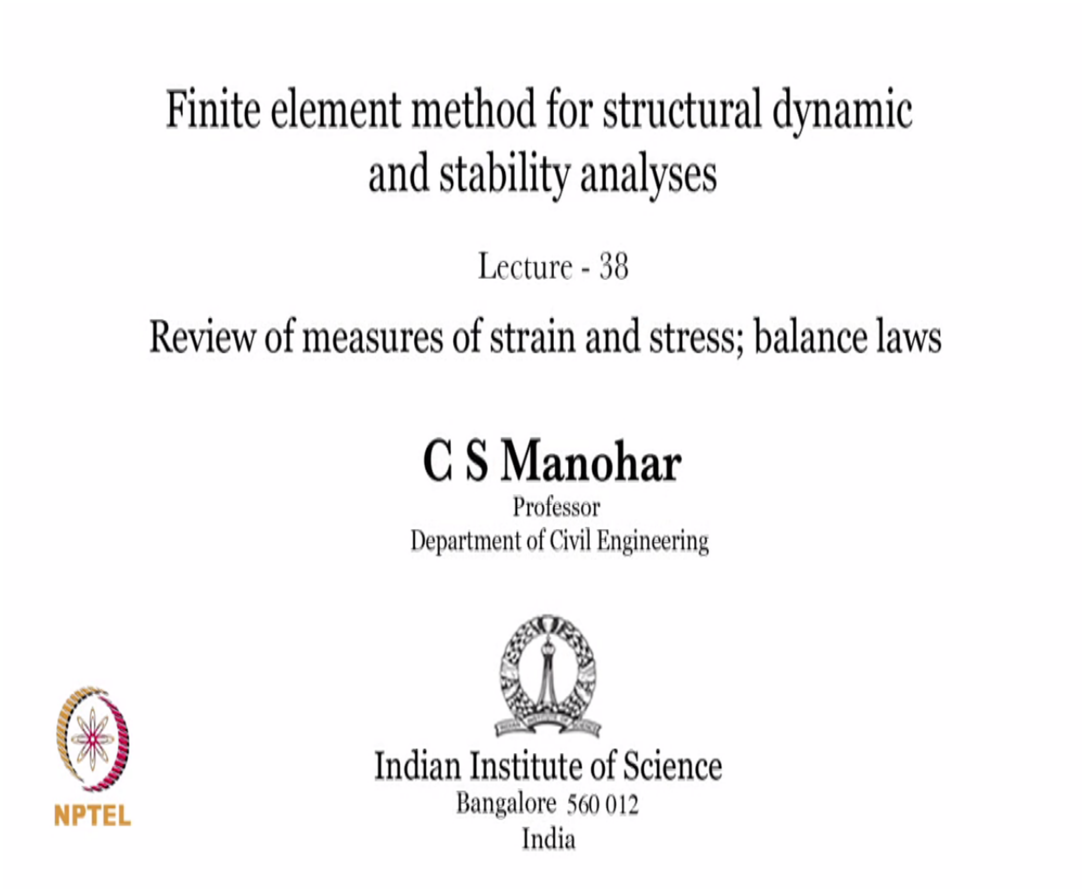
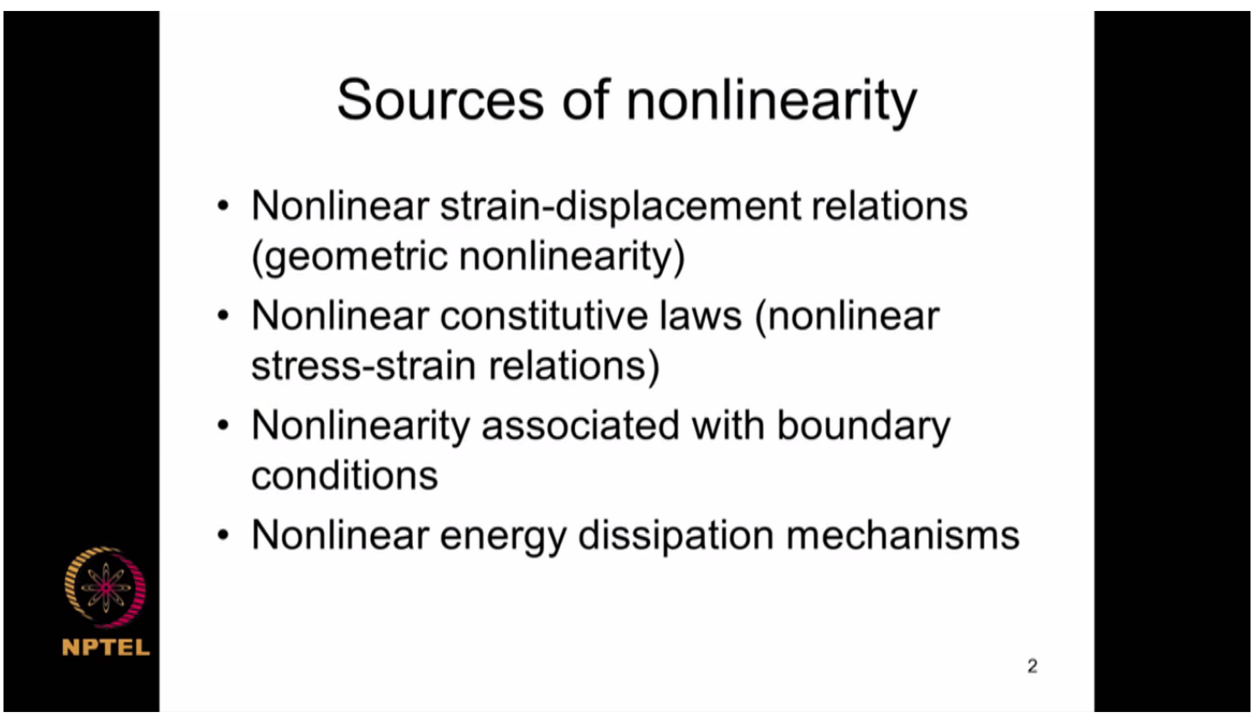
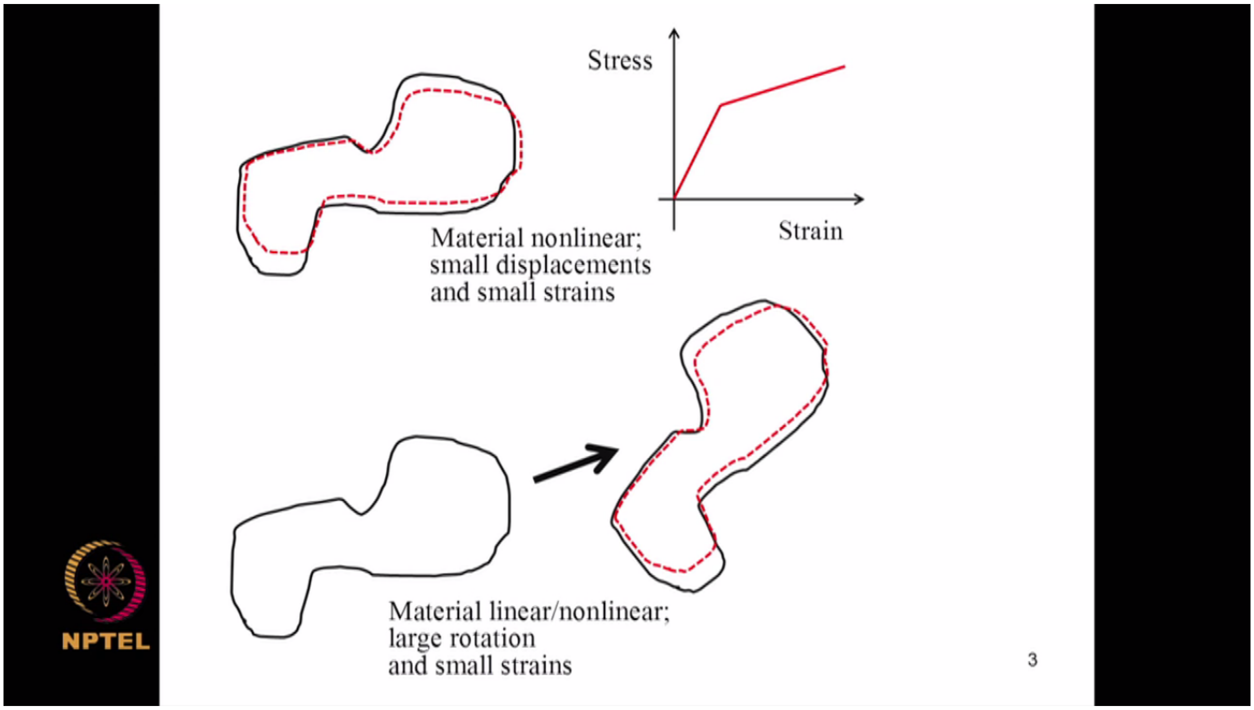


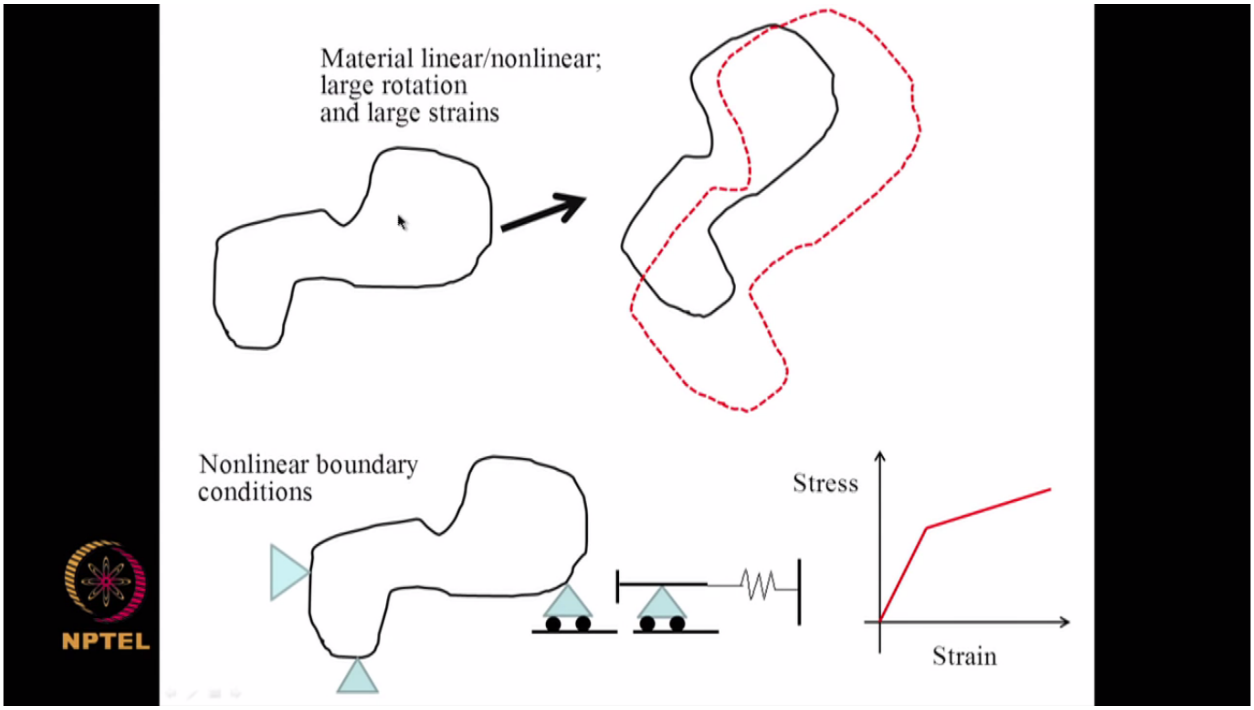
The previous lecture we started talking about nonlinear finite element model development.



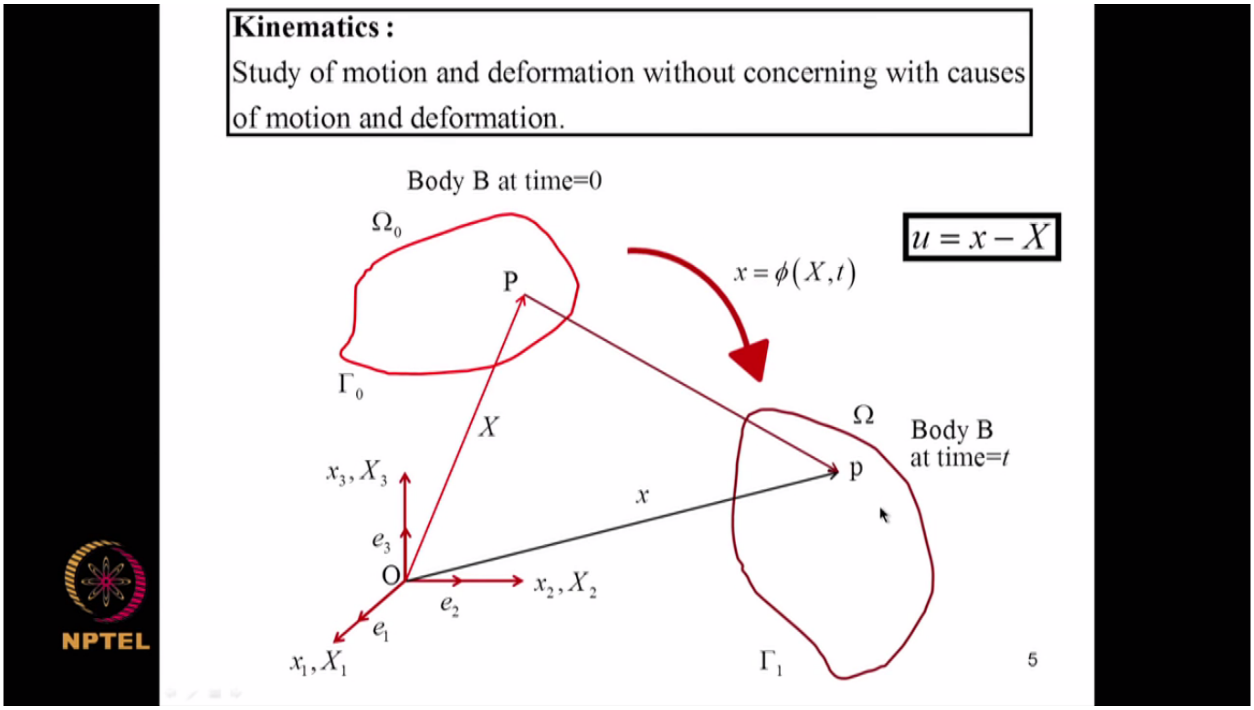
We started by discussing sources of non-linearity and we saw that the main source of non-linearity could be due to strain displacement relations being nonlinear. That is called geometric non-linearity and the stress-strain relations could be nonlinear that is material non-linearity. There could be non-linearity associated with boundary conditions or energy dissipation mechanisms.



So various schemes of classification are possible. All these nonlinearities can coexist in a problem. So in this case what is shown here is small displacements and small strains but there is material non-linearity here, whereas, in this case there is material could be linear or non-linear but there are large rotations but small strains.

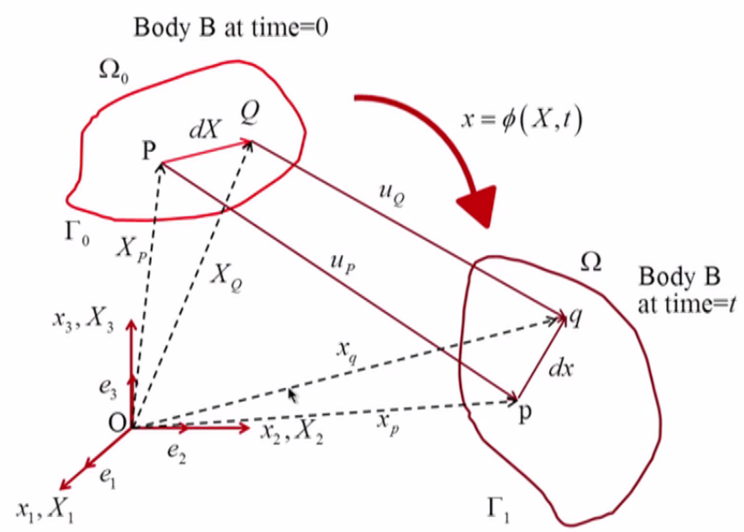


Here material again could be linear or non-linear. There is large rotation and large strains and this is a schematic of a situation where there could be non-linearity associated with boundary conditions. So this spring will come into action only when this displacement here exceeds this threshold in which case the stiffness of the system increases. Here the loading and unloading path will be tracing each other whereas in material non-linearity the unloading path will be different from the loading path.



After reviewing few details about qualitative feature of nonlinear system response and how it differs from response of linear systems we started talking about elements of continuum mechanics and I briefly started talking about kinematics that is study of motion and deformation without concerning the causes of motion and deformation. So we have a configuration at time equal to 0. It is here and this could be the reference configuration and this is a Cartesian coordinate system in which the position of particles in the material points in this body are described. So P, the position vector of P is the X and after deformation this point P with position vector X gets mapped to point P with position vector lower X through this transformation. And P-P is the displacement vector defined as U of X minus X. So in Lagrangian descriptions we treat the coordinates of capital P that is capital X1, X2, X3 as the independent variables whereas in Eulerian system we treat the current position, the position of particle P in the current configuration that is lower case x1, x2, x3 as the independent variables. That is Eulerian system. So in solid mechanics problem often Lagrangian coordinate system is used.

Deformation gradient



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Now let us consider a line segment PQ in the body be at in the reference configuration and due to this deformation, this point PQ moves to this position pq as shown here and the initial length is d capital X, this is d lowercase x. So various position vectors of PQ in the T equal to 0 and at some time T are shown here.

$$x \equiv x(X,t)$$

$$x_1 = x_1(X_1, X_2, X_3, t) \Rightarrow dx_1 = \frac{\partial x_1}{\partial X_1} dX_1 + \frac{\partial x_1}{\partial X_2} dX_2 + \frac{\partial x_1}{\partial X_3} dX_3$$

$$x_2 = x_2(X_1, X_2, X_3, t) \Rightarrow dx_2 = \frac{\partial x_2}{\partial X_1} dX_1 + \frac{\partial x_2}{\partial X_2} dX_2 + \frac{\partial x_2}{\partial X_3} dX_3$$

$$x_3 = x_3(X_1, X_2, X_3, t) \Rightarrow dx_3 = \frac{\partial x_3}{\partial X_1} dX_1 + \frac{\partial x_3}{\partial X_2} dX_2 + \frac{\partial x_3}{\partial X_3} dX_3$$

$$\begin{Bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{Bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \begin{Bmatrix} dX_1 \\ dX_2 \\ dX_3 \end{Bmatrix} \Rightarrow \text{or } dx = F.dX$$

$\{dx\} = [F]\{dX\}$ or $dx_i = F_{ij} dX_j$

F = Deformation gradient tensor

$$F = \left(\frac{\partial x}{\partial X}\right)^t = (\nabla_0 x)^t ; J = |F| = \text{Jacobian}$$

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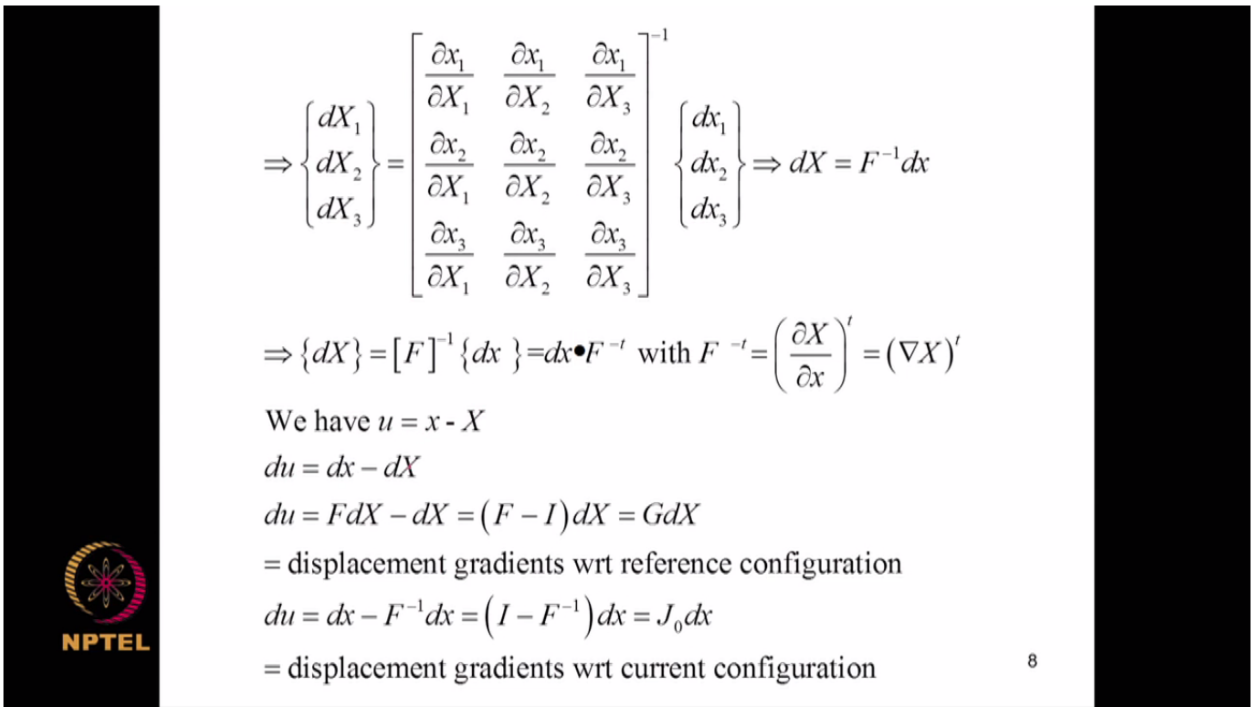
Now the position vector in the current configuration is a function of the position in the original configuration and I can write X1, X2, X3 in the long hand in this form and from this I deduce dX1 is dX1 by dX1 into dX1 etcetera that is shown here. So this set of equations where I get dX1,dX2 dX3 which are components of position vector in the deformed configuration which are related to the components of position vector in undeformed configuration through this matrix and this matrix is known as deformation gradient tensor.

So we get this equation in matrix notation it is dx is equal to F dX in in the indicial notation dXi is Fidxj or in tensile notation is f.dX. The determinant of this matrix F is known as Jacobian.



By inverting this relation I can also here the components of line segments in the deformed configuration are related to component of position vector in the original configuration. This can be inverted and I can get this relation and dX is f inverse dx. We have displacement vector as U is X minus x and from which I get to use du =dx-dX and du I can write therefore as for dX I will write FdX so this is (F-I)dx and this matrix -I I call it as capital G and this is a displacement gradient with respect to reference configuration.

Alternatively I can express dX in terms of dx through this relation and I get this is I minus F inverse into dX and this matrix is known as J naught. So these are displacement gradient with respect to current configuration.

$$G=\begin{bmatrix}\frac{\partial x_1}{\partial X_1}-1 & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3}\\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2}-1 & \frac{\partial x_2}{\partial X_3}\\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3}-1\end{bmatrix}=\begin{bmatrix}\frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} & \frac{\partial u_1}{\partial X_3}\\ \frac{\partial u_2}{\partial X_1} & \frac{\partial u_2}{\partial X_2} & \frac{\partial u_2}{\partial X_3}\\ \frac{\partial u_3}{\partial X_1} & \frac{\partial u_3}{\partial X_2} & \frac{\partial u_3}{\partial X_3}\end{bmatrix}$$

$$J_0=\begin{bmatrix}1-\frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3}\\ \frac{\partial x_2}{\partial X_1} & 1-\frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3}\\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & 1-\frac{\partial x_3}{\partial X_3}\end{bmatrix}=\begin{bmatrix}\frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} & \frac{\partial u_1}{\partial X_3}\\ \frac{\partial u_2}{\partial X_1} & \frac{\partial u_2}{\partial X_2} & \frac{\partial u_2}{\partial X_3}\\ \frac{\partial u_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3}\end{bmatrix}$$

$$G=F-I=(I-J_0)^{-1}-I$$

$$J_0=I-F^{-1}=I-(G+I)^{-1}$$

For small deformation, $G\approx J_0^{-1}$ & $J_0\approx G^{-1}$

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So this G and J naught matrices are shown here. This is with respect to displacements whereas this is with respect to the position coordinates of the position vector.

So G is this. J naught is this but for small deformation G and J naught related through this relation. So small deformation this relations apply.

Examples

1. $x=\lambda e_1X_1+\lambda e_2X_2+\lambda e_3X_3$

$$\left.\begin{array}{l}x_1=\lambda X_1\Rightarrow u_1=(\lambda-1)X_1\\ x_2=\lambda X_2\Rightarrow u_2=(\lambda-1)X_2\\ x_3=\lambda X_3\Rightarrow u_3=(\lambda-1)X_3\end{array}\right\}\Rightarrow F=\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix}$$

This deformation is called pure dilatation.

2. $x=(1+\alpha)e_1X_1+e_2X_2+e_3X_3$

$$\left.\begin{array}{l}x_1=(1+\alpha)X_1\Rightarrow u_1=\alpha X_1\\ x_2=X_2\Rightarrow u_2=0\\ x_3=X_3\Rightarrow u_3=0\end{array}\right\}\Rightarrow F=\begin{bmatrix}1+\alpha & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

This deformation is called pure extension.

3. $x=AX+c\Rightarrow u=(A-I)X+c; F=A$

When F is independent of X, we say that the deformation is homogeneous.

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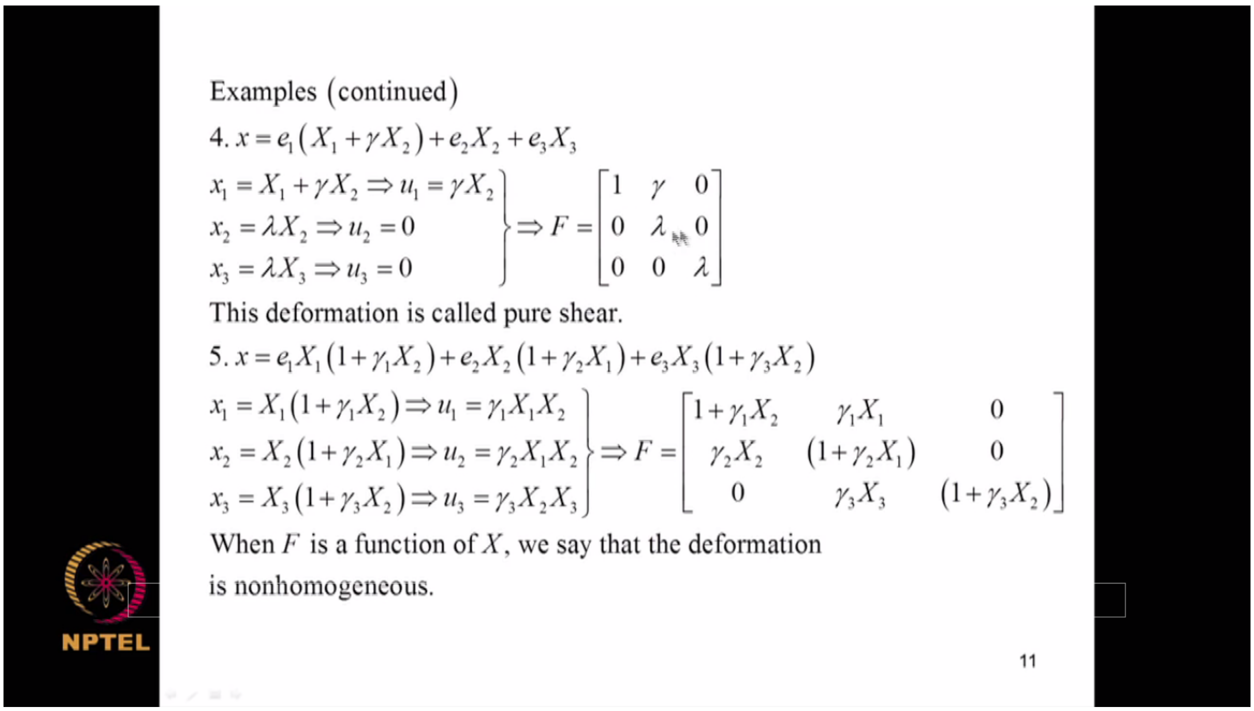
We can consider a few examples. Suppose I have the displacement field given by lambda e1X1, lambda e2X2, lambda e3X3 so that x1, x2, x3 are respectively lambda X1, lambda X2 and lambda X3. The displacement e1, e2, e3 will be lambda-X1, X1 and lambda- X2, lambda-1 X3. From this I get the deformation this matrix F as this and this type of deformation is called pure dilatation.

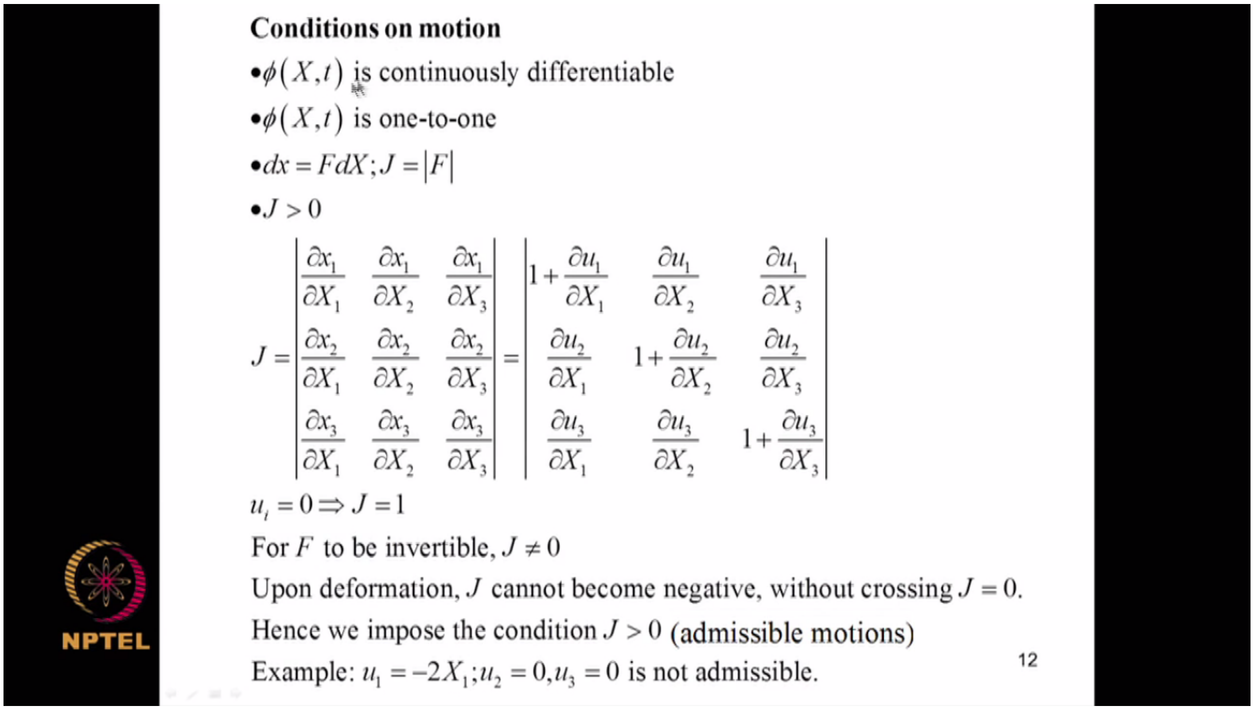
Now another example I will consider x as 1+alpha e1X1+e2X2+e3X3 so that X1 is 1+alphaX1 from which I get e1 is alphaX1 and X2 and X3 are such that e2 and e3 are zero from which I get F to be given by this.

So this deformation is called pure extension. Now I can consider X to be A into X plus C from this it follows u is (A - I)X plus C and F is simply A. when the matrix F is independent of X, we say that the deformation is homogeneous.



A slightly more involved another example here x1 is X1 plus gamma X2, x2 is lambda X2, x3 is lambda X3 so that the F matrix will be given by this and this deformation is called pure shear.

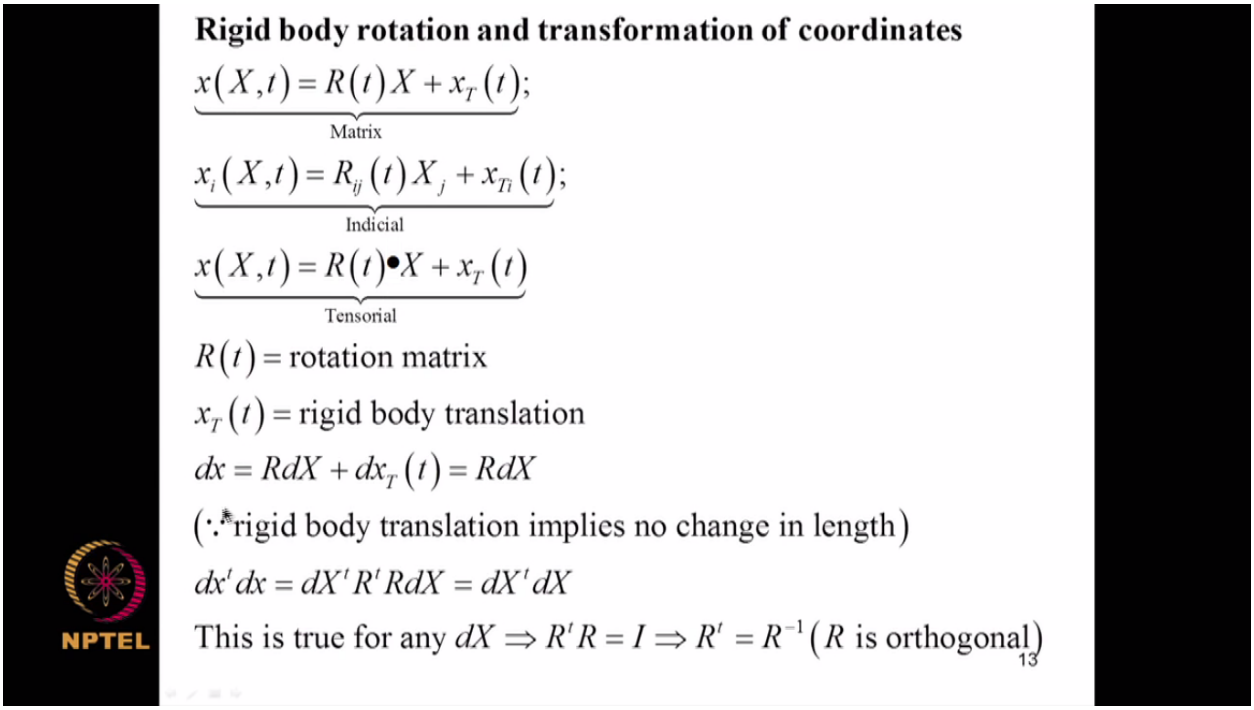
Now here I have a displacement field where there are non-linear terms X1 into X2 and so on and so forth. So u1 will be gamma 1 X1X2 u2 is X2 is gamma 2 X1X2 and X3 is gamma 3 X2X3 from which I get F to be this which is a function of now X1X2X3 and such deformation are called non-homogeneous. So there are simple illustrations.



Now this Phi of X comma T which maps the position vector X to lowercase x it has to satisfy certain conditions. This function is continuously differentiable and this is one-to-one and that means F matrix can be inverted so dX is FdX and J is the Jacobian determinant of F. We impose the condition J to be greater than 0. You can see that the determinant can be expressed in terms of displacements as shown here if all displacements are 0 then I get J equal to one. So for F to be invertible J must not be equal to zero because it is determinant of F, therefore F inverse to exist. Determinant of F must not be zero.

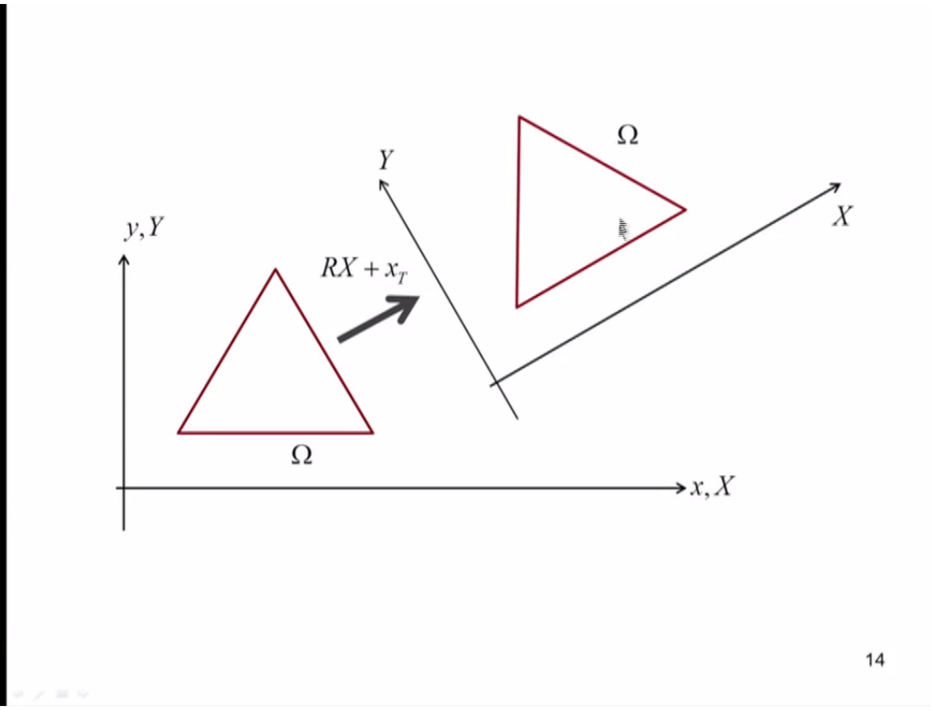
So upon deformation J cannot become negative without crossing J equal to zero. So hence we impose the condition that J must be greater than or equal to zero and these are such motions are admissible motions.

Now I can – an example of a motion which is not not admissible is shown here you can verify that this is not admissible.



I mentioned in the previous lecture that rotations play a crucial role in analysis of nonlinear systems. So let us consider the displacement field where there is a rigid body, translation, and a rotation.

This R matrix is a rotation matrix. So this relation is expressed in matrix form here, indicial form here and a tensorial form as shown here. This R of T is a rotation matrix, XT of T is a rigid body translation. Now from this equation I find dX it will be RdX plus dxt of t dxt of t is 0 because rigid body translation implies no change in length. Therefore this is RdX. Now if you find the length dxt square of the length dxtdx and if for dxt if I write this equation I get dX transpose R transpose RdX. Since R transpose R is a identity matrix because R is a rotation matrix I get this. So this length remains unchanged therefore I mean I have to verify that R transpose R is I. Therefore the argument is the length in a rigid body rotation, and translation length does not change. Therefore dxt transpose dX must be equal to dxtdx therefore this is true for any dX. Consequently I get the relation that R transpose R is I or in other words R transpose R inverse that is R is orthogonal.



Now how does pictorially it looks like? Suppose you'd consider a triangular domain upon this transformation RX+xT this R, effect of R is to rotate this as shown here and xT the effect of xT is to translate. So this is how the element looks upon undergoing this deformation.

Consider two coordinate systems x_j & x'_j such that

$x = x_j e_j = x'_j e'_j$ where e_i are the unit vectors in x_j system and

e'_j are the unit vectors in x'_j system. Clearly, $e_i \bullet e_j = \delta_{ij}$ & $e'_i \bullet e'_j = \delta_{ij}$

$$x_j e_j = x'_j e'_j \Rightarrow x_j e_j \bullet e_i = x'_j e'_j \bullet e_i$$

$\Rightarrow x_i \delta_{ij} = x'_j e'_j \bullet e_i \Rightarrow x_i = R_{ji} x'_j$ where $R_{ji} = e'_j \bullet e_i$

Similarly, $x_j e_j = x'_j e'_j \Rightarrow x_j e_j \bullet e'_i = x'_j e'_j \bullet e'_i$

$$\Rightarrow x'_j = x_j e_j \bullet e'_i = R_{ij} x_j$$

$$\{x\} = [R]^t \{x'\}; \{x'\} = [R]\{x\}$$

Similarly, it can be shown that $\sigma' = R^t \sigma R$

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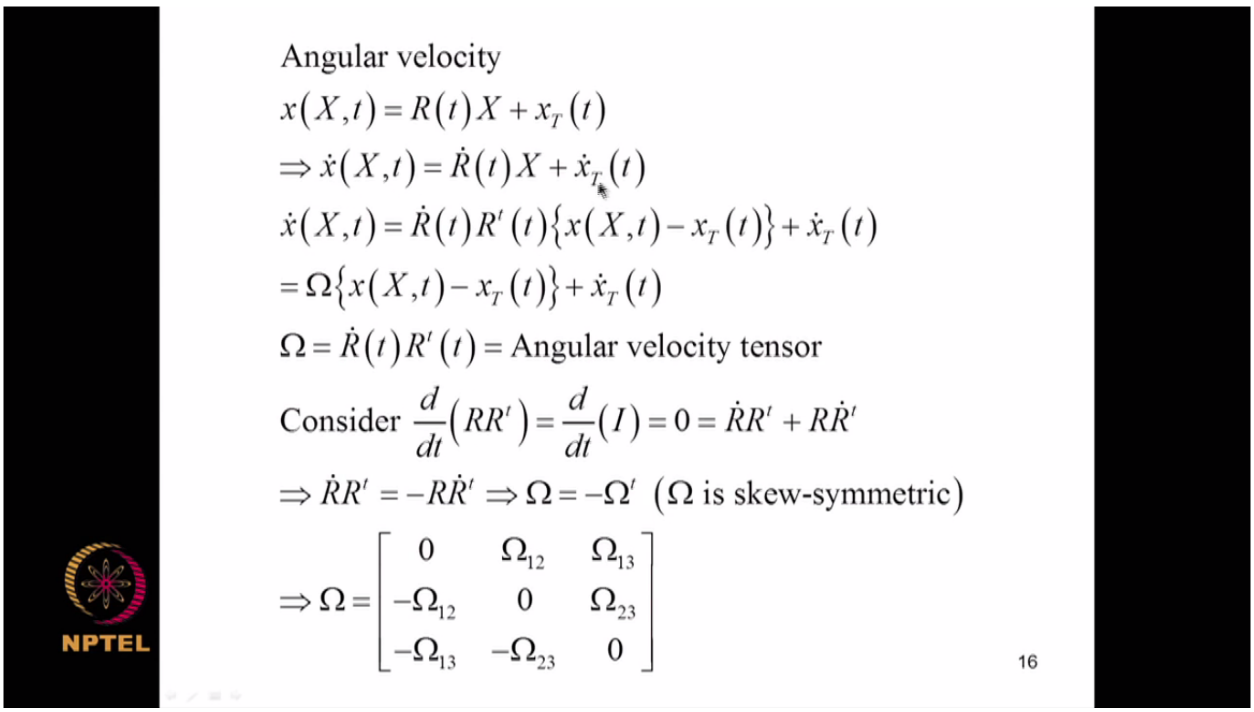
Now if we consider two coordinate systems xj and xj prime such that a position vector in xj system is xjej and position vector in xj prime coordinate system xjej prime where ei unit vectors in xj system and ej prime are the unit vectors in xj prime system. Now repeated indices imply summation here. Now clearly this dot product of eiej is Delta ej and since this is equal I get this relation. I will dot product with ei and I get this and from this denote ej prime dot ei Rji and this gives me the relationship between xi and xj prime and similarly I consider this equation and again do another dot product. This time with the ei prime I get this relation. So from this analysis we get that X is R transpose x Prime and x prime is Rx. So this is how a vector undergoes transformation due to coordinate transformation. By using similar arguments we can show that a tensor like stress undergoes transformation following this role.



We introduce another quantity known as angular velocity. So we consider again the rigid body translation and rotation as shown here and I differentiate this with respect to time I get X dot is R.X plus X.T Twitter. Now for X I will write using this relation, it is X- xT of t into R inverse of that and R inverse is RT and I get this relation. So from this I get this equation. I rewrite in this form by denoting R. R transpose by capital Omega and this quantity is known as angular velocity tensor.

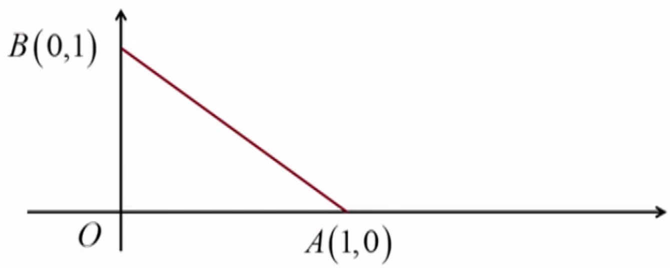
Now we can show that this angular velocity tensor is a skew symmetric matrix to show that we consider D by DT of RR transpose since RR transpose is identity matrix this must be equal to 0 so that means r dot r transpose plus RR dot transpose must be 0 from which i get the relation Omega must be equal to minus Omega transpose therefore Omega is skew symmetric and this will have this form.

Example: The motion of the element OAB is described by

$x = 2X + 3Yt$

$y = 2Xt + 3Y$

- Examine the admissibility of the motion
- Sketch the configuration of the element at $t = 0.5$
- Determine the displacement, velocity and acceleration fields



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A simple example I consider x equal to 2X plus 3Yt and y is 2Xt plus 3Y. So the question is examine the admissibility of the motion sketch the configuration of the element at T equal to 0.5 second and determine the displacement velocity and acceleration fields. So the domain is OAB is described here. This length is 1. this length is also 1.

$x = 2X + 3Yt$

$y = 2Xt + 3Y$

Deformation gradient: $F = \begin{bmatrix} 2 & 3t \\ 2t & 3 \end{bmatrix}$

$J = |F| = 6 - 6t^2$

For motion to be admissible, $J > 0 \Rightarrow t < 1$

At $t = 1/2, x = 2X + \frac{3}{2}Y$ & $y = X + 3Y$

X	Y	x	y
0	0	0	0
1	0	2	1
0	1	$\frac{3}{2}$	3

$u = x - X = 2X + 3Yt - X = X + 3Yt \Rightarrow \dot{u} = 3Y, \ddot{u} = 0$

$v = y - Y = 2Xt + 3Y - Y = 2Xt + 2Y \Rightarrow \dot{v} = 2X, \ddot{v} = 0$

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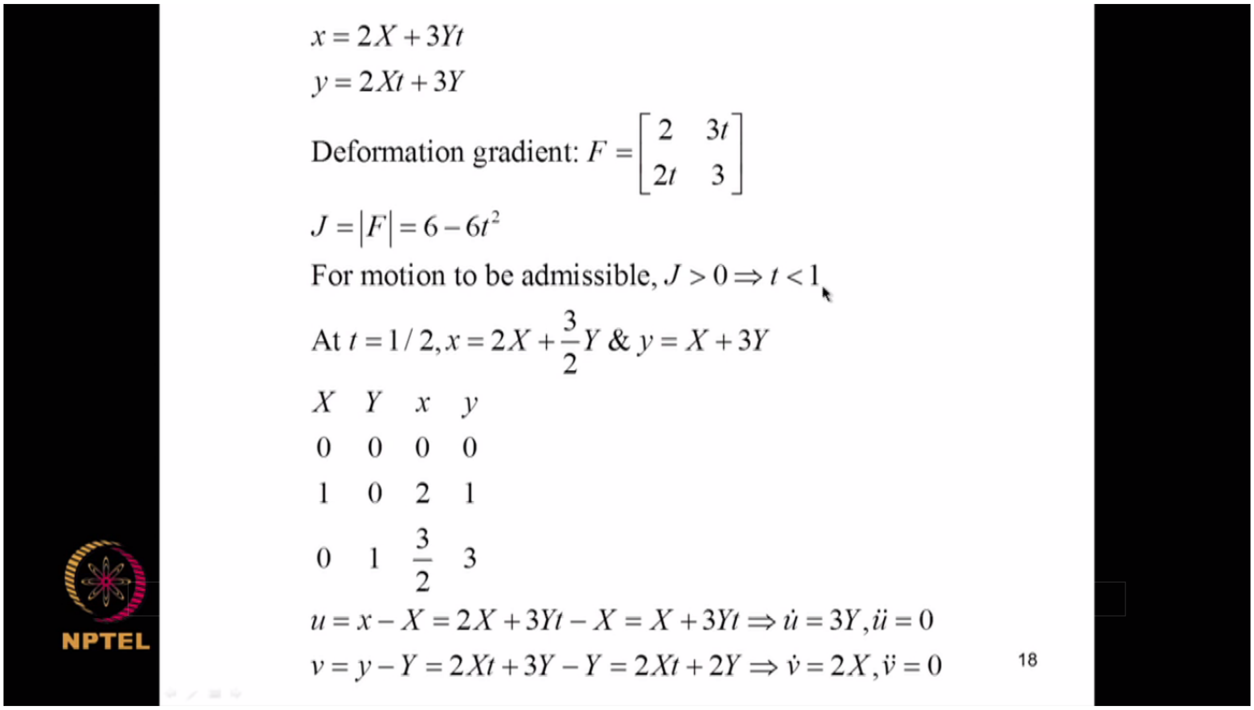
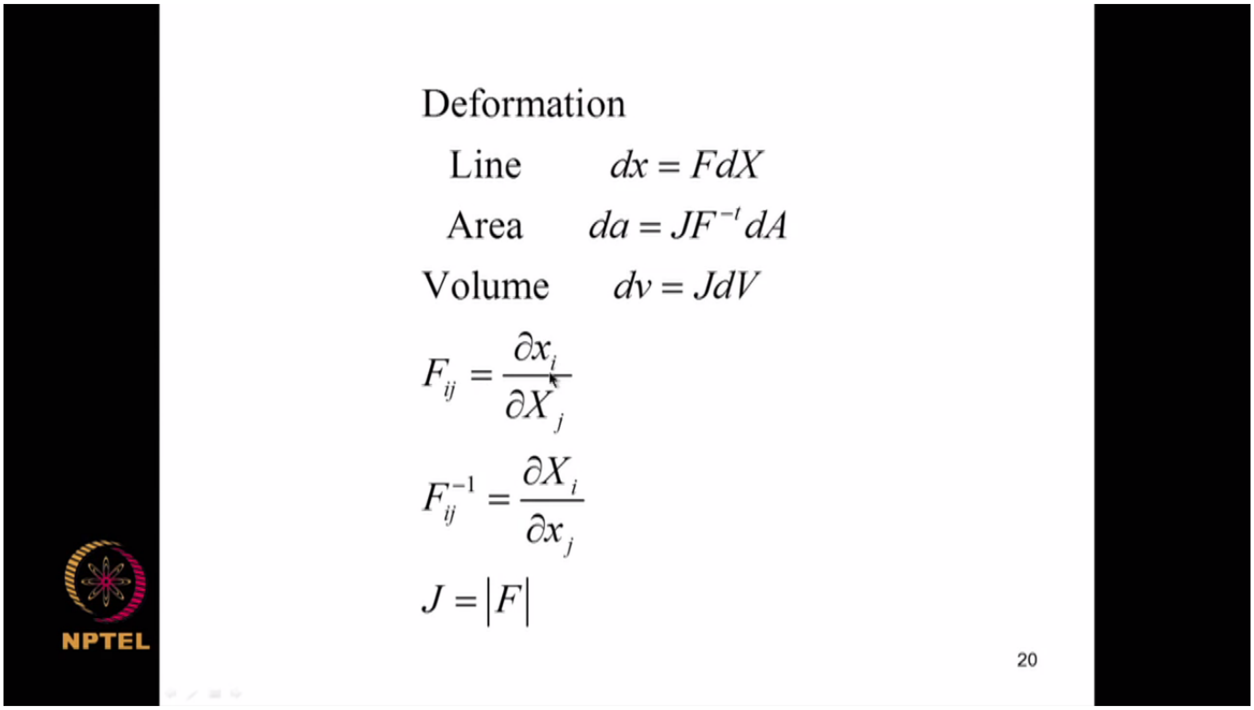
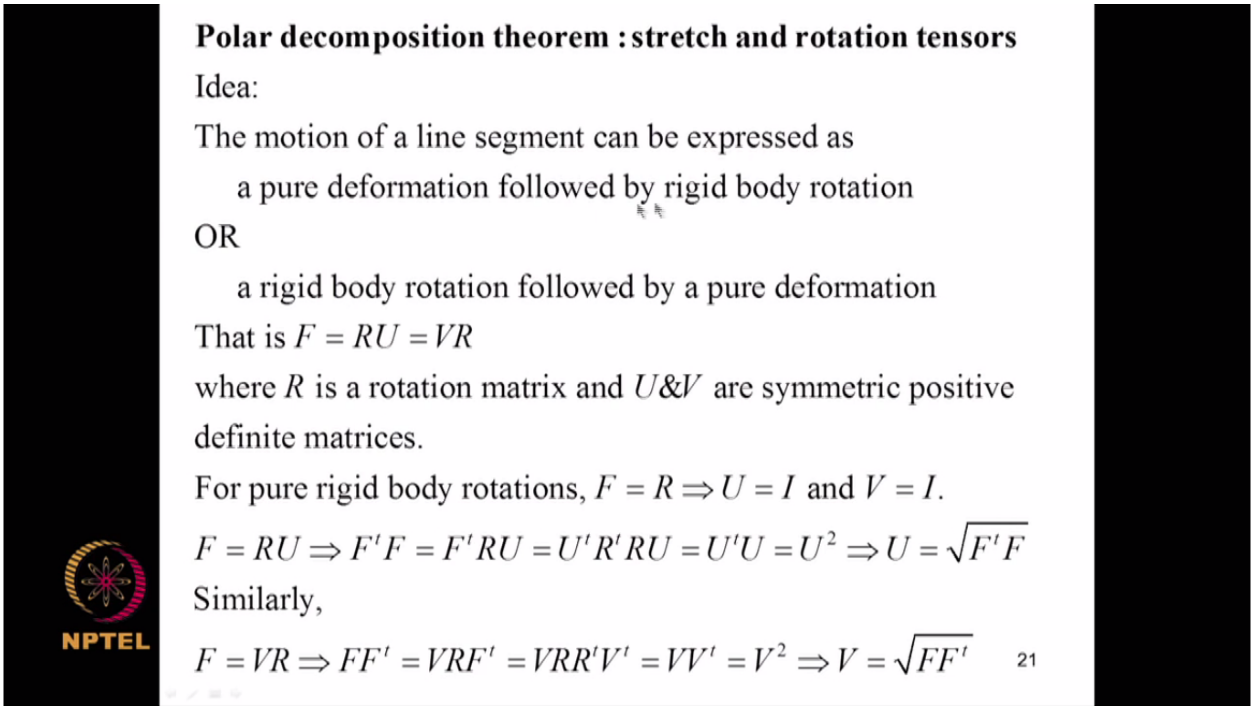
So deformation gradient you can easily evaluate it to be this and the Jacobian will be 6 minus 6t square and this has to for motion to be admissible this has to be greater than 0 therefore this motion is admissible only for t less than 1 at T equal to 1/2 I will put for T 1/2 I get this displacement field and I can map the points, the three points here O, A, B, and I will plot them in the transform coordinate and find out the displacement field and velocity field and acceleration field.



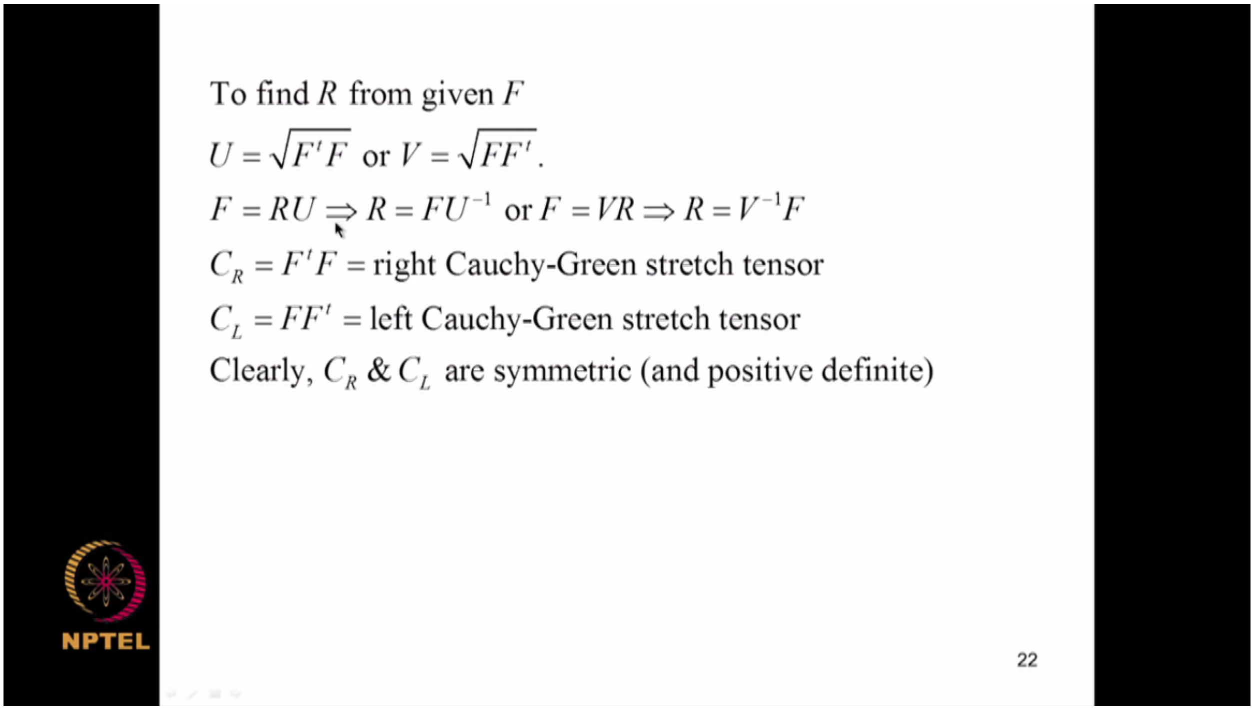
And this triangle is the O prime, B prime is the – this is how the triangle looks upon deformation at T equal to half. Okay.



Now there are other few things that I am stating without proof. Some of this we have done but other couple of things I am not doing. Upon deformation dX becomes FdX this we have shown. Similarly area element and volume element undergo transformation as shown here da is JF transpose inverse to da and volume is dv is JdV. Fij we already discussed duX by duXJ and inverse of this is this and J is the Jacobian --so is the determinant of F called the Jacobian. So these two results I am stating without proof. You can with some effort you will be able to show that.



There is an important concept known as polar decomposition theorem and that leads to notion of stretch and rotation tensors. The main idea is this the motion of a line segment can be expressed as a pure deformation followed by a rigid body rotation or a rigid body rotation followed by a pure deformation. That is F can be written as R into U or V into R where R is a rotation matrix and U and V are symmetric positive definite matrices. So for pure rigid body rotations F is R and consequently U will be I and V will be I. Now for other situations we can work out how you can be determined is as follows. We consider F equal to RU and from this I get F transpose F is F transpose RU and for F transpose I write U transpose R transpose and this becomes U transpose U which is U square. So from this I get U to be square root of F transpose F. Similarly if I consider the second relation F equal to VR I get post-multiplying by F transpose I get FF transpose is VRF F transpose and again by making the substitutions we show that we square root of FF transpose.



So how to find R from a given F? U is first we find U and V by square root of F transpose F and FF transpose square root of a of transpose F. F is RU therefore R is F U inverse or alternatively F is VR, R is V inverse F. Now this quantity F transpose F is known as denoted by CR and it is known as right Cauchy green stress tensor and CL which is FF transpose is left Cauchy green stress tensor. Clearly they are symmetric and positive-definite. So that can be verified by inspection here.

How to find square root of a symmetric, positive definite matrix?

Let A be a $n \times n$ symmetric, positive definite matrix.

Consider the eigenvalue problem

$$A\phi = \lambda\phi$$

Let Φ be the matrix of eigenvectors such that

$$\Phi' A \Phi = \Lambda \ \& \ \Phi'\Phi = I \text{ with } \Lambda = \text{Diag}\left[\lambda_1, \lambda_2, \cdots \lambda_n\right]$$

$\lambda_1, \lambda_2, \cdots \lambda_n > 0 \ \&$ Φ is real valued.

$$\Phi' A \Phi = \Lambda \Rightarrow A = \Phi \Lambda \Phi'$$

Let $B = \sqrt{A}$

Consider $B = \Phi D \Phi'$

$$\Rightarrow B^2 = \Phi D \Phi' \Phi D \Phi' = \Phi D^2 \Phi' = A \Rightarrow D^2 = \Lambda \Rightarrow D = \sqrt{\Lambda}$$

$$B = \sqrt{A} = \Phi \sqrt{\Lambda} \Phi'$$

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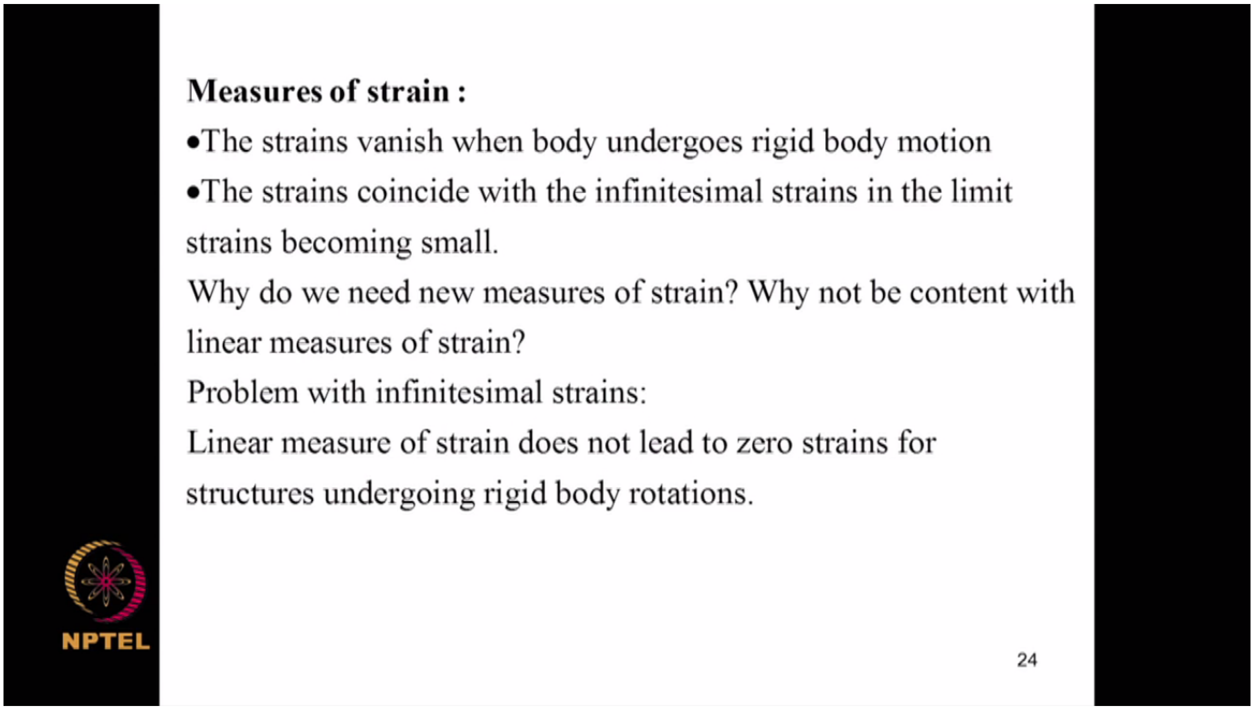
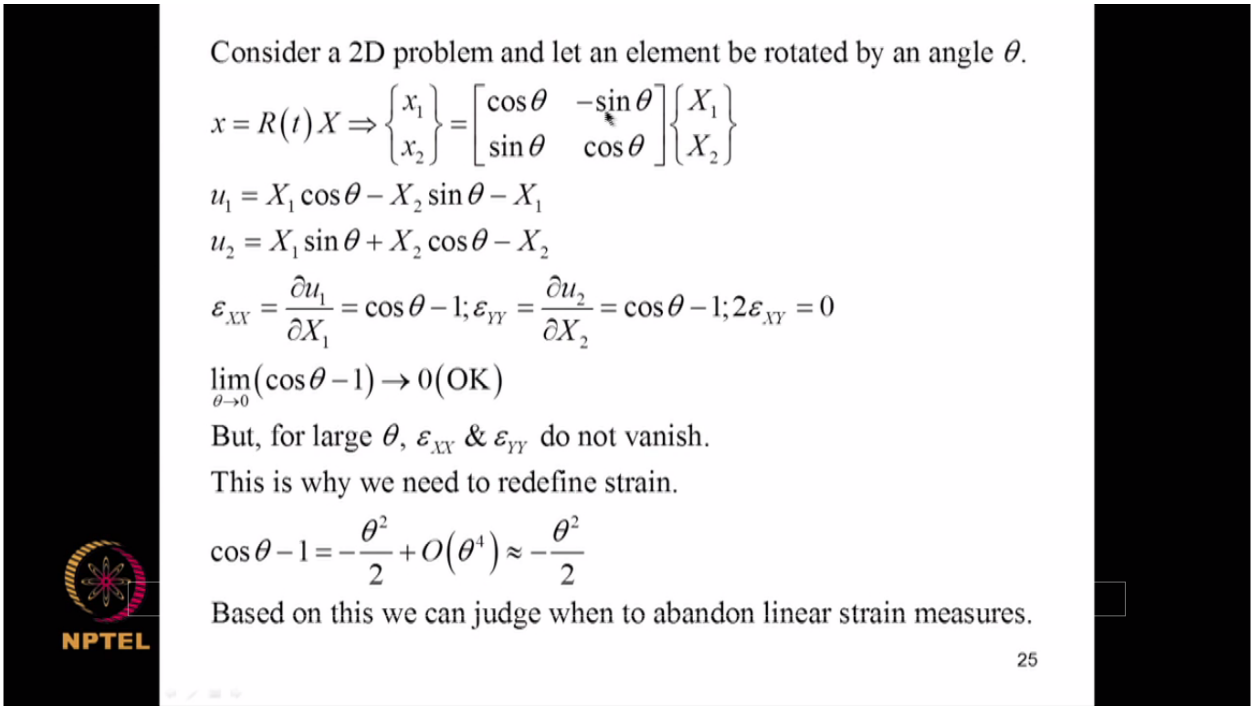
Now how do you find a square root of a symmetric positive definite matrix? Quickly we can recall. Let A be a n by n symmetric positive definite matrix. Let us consider the eigenvalue problem a phi equal to lambda phi. Let Phi be the matrix of eigenvectors such that it is Phi transpose Phi is capital lambda and Phi transpose Phi is I. So this lambda is a diagonal matrix of eigenvalues obtained by solving this problem. And we know that these lambdas are non-positive and Phi is real valued. So from this I can write A as Phi lambda phi transpose. Now let B be square root of A. That's what I wish to find out. So and I consider B equal to Phi D Phi transpose. I do not know what is D. So from this I get B square is Phi D transpose Phi D Phi transpose into Phi D Phi transpose but Phi transpose Phi is I that is how we have normalized. This will therefore will be Phi D square Phi transpose. This is equal to A because B square is A and consequently by comparing these two relations I get lambda to be D square it will be equal to lambda capital lambda and therefore D square root lambda. So B is therefore obtained as Phi square root lambda phi transpose.

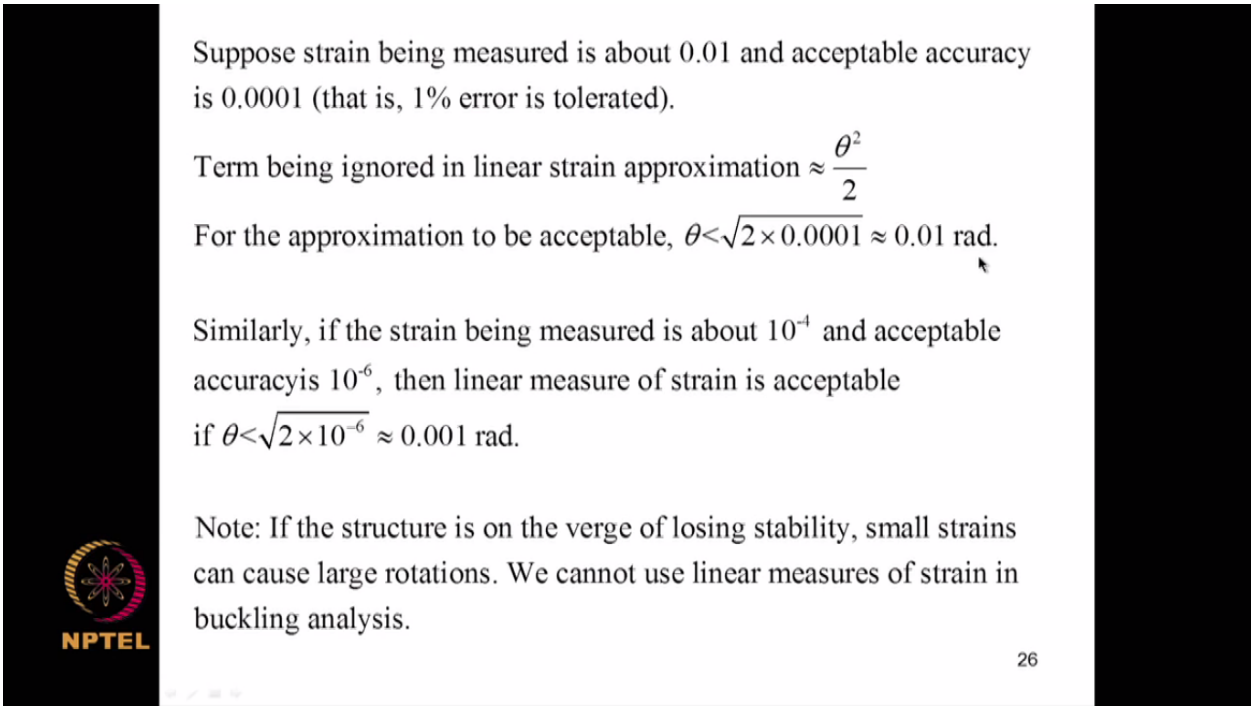
So this gives interpretation of what is the square root of A matrix.



Now we will now talk about measures of strain. Now there are two requirements that we need to satisfy. First is the strains must vanish when body undergoes rigid body motion. Secondly the strains coincide with the infinitesimal strains in the limit of strains becoming small. So any measure of strain that we develop should conform to this and we can ask – begin by asking the question why do we need new measures of strain, why not be content with linear measures of strain. That is infinitesimal strain. Now the problem with infinitesimal strains is that linear measure of strain does not lead to zero strains for structures undergoing rigid body rotations. How do we see that?

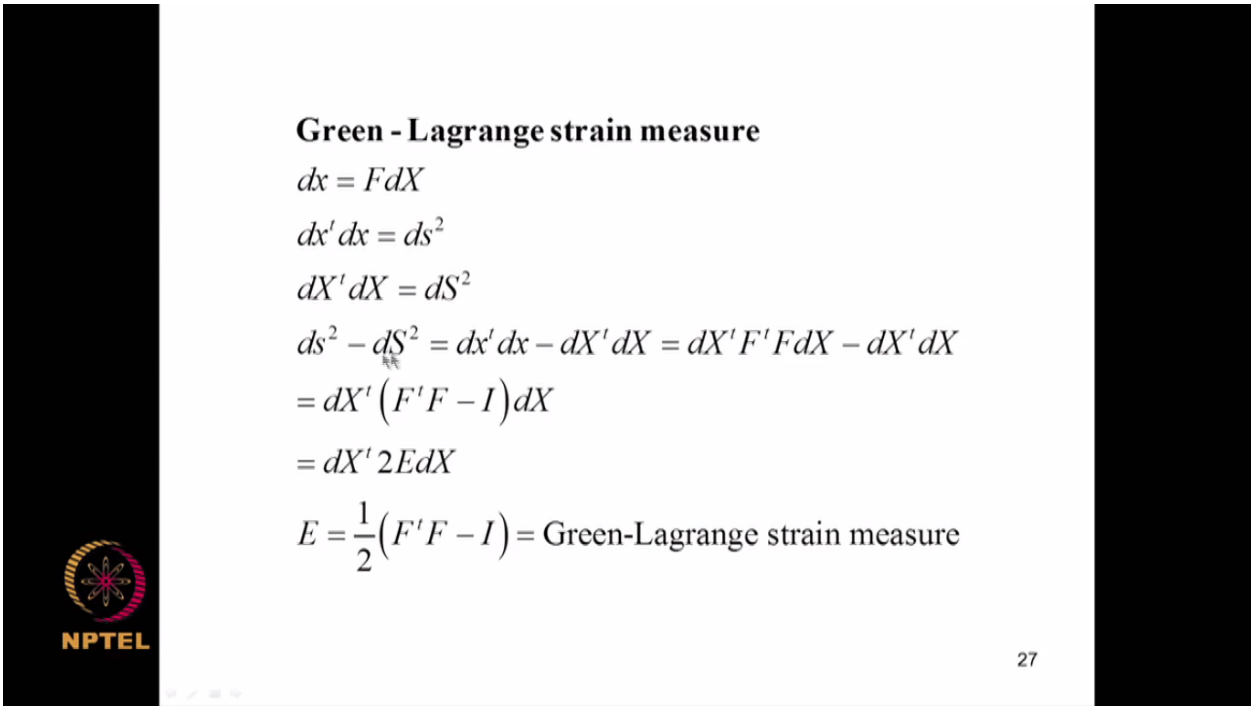


We can consider a small example. You consider a two-dimensional example of an element which is rotated by angle theta. It is a rigid body rotation. So the deformation is given by X is equal to x1x2 is cos theta minus sine theta sine theta cos theta x1x2. So for this deformation for no any value of theta we expect that there won't be any – the strains would be zero. Now we can find the displacement field u1 is X – and this and u2 is this and from which I compute the strain infinitesimal strain component du1 by du x1 is cos theta minus one, epsilon YY is du2 by du2X which is cos theta minus 1 and shear strain is 0. Now you look at epsilon XX and epsilon YY they are not 0 but of course as theta goes to 0 this goes to 0 that is okay but for large theta epsilon XX and epsilon YY do not vanish. This is why we need to redefine strain when you are considering large problems with large displacements. Now to examine this in a slightly greater detail we can consider cos theta minus 1 this can be shown to be equal to minus theta square by 2 plus order of theta to the power of 4 and approximately we can take it as theta square by 2. So based on this we will be able to judge when to abandon linear strain measures. So when theta is large we need to abandon.



So just to quickly see that suppose the strain being measured is about point naught 1 and acceptable accuracy is about 1% of that. So this is accuracy. Now the term that we are ignoring in linear strain approximation is of the order theta square by 2. We assume theta to be small so that theta square by 2 can be ignored. So for approximation to be acceptable theta has to be less than this which is about point naught one radians. So if you want if you are dealing with strains of about point naught one and you want to characterize with 1% accuracy the rotation should not cross this. Similarly if the strain being measured is about ten to the power of minus four and acceptable accuracy is ten to the power of minus 6 that is again 1% percent error. The linear measure of strain is acceptable if theta is less than point naught naught radian. Now obviously if theta exceeds this the major infinitesimal strain measures are not acceptable.

Now it is important to note that if the structure is on the verge of losing stability small strains can cause large rotations. So we cannot use linear measures of strain in buckling analysis that is why we, if you recall, we use nonlinear strain displacement relations when we did buckling analysis.



Now equipped with this we can introduce the first strain measure that is Green-Lagrange strain measure. we have seen this earlier but in a slightly different notation. dX is F into dX and therefore length of an element dX square is dX transpose dX. This is in the deformed configuration and in the original configuration it is this. So the change in square of the length say ds square minus dS square and this I can write in this form now. For dX if I use the relation FdX, I can rewrite this as DX transpose F transpose FdX minus dX transpose dX. So I will write this quantity in this form dX transpose F transpose F minus I into dX. This quantity in the parentheses I call it as 2 into tensor E defined as half of F transpose F minus I. This quantity is known as Green-Lagrange strain measure.

$$E = \frac{1}{2}\left(F^t F - I\right)$$

Recall: $u = x - X \Rightarrow \dfrac{\partial u}{\partial X} = \dfrac{\partial x}{\partial X} - I$

$$\Rightarrow G = F - I$$

$$E = \frac{1}{2}\left(F^t F - I\right) = \frac{1}{2}\left[\left(G + I\right)^t \left(G + I\right) - I\right] \Rightarrow E = \frac{1}{2}\left(G + G^t + G^t G\right)$$

For rigid body motions

$$x = R(t) X + x_T(t) \Rightarrow F = R(t)$$

$$E = \frac{1}{2}\left(F^t F - I\right) = E = \frac{1}{2}\left(R^t R - I\right) = \frac{1}{2}\left(I - I\right) = 0(\text{OK})$$

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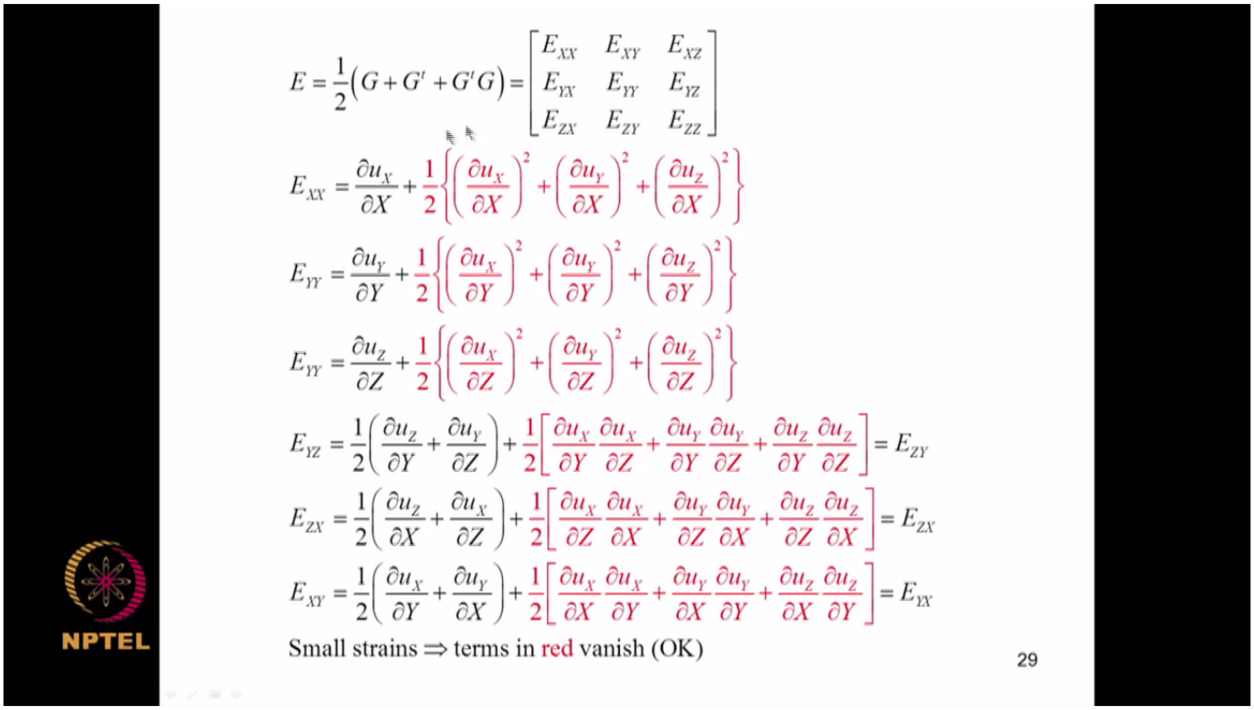
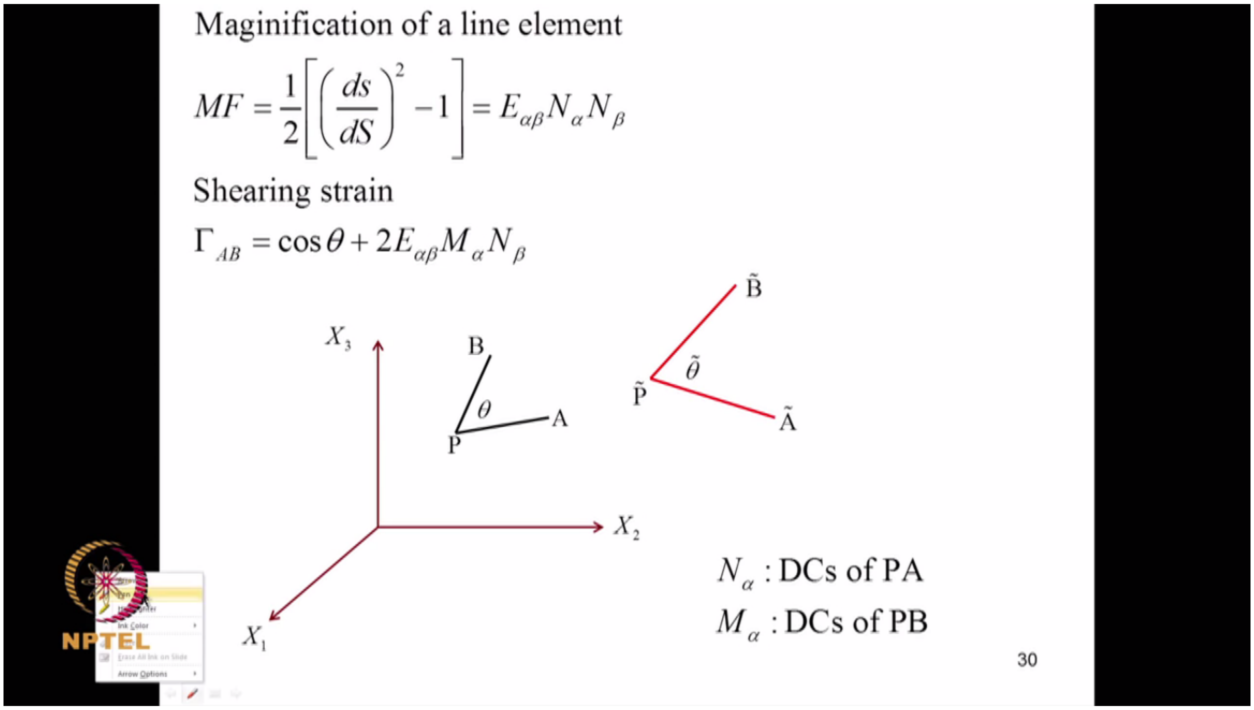
Now we can relate this to displacement, gradients of the displacement. So we have U is x minus X that is du by duX is dux by duX minus I and we have G is equal to F minus I and for F if I write now I plus G, I will be able to get that and upon slight simplification I get E as this. Now one of the required that we stipulated is under rigid body motions the strain measure should go to zero. So we can verify whether that is true here. So I again consider rigid body motions as R into X plus X T of T . So this is translation. This is rotation. R is a rotation matrix. So F is R of T in this case and if I substitute that into this I get F transpose is R transpose. Here it is R transpose R minus I and R transpose R is I, therefore, this is 0. So unlike the infinitesimal strain measures this is 0 for any rotation any rigid body translation and rotation.



Now how about the other requirement that when strains are small we should recover the infinitesimal strain components. So to be able to do that we expand this and write in terms of all the terms we write in longhand and quantities that are shown in the red are the nonlinear terms and the quantities in black are the infinitesimal strengths. So for small strains you can clearly see that quadratic terms can be ignored. So we recover back the – all the terms in the red vanish and we recover the infinitesimal strain components. So this is – so therefore this definition is acceptable by the two [Indiscernible] [0:25:46] that we stipulated.



Now we can also show that the magnification of a line segment that is in the - if there is a line segment PQ with direction cosines N alpha upon deformation if I define a quantity known as magnification factor as ds by dS whole square minus 1 that you can see that this is nothing but ds minus dS whole square divided by dS square and this is defined as a magnification factor of a line element. We can show that in terms of the Green-Lagrange strain tensor this magnification factor is given by this. So clearly here if the line segment is such that it aligns along with X1 axis this is a repeated index imply summation. So the line segment that is lying along X axis is magnified by the quantity E11 and a line segment which is aligned with X2 axis is magnified by E22 and the line segment along this is magnified by E33. Now similarly if you take two line segments which bear an angle theta before deformation and deform to this configuration, we can show that a measure of shearing strain this again we have discussed in the previous one of the previous lectures, is given by this and the strain E appears here. So here again if theta is PI by 2 and a line segment is aligned with X-axis and Y-axis here there are 2 direction cosines N alpha is direction cosines of PA and M alpha direction cosines of PB. So if PA aligns along one of these axis and PB aligns follow along one of this axis then for example epsilon e12 will be the shearing strain as per this definition between these two line segments. So the Green-Lagrange strain has this interpretation.

Almansi - Hamel (Eulerian) strain

$$dx = F dX$$

$$dX = F^{-1} dx$$

$$dx^t dx = ds^2$$

$$dX^t dX = dS^2$$

$$ds^2 - dS^2 = dx^t dx - dX^t dX = dx^t dx - dx^t F^{-t} F^{-1} dx$$

$$= dx^t \left(I - F^{-t} F^{-1} \right) dx = 2 dx^t e dx$$

$$e = \frac{1}{2}\left(I - F^{-t} F^{-1} \right)$$

e = Almansi - Hamel (Eulerian) strain measure.

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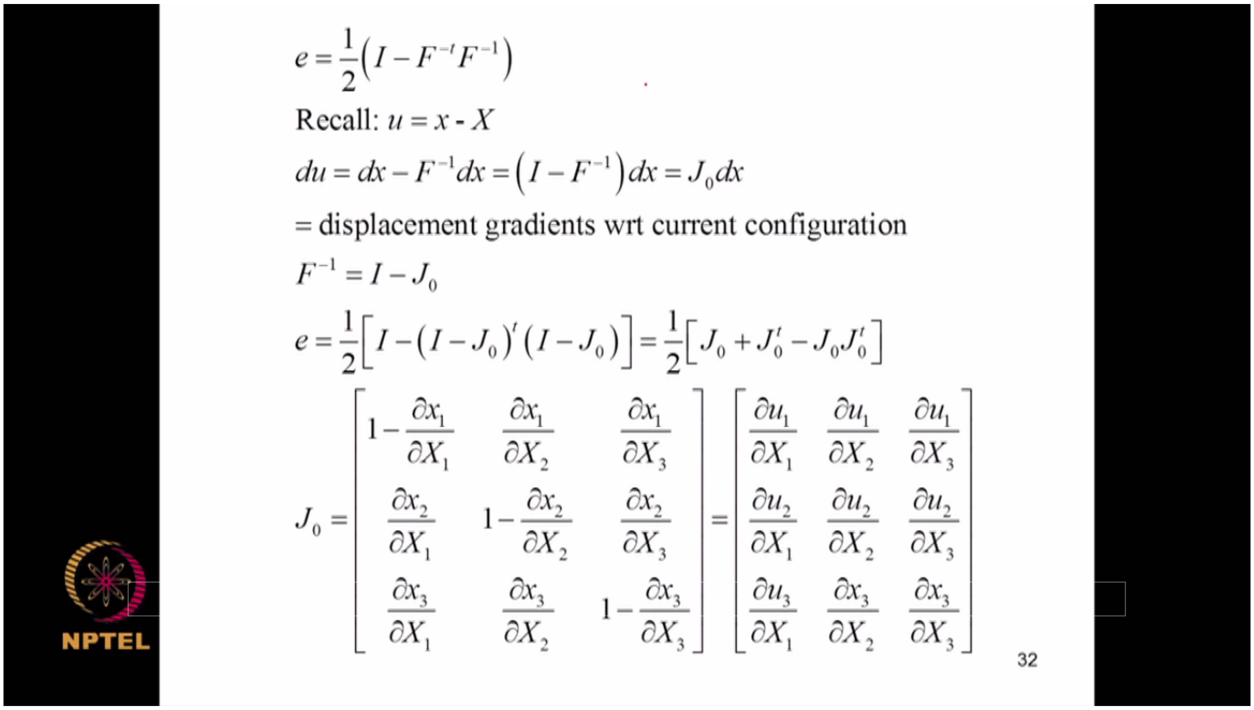
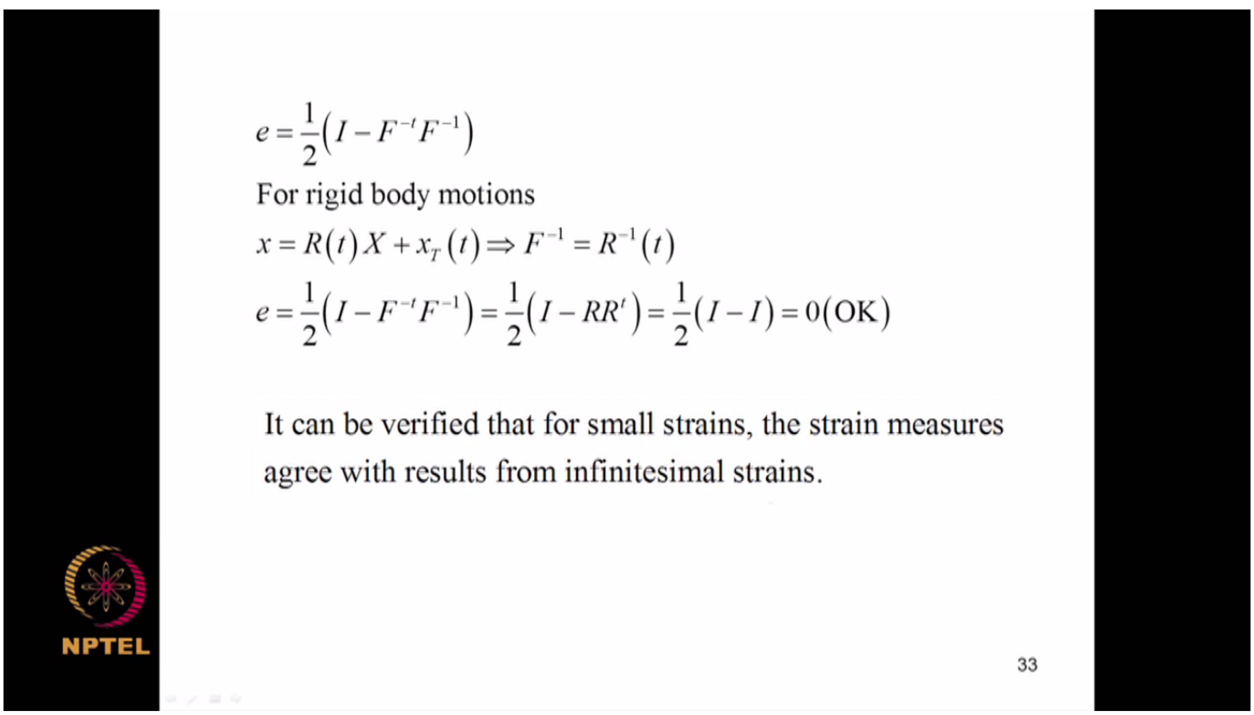
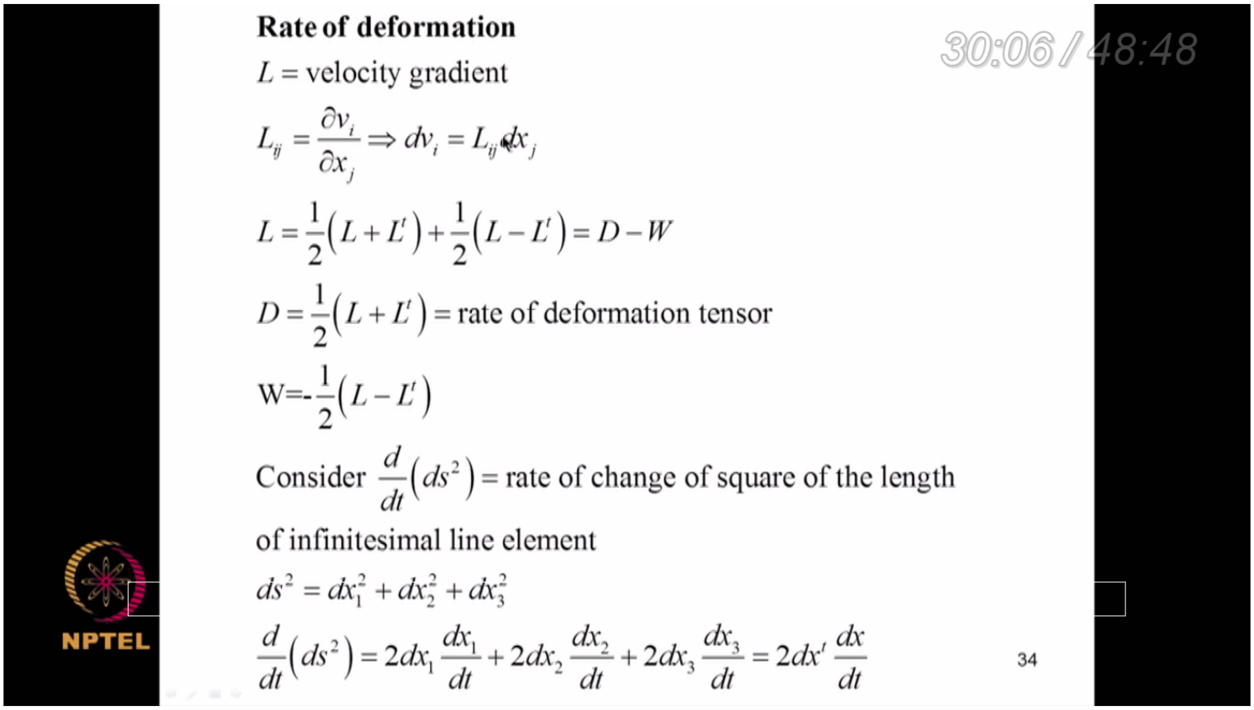
There's another strain measure known as Almansi-Hamel or Eulerian strain. Here instead of eliminating capital dX we eliminate – here in this case if you see here we obtained the difference in square of the length in terms of dX in the original configuration. A similar equation can be derived by using dX in the current configuration. So that definition takes us to the Almansi-Hamel Eulerian strain and this is defined with notation small e and here ds square minus dS square is written in terms of lowercase dX and we get in this form and this quantity I minus F inverse transpose F inverse is defined as e. This is the Almansi-Hamel strain measure.



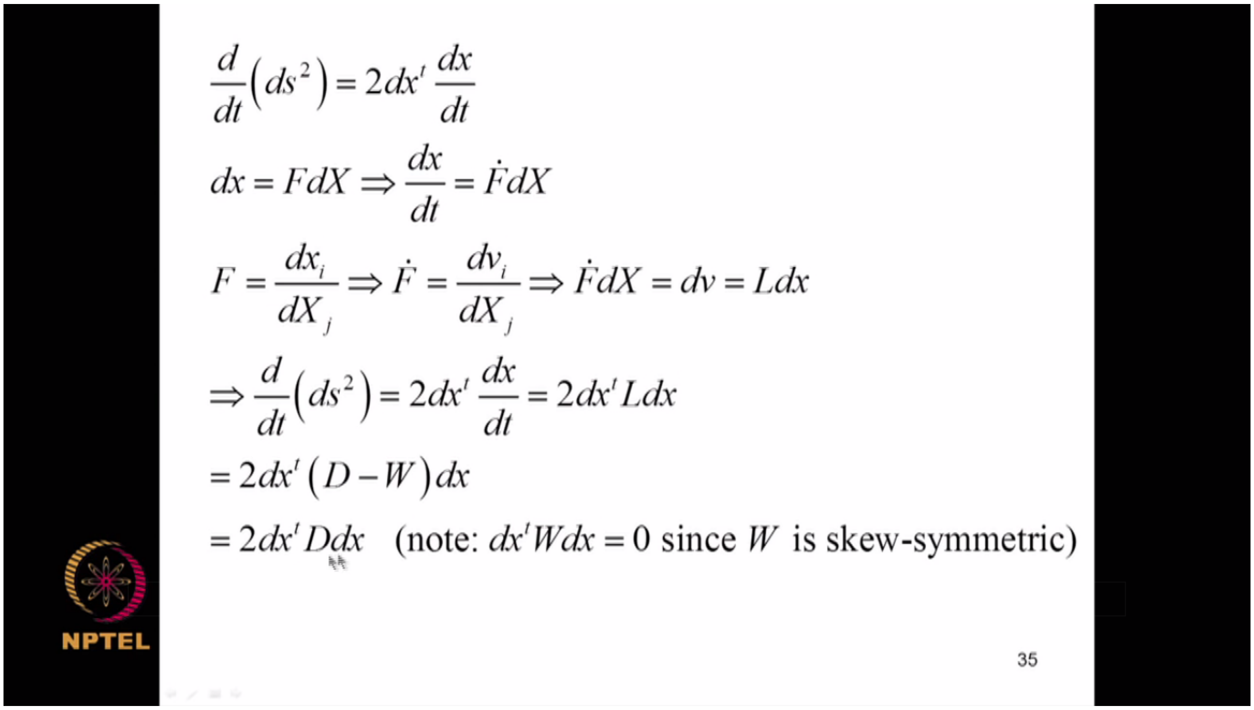
Here again if you take a rigid body rotation we can show that we can first derive the strain components in terms of displacement and we get in terms of J naught matrix is expressed in terms of J naught matrix as shown here. This can be verified.



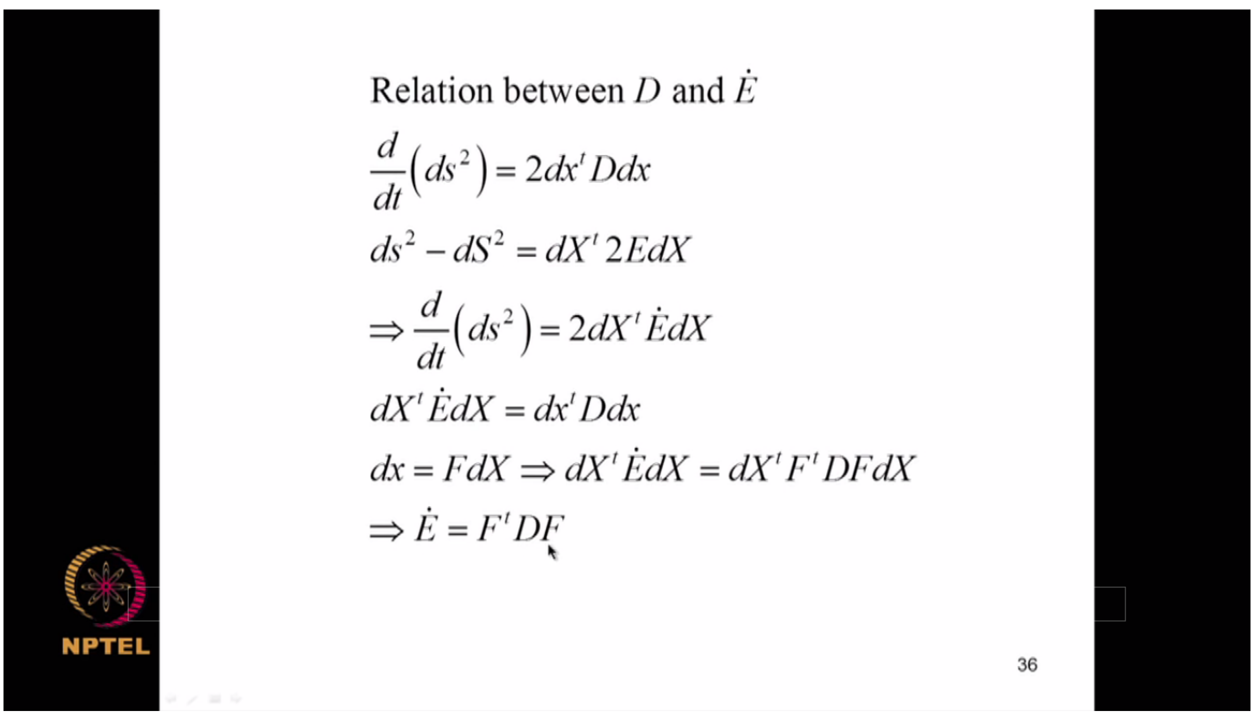
If you consider now rigid body motions X as RX plus XT of T again we can show that E becomes 0 and by expanding the terms we can again show that it can be verified that for small strain the strain measures agree, measures agree with results from infinitesimal strains. Two definitions of strain measure.



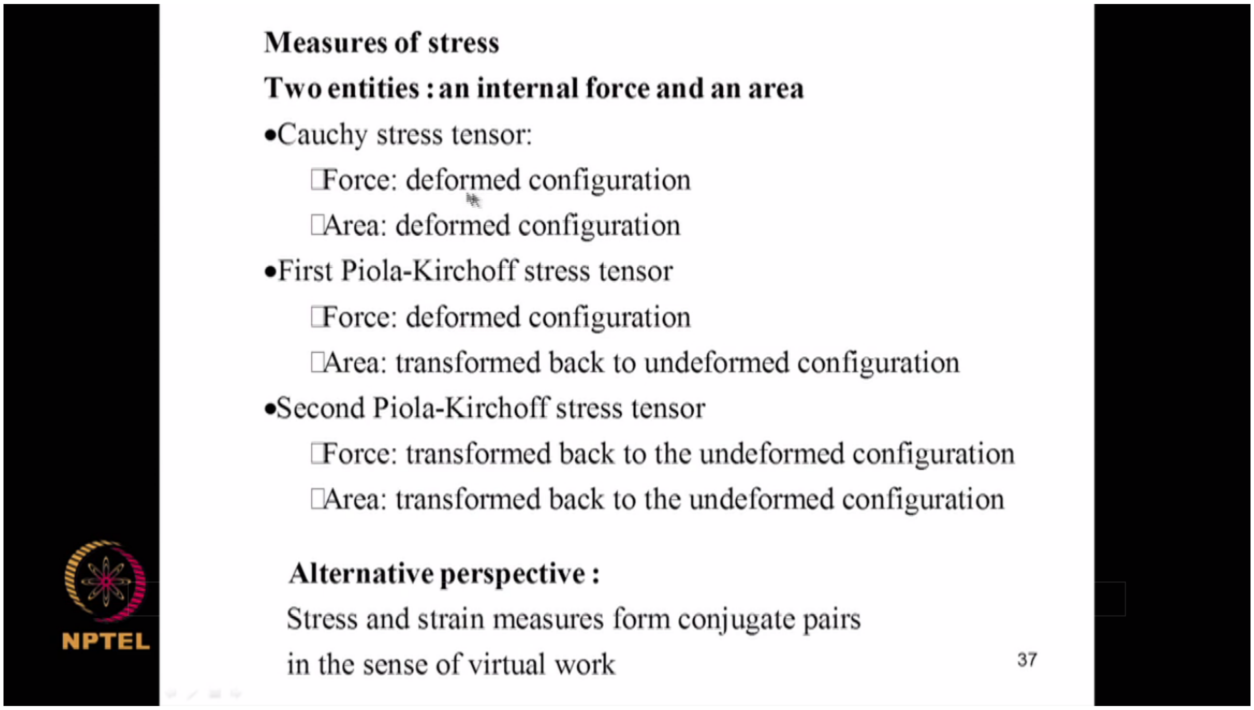
We also talk about what is known as the rate of deformation. We call capital L as velocity gradient where Lij is defined as duvi by duxj. So that means dvi is Lij dxj. This L matrix furthermore we write it as sum of a symmetric matrix and an anti-symmetric matrix and this symmetric component of that is known as rate of deformation tensor and W is given by this. Now if you consider the rate of change of the line segment ds square rate of change of square of the length of infinitesimal line element if you consider this you can begin by noting that ds squared dX1 square plus dX2 squared plus dX3 square and by writing this in this form we will be able to see that it is 2dxt dx transpose dx by dt.



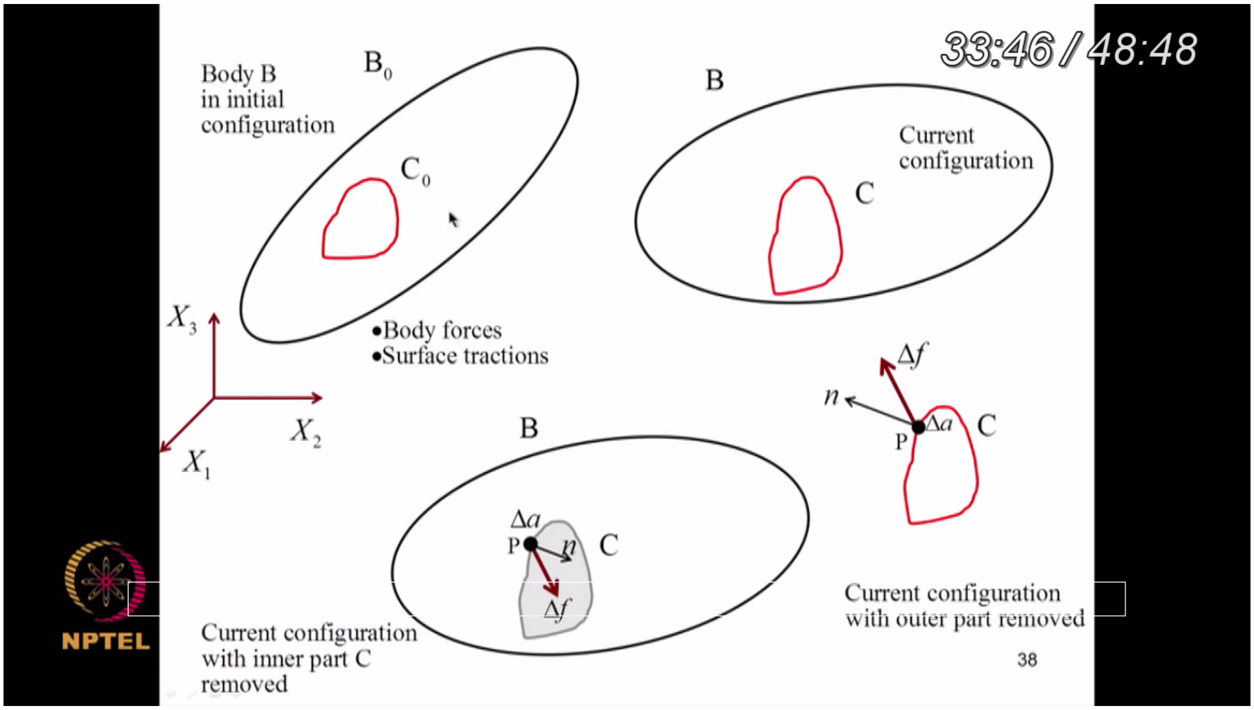
And we can rearrange these terms and use this identity and we can actually show that the F matrix and d matrix are related through this. So this F dot matrix. So this is some discussion on rate of deformation.



The relationship between D and derivative of the Green-Lagrange tensor can also be derived. I have indicated the steps here and we can show that E dot is F transpose DF. So I leave it as an exercise for you to verify this.

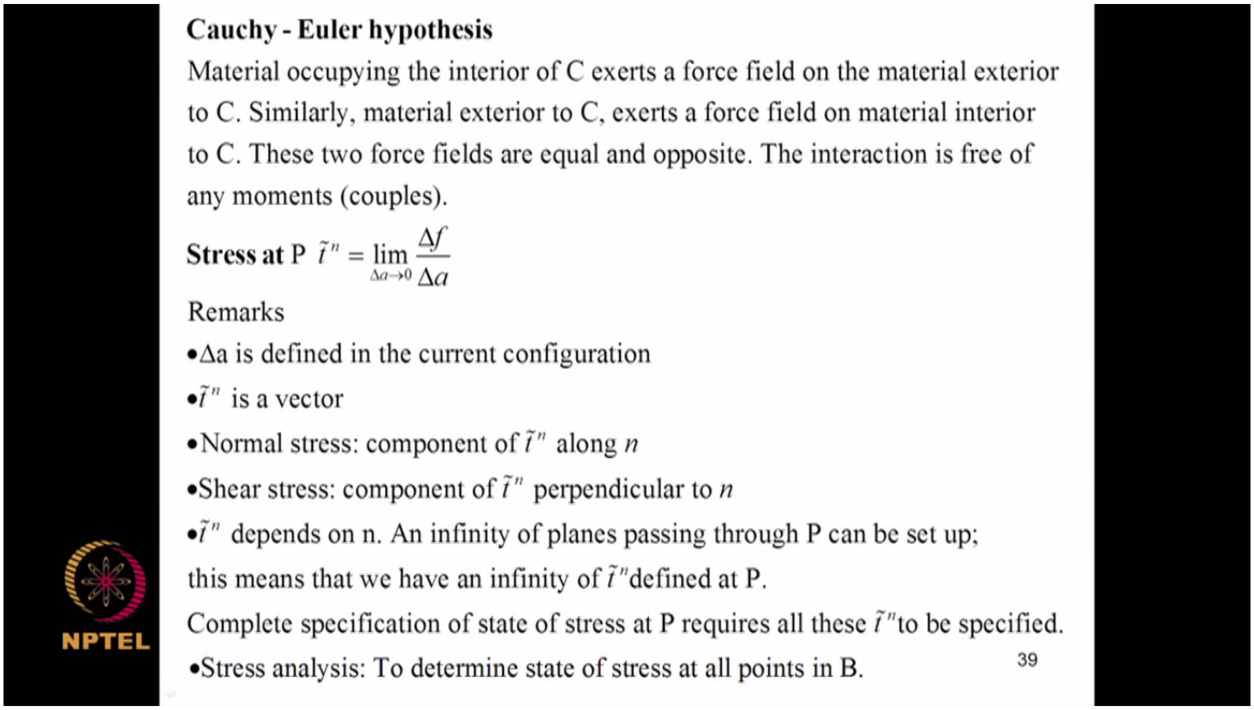


Now how about measures of stress? We have talked about measures of strains. While defining stress there are two alternative perspectives. In the first perspective we think of an internal force and an area over which this force acts. In Cauchy stress tensor with which we are all familiar, the force is a deform -- force is reckoned with respect to deform configuration and area is also reckoned with respect to deform configuration. Now this Cauchy stress tensor is difficult to use because beforehand we will not know the properties of the deformed configuration. So that is what makes us to think of alternative measures of stress. So in first The first Piola-Kirchhoff stress tensor the force is measured with respect to a deformed configuration but the area is transformed back to the undeformed configuration. So it is force internal force reckoned with respect to deformed configuration and expressed with respect to area, the distress is expressed with respect to area in the undeformed configuration. In the second Piola-Kirchhoff stress tensor the force is also transformed back to the undeformed configuration. Area is also transformed back to the undeformed configuration. So this is one way of looking at stress but the other alternative is to look at stress and strain measures as conjugate pairs which combine together to produce an expression for internal work done. So in terms of virtual work concept we have seen that a strain energy stored in a body is expressed as product of integral of a product of stress and strain. So we can think of stress as something that is a conjugate of a strain measure so that along with the associated with strain measure it leads to a proper definition of internal work done due to deformation.



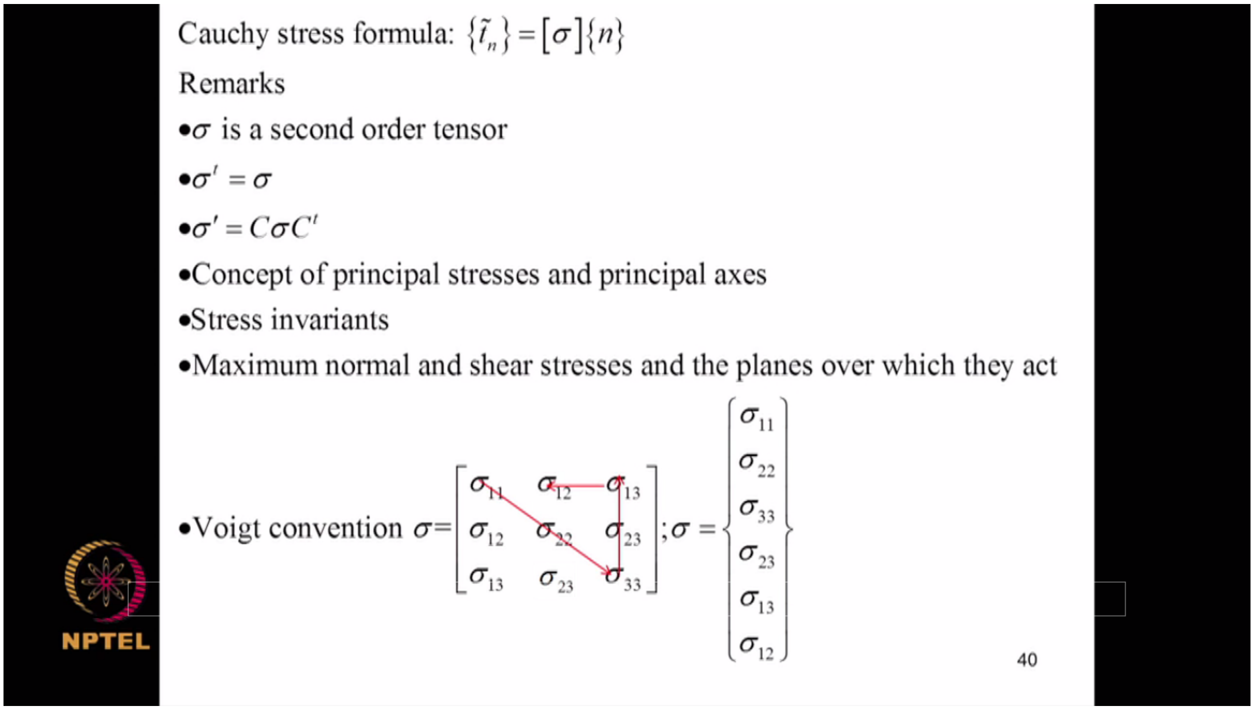
Now let's quickly recall the Cauchy stress. Definition of Cauchy stress. So we have an object in the initial configuration body B I call it as B naught and this is a coordinate system and this is acted upon by body forces that is forces which are proportional to the volume and surface tractions which are forces which are proportional to the area, and upon application of these forces the body deforms and the process of deformation is opposed by an internal set of forces set up in the body. And that internal set of forces is what creates stress in the body. To characterize that what we do is we consider the body in the current configuration that is after the application of these body forces and surface tractions the body has deformed and internal force system has been developed.

So what I do is I consider an imaginary region C and I cut this out from this configuration. So this picture represents the current configuration with inner part of C removed and this configuration, this figure represents the current configuration with outer part removed. So what I do is at a point P I consider an area element say Delta a and that element has a in this figure it has an outer unit normal n and the internal force acting on this Delta a produces a vector and that is Delta f. It need not coincide with the N nor it should be need to be parallel to the surface area. This force system when imagined for this part there is a hole here and n is a unit outward normal and Delta f is the force acting on the elementary area Delta a at P. Now the fact that such internal force system exists is the Cauchy-Euler hypothesis.



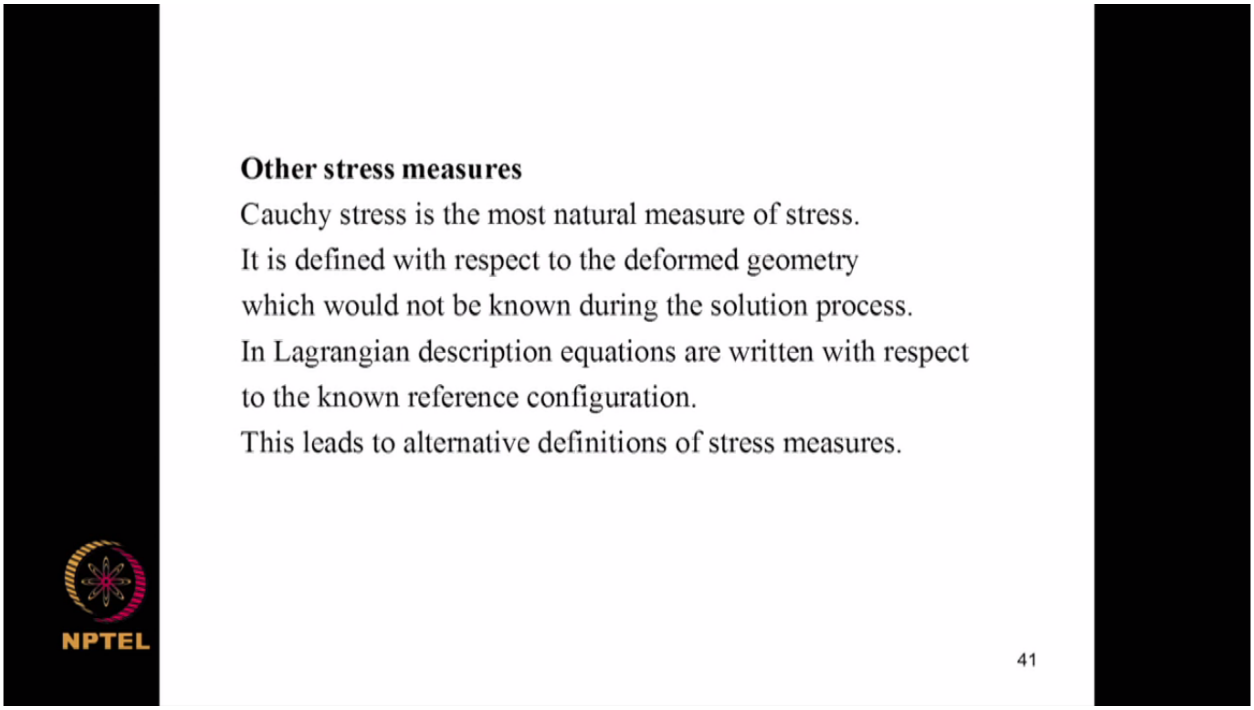
So what it says is material occupying the interior of C exerts a force field on the material exterior to C. Similarly material exterior to C exerts a force field on material interior to C. these two force fields are equal and opposite. The interaction is free of any moment. There are no couples. Okay. It's only, the Delta f is only a force. There is no moment there. Okay. This is an assumption that we make and under these conditions we define stress at P I call it as t tilde n there is N is a unit outward normal tilde is a denotes that is a vector. This is limit of Delta a going to zero Delta f by Delta a. Delta a is defined in the current configuration and t tilde n is a vector. Normal stresses components of t tilde n along n and shear stress components of t tilde n perpendicular to n. t tilde n clearly depends on n. That means passing through this point P I can select so many area segments that means this the way I have cut this is not the only way. I can cut it in many ways. So the direction of unit outward normal can vary. So passing through point P I can draw an infinity of planes with unit outward normal n and we need to if you want to define state of stress at point P I need should be able to specify what is t tilde n for any choice of the orientation of n. So complete specification of state of stresses at P requires all these t tilde n to be specified for any choice of n.

Now stress analysis is determination of stress analysis consists of to determine state of stress at all points in B. So this looks like a tall order at any one point I need some infinity of vectors and there are infinite points in B. So how do we proceed?

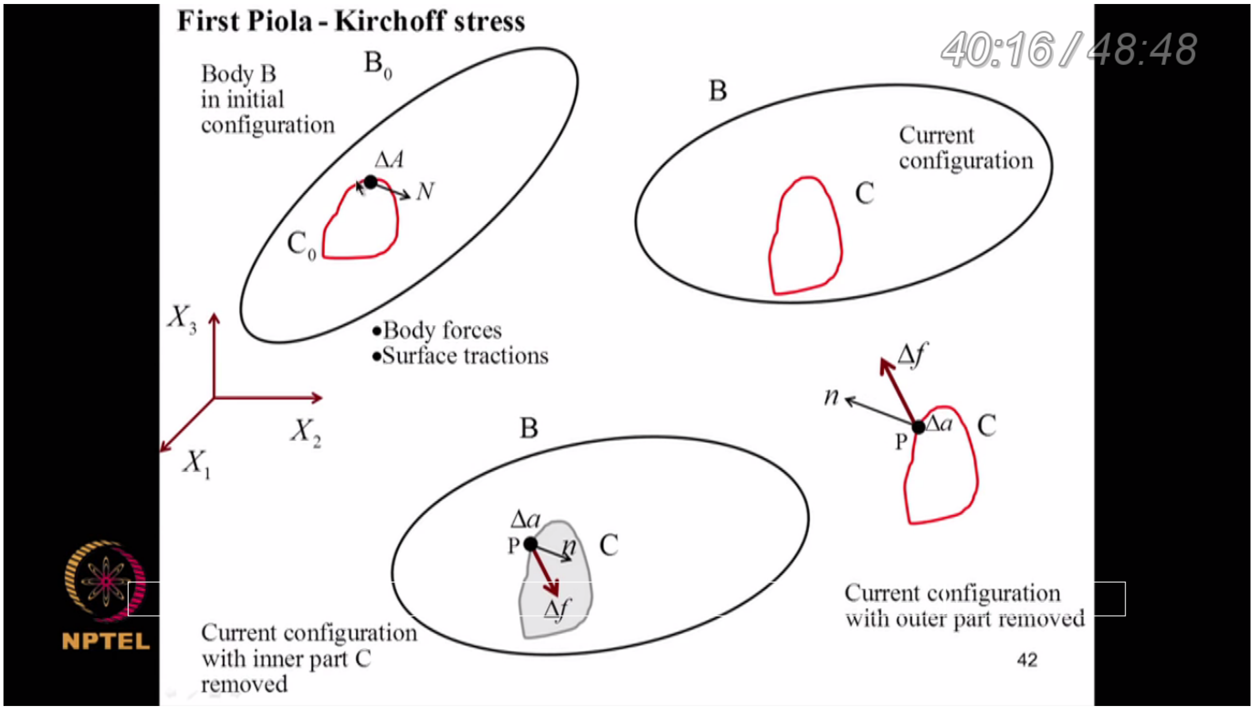


So here what we do is we select a cardinal coordinate system and erect three planes which are mutually perpendicular passing through P and define the stress vector on these three planes and knowing that we will be able to specify stress on any plane that is inclined to this Cardinal plane. So according to Cauchy stress formula this sigma is the stress tensor. This is t tilde n is the stress vector with the unit outward normal n and this is given by this.

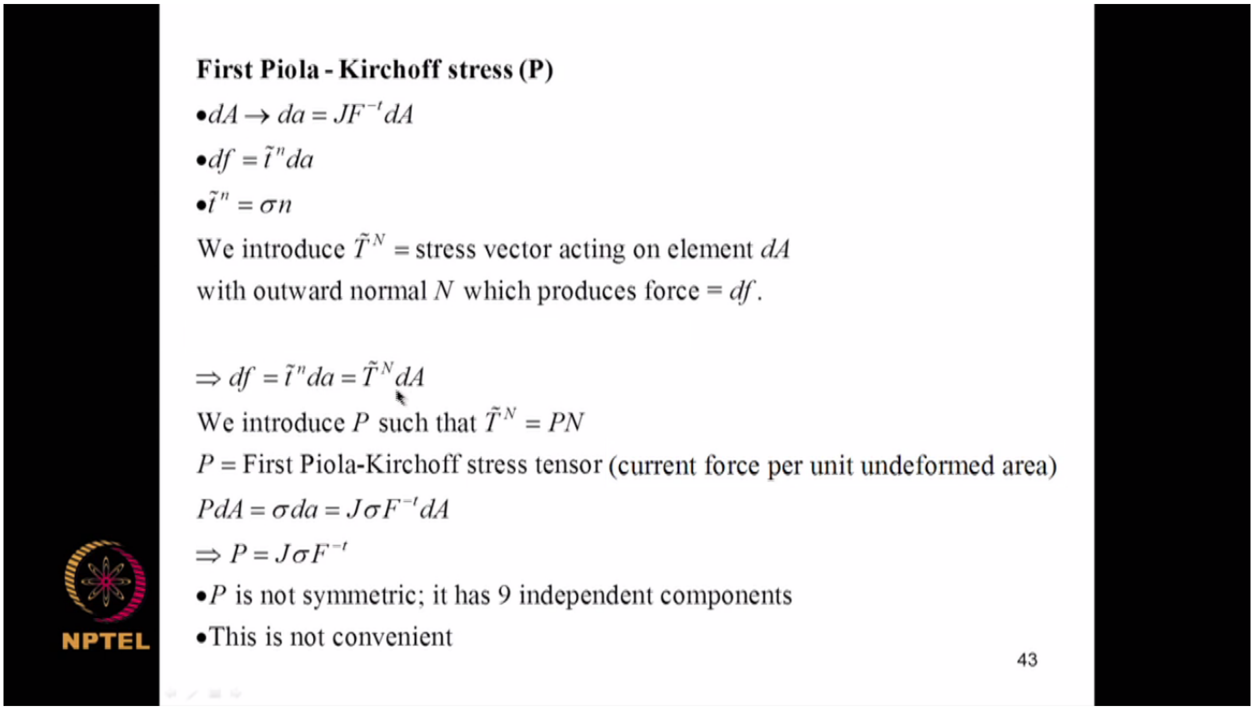
So this sigma is a second order tensor and this is symmetric because there are no interacting moments and if you change coordinate system sigma prime is given by C sigma C transpose where C is the transformation matrix and this I am quickly recalling. I expect that you have – this is not the first time you are hearing about all this. So this leads to the concept of principal stresses and principal axis then stress invariants and we will be able to find out maximum normal and shear stresses and the planes for which they act and when writing stress in finite element formulations as you have seen stress can be written either as a three by three matrix which is symmetric or as a column vector by using what is known as white convention. So we select elements in this order. These diagonals this and this and I have sigma 1-1, 2-2, 3-3 then 2-3, 1-3 and 1-2. So this is how we arrange the column vector.



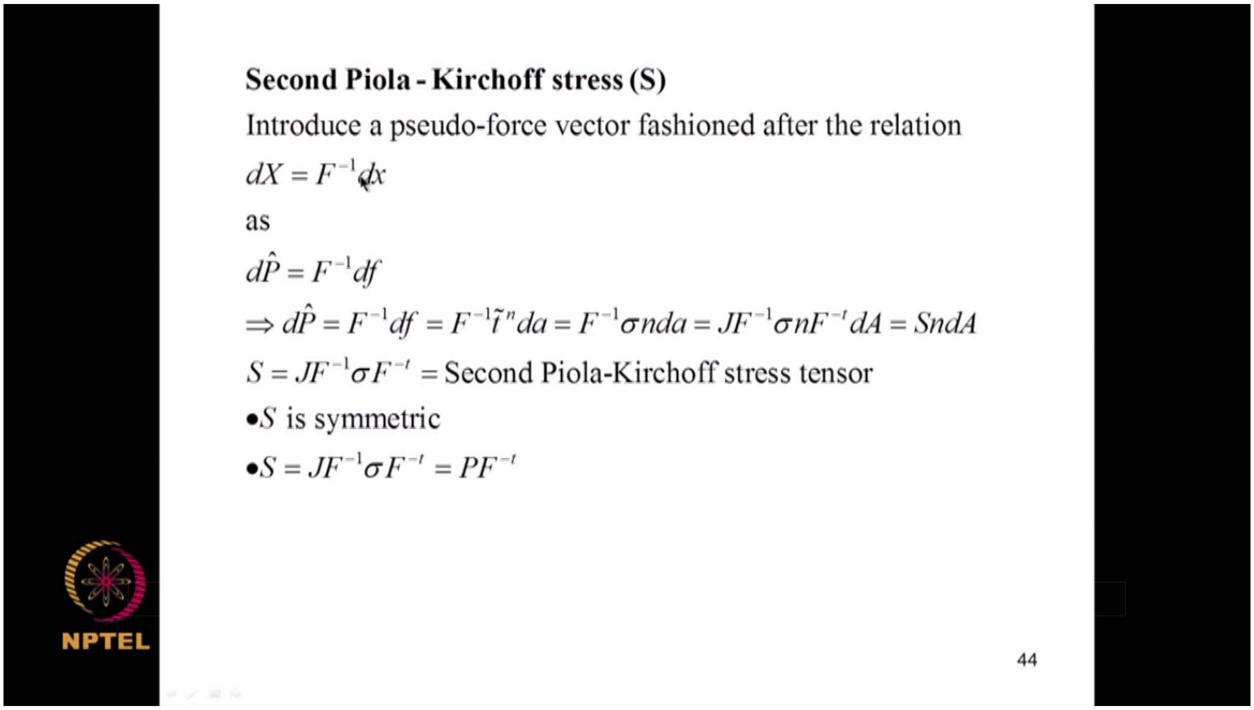
Now Cauchy stress is the most natural measure of stress because it finds how – it considers the body in the deformed configuration when the internal force system has been set up and it describes the state of force per unit area in some sense. That area is also recorded with respect to deformed configuration. But that itself leads to a certain difficulties that is it is defined with respect to deform geometry which would not be no during the solution process. So in Lagrangian description equations are written with respect to the known reference configuration. So this – there is a contradiction between these two and consequently treats the requirement that we need alternative definitions of stress measures.



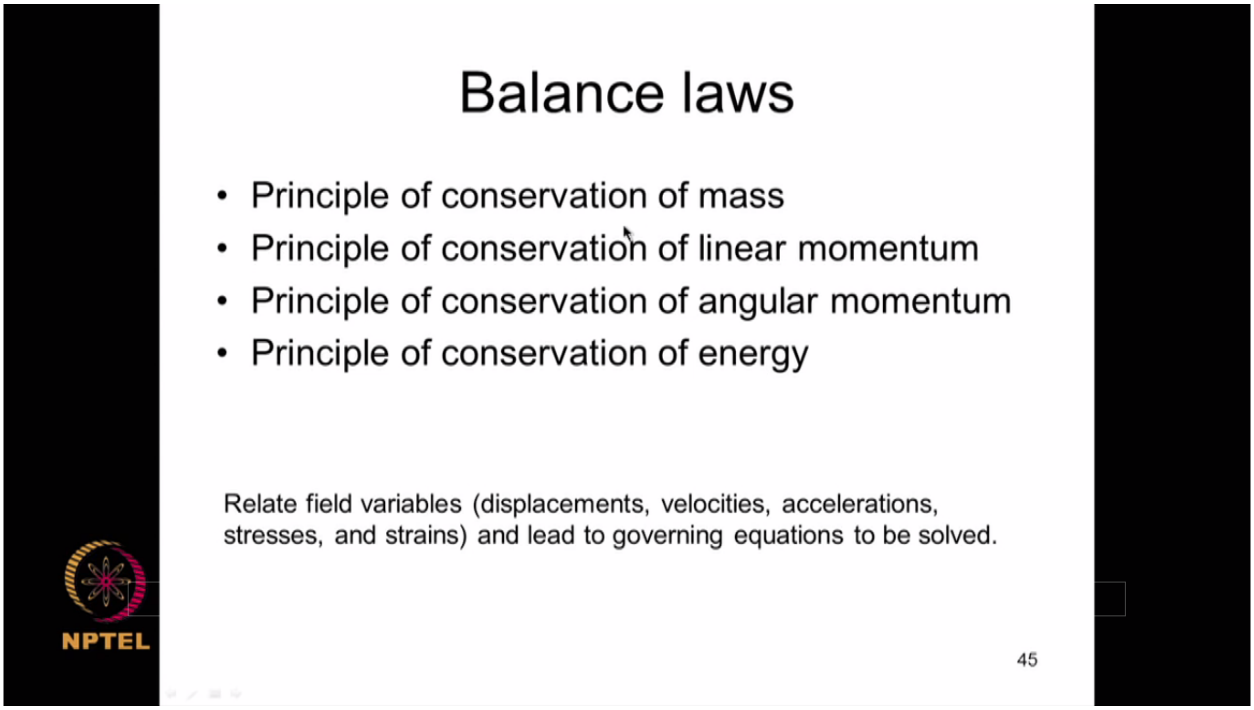
So this leads to a couple of definitions for stress. The first is known as first Piola-Kirchoff stress. So here what we do is this is – the description is quite similar to what I talked about. Now this stress is defined what we do is this is the unit outward normal here and this is the force vector and I consider this area Delta a I map it back to in the -- what it would be in the undeformed configuration. So I will consider this Delta f and this area in the undeformed configuration and set up a definition for stress. How do I do that?



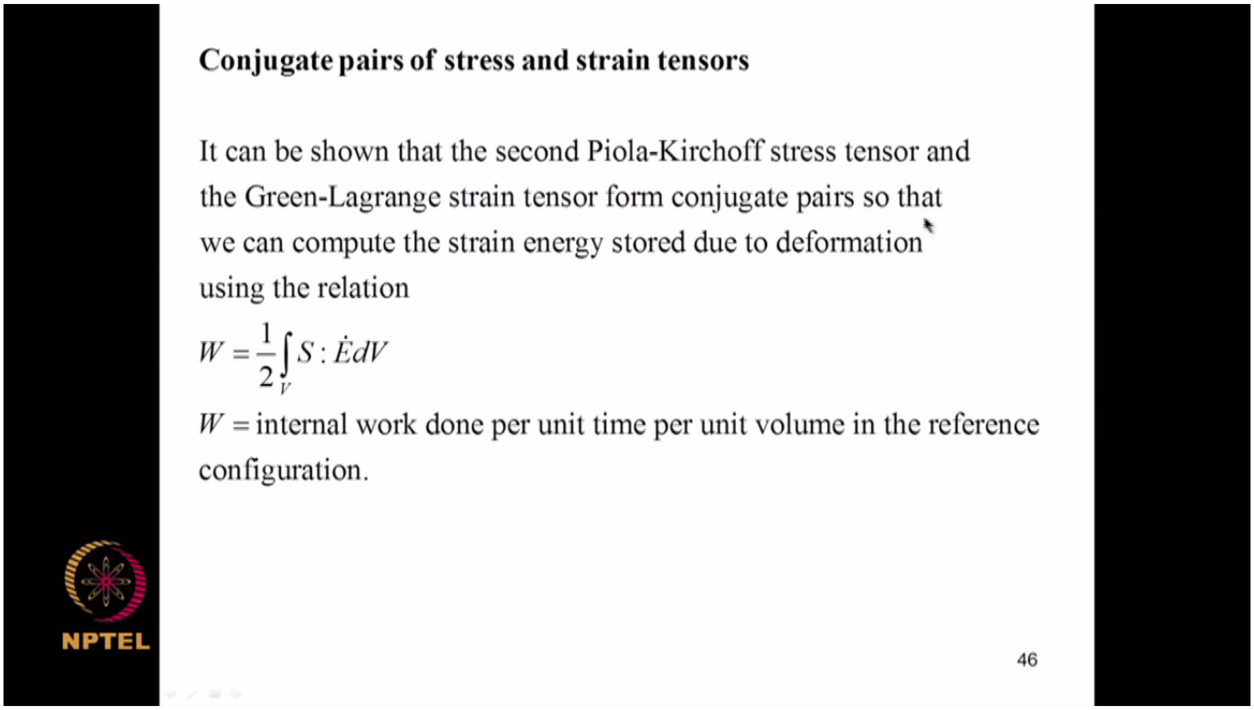
So we have seen that the rule for transformation of areas is dA JF inverse transpose dA and we have df is t tilde n da t tilde n itself is sigma n. These results are known. Now what I do is we introduce capital T tilde n I call it a stress vector acting on element dA. That is this in this. Okay. It is introduced in such a way that it produces the force df that force is this df. So I have df as T tilde n da and this must be equal to capital T tilde n dA. So this and consistent with this definition T tilde n is sigma n I introduce another matrix P such that T tilde n is P into N where capital N is the vector of unit outward normal. So this quantity capital P is known as first Piola-Kirchoff stress tensor. This is a current force per unit undeformed area. So we need not know the deformed configuration to work with the first Piola-Kirchoff stress. There is a problem here and you can relate the the first Piola-Kirchoff stress to the Cauchy stress through this relation using the relation between you know da and capital dA as shown here. If you observe this matrix carefully we see that P is not symmetric and it has 9 independent components and when working with constitutive laws with symmetric strain matrices this becomes inconvenient. So this is not going to be convenient for our modeling purposes.



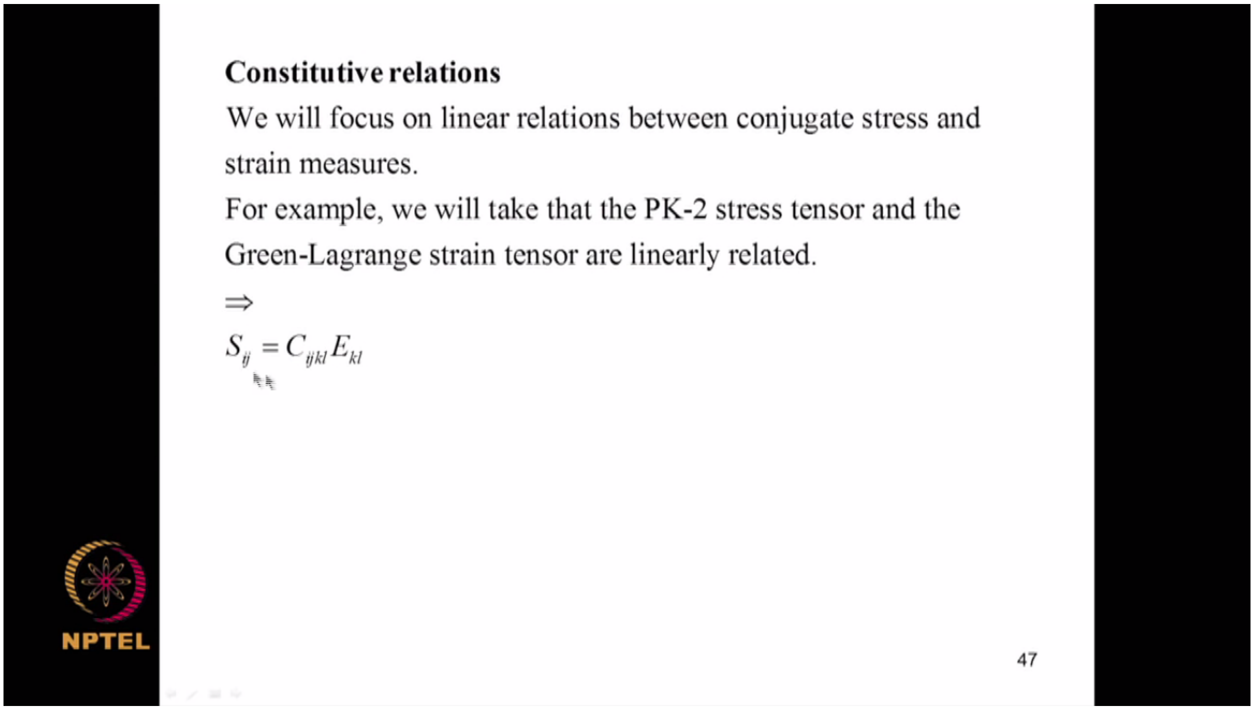
So that leads us to introduction of an another stress measure known as second Piola-Kirchoff stress. Now here what we do is we introduce a pseudo force vector fashioned after the relation dX is F inverse dx. I define DP cap as F inverse df. See I have here this df and I define with respect to the undeformed configuration another force vector, see a line segment which is again a vector gets transformed through this relation. So using – this is a vector and force is also a vector using the same transformation I define a force vector DP cap as F inverse df. So this DP cap is F inverse df and using our relations or definition of df I can write this as F inverse for df I will write t tilde n da and again for t tilde n if I write sigma nda I can rearrange the terms and I get a quantity know SndA where S is given by jf F inverse sigma F inverse transpose. This quantity is known as second Piola-Kirchoff stress tensor. And as you can see S will be symmetric here. Sigma is symmetric and there is a F inverse and F inverse transpose coming here. So if you find S transpose of S it will be same as S. So this is the second Piola-Kirchoff stress. So the relationship between second Piola-Kirchoff stress Cauchy stress and the first Piola-Kirchoff stress is through these three relations. Okay.



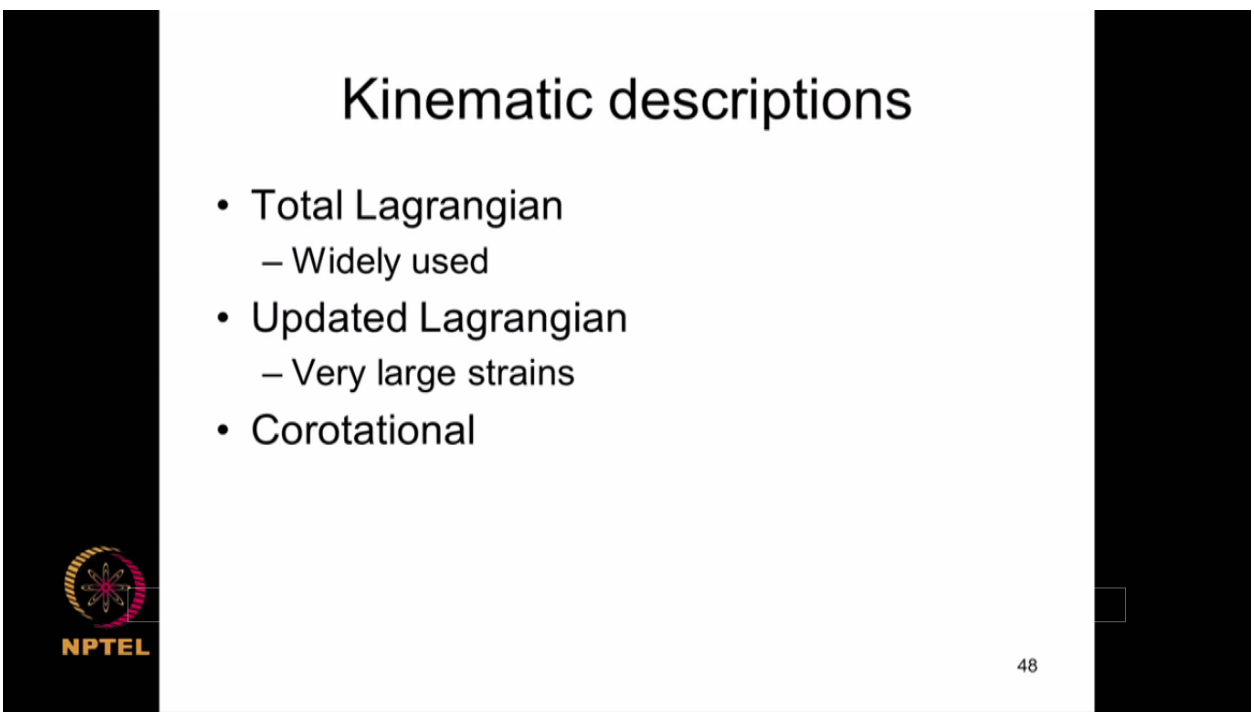
Now to proceed further we need to set up the physical laws which are expressed as what are known as balanced laws that is principle of conservation of mass, principle of conservation of linear momentum, principle of conservation of angular momentum, and principle of conservation of energy. So these form the backbone of our mathematical formulation of problems of continuum mechanics and these basically relate the field variables like displacement, velocity, acceleration, stresses and strains, and to the body geometry applied surface, tractions and body forces, boundary conditions, etc. and they lead to the governing equations to be solved.



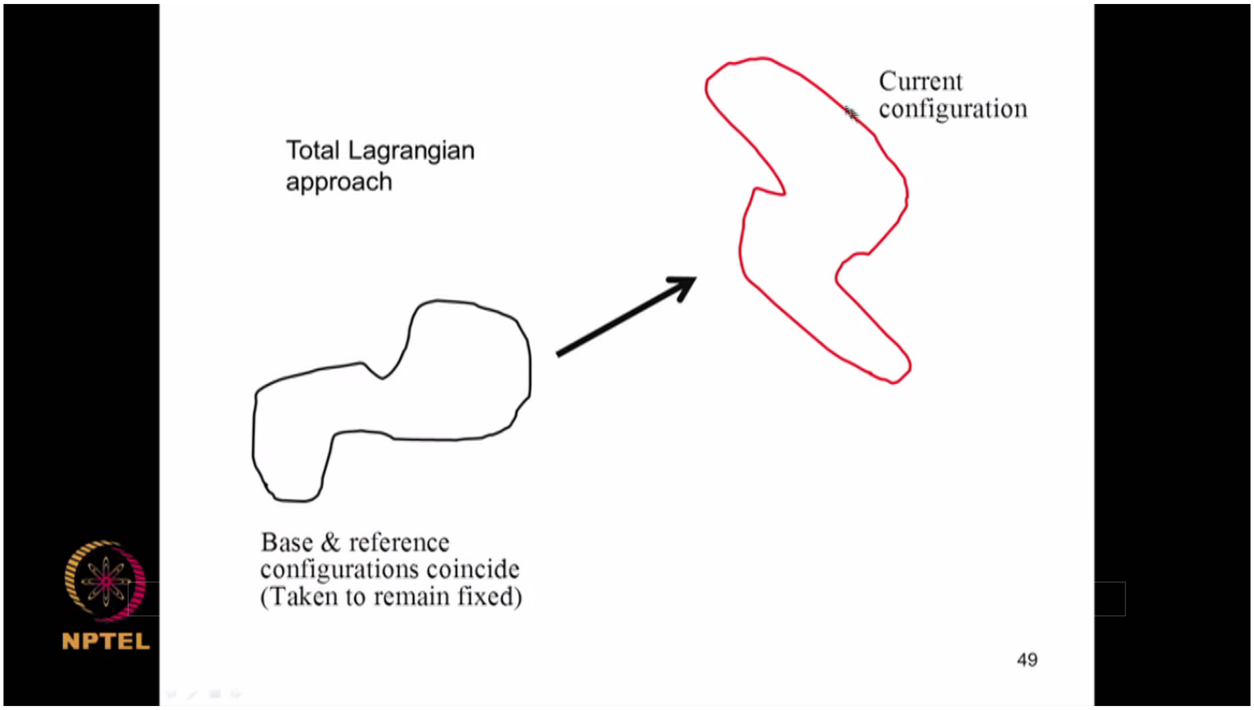
Now we need to elaborate on that but before that we can make some observations. When introducing the notion of stress I talked about conjugate pairs of stress and strain. We can show that the second Piola-Kirchoff stress tensor and the Green-Lagrange strain tensor form conjugate pairs so that we can compute the strain energy stored due to deformation using this relation. This is S, double dot, E dot dv, where v is the volume. So W is the internal work done per unit time per unit volume in the reference configuration. So we start with this expression such as this and use either principle of virtual work or variational approaches and we will be able to express S and E in terms of the displacement fields and those displacement fields will be interpolated within an element and we derive the governing structural matrices and vectors. So that we need to do.



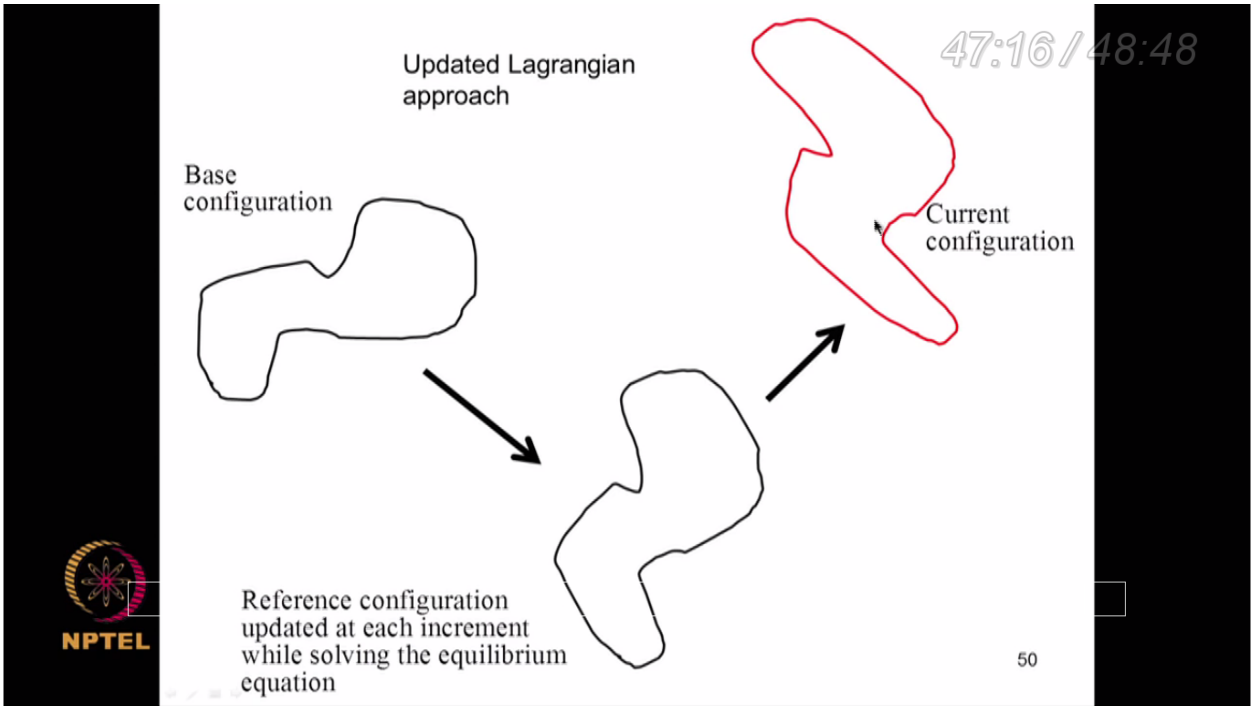
To begin our discussion what we will do is we will focus on linear relationship between conjugate stress and strain measures. So for example we will take that the PK-2 that is Piola-Kirchoff second stress tensor and the Green-Lagrange strain tensor are linearly related. It's a like Hooke's law between stress and strain but the stress is now not the Cauchy stress and strain is not the Green-Lagrange, this Green-Lagrange but stress is second Piola-Kirchoff. So this is what we will start doing and if there is material non-linearity of course this will be more involved.



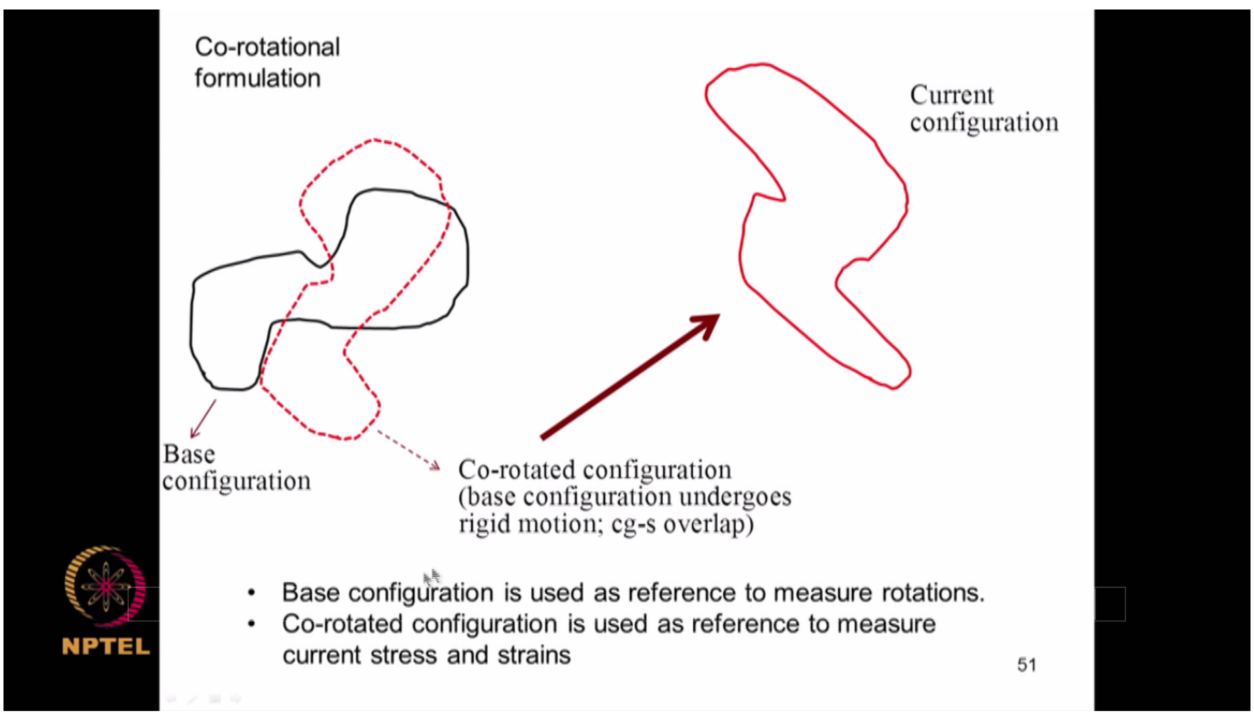
Now in our development of finite element formulations there are three alternative kinematic descriptions that are possible. So they are known as total Lagrangian approach, updated Lagrangian approach, and co-rotational formulation.



So I will just explain what these are then we will consider the more details in the following lectures. In the total Lagrangian approach the base and reference configurations coincide and it is taken to remain fixed and the current configuration is this. So we describe the reference configuration to be the undeformed or the initial configuration. This is total Lagrangian approach.



In updated Lagrangian approach the reference configuration is updated at each increment of loading while solving the equilibrium equations. So this is a current configuration and this is the reference configuration. This is the undeformed configuration. So this reference state gets updated at every time as the load is incremented.



In co-rotational formulation we start with the base configuration and it is used as a reference to measure rotations and co-rotated configuration is used as a reference to measure current stress and strength. So it will – this this – all these figures are very exaggerated. Here the co-rotated configuration undergoes rigid motion. For example, the CG of these two configurations will coincide. So it is a rigid body motion and then from this rotated configuration we characterize a current configuration.

So what we will do is in the following lectures we will elaborate on this and try to develop finite element formulations with whatever time that is left probably we will deal with total Lagrangian formulation for simple line elements like bars and beams and I will also outline how to proceed for two-dimensional and other problems. So that we will take up in the following lectures. So at this stage we will close this lecture

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