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Course Title

Finite element method for structural dynamic
And stability analyses

Lecture – 37

Introduction and review of continuum mechanics

By

Prof. CS Manohar
Professor
Department of Civil Engineering
Indian Institute of Science,
Bangalore-560 012
India

Finite element method for structural dynamic and stability analyses

Module-11

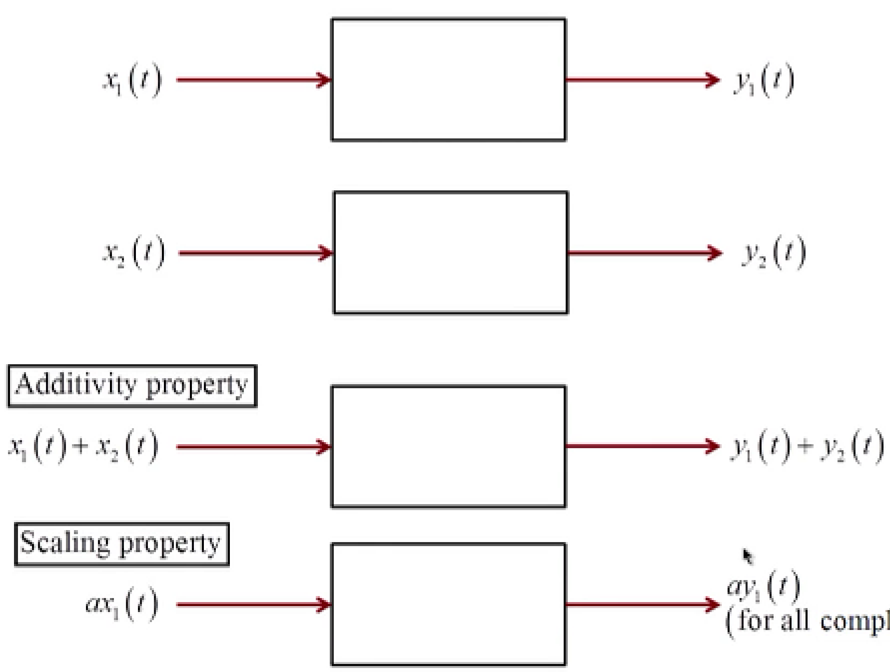
Nonlinear FE Models

Lecture-37 Introduction and review of continuum mechanics

Prof C S Manohar
Department of Civil Engineering
IISc, Bangalore 560 012 India

In this lecture and for the remaining part of this course we will discuss some issues related to development of Nonlinear Finite Element Models. So we will begin in today's lecture a brief introduction of the, what are the main issues and we will start reviewing principles of continuum mechanics as much as is needed for purpose of illustrating the model development.

Linear systems & principle of superposition



So first we can ask our self the question what are nonlinear system, so a simple answer would be to begin by defining what is linear system, and a negation of that definition would tell us what is nonlinear system, so if you have a system with input X1 producing output Y1, and X2(t) producing output Y2(t), if we were to apply X1 and X2 together the response would be some of Y1 and Y2, and this is known as Additivity property. Similarly if we apply an input which is a scalar multiple A of X1(t) the response will be A of Y1 (t), so if these two properties are satisfied we say that the system is linear.

Nonlinear system : a system that is not linear

- **A nonspecific description**

$$y = mx + c$$

$$y_1 = max_1 + c$$

$$ay = amx_1 + ac \neq y_1$$

System is not linear since it does not obey scaling property.

Zero input produces zero output in a linear system.

This is not satisfied in this case.

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A non-linear system is a system that is not linear, so one of these conditions would not be satisfied so this is a non-specific description, it doesn't tell you what exactly is the nature of non-linearity, it simply tells that linear property is not obeyed. Now some simple examples, suppose if you consider a system whose input is X and output is Y, and they are related through this equation Y = MX + C, it's a scalar equation, so if X1 is the input let Y1 be the output, now if I multiply the input by a factor A we see that this is not equal to the principle of superposition is not valid here, suppose Y1 is response to AX1, and if you consider the response AY which is not equal to Y1, so the system is not linear since it does not obey the scaling property, so lesson from this is that 0 input produces 0 output in a linear system, so this is not satisfied in this case.

$$\ddot{y}+2\eta\omega\dot{y}+\omega^2 y=x(t);y(0)=0,\dot{y}(0)=0$$
$$\ddot{y}_1+2\eta\omega\dot{y}_1+\omega^2 y_1=x_1(t);y_1(0)=0,\dot{y}_1(0)=0$$
$$\ddot{y}_2+2\eta\omega\dot{y}_2+\omega^2 y_2=x_2(t);y_2(0)=0,\dot{y}_2(0)=0$$
$$\Rightarrow(\ddot{y}_1+\ddot{y}_2)+2\eta\omega(\dot{y}_1+\dot{y}_2)+\omega^2(y_1+y_2)=x_1(t)+x_2(t)$$
$$\Rightarrow(a\ddot{y}_1)+2\eta\omega(a\dot{y}_1)+\omega^2(ay_1)=\left[ax_1(t)\right];ay_1(0)=0,a\dot{y}_1(0)=0$$
$$\Rightarrow \text{system is linear}$$

$$\ddot{y}+2\eta\omega\dot{y}+\omega^2 y+\alpha y^3=x(t);y(0)=0,\dot{y}(0)=0$$
$$\ddot{y}_1+2\eta\omega\dot{y}_1+\omega^2 y_1+\alpha y_1^3=x_1(t);y_1(0)=0,\dot{y}_1(0)=0$$
$$\ddot{y}_2+2\eta\omega\dot{y}_2+\omega^2 y_2+\alpha y_2^3=x_2(t);y_2(0)=0,\dot{y}_2(0)=0$$
$$\Rightarrow(\ddot{y}_1+\ddot{y}_2)+2\eta\omega(\dot{y}_1+\dot{y}_2)+\omega^2(y_1+y_2)+\alpha(y_1^3+y_2^3)=x_1(t)+x_2(t)$$
$$\ddot{y}_3+2\eta\omega\dot{y}_3+\omega^2 y_3+\alpha y_3^3=x_1(t)+x_2(t);y_3(0)=0,\dot{y}_3(0)=0$$
$$\text{Clearly, } y_3\neq(y_1+y_2)\Rightarrow \text{system is nonlinear}$$

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Now to fix the idea we can consider a simple single degree linear vibrating system with input X(t) and output Y(t), so let us assume that system starts from rest, suppose Y1 is a response to X1, and Y2 is response to X2, if we now add these 2 equations we will see that the output gets added up X1 + X2, so by inspection we see that Y1 + Y2 is a solution of this equation therefore principle of first condition is satisfied. Similarly if I multiply this equation input by A, we can again by inspection we see that AY1 is a solution, so the system is linear.
Now if you now include at cubic nonlinear term alpha Y cube, let under X1 let Y1 be the response, and under X2 to let Y2 to be the response, let us assume for the purpose of illustration that we have 0 initial conditions. Now again as before if I add the inputs I mean add these two equations I see that I get this equation, whereas if I now apply an excitation X2 + X3, that is X1 + X2 if the response is denoted Y3 we see that Y3 is not equal to Y1 + Y2 therefore system is non-linear.

Exercise
Examine the previous two examples by including non-zero initial conditions

$$y(t) = \frac{1}{x(t)}\left(\frac{dx}{dt}\right)^2$$

$$x(t) \to ax(t) \Rightarrow y_1(t) = \frac{a^2}{ax(t)}\left(\frac{dx}{dt}\right)^2 = ay(t)$$

Scaling property is satisfied.
Additivity property is not satisfied (verify)
System is not linear.

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Now as an exercise we can examine the two examples that is a single degree linear system and single degree nonlinear system by including the effect of nonzero initial conditions, so you can see what happens. Another system X(t) is the input and Y(t) is output, so if I now supply an input which is a scalar multiple of X(t) say AX(t) we see that Y1(t) will be AY(t) that means scaling property is satisfied, but additivity property is not satisfied, therefore the system is still nonlinear, it is not linear, although one of the property is satisfied.

Qualitative features of nonlinear dynamic response

- Response to all the loads need to be analyzed simultaneously
- Undamped free vibration: frequency of oscillations depends upon initial conditions.
- Harmonic inputs at Ω can produce
  - harmonic response at frequencies $\neq \Omega$
  - non-harmonic responses
  - aperiodic responses

  primary, subharmonic and super harmonic resonances
- Reciprocity relations are not valid

Now we can start discussing about what's the implication of system being nonlinear on response analysis, so response to all the relevant loads need to be analyzed simultaneously, so you cannot analyze response to individual loads and superpose the response, because principle of superposition is not valid. Now I will illustrate with some simple cases but I will just enunciate some of the properties, in undamped free vibration frequency of oscillations depend on initial conditions which is again an unusual feature if you are thinking this property for linear systems. Then harmonic inputs at frequency omega can produce harmonic response responses at frequencies not equal to omega, it can produce non harmonic responses and it can also produce aperiodic responses, so we can think of primary sub harmonic and super harmonic resonances, whereas in a linear single degree freedom system there is one natural frequency and one resonant frequency, whereas for a single degree freedom nonlinear system under a single frequency excitation there can be several resonances.
Then reciprocity relations which are one of the major features of linear system are not valid for non-linear system, so this property helps us to verify in a laboratory, for example if a system is behaving linearly or not, then so large responses can occur at frequencies other than the driving

- Large responses can occur at frequencies other than the driving frequencies
- Steady state responses depend upon initial conditions
- Multiplicity of steady state solutions are possible
- System can possess multiple equillibrium states and display a wide range of bifurcations
- Concept of normal modes, natural frequencies, and natural coordinates no longer applicable
- Band limited excitations can produce responses with frequency content outside the bandwidth of the excitation.

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frequency, this again refers to the sub harmonic and super harmonic resonances that I mentioned.
Then steady state responses depend on initial conditions, whereas in a linear systems under harmonic excitations steady state when they exist they are independent of initial conditions, whereas this is not true here, then there can be multiplicity of steady-state solutions for nonlinear system, if you simply consider an algebra quadratic equation AX square + BX + C = 0 even for such simple algebraic equation we already know there are two solutions, so if it is a differential equation with a non-linear terms one can still expect that there will be multiplicity of solutions that indeed happens, then system can process multiple equilibrium states and

display a wide range of bifurcations, this we have discussed when we talked about stability problems.

Then concept of normal modes, natural frequencies, and natural coordinates are no longer applicable for nonlinear systems, so you cannot uncouple the equations of motion through a transformation. Then band limited excitation can produce responses with frequency content outside the bandwidth of excitation, so this is again is a very important aspect of non-linear system behavior both in response analysis as well as in experimental work, so in fact a narrowband, for example a single frequency harmonic excitation can produce a very broad band response and that type of behavior is known as chaos, we will not be touching upon that I am just mentioning for sake of completeness.

Example

$$\ddot{x} + \omega_n^2 x + \mu x^3 = 0; x(0) = A \;\&\; \dot{x}(0) = 0$$

$\mu = 0 \Rightarrow x_0(t) = A\cos\omega_n t$

$\mu \neq 0 \Rightarrow$ The frequency of oscillation and the nature of oscillations change.

$$x(t) = x_0(t) + \mu x_1(t) + \mu^2 x_2(t) + \cdots$$

$$\omega^2 = \omega_n^2 + \mu\alpha_1 + \mu^2\alpha_2 + \cdots$$

Unknowns: $\{x_1(t), x_2(t), \cdots\} \;\&\; \{\alpha_1, \alpha_2, \cdots\}$

$$\{\ddot{x}_0(t) + \mu\ddot{x}_1(t) + \mu^2\ddot{x}_2(t) + \cdots\} +$$

$$[\omega^2 - \mu\alpha_1 - \mu^2\alpha^2 + \cdots]\{x_0(t) + \mu x_1(t) + \mu^2 x_2(t) + \cdots\} +$$

$$\mu\{x_0(t) + \mu x_1(t) + \mu^2 x_2(t) + \cdots\}^3 = 0$$

$$\mu^0 : \ddot{x}_0(t) + \omega^2 x_0(t) = 0$$

$$\mu^1 : \ddot{x}_1(t) + \omega^2 x_1(t) - \alpha_1 x_0(t) - x_0^3(t) = 0$$

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Now some simple analytical solutions to illustrate some of the features, this is not an exhaustive coverage but just to motivate you to special features that you should expect when you are dealing with nonlinear problems. Suppose we consider an undamped single degree freedom system with cubic nonlinear terms and it starts with an initial condition A, X(0) is A and the initial velocity is 0, mu is a parameter, if mu is 0 system is linear so we can see that solution is the A cos omega NT, but omega N is this frequency.

Now if mu is not 0 one can expect that the frequency of oscillation and the nature of oscillation would change, it need not be at omega N, it will be influenced by mu, so what I will do is I will represent the solution in a series X naught(t) + mu X1 + mu square X2 + so on and so forth, then omega square which is the frequency of oscillation I will again expand in a similar series, now the unknowns here are X naught and omega n are known, X1, X2, alpha 1, alpha 2 they are not known, what we do is we substitute this assumed solution into this form, into the given equation and then collect terms you know on both sides which are powers of mu, so if you collect terms on mu to the power of 0 I get this equation, mu to the power of 1 I get this equation, now we can examine this solution to these equations, so if we consider the first equation it is X naught double dot + omega square X naught = 0 from which we get X naught(t) is A cos omega T.

$$\ddot{x}_0(t) + \omega^2 x_0(t) = 0 \Rightarrow x_0(t) = A\cos\omega t$$

$$\ddot{x}_1(t) + \omega^2 x_1(t) = \alpha_1 A\cos\omega t + A^3\cos^3\omega t$$

$$= \left(\alpha_1 - \frac{3}{4}A^2\right) A\cos\omega t - \frac{A^3}{4}\cos 3\omega t$$

$$\text{If } \left(\alpha_1 - \frac{3}{4}A^2\right) \neq 0, \lim_{t\to\infty} x_1(t) \to \infty$$

$$\text{This is physically not valid} \Rightarrow \left(\alpha_1 - \frac{3}{4}A^2\right) = 0 \Rightarrow \alpha_1 = \frac{3}{4}A^2$$

$$x_1(t) = C_1\sin\omega t + C_2\cos\omega t + \frac{A^3}{32\omega^2}\cos 3\omega t$$

$$x_1(0) = 0 \;\&\; \dot{x}_1(0) = 0 \Rightarrow x_1(t) = \frac{A^3}{32\omega^2}(\cos 3\omega t - \cos\omega t)$$

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Now X1, the equation for X1 has X naught(t) on the right hand side or here, it is present here, so if we now take that into account and solve the next level of equation which is again linear, all these equation for X naught, X1, etcetera will be linear, but solution at the previous type drives the solution at the next step, so if we solve this problem we get X1(t) to be given by something like this.

Now if, that means I have expanded cos cube omega T I'm writing the right hand side, now if we examine this equation if this omega, and this omega are the same, so as time becomes large the solutions will become unbounded, because we are in a resonant kind of situation, but system is undamped it is conservative therefore this type of solutions are not possible so we demand that this multiplicating term alpha 1 - 3/4th A square must be 0 for simulating the expected qualitative behavior of the system so we get alpha 1 to be 3/4th of A square, so from based on this I can construct this solution and using the initial conditions I will be able to write this

$$x(t) = A\cos\omega t + \frac{\mu A^3}{32\omega^2}(\cos 3\omega t - \cos\omega t)$$

$$\omega = \omega_n\sqrt{1 + \frac{3\mu}{4}A^2}$$

Remark

- Solution is periodic (but not harmonic)
- Period depends upon initial condition
- $\mu > 0 \Rightarrow \omega > \omega_0$ (hardening)
- $\mu < 0 \Rightarrow \omega < \omega_0$ (softening)
- Solution can be improved by including higher order terms

solution for X1(t), so if we restrict our analysis to only the first two terms I get X(t) as A cos omega T + mu A cube/32 omega square this cos 3 omega T - cos omega T, omega curiously is given by this, this omega depends on A, what is A? A is the initial condition, so this is what I was telling the frequency of free vibration can depend on initial conditions, and the response here is periodic but it is not harmonic, okay, so period depends on initial condition, if mu is greater than 0, omega will be greater than omega naught we call that hardening behavior, and mu less than 0 we call softening here. Now we can of course include higher order terms and do this, the objective of this discussion is not to illustrate the solution method but to highlight the qualitative feature of the response.

Example

$$\ddot{x} + c\dot{x} + \omega_n^2 x + \mu x^3 = F\cos(\omega t + \phi); x(0) = x_0 \ \& \ \dot{x}(0) = \dot{x}_0$$

Consider steady state response of the form

$$x_0(t) = A\cos\omega t$$

$$\Rightarrow -A\omega^2\cos\omega t - cA\omega\sin\omega t + \omega_n^2 A\cos\omega t + \mu A^3\cos^3\omega t$$

$$= F(\cos\omega t\cos\phi - \sin\omega t\sin\phi)$$

$$\Rightarrow \left[\left(\omega_n^2 - \omega^2\right)A + \frac{3}{4}\mu A^3\right]\cos\omega t - c\omega A\sin\omega t + \frac{\mu A^3}{4}\cos 3\omega t$$

$$= A_0\cos\omega t - B_0\sin\omega t$$

$$\Rightarrow \left[\left(\omega_n^2 - \omega^2\right)A + \frac{3}{4}\mu A^3\right] = A_0 \ \& \ c\omega A = B_0$$

$$\Rightarrow F^2 = \left[\left(\omega_n^2 - \omega^2\right) + \frac{3}{4}\mu A^3\right]^2 + \left[c\omega A\right]^2$$

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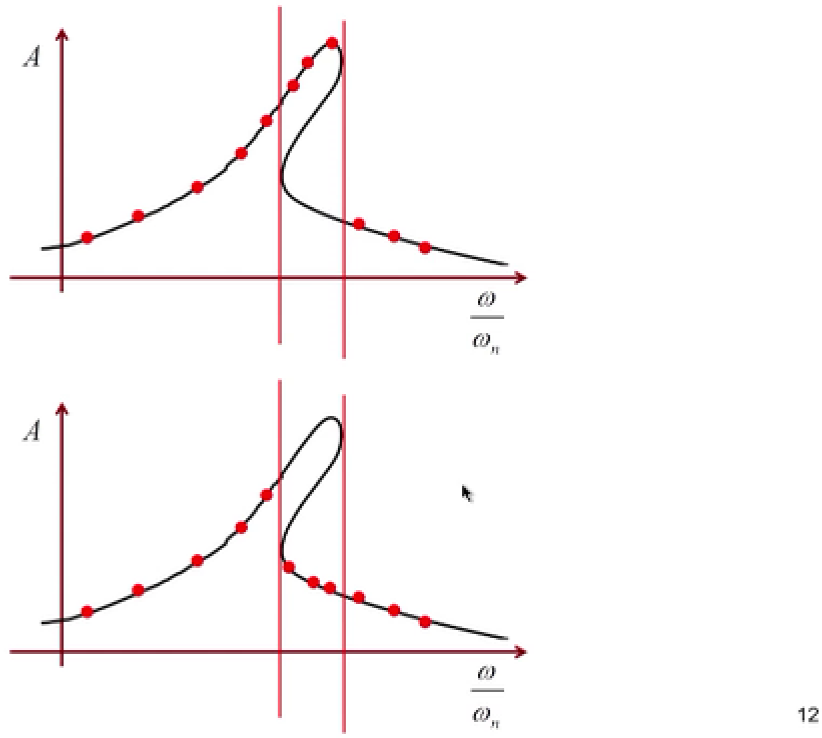
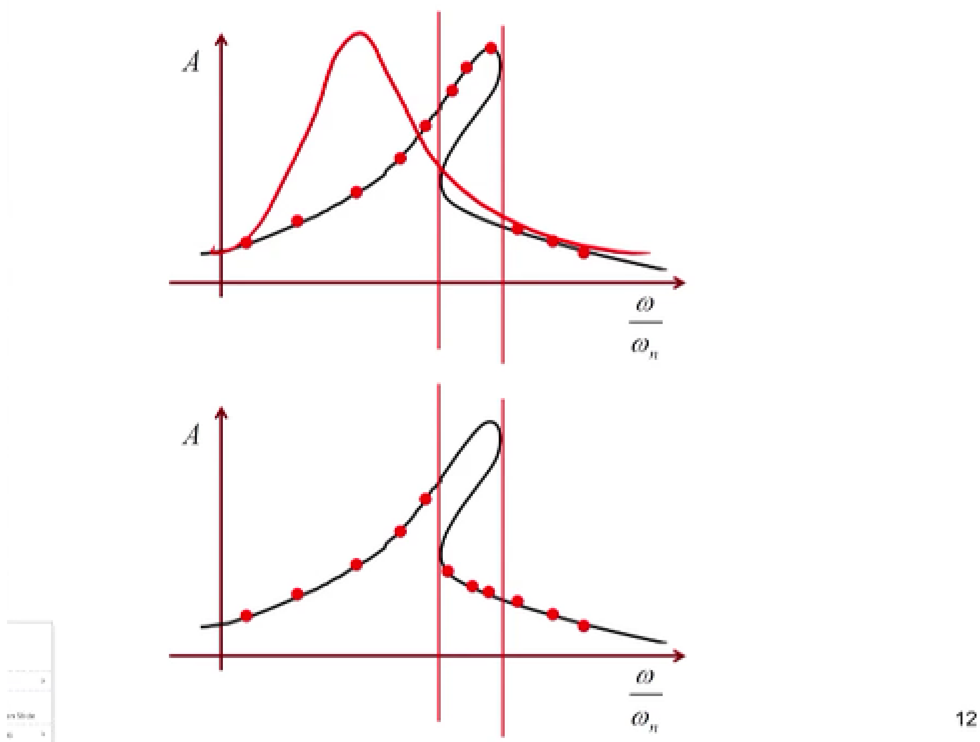
Now a similar system under harmonic excitation it is damped nonlinear system and a harmonic excitation, if mu is 0 we know that response is harmonic for large times, and that is the harmonic steady state, now we want to examine what should be the solution, so if mu is 0 I have X naught(t) is A cos omega T, now I will assume that for some A and omega, for some A this could be the solution so I will substitute this assume solution into the equation of motion, this is only solution in the steady state, so I will to collect terms containing sine cosine terms, do a what is known as harmonic balance and I get a pair of equations which are given by this. Now this resulting equation is the frequency response function of the system, so this is forcing function F amplitude, omega N is a natural frequency, omega is the driving frequency, A is the amplitude of response. In linear systems this, on the right hand side we will not get these A cube term, okay, so now if I plot this frequency response function we start getting curves like,



in a linear system we know that the typically this curve will look like this, now in a non-linear



system things change in certain bandwidth of excitation frequencies there are 3 possible routes and which one is actually realized in a solution depends on initial conditions, and all 3 solutions need not be stable, okay, so now the question is we have to examine stability of these multiple steady-state solutions and only those solutions which are stable will be realized. Now if you were to do a thought experiment where you are increasing the driving frequency and you are

starting with initial conditions which are on this branch, so as you progress along this line when you come here the response suddenly drops, this is steady state response, this you won't see in the time history of response, it is two different time histories, but with initial conditions in the neighborhood of this response, so it drops and it will follow this path, so this branch will not be realized in such a calculation.
But on the other hand if you start with this side of the solution and you progress this way keeping initial conditions in the neighborhood of the steady-state solutions here, then you will see that you will move along this path and there will be an upward jump and it will be like this, so if you ignore now presence of non-linearity you may think that this is the resonant amplitude frequency whereas it will be this and that can lead to un-conservative estimate of the response.

Stability of solutions

$$u(t) = x_0(t) + \xi(t)$$

$$\Rightarrow \ddot{x}_0(t) + \ddot{\xi}(t) + c\left\{\dot{x}_0(t) + \dot{\xi}(t)\right\} + \omega_n^2\left\{x_0(t) + \xi(t)\right\} + \mu\left\{x_0(t) + \xi(t)\right\}^3$$

$$= F\cos(\omega t + \phi)$$

$$\Rightarrow \ddot{\xi} + c\dot{\xi} + \xi(t)\left\{\omega_n^2 + 3\mu A^2\cos^2\omega t\right\} = 0$$

Time varying system with periodic coefficient

Use Floquet's theory to infer stability of the solutions.

Steady state solutions can be multivalued.

The steady states are functions of initial conditions.

Now how do you see, in regions where there are multiple solutions or even elsewhere how do you know that the solutions are realizable, so what we can do we did this stability analysis earlier, what we will do is we will perturb the assume solution by a small perturbation, substitute back into this, and we get now a time varying system where the assumed periodic solution appears as a coefficient, so for a given value of A from this graph, for a given pair of values of omega and A you come here and do a Floquet’s analysis and find out whether the perturbations grows in time or not, so that establishes whether the assume solution is stable or not.

Alternative approach (A H Nayfeh and D T Mook, 1979, Nonlinear oscillations, John Wiley, New York)

$$\ddot{x} + 2\varepsilon\mu\dot{x} + \omega_0^2 x + \varepsilon\alpha x^3 = F\cos\Omega t;\ x(0) = x_0\ \&\ \dot{x}(0) = \dot{x}_0$$

$$\Omega \approx \omega_0$$

$$\Omega = \omega_0 + \underbrace{\varepsilon\sigma}_{\text{Detuning}}$$

$$\ddot{x} + 2\varepsilon\mu\dot{x} + \omega_0^2 x + \varepsilon\alpha x^3 = \varepsilon k\cos\Omega t$$

Method of multiple scales $\Rightarrow$

$$x(t) = a(t)\cos\left[\omega_0 t + \beta(t)\right]$$

$$a' = -\mu a + \frac{1}{2}\frac{k}{\omega_0}\sin(\sigma\varepsilon t - \beta)$$

$$a\beta' = \frac{3}{8}\frac{\alpha}{\omega_0}a^3 - \frac{1}{2}\frac{k}{\omega_0}\cos(\sigma\varepsilon t - \beta)$$

Let $\gamma = (\sigma\varepsilon t - \beta)$

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So if there are multiple steady-state solutions possible then which one will be realized is a question that we have to answer, and in fact it turns out that in regions where there are multiple steady-state solutions the responses will be dependent on initial conditions, so this again is a newer feature.

Now there is another way of looking at this problem I will quickly run through this, we again consider a similar system, system with cubic nonlinearity and I will seek the solution in the neighborhood of capital Omega being in the neighborhood of omega naught, so what we do is I perturb omega naught by a small parameter known as detuning, okay, this is called as detuning, epsilon is a small parameter. Now I will arrange the amplitude of excitations and damping in this manner, and this method is known as method of multiple scales the discussion on this is not focused on how to implement the method but what the method tells us if you do it, so I will skip the details of this implementation of the solution, but what I will show is in this method I get solution in the form A(t) cos omega naught T + beta(t), where A and B are slowly varying functions of time which are governed by these 2 differential equation A and beta, okay.

$$\lim_{t\to\infty} a' \to 0 \,\&\, \lim_{t\to\infty} \beta' \to 0 \Rightarrow$$

$$\left.\begin{aligned} -\mu a + \frac{1}{2}\frac{k}{\omega_0}\sin\gamma = 0 \\ \frac{3}{8}\frac{\alpha}{\omega_0}a^3 - \frac{1}{2}\frac{k}{\omega_0}\cos\gamma = 0 \end{aligned}\right\} \Rightarrow \underbrace{\left[\mu^2 + \left(\sigma - \frac{3}{8}\frac{\alpha}{\omega_0}a^2\right)^2\right]a^2 = \frac{k^2}{4\omega_0^2}}_{\text{Multi-valued frequency response function}}$$

Stability

Investigate the fixed points and stability of the following system of equations:

$$a' = -\mu a + \frac{1}{2}\frac{k}{\omega_0}\sin(\sigma\varepsilon t - \beta)$$

$$a\beta' = \frac{3}{8}\frac{\alpha}{\omega_0}a^3 - \frac{1}{2}\frac{k}{\omega_0}\cos(\sigma\varepsilon t - \beta)$$

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Now I can rewrite this equation in this form and if a steady state exists you can see that A and beta should become constant, so in which case A prime and B prime must be equal to 0, so the condition for steady state is that these are 0, these are nothing but fixed points of these 2 differential equations, so corresponding to each fixed point of the equation for amplitude and phase there is one steady state possible, and this again turns out to be multiple valued, and how do we study stability here, what we do is we study the stability at the fixed point that also we have discussed earlier, so in these 2 approaches, in the first approach we are using Floquet's theory and the system that we are studying is a system with time-varying coefficients, whereas here in the next approach we get the steady state as fixed points of certain simplified set of differential equations, and we investigate the stability of the solution by studying the stability of the fixed points.

Importance of nonlinear analysis

- Earthquake engineering:
  - Structures are to be designed to display controlled inelastic responses; there are certain preferred modes of failures and certain failure modes are not preferred.
  - Use of snubbers, restraint devices, isolators, and nonlinear energy dissipation devices (e.g., slotted bolt connections)
- Wind engineering: wind structure interactions; across wind oscillations; galloping.
- Materials like concrete and soil display nonlinear behaviour even at low values of strains. Differing behaviour in tension and compression. Response depends upon entire time history, duration over which the load is applied and ambient effects such as temperature.
- Vibration of cracked structures.
- Study of failures.
- Loss of stability
- Prototype testing using nonlinear FE models (e.g., crash analysis in automotive design, simulation of drop test in electronics industry)

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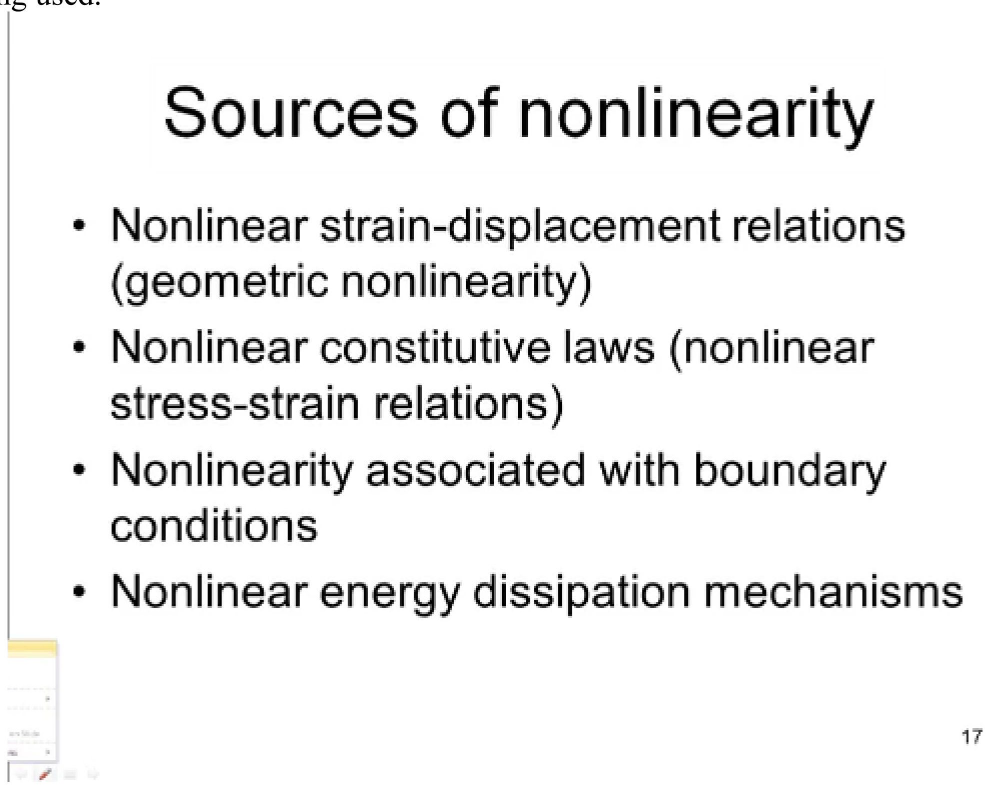
Now this gives you a kind of a brief overview of what are the qualitative feature that you could expect in a typical nonlinear system, this is just a tip of a iceberg, there are many more complicating issues, but given that you have an understanding of linear system behavior, this is a list of items that you should start thinking about.

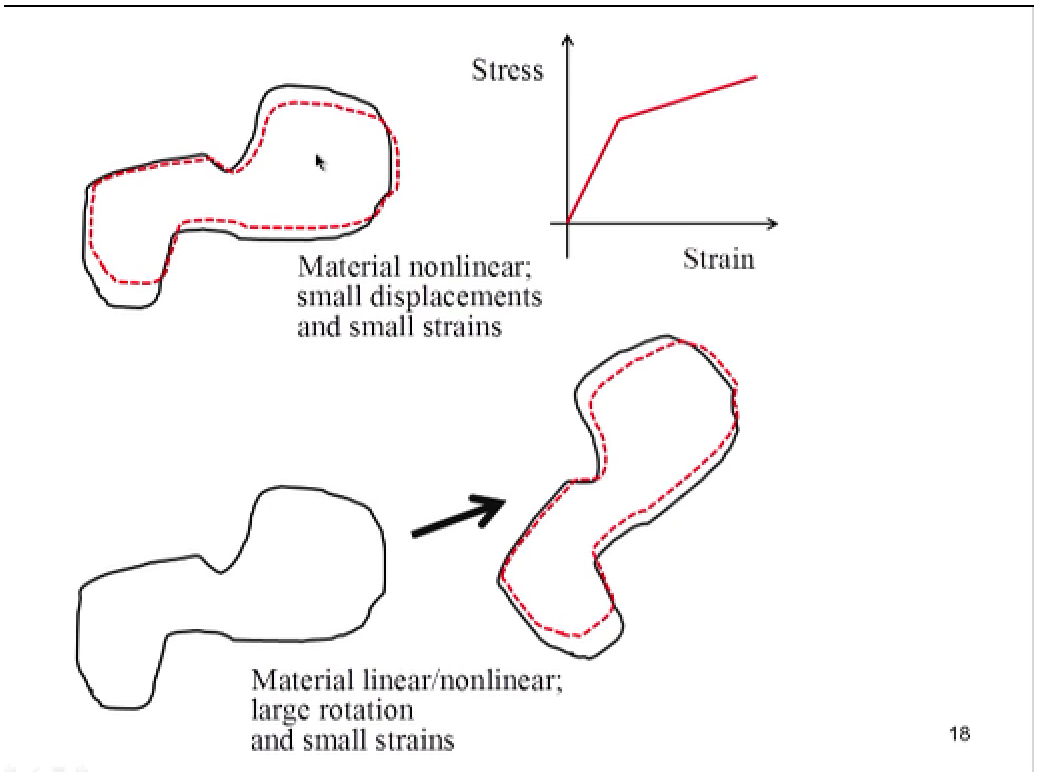
Now why nonlinear analysis is important? There are many reasons, although many engineering systems are designed to behave linearly, one might think that nonlinearity is not that important in engineering design, but that is not factually correct, for example in earthquake engineering we design the structures to display controlled in elastic response, there are certain preferred modes of failures and certain failure modes that are not preferred that means we design the structure to fail in a particular mode, for example we want to have strong columns and weak beams that means if a multi-story building is going to fail under an earthquake, the failure should begin with failure of beams, that is slabs and things like that, and not for example the ground floor column, if a ground floor column fails no matter how strong the structure is the whole structural collapse.

Then similarly for industrial structures like piping systems and things like that we use certain supporting devices known as snubbers, and nonlinear energy dissipation devices and things like that, these again you know impart nonlinearity to the system, for example heat exchanger or pipe in a nuclear reactor conveys fluid at hot temperature, so in the normal course of its operation it should be able to withstand the thermal loads because of the conveyance of hot fluids, but in the event of an earthquake there will be additional support motions, to cope up with thermal loads the structure needs to be flexible but that very flexibility brings the natural frequency of the system into the range of earthquake excitation frequencies, so supports likes snubbers in the event of an earthquake they lock the structure and reduce the spans of the piping and increase the natural frequency so that it attracts lesser seismic loads.

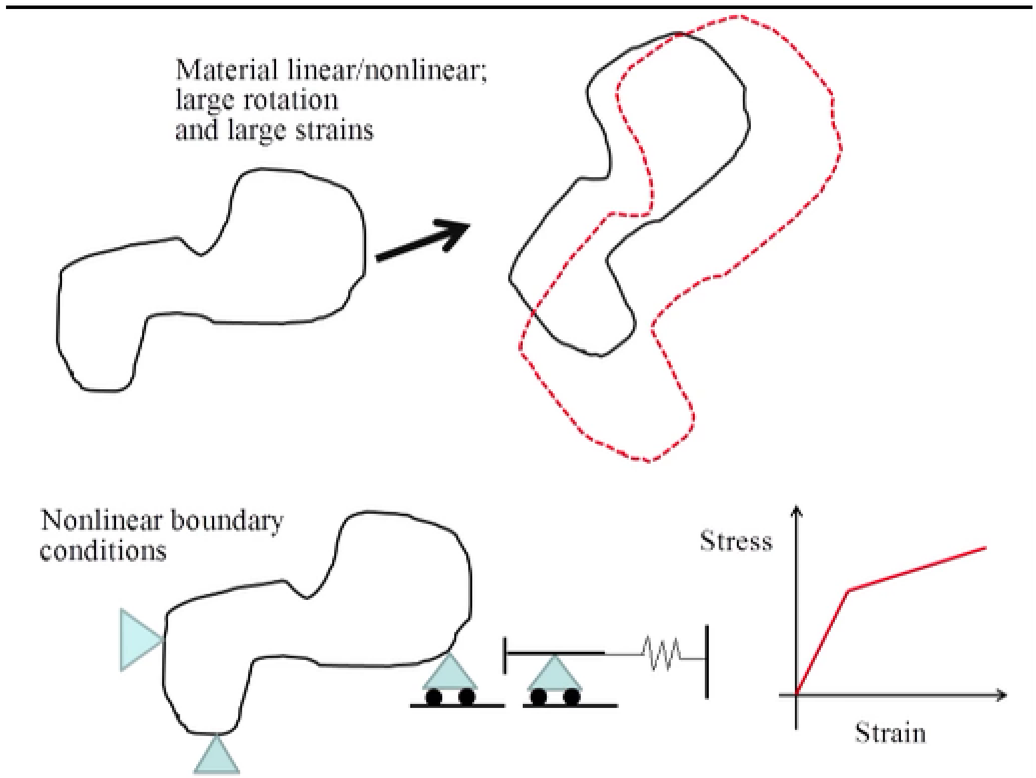
Similarly in wind engineering the interaction between structures and the flow past vibrating structures induces highly nonlinear forcing functions that I will just briefly mention in the next one of the next slides. Materials like concrete and soil which are important in civil engineering application display nonlinear behavior even at low strengths, so the response need not be very large before nonlinearity switched on, they also display differing behavior in tension and compression, then response depends upon entire time history duration over which the load is applied and ambient effects such as temperature.
Next if a structure is cracked and it starts vibrating, the closing and opening of cracks induces certain type of nonlinearities that we have to think of, and as engineers we are always interested in steady of failures because we want to design structures to prevent failures, so to understand failure we have to enter non-linear regimes of responses, a linear system in principle can never fail you know it can withstand infinite stresses, there could be problems of loss of stability this we have discussed like buckling and snap through and so on and so forth. In many application, modern application prototype testing using nonlinear FE models have become popular, for example crash analysis in automotive design, simulation of drop tests in electronic industry etcetera, instead of performing costly experimental studies cheaper finite element simulations are being used.



The sources of non-linearity in structural mechanics problems can originate from nonlinear strain displacement relations, this type of nonlinearity is called geometric nonlinearity, or the relation, the constitutive relations typically relating stress and strain it could be temperature as well could be nonlinear, then non-linearity associated with boundary conditions as in contact related problems and so on and so forth, and similarly nonlinear energy dissipation mechanisms, these are some sources of nonlinearity. So a structure can behave in different



ways, for example the material of the structure could be nonlinear but there could be small displacements, so the typical force response diagram may look like this, so the deformations are not large but the material of the structure has already entered a nonlinear regime so it is nonlinearity in constitutive relations. In the other type of behavior material may be linear or nonlinear but large rotations will occur and there are small deformations, so there are small strains but large rotations.



Now of course one can have large rotations as well as large strains, so these are the most difficult problems to deal with, there could be special conditions like for example the same structure is supported through a gap and a loaded spring like this, so till the time this gap is negotiate, the spring won't come into action and moment that happens there will be a by linear nonlinearity, and this type of boundary conditions create what are known as nonlinear boundary condition.

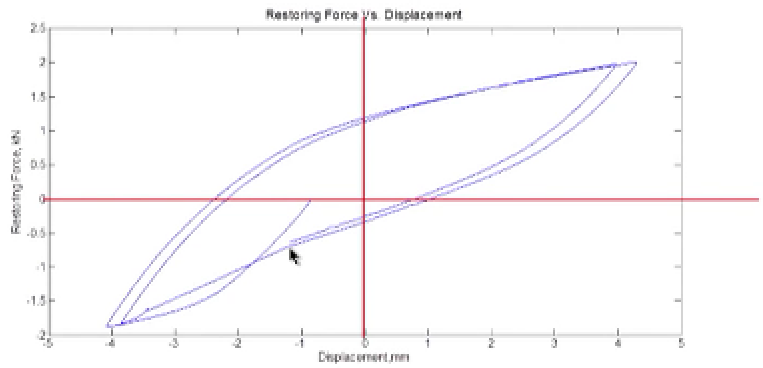
Nonlinearly elastic systems and systems with hereditary nonlinearities

$$m\ddot{x}+c\dot{x}+kx+g\left[x(t),\dot{x}(t),t\right]+h\left[x(\tau),\dot{x}(\tau);0\le\tau\le t\right]=f(t);$$

$x(0)$ & $\dot{x}(0)$ specified

$g\left[x(t),\dot{x}(t),t\right]=$ Nonlinear function of instantaneous values of $x(t)$ & $\dot{x}(t)$.

$h\left[x(\tau),\dot{x}(\tau);0\le\tau\le t\right]=$ Nonlinear function of response time histories up to time t.



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If you write the equation of motion for a simple case again say MX double dot + CX + a nonlinear term, the function G is a function of instantaneous value of displacement and velocity, this is inelastic, this is non-linearly elastic system, that means upon unloading this force would go to 0, and there won't be any residual displacement, the loading and unloading path will be tracing each other. On the other hand there could be forces of nonlinear forces which are dependent on entire history of the response up to the current time, so this typically originates from material nonlinearity, and this originates from geometric nonlinearity, so a typical force displacement you know graph for a system exhibiting so called hereditary nonlinear behavior or hysteretic nonlinear behavior is shown here, so this is something like a force and a displacement, and for a given value of displacement you can see that there are multiple forces possible, which one will be realized depends on how you have reached that point, that means it depends on the history of the response up to that time, so this type of systems are more difficult to analyze as you could expect then this model, if this are difficult to model as well as to more difficult to analyze.

Nonlinear effects in wind induced oscillations

e.g., Across wind oscillations of chimneys

$$m\ddot{x} + c\dot{x} + kx - k_a\dot{x}\left(1 - \varepsilon\dot{x}^2\right) = F\left(t\right)$$

- Limit cycles
- Entrainment

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Now there are nonlinear effects in I mentioned about wind engineering problems so if you imagine that there is a chimney which is subjected to say flow, this is planar view, the chimney is something like this, and the flow is taking place like this, the flow past this chimney will create a pressure field on the object, and if you integrate the pressure field over the surface area you get a force which can be resolved in line and across the flow directions, and we can show that for certain flow velocities and certain geometries there will be vertices that will be shed, and because of that there will be dynamic excitations predominantly harmonic on this chimneys, and these are known as across wind oscillations, so if the chimney is flexible the nature of these excitations become fairly complicated, so flow past a flexible object can create severe interaction, fluid structure interaction and for that type of systems there will be a special type of non-linearity as shown here I discuss this when we discuss limit cycle oscillations in one of the earlier lectures, you can see here that for small X dot, the term inside the parentheses will be less than 1, and the net effect of this term will be negative, it is like a negatively damped system and small oscillations tend to grow.

For large amplitude oscillations the term here becomes negative and induces a positive sign on this, and large amplitudes tend to decay, so in free vibration the system displays what is another limit cycle behavior, so that is a periodic solution, it is an isolated periodic solution, and when such systems are driven by external excitations there can be complex interactions between the periodic solutions which are highly nonlinear periodic solutions in free vibration and the components due to external excitation, and there are very many complex behavior something known as entrainment and things like that these are again characterizes nonlinear resonances, and if you want to understand peak response in such systems you have to understand the basic entrainment phenomena.

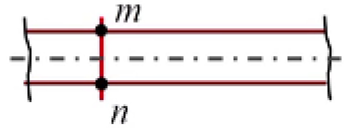
Objectives

- A brief review of background concepts
- Present a flavour of treatment of nonlinear structural mechanics problems using finite element method.
- Focus on geometrically nonlinear problems

Now so with this brief background we will come to the objectives of the discussion on nonlinear systems, so the idea here is the subject is very vast, it cannot be covered in few lectures that I am planning to you know dedicate to this topic, the idea here is to provide a brief review of background concepts and present a flavor of treatment of nonlinear structural mechanics problems using finite element method, the focus is on geometrically nonlinear problems, we will not be talking about material non-linearity by and large we will be focusing on geometrically nonlinear problems.

Planar beam element

- Large tranverse displacements
- Small strains
- Moderate rotations



Changes in geometry due to deformation is not accounted for in defining stress.

Euler-Bernoulli beam: Upon deformation *mn* remains straight and normal to the neutral axis and its length does not change.

$$\left.\begin{aligned} u_1 &= u_0(x) - z\frac{dw_0}{dx} \\ u_2 &= 0 \\ u_3 &= w_0(x) \end{aligned}\right\} \begin{aligned} &u_1, u_2, u_3 = \text{displacement along } x, y, z \text{ reespectively} \\ &u_0(x) = \text{axial displacement of a point on neutral axis} \\ &w_0(x) = \text{transverse displacement of a point on neutral axis} \end{aligned}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}$$

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We can begin with whatever background we have without asking too many newer questions, for example if you are talking about a planar Euler-Bernoulli beam, if we assume that there are large transverse displacement but small strains and there are moderate rotations, then the changes in geometry due to deformation need not be accounted for while defining stress, so the point here is if a structure undergoes large amplitude oscillations, the geometry of the structure also would change, so that needs to be taken into account while defining stress, so that modifies many of the basic formulations that we'll use for analyzing this system, but suppose we don’t get into that under these conditions probably one can overlook those complications, then we will be able to proceed with whatever background we have, for example if we assume the invoke the Euler-Bernoulli hypothesis that upon deformation the line segments MN remains straight and normal to the neutral axis, and its length does not change, we get the displacement field, this we have discussed a few times where U1, U2, U3 are displacement along X, Y, Z respectively, and U naught(x) and W naught(x) are the displacement of the point on neutral axis.

The strain displacement relations and that we will be using are nonlinear, we are not using the infinitesimal definition of the strain we are including the nonlinear terms as well, this I have discussed, the definitions I have discussed in one of the previous lectures, so the nonlinear

$$u_1 = u_0(x) - z\frac{dw_0}{dx}; u_2 = 0; u_3 = w_0(x)$$

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2\right] = \left(\frac{du_0}{dx} - z\frac{d^2 w_0}{dx^2}\right) + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_2}\right)^2 + \left(\frac{\partial u_2}{\partial x_2}\right)^2 + \left(\frac{\partial u_3}{\partial x_2}\right)^2\right] = 0$$

$\left(\frac{\partial u_1}{\partial x_1}\right)^2$ ignored (small strains)

$\left(\frac{\partial u_3}{\partial x_1}\right)^2$: rotation included

$$\varepsilon_{33} = \frac{\partial u_3}{\partial x_3} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_3}\right)^2 + \left(\frac{\partial u_2}{\partial x_3}\right)^2 + \left(\frac{\partial u_3}{\partial x_3}\right)^2\right] = 0$$

$$2\varepsilon_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_2} = 0$$

$$2\varepsilon_{13} = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} + \frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_3} + \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_3} = 0$$

$$2\varepsilon_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} + \frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_3} + \frac{\partial u_2}{\partial x_2}\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\frac{\partial u_3}{\partial x_3} = 0$$

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strain displacement relations are again displayed here the quantities that appear in the red are the nonlinear terms, so for this assume displacement field I get epsilon1 1, I will retain the first term dou U1/dou X1 but this quadratic term we are not including, because we assume strains are small but dou U3/dou X1 is a rotation that is included, so I get this term for epsilon 1 1, that mean dou U1/dou X1 whole square is ignored, but dou U3/dou X1 is retained, now only this strain will be nonzero, all other strains will be 0. And stress again I assume isotropic elastic

$$u_0(x) \to u(x,t); w_0(x) \to w(x,t)$$

$$u_1(x,t) = u(x,t) - z\frac{\partial w_0}{\partial x}; u_2 = 0; u_3(x,t) = w(x,t)$$

$$\varepsilon_{11} = \left(\frac{\partial u}{\partial x} - z\frac{\partial^2 w_0}{\partial x^2}\right) + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2$$

$$\sigma_{11} = E\varepsilon_{11}$$

$$U = \frac{1}{2}\int_V E\varepsilon_{11}^2 dV \;\&\; T = \frac{1}{2}\int_V \rho\left(\dot{u}^2 + \dot{w}^2\right)dV$$

$$U = \frac{1}{2}\int_A\int_0^l E\left\{\left(\frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}\right) + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right\}^2 dAdx$$

$$= \frac{1}{2}\int_A\int_0^l E\{\left(\frac{\partial u}{\partial x}\right)^2 + z^2\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \frac{1}{4}\left(\frac{\partial w}{\partial x}\right)^4 - 2z\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x^2} + \frac{\partial u}{\partial x}\left(\frac{\partial w}{\partial x}\right)^2$$

$$-z\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2\}dAdx$$

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material, so stress is related to strain through this relation, I will be able to write the expression for strain energy and the kinetic energy, using the assumed you know form of displacement and the consequence strains, and I get strain energy in this form, and if I now use the assume displacement form I get this, you see now there are quadratic terms in displacement in the expression for strain energy, so the terms appearing in red are the newer terms.

$$U = \frac{1}{2}\int_A\int_0^l E\{\left(\frac{\partial u}{\partial x}\right)^2 + z^2\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \frac{1}{4}\left(\frac{\partial w}{\partial x}\right)^4 - 2z\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x^2} + \frac{\partial u}{\partial x}\left(\frac{\partial w}{\partial x}\right)^2$$

$$-z\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2\}dAdx$$

Consider beam to possess symmetric cross section.

$$\Rightarrow U = \frac{1}{2}\int_0^l AE\left(\frac{\partial u}{\partial x}\right)^2 dx + \frac{1}{2}\int_0^l EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx$$

$$\underbrace{+\frac{1}{8}\int_0^l AE\left(\frac{\partial w}{\partial x}\right)^4 dx + \frac{1}{2}\int_0^l AE\frac{\partial u}{\partial x}\left(\frac{\partial w}{\partial x}\right)^2 dx}_{\text{New terms due to presence of nonlinearity}}$$

$$T = \frac{1}{2}\int_V \rho\left(\dot{u}^2 + \dot{w}^2\right)dV = \frac{1}{2}\int_0^l m\left(\dot{u}^2 + \dot{w}^2\right)dx$$

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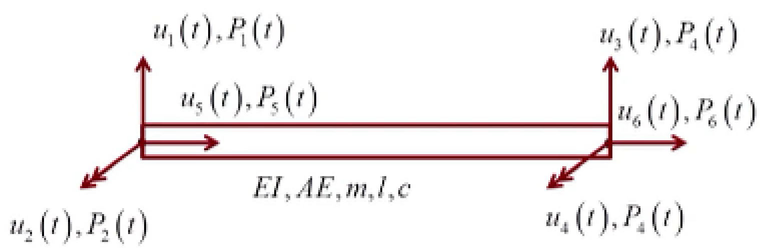
Now if we assume the beam to possess symmetric cross-section, the terms appearing in red are not the, I mean it does not convert a nonlinear terms but the terms which will go to 0 if you assume beam section to be symmetric, so if that happens these 2 terms will go to 0, and I am left with U which is this, and the second set of terms are the new terms due to presence of nonlinearity. Kinetic energy is given by this I can write the Lagrangian, and again I will take a 2

$$\boxed{\begin{aligned} L = {} & \frac{1}{2}\int_0^l m\left(\dot{u}^2+\dot{w}^2\right)dx-\frac{1}{2}\int_0^l AE\left(\frac{\partial u}{\partial x}\right)^2 dx-\frac{1}{2}\int_0^l EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx \\ & -\frac{1}{8}\int_0^l AE\left(\frac{\partial w}{\partial x}\right)^4 dx-\frac{1}{2}\int_0^l AE\frac{\partial u}{\partial x}\left(\frac{\partial w}{\partial x}\right)^2 dx \end{aligned}}$$



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noded element with 3 degrees of freedom per node, and we will again interpolate the axial displacement using linear interpolation functions and transverse displacement using Hermite polynomials I get the set of equations, and if I run through the Lagrange's equation I will get

$$L = \frac{1}{2}\int_0^l m\left(\dot{u}^2 + \dot{w}^2\right)dx - \underbrace{\frac{1}{2}\int_0^l AE\left(\frac{\partial u}{\partial x}\right)^2 dx - \frac{1}{2}\int_0^l EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx}_{\text{Lead to the structural element mass and stiffness matrices deduced earlier}}$$

$$\underbrace{-\frac{1}{8}\int_0^l AE\left(\frac{\partial w}{\partial x}\right)^4 dx - \frac{1}{2}\int_0^l AE\frac{\partial u}{\partial x}\left(\frac{\partial w}{\partial x}\right)^2 dx}_{\text{Newer terms due to nonlinearity}}$$

Consider $L_1 = \frac{1}{8}\int_0^l AE\left(\frac{\partial w}{\partial x}\right)^4 dx = \frac{1}{8}\int_0^l AE\left\{\sum_{i=1}^4 u_i(t)\phi_i'(x)\right\}^4 dx$

$$\frac{\partial L_1}{\partial u_k} = \frac{1}{2}\int_0^l AE\left\{\sum_{i=1}^4 u_i(t)\phi_i'(x)\right\}^3 \phi_k'(x)dx = \frac{1}{2}\sum_{i=1}^4\sum_{j=1}^4\sum_{m=1}^4 u_i(t)u_j(t)u_m(t)I_{ijmk}$$

$$I_{ijmk} = \int_0^l\int_0^l\int_0^l AE\phi_i'(x)\phi_j'(x)\phi_m'(x)\phi_k'(x)dx; k = 1,2,3,4$$

the required equations of motion, so if you examine the Lagrangian the first few terms were already encountered when you do the linear analysis, so they lead to typical mass and stiffness matrices that we have already derived, and the newer terms will originate from these 2 nonlinear term, so I suppose if you will focus on one of this, suppose the first term 1/8, AE, Dou W/Dou X 4 DX and substitute for the assumed displacement form I get this, and when we run the Lagrangian on this, we will get cubic terms here, and this I IJMK is a new integral that has to be determined, so here we won't get matrices, we will get vectors.

$$\text{Similarly, consider } L_2 = \frac{1}{2}\int_0^l AE\frac{\partial u}{\partial x}\left(\frac{\partial w}{\partial x}\right)^2 dx$$

$$L_2 = \frac{1}{2}\int_0^l AE\left(\sum_{i=5}^{6} u_i(t)\phi_i'(x)\right)\left(\sum_{i=1}^{4} u_i(t)\phi_i'(x)\right)^2 dx$$

$$\frac{\partial L_2}{\partial u_k} = \int_0^l AE\left(\sum_{i=5}^{6} u_i(t)\phi_i'(x)\right)\left(\sum_{i=1}^{4} u_i(t)\phi_i'(x)\right)\phi_k'(x)dx; k = 1,2,3,4$$

$$= \sum_{i=5}^{6}\sum_{j=1}^{4} u_i(t)u_j(t)\int_0^l AE\phi_i'(x)\phi_j'(x)\phi_k'(x)dx; k = 1,2,3,4$$

$$= \sum_{i=5}^{6}\sum_{j=1}^{4} u_i(t)u_j(t)J_{ijk}\, k = 1,2,3,4$$

$$J_{ijk} = \int_0^l AE\phi_i'(x)\phi_j'(x)\phi_k'(x)dx; i = 5,6; j,k = 1,2,3,4$$

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So similarly the other term involving the other nonlinear term which is this, if we do that again we get newer nonlinear terms which could be quadratic or cubic, so I again name some of the

$$L_2 = \frac{1}{2}\int_0^l AE\left(\sum_{i=5}^{6} u_i(t)\phi_i'(x)\right)\left(\sum_{i=1}^{4} u_i(t)\phi_i'(x)\right)^2 dx$$

$$\frac{\partial L_2}{\partial u_k} = \frac{1}{2}\int_0^l AE\left(\sum_{i=5}^{6} u_i(t)\phi_i'(x)\right)\left(\sum_{i=1}^{4} u_i(t)\phi_i'(x)\right)^2 \phi_k'(x)dx; k = 5,6$$

$$= \frac{1}{2}\int_0^l AE\sum_{i=5}^{6}\sum_{r=1}^{4}\sum_{s=1}^{4} u_i(t)u_r(t)u_s(t)\phi_i'(x)\phi_r'(x)\phi_s'(x)\phi_k'(x)dx; k = 5,6$$

$$= \frac{1}{2}\sum_{i=5}^{6}\sum_{r=1}^{4}\sum_{s=1}^{4} u_i(t)u_r(t)u_s(t)K_{irsk}$$

$$K_{irsk} = \int_0^l AE\phi_i'(x)\phi_r'(x)\phi_s'(x)\phi_k'(x)dx; r,s = 1,2,3,4; i,k = 5,6$$

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integrals that appear here through notation KIRSK etcetera, so the final form of equation of

Form of the element equation of motion

$$[M]_e\{\ddot{u}\}_e+[K]_e\{u\}_e+\{\text{Vector of quadratic and cubic terms in } u(t)\ \}_e=0$$

Remarks

- Assembly of element level matrices and vectors can be done as before to obtain the global equations of motion.
- Derivation of external forces and imposition of BCs again follows the earlier developed procedure.
- The resulting equations of motion would be of the form
$M\ddot{u}+C\dot{u}+Ku+g[u]=F(t);u(0)\,\&\,\dot{u}(0)$ specified.
- These equations can be integrated using numerical procedures discussed earlier (see Lecture 16).

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motion at the element level will be ME UE double dot + KE UE + vector of quadratic and cubic terms in U(t), okay and the multipliers that appear here are properties of the structural system, okay so this is the equation, so in free vibration this will be 0.

Now energies in different elements can be added, so Lagrangian can be constructed for the built-up structure using the approach that we have used, there is no change in that aspect of our work, so assembly of element level matrices and vector can be done as before to obtain global equations of motion, then derivation of external forces and imposition of boundary conditions again follows the earlier developed procedure there is nothing new there, the resulting equation of motion for the structure after imposing boundary conditions and after computing the external forces will be of this form in this case, so this G(u) is the nonlinear term that is arising in this model, and this will as we have seen it will have cubic and quadratic terms in U.

Now we've already discussed how to solve this equation, see for example during earlier lecture, lecture number 16 we have developed a you know operator splitting methods and other methods to tackle these equations so that can be used, I am not going to discuss the solution procedures at this juncture again.

Timoshenko beam element

$$u_1 = u_0(x) + z\phi(x)$$

$$u_2 = 0$$

$$u_3 = w_0(x)$$

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2\right] = \left(\frac{du_0}{dx} + z\frac{d\phi}{dx}\right) + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2$$

$$\varepsilon_{22} = 0$$

$$\varepsilon_{33} = 0$$

$$2\varepsilon_{12} = 0$$

$$2\varepsilon_{13} = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} = \phi(x) + \frac{dw_0}{dx}$$

$$2\varepsilon_{23} = 0$$

$$U = \frac{1}{2}\int_V (\sigma_{11}\varepsilon_{11} + \varepsilon_{13}\sigma_{13})\, dV$$

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Similar analysis can be done for Timoshenko beam, this I leave as an exercise so this is the assume displacement field and this will be the strain, and there will be one more strain which is epsilon 1 3, and you will have to use this expression and construct the Lagrangian, now you have to include kinetic energy due to translation and the rotation, the rotor inertial effect also has to be included here, so once you do that following a similar procedure you will be able to derive the equation for Timoshenko beam.

How about a more general theory?

- Allow measures of strain and stress to be defined consistent with deformations.
- Allow for material nonlinearity

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Now this is alright, but how about a more general theory? Now in a more general theory we need to allow for measures of strain and stresses to be defined consistent with the deformations, and also we need to allow for material nonlinearity, this will not be doing but this we will discuss now, as a structure undergoes large deformation the cross sectional properties might change, so when we define stress, we will have questions on some kind of a force divided by an area, which area are you talking about, is it the structure in his un-deformed configuration or in the deformed configuration, if you say it the area is to be computed based on deform configuration when you are defining stress you would not know, what the deform configuration is, right, so and then similarly when you define strains, you have to think about large rotations, I will show either during this lecture or the next lecture that if you use in the infinitesimal definition for strains, a structure undergoing rigid rotation the strains won't be 0, so that is not acceptable, so say you can't stick with infinitesimal strain definition that also needs to be modified, so some of these issues need to be addressed and to do all this systematically we need

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to get into the subject of continuum mechanics, and develop all the language and the notations in a systematic way before we even we can address as a simple class of problems, this subject is very vast as I already said, we are not going to discuss many aspects of this, we will not be discussed, I have given a list of references which cover this subject in good detail and I will be using some of these references during the lecture.

Notations

- Indicial notations
- Algebraic notations
- Matrix notations
- Tensor notations

Now the subject of nonlinear analysis of structures is mathematically lot more refined than a linear analysis, there are many issues associated with notations and as I already said definition of stress, strain, and the balance loss, all of them we need to revisit so there are issues about notations, there are four sets of notations that one has to use, one has to understand, to

Indicial notations

- A set of variables $x_1, x_2, \cdots, x_n$ is denoted as x_i.

The range of values taken by the index i, needs to be specified.

Typically, $i = 1, 2, 3$.

- Repeated indices implies summation

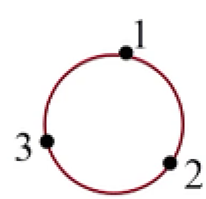
$\alpha = \sum_{i=1}^{n} a_i x_i$ is written as $\alpha = a_i x_i$ $(i = 1, 2, \cdots, n)$ (Note: $\alpha = a_i x_i = a_s x_s$)

$U = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} u_i u_j$ is written as $U = \frac{1}{2} k_{ij} u_i u_j$ $(i, j = 1, 2, \cdots, n)$

- Kronecker delta $\delta_{ij} = \begin{matrix} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{matrix}$

$ds^2 = dx_1^2 + dx_2^2 + dx_3^2$ is written as $ds^2 = \delta_{ij} dx_i dx_j$

- Permutation symbol ε_{ijk}



$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$

$\varepsilon_{213} = \varepsilon_{132} = \varepsilon_{321} = -1$

$\varepsilon_{111} = \varepsilon_{222} = \varepsilon_{333} = \cdots = 0$

understand literature on the subject the indicial notations I will quickly review this, a set of variables X1, X2, XN is simply denoted as XI, that means the indicial, here in the name refers to the index to the variables that we assign, the range of values taken by the index I needs to be specified, typically I runs from 1, 2, 3, if they are in a Cartesian space but it need not be so. Now repeated indices implies summation, for example if I have a term like alpha = I = 1 to N or AI XI, this is simply written as alpha AI XI and I, I have to specify what range it has to be used, I is 1 to N, see this I is a dumb index, instead of writing AI XI as well I can write AS XS, so that I or S is not very important, it is a dummy index. Similarly a term like this is written as 1/2 KIJ UI UJ, you can see that I and J are repeated, therefore a summation on I and J are implied from I = 1 to N, this Kronecker delta is a symbol that is used delta IJ is 1, If I = J otherwise it is 0, so using that for example the length of an infinitesimal element DS square which is given by DX1 square + DX2 square + DX square is written as delta IJ, DXI, DXJ, the symbol known as permutation symbol epsilon IJK it is defined as shown here, so you can keep this figure in mind, if you run from 1, 2, 3, or 2, 3, 1, or 3, 1, 2 epsilon IJK is 1. On the other hand you run in the other way 1, 3, 2, 3, 2, 1 or 2, 1, 3 it is – 1, for all other combinations it is 0.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}; |A| = \varepsilon_{ijk} a_{i1} a_{j2} a_{k3}$$

- $\varepsilon - \delta$ identity: $\varepsilon_{ijk}\varepsilon_{rst} = \delta_{js}\delta_{kt} - \delta_{jt}\delta_{ks}$
- Differentiation

$$f \equiv f(x_1, x_2, \cdots, x_n)$$

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \cdots + \frac{\partial f}{\partial x_n}dx_n \text{ is written as } df = \frac{\partial f}{\partial x_i}dx_i \quad (i = 1, 2, \cdots, n)$$

- The , symbol

Consider

$$f_1 \equiv f_1(x_1, x_2, x_3), f_2 \equiv f_2(x_1, x_2, x_3), f_3 \equiv f_3(x_1, x_2, x_3)$$

$$f_{i,j} = \frac{\partial f_i}{\partial x_j}$$

Similarly $\sigma_{ij,k} = \frac{\partial \sigma_{ij}}{\partial x_k}$

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Now if A is a 3 by 3 matrix the determinant of A can be written using the permutation symbol in this way, there is a small exercise, there is an identity epsilon delta identity you can show that epsilon IJK and kronecker delta related through this identity. Now there is a symbol for differentiation, suppose if you consider a function which is F(x1, x2, up to Xn) and if DF is wha I am looking at, it is given by this, this is written compactly as DF = dou F/dou XI into DXI, the index I repeats and it has to be summed over 1 to N, and this comma symbol that is a differentiation symbol, if I again have this function F1, F2, F3, to be functions of X1, X2, X3, if I write F(i,j) it is dou FI/dou XJ, similarly sigma IJ,K is delta sigma IJ/dou XK, this comma K means you have to differentiate with respect to the K-th variable, so this is like a mathematical shorthand for writing the long expressions, physics will get buried inside these notations, so it is not very convenient if you are understanding the subject for the first time, but it is very useful in compactly expressing the results.

Algebraic notations

Vectors and tensors are represented by single letters (bold face)

$$x = (x_1, x_2, x_3)$$

$$y = (y_1, y_2, y_3)$$

$$x \bullet y = x_1 y_1 + x_2 y_2 + x_3 y_3$$

$$z = x \times y = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = e_1 z_1 + e_2 z_2 + e_3 z_3$$

$$z_i = \varepsilon_{ijk} x_j y_k$$

$$\text{grad} = \nabla = e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3} = e_i \frac{\partial}{\partial x_i}$$

$$f \equiv f(x_1, x_2, x_3) = \text{scalar function}$$

$$\text{grad } f = e_1 \frac{\partial f}{\partial x_1} + e_2 \frac{\partial f}{\partial x_2} + e_3 \frac{\partial f}{\partial x_3} = e_1 f_1 + e_2 f_2 + e_3 f_3$$

$$f_i = \frac{\partial f}{\partial x_i}$$

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The algebraic notations suppose I have a vector with components X1, X2, X3 and Y1, Y2, Y3, X dot Y is X1 Y1 + X2 Y2 + X3 Y3, and X x Y is given by this, and this is written as ZI is epsilon IJK XJ YK, then grad that is this delta inverted delta is this operation E1 dou/dou X1 + E2 dou/dou X2 + this it, this is written IE dou/XI, now this is a scalar function if you take a grad of a scalar function it is given by dou F/dou XI, that I-th component is this. Now if you apply the grad operation on a vector-valued function F we can define what is known as divergence which is Del dot F which is given by this, this is a dot symbol, the curl is delta x F and that is given by this.

$$\nabla = e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3} = e_i \frac{\partial}{\partial x_i}$$

Consider a vector valued function F

$$\text{div} F = \nabla . F = \left(e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3} \right) \bullet \left(e_1 F_1 + e_2 F_2 + e_3 F_3 \right)$$

$$= \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} \right) = \frac{\partial F_i}{\partial x_i}$$

$$z = \text{Curl } F = \nabla \times F = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{vmatrix} = e_1 z_1 + e_2 z_2 + e_3 z_3$$

$$z_i = \varepsilon_{ijk} \frac{\partial F_k}{\partial x_j}$$

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Now if you apply a grad function on a vector-valued function is this is written delta, F and this is given by this, and F itself is a vector therefore I had to write this, so you can see that grad of a

$$\nabla = e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3} = e_i \frac{\partial}{\partial x_i}$$

Consider a vector valued function F

$$\text{grad } F = (\nabla, F) = e_1 \frac{\partial F}{\partial x_1} + e_2 \frac{\partial F}{\partial x_2} + e_3 \frac{\partial F}{\partial x_3}$$

$$= e_1 \frac{\partial}{\partial x_1} \left(e_1 F_1 + e_2 F_2 + e_3 F_3 \right) + e_2 \frac{\partial}{\partial x_2} \left(e_1 F_1 + e_2 F_2 + e_3 F_3 \right)$$

$$+ e_3 \frac{\partial}{\partial x_3} \left(e_1 F_1 + e_2 F_2 + e_3 F_3 \right)$$

Thus grad F can be described by the following matrix

$$\begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_2}{\partial x_1} & \frac{\partial F_3}{\partial x_1} \\ \frac{\partial F_1}{\partial x_2} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_3}{\partial x_2} \\ \frac{\partial F_1}{\partial x_3} & \frac{\partial F_2}{\partial x_3} & \frac{\partial F_3}{\partial x_3} \end{bmatrix}$$

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vector valued function will have these gradients presented in their representation, in matrix

Matrix notations

$$x = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}; r^2 = x'x; \sigma = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{zy} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix}; \varepsilon = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{zy} \\ 2\varepsilon_{zx} \\ 2\varepsilon_{xy} \end{Bmatrix}; U = \frac{1}{2}\sigma'\varepsilon = \frac{1}{2}\varepsilon' C\varepsilon$$

No explicit mention of connective symbols (multiplication)

Tensor notations

- indices not shown
- applicable to Cartesian and other coordinate systems

$x_i y_i \equiv \mathrm{x.y}$ (dot denotes contraction of inner indices)

$\mathrm{A}_{ij} B_{ij} \equiv A : B$ (colon denotes contraction of a pair of repeated indices)

$\sigma_{ij} = c_{ijkl}\varepsilon_{kl} \equiv \sigma = C : \varepsilon$

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notations we arrange the components of vector in a column like this and even stress is arranged as a column like this, right, so stress is a tensor represented 3 by 3 matrix but in the so-called white notations, we write it as 6 x 1 vector, similarly strain, so the strain energy is written as 1/2 sigma transpose epsilon, so we represent the terms like this, for example R square is written as X transpose X and so on and so forth, so when doing this we don't write explicitly the connective symbol, for example when I say X transpose X, I am not putting in between any dot or a multiplication or any other symbol.

In tensor notations indices are not shown, this is applicable to Cartesian and other coordinate systems XI YI that means it is summation of is X1 Y1 + X2 Y2 + X3 Y3 is simply written X dot Y, AIJ, BIJ is written as A double dot B, this is a new symbol that we will use, colon denotes contraction of pair of repeated indices whereas a dot denotes, a single dot denotes contraction of inner indices whereas this double dot denotes contraction of pair of repeated indices, so this relation sigma IJ, CIJ, KL epsilon KL is a tensor notation is written as C double dot epsilon, so I have given some examples of writing different expressions in alternative

$$\underbrace{\phi.\mathrm{K}.\phi}_{\text{Tensor}} = \underbrace{\phi^t \mathrm{K} \phi}_{\text{Matrix}} = \underbrace{\phi_i K_{ij} \phi_j}_{\text{Indicial}}$$

$$\underbrace{\frac{1}{2}\varepsilon : \mathrm{C} : \varepsilon}_{\text{Tensor}} = \underbrace{\frac{1}{2}\varepsilon^t C \varepsilon}_{\text{Matrix}} = \underbrace{\frac{1}{2}\varepsilon_i C_{ij} \varepsilon_j}_{\text{Indicial}}$$

$$\underbrace{\sigma' = C\sigma C^t}_{\text{Matrix}} \quad \underbrace{\sigma'_{ij} = C_{im}\sigma_{mn}C_{jn}}_{\text{Indicial}}$$

$$\underbrace{\sigma_{ij,j} + b_i = 0}_{\text{Indicial}} \equiv \underbrace{D^t\sigma + b = 0}_{\text{Matrix}} \equiv \underbrace{\begin{matrix} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} + \dfrac{\partial \sigma_{13}}{\partial x_3} + b_1 = 0 \\ \dfrac{\partial \sigma_{21}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} + \dfrac{\partial \sigma_{23}}{\partial x_3} + b_2 = 0 \\ \dfrac{\partial \sigma_{31}}{\partial x_1} + \dfrac{\partial \sigma_{32}}{\partial x_2} + \dfrac{\partial \sigma_{33}}{\partial x_3} + b_3 = 0 \end{matrix}}_{\substack{\text{Full notation} \\ \text{Notation of last resort}}}$$

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notations, this you can examine, so if we have something like phi transpose K phi, in tensor notation it is written as phi dot K dot phi, in indicial it is phi I, KIJ, phi J, similarly you have 1/2 epsilon transpose C epsilon this is written as 1/2 epsilon double dot C double dot epsilon, whereas this epsilon I CIJ epsilon J, so on and so forth this, and equilibrium equations for elasticity is in the long hand in full notation form it is given by this, in matrix form it is given by this, in indicial is given by this. Actually the full notation is a notation of last result where everything is spelt out without cutting on any, you know, you write all the terms and this clearly becomes cumbersome if you have to deal with this type of equations to of.

Continuum hypothesis

- Matter is infinitely divisible.
- Each infinitesimal element retains all the properties of the material.
- Newtonian mechanics is directly applicable.
- Calculus works (governing equations can be derived as PDE-s or ODE-s; variational approaches can be adopted to describe system behavior)
- Attention is limited to characteristic dimensions > about 10^{-6} cm (diameter of a water molecule $\approx 10^{-8}$ cm)
- The theory is valid for both solids and fluids
- Notions of density, temperature, pressure, etc., at a point make sense.
- Primary aim: to model macroscopic behavior of solids and fluids.

- Ignores the atomic structure of matter. (Matter consists of discrete particles which are perpetually in motion)
- Questions on treatment of molecular, grain or crystal structure are not addressed.

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Now I will start now a quick review of continuum mechanics, there is what is known as Continuum Hypothesis, so according to this hypothesis material is infinitely divisible, and each infinitesimal element retains all the properties of the material, so that Newtonian mechanics is directly applicable, that means calculus works that means concept of elementary, strip, and things like that work, and we can derive the governing physical laws can be expressed as partial differential equations or as ordinary differential equations or through variational arguments. Now obviously we know matter is not infinitely divisible it breaks down to elementary particles if we do that, but that we are ignoring in this hypothesis, so consequently we need to focus our attention to characteristic dimensions which is about greater than about 10 to the power of - 6 centimeter, so if you are dealing with dimensions less than this then continuum hypothesis needs to be, I mean you have to look at other possible effects that are present in the physics of the problem, just to give in this context diameter of a water molecule is about 10 to the power of - 8 centimeter, so if you are dealing with fluid mechanics problem in which the medium is water, the fluid is water, then you cannot think of sizes less than 10 to the power of - 6 characteristic length less than this. The continuum mechanics theory is valid for both solids and fluids, it doesn't distinguish between the two and due to the assumption of existence of continuum notions of density, temperature, pressure at a point makes sense. The primary aim is to model macroscopic behavior of solids and fluids, so just to emphasize again it ignores the atomic structure of the matter and also matter consists of discrete particle which are perpetually in motion, even this motion is not included in our analysis.

Three themes

- Kinematics : Motion and deformation
- Kinetics: Concept of stress
- Balance laws (common to fluids and solids)

Understanding of

- rotations
- alternative definitions of stress and strain
- treatment of material nonlinear behavior

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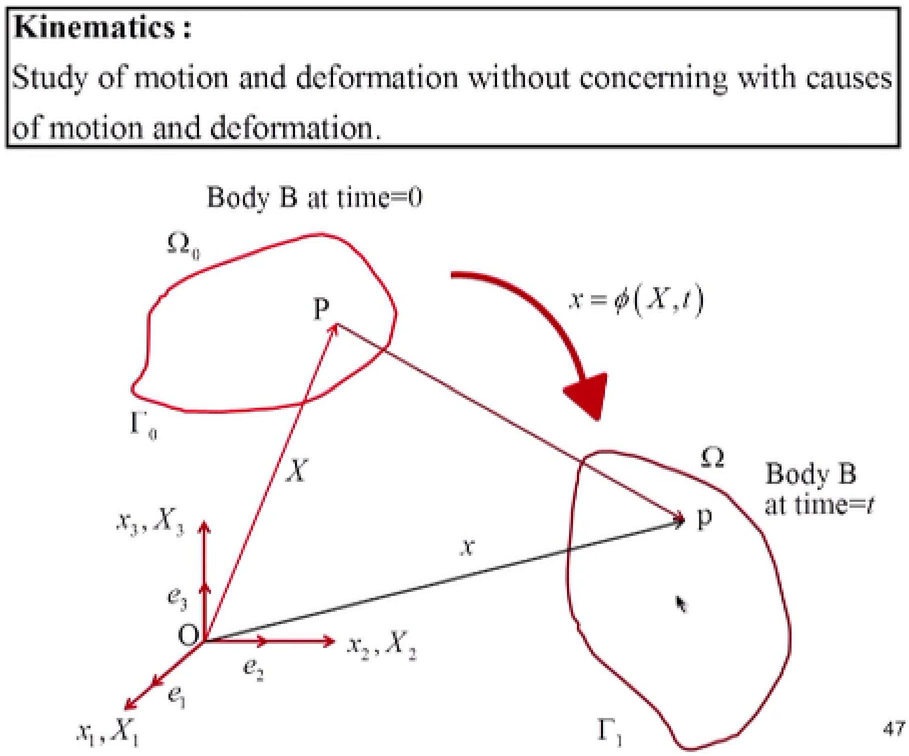
Then questions on treatment of molecular, grain, or crystal structure are not addressed in continuum mechanics, there are different themes in continuum mechanics we talk about kinematics where we talk about motion and deformation, kinetics where we talk about concept of stress, and there are different balance laws which basically enunciates certain physical laws, I will come to some of them which are common to both fluids and solids.

In the context of nonlinear structural mechanics problem what is crucial to gain a reasonable understanding of the subject is to understand how rotations are dealt with, rotations are very crucial in problems of nonlinear analysis, and what is the need for defining alternatively, what is the need for alternative definitions for stresses and strains, and then how to treat material nonlinear behavior.



So we will start some simple questions about kinematics, kinematics study of motion and deformation without concerning with causes of motion and deformation, we don't talk about forces which create the motion and deformation, we simply focus on geometry, so here we talk about a reference configuration say let's assume body B at time 0, this omega naught is a domain, gamma naught is the boundary, and we consider a Cartesian coordinate system, so capital X1, X2, X3 is for body at time T = 0.
Now during the process of deformation every point here with position vector OP which is X gets mapped to another point P whose position vector is X, this X is related to capital X through this relation, so this is a mapping of the deformation.

Reference frame: origin O; orthonormal basis: e_1, e_2, e_3

Body B occupies different regions $\Omega_0, \cdots, \Omega$ at time instants $0, \cdots, t$.

The regions $\Omega_0, \cdots, \Omega$ occupied by the body at different time instants $0, \cdots, t$ are known as configurations of the body at the respective time instants.

time $t = 0$

Ω_0 = initial state of the body; initial configuration.

Could also be taken as the reference configuration with respect to which motion is described.

Undeformed configuration: is an idealization.

Γ_0 = boundary of the initial configuration.

time t

Ω = current state of the body; current (deformed) configuration.

Γ_1 = boundary of the current configuration.

The reference frame, the origin is at 0 and there is orthonormal basis E1, E2, E3, this is a coordinate system, the body B occupies different regions omega naught, omega 1, etcetera, omega at times T = 0, 1, 2, T1, T2, T3 and T, the regions omega naught, omega 1 etcetera omega occupied the body at different time instants are known as configurations of the body at the respective time instants. At time T = 0, we say that omega naught is the initial state of the body or the initial configuration, it could also be taken as reference configuration with respect to which motion is described. There are other names like it is taken to be un-deformed configuration, it is an idealization nobody is truly un-deformed because the gravity and things like that always act on them, so what you see as a reference configuration is already deformed due to someone or the other effects.

Now gamma naught is a boundary of the initial configuration, at time T this is a current state of the body, the current deformed configuration, gamma is the boundary of the current

Eulerian (spatial) and Lagrangian (material) coordinates

In Lagrangian description we take (X_1, X_2, X_3, t) as independent variables.

In Eulerian description we take (x_1, x_2, x_3, t) as independent variables.

$X = X_i e_i \left(\equiv X_1 e_1 + X_2 e_2 + X_3 e_3\right)$

- Position vector of a material point in the initial configuration.
- This does not change with time.
- Labels all material points.

$x = x_i e_i \left(\equiv x_1 e_1 + x_2 e_2 + x_3 e_3\right)$

- Provides the position of a point in the current configuration.
- Changes as configurations evolve in time.

In problems of solid mechanics, Lagrangian description is often used.

Lagrangian description is also known as material description and

Eulerian description is also known as spatial description.

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configuration, so this is gamma 1, there are 2 coordinate system that we can think of using to describe the problem, one is known as Eulerian, the other is Lagrangian, in Lagrangian description we take X1 capital (X1, X2, X3,T) as independent variables, that means the point P is described by its position in the initial configuration, that is capital X1, X2, X3 those X1, X2, X3, are taken as independent variables. In Eulerian description we take the lower case x1, x2, x3 as independent variables.

Now X is returned position vector X is written as XI EI which is nothing but X1 E1 + X2 E2 + X3 E3, so this is a position vector of a material point in the initial configuration, this does not change with time, because initial configuration is some reference position that doesn't change with time, it labels all material points, whereas x the lower case x which is again XI EI this provides the position of a point in the current configuration, changes as configurations evolve in time. In problems of solid mechanics we adopt Lagrangian descriptions, the Lagrangian description is also known as material description, and the Eulerian description is also known as spatial description, the motion itself that is this function Phi (X, T) is defined, motion that is a

Motion

$$x(X,t)=\phi(X,t)\Rightarrow x_i=\phi_i(X,t), i=1,2,3\Rightarrow \begin{array}{l} x_1=\phi_1(X_1,X_2,X_3,t)\\ x_2=\phi_2(X_1,X_2,X_3,t)\\ x_3=\phi_3(X_1,X_2,X_3,t)\end{array}$$

When reference and initial configurations coincide we get

$x(X,0)=X=\phi(X,0)$

$X_i=x_i(X,0)=\phi_i(X,0)\Rightarrow\phi(X,0)$ is an identity transformation.

Material coordinates

Displacement : $u(X,t)=x-X=\phi(X,t)-\phi(X,0)=\phi(X,t)-X$

Velocity : $v(X,t)=\dfrac{\partial\phi(X,t)}{\partial t}=\dfrac{\partial u(X,t)}{\partial t}\equiv\dot{u}$

Acceleration : $a(X,t)=\dfrac{\partial v(X,t)}{\partial t}=\dfrac{\partial^2 u(X,t)}{\partial t^2}\equiv\ddot{u}$

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coordinate in the current configuration, a point in the current configuration that is the position vector of a point in the current configuration is related to where the point was in the reference configuration through this function, so this is in long hand that is, there are 3 functions X1, X2, X3, phi 1, phi 2, phi 3 that relate the capital X1, X2, X3, to the lowercase x1, and x2 and x3. When reference and initial configurations coincide at T = 0, X(x,0) is capital X, so that would mean which is phi(x,0), so then XI(XI, X,0) this is the definition which is phi I(x,0) and this is an identity transformation.
So in material coordinates displacement is given by X - capital X, which is nothing but phi(x,t) – phi(x,0) or phi(x,t) – capital X, velocity is its time gradient, capital X does not change with time therefore the gradient is simply DU/DT as shown here, there is no dou U/dou X term which gets multiplied by dou X/dou T that is not there because capital dou of capital X/dou T is 0, so similarly acceleration also can be defined.

Spatial coordinates

$$\frac{D}{Dt}\phi(x,t)=\frac{\partial \phi}{\partial t}+\frac{\partial \phi}{\partial x_j}\frac{dx_j}{dt}$$

$$a_i=\frac{\partial v_i}{\partial t}+\frac{\partial v_i}{\partial x_j}\frac{dx_j}{dt}=\frac{\partial v_i}{\partial t}+\sum_{j=1}^{3}\frac{\partial v_i}{\partial x_j}\frac{dx_j}{dt}; i=1,2,3$$

$$a=\frac{\partial v}{\partial t}+v\bullet\text{grad } v$$

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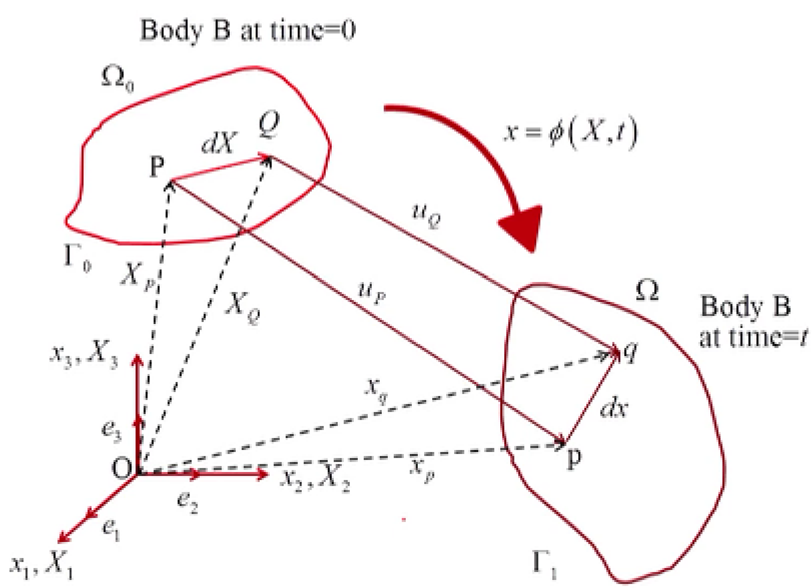
On the other hand in the spatial coordinates if you want to define the gradient of a function say phi(x,t), this is dou phi/dou T + dou phi/dou XJ and DXJ/DT, so there will be a new term, so acceleration gets defined like this, and this is a definition of acceleration if you are looking at spatial coordinates.

Deformation gradient



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Now a primary quantity of interest in discussing deformation is what is known as deformation gradient, so the problem is, the question is, the issue is this, this is a configuration of the body at time T = 0, and PQ is a line segment, upon deformation capital P goes to small p, and capital Q goes to capital Q and this line segment DX gets mapped to this lowercase dx, the question is how this dx is related to this capital DX, and that will be through a matrix known as deformation gradient. So we will take up this discussion on deformation gradient and follow up this topic in the next lecture, so we will close this lecture at this point.

<u>Programme Assistance</u>

Guruprakash P

Ramchandran MV

Dipali K Salokhe

Technical Supervision

B K A N Singh

Gururaj Kadloor

Indian Institute of Science

Bangalore