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Course Title
Finite element method for structural dynamic
And stability analyses
Lecture – 35
Inverse response sensitivity analysis
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Finite element method for structural dynamic and stability analyses

Module-10

FE Model updating

Lecture-35 Inverse response sensitivity analysis

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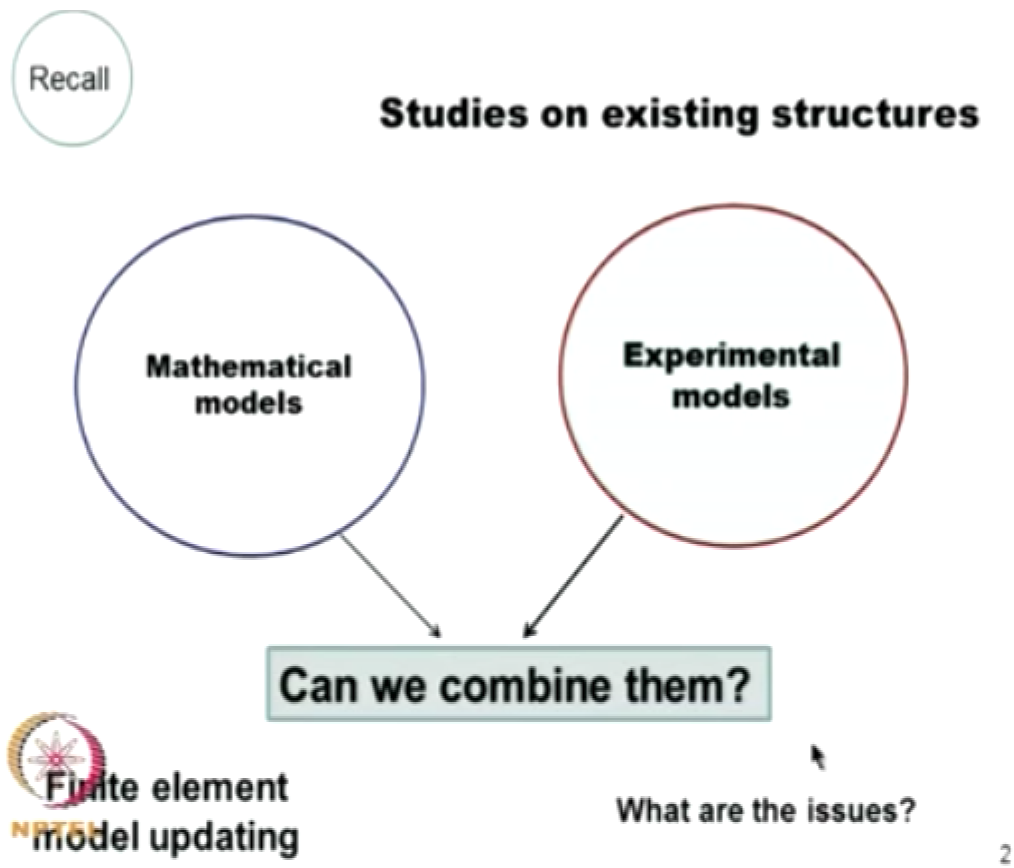
Towards the end of the previous lecture we started discussing about problems of finite element model updating, I provided you with the basic motivations for considering this type of problems, so we will continue with that discussion.



So if you recall the question of finite element modeling arises when we deal with structures which have already come into existence, so for such type of structures one can make of course mathematical models also it becomes possible to measure the performance of the structure under operational loads or diagnostic loads, so the predictions from experimental models most often would not may come, you know, agree with predictions from a mathematical model, so the question arises how can we combine these 2 models? Both models are prone to errors imperfections there are various assumptions that we make in making mathematical models pertaining to constitutive laws, we may assume isotropy, homogeneity, linearity, geometric non-linearity, hereditary non-linearity, and so on and so forth, and we postulate certain idealized boundary conditions and in structures with jointed elements we assume certain features associated with the joint behavior, so there are many, and damping is another major issue where a significant idealization is done, so in an experimental study none of these issues are primarily compromised, the constitutive laws, joint behavior, boundary conditions, presence of non-linearity, all that are captured without any compromise but the imperfection associated with experimental work are associated with the process of a measurement that is calibration errors associated with sensor, the sensor structure interaction, the actuator structure interaction, and problems with data acquisition that may can have problems of aliasing so on and so forth, so both these models therefore are imperfect so we need to somehow allow for these things when we try to reconcile predictions from these two models.

Morphology of FE model updating

- An existing instrumented structure whose response has been measured under operational or diagnostic loads
- A FE model for the structure on which the above measurement conditions can be simulated
- A strategy to adjust the parameters of the FE model to reconcile the predictions from FE model and the experimental observations on system response.

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Now a typical element of a finite element model updating process has, as I already said an existing instrumented structure whose response has been measured under operational or diagnostic loads, by diagnostic loads we can assume that we know what the loads we are applying. And we have a finite element model for this structure on which the above measurement conditions can be simulated. Then we need to have a strategy to adjust the parameters of the FE model to reconcile the prediction from FE model and the experimental observations on the system response, here we should take into account various features like mismatch of degrees of freedom between finite element model and experimental model, presence of noise in sensor experimental work, as well as imperfections in finite element model so on and so forth.

Methods

- Direct matrix methods
- Inverse sensitivity analysis
- Response function methods
- Time domain methods
- Bayesian filtering methods

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Systems
- Linear time invariant
- Linear time variant
- Nonlinear

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There are several methods which have been developed over the last few decades, they can be broadly classified as direct matrix methods, inverse sensitivity methods, response function methods, and certain methods which operate only in time domain, and the Bayesian filtering methods, so the various scope of these methods are, to examine the scope of these methods we need to also consider if the system is linear and time-invariant, or linear time-invariant, or non-linear problems are obviously more difficult to handle. So what we will do is in this introductory lectures we will focus our attention on linear time-invariant methods, and we will not be spending time on all these methods will basically focus on inverse sensitivity analysis, and just to give you a comparison of these methods among these various method the Bayesian filtering methods are the most powerful methods, but the background to develop these methods require you know developments in theory of probability, statistics, and random processes, and Bayes theorem in particular Markov process theory and so on and so forth, since this has not been addressed in this course we will not be considering questions on Bayesian filtering methods, other methods I will briefly touch upon but the main focus of our discussion will be on inverse sensitivity analysis. And we will focus on linear time-invariant dynamical systems.

Inverse sensitivity analysis

Focus: linear time invariant dynamical systems

- Response descriptors
- Natural frequencies
- Mode shapes
- FRF-s
- IRF-s
- Time histories of responses

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Now when we talk about inverse sensitivity analysis it is basically inverse response sensitivity analysis, so we considered various response descriptors like natural frequency, mode shapes, frequency response functions, impulse response functions, time histories of responses under measured or unmeasured applied loads.

General features

• $\{p_i\}_{i=1}^{n}$ = set of generic system parameters

(that could be associated with mass, stiffness or damping characteristics of the system)

• $\Gamma_k(p_1, p_2, \cdots, p_n), k = 1, 2, \cdots, N_k$ = set of observed dynamic properties of the system

(e.g., system natural frequencies, mode shapes, FRF, IRF)

• $p_u = \{p_{ui}\}_{i=1}^{n}$ = Initial guess on the values of system parameters before measurements are made.

• $p_{di} = p_{ui} + \Delta_i$ so that Δ_i $, i = 1, 2, \cdots, n$ are the changes to be determined so that the predictions from experiments and FE model on measured response characteristics are reconciled.

So the general features of inverse sensitivity methods consist of, we have a set of N generic system parameters this could be associated with mass, stiffness, or damping, or even forcing characteristics, and all these have been parameterized and we have a set of N parameters, and we have a set of NK observed dynamic properties of the system gamma K, for example system natural frequency, mode shapes, frequency response function, impulse response function, etcetera, so these are obviously functions of the system parameters P1 to PN.

Next we make an initial guess on the values of system parameters before measurements are made, I will call them as PU, and they are again N in number. Now after the measurements are made we don't know what is the values of the system parameters from which the measurements have emanated, I call them as PDI so if we apply a correction delta I to the initial guess that we make on system parameters we postulate that we will arrive at the unknown, the system parameters as indicated by the experimental results, so these delta 1, delta 2, N are the changes to be determined so that the prediction from the experiments and FE model on measured response characteristics are reconciled, so I have to clarify what this reconciliation mean, it has to be quantified so we will see what it means.

Taylor's expansion

$$\Gamma_k\left[p_{u1}+\Delta_1, p_{u2}+\Delta_2, \cdots, p_{un}+\Delta_n\right] = \Gamma_k\{p_{u1}, p_{u2}, \cdots, p_{un}\} + \sum_{i=1}^{n}\frac{\partial \Gamma_k}{\partial p_{ui}}\Delta_i$$

$$+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 \Gamma_k}{\partial p_{ui}\partial p_{uj}}\Delta_i\Delta_j + \cdots$$

$$\Rightarrow \Delta\Gamma_k = \Gamma_k\{p_{d1}, p_{d2}, \cdots, p_{dn}\} - \Gamma_k\{p_{u1}, p_{u2}, \cdots, p_{un}\}$$

Difference between predictions from FE model and the experiments

$$= \sum_{i=1}^{n}\frac{\partial \Gamma_k}{\partial p_{ui}}\Delta_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 \Gamma_k}{\partial p_{ui}\partial p_{uj}}\Delta_i\Delta_j + \cdots$$

First order method

$$\Delta\Gamma_k = \sum_{i=1}^{n}\frac{\partial \Gamma_k}{\partial p_{ui}}\Delta_i; k = 1, 2, \cdots, N_k$$

Second order method

$$\Delta\Gamma_k = \sum_{i=1}^{n}\frac{\partial \Gamma_k}{\partial p_{ui}}\Delta_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 \Gamma_k}{\partial p_{ui}\partial p_{uj}}\Delta_i\Delta_j; k = 1, 2, \cdots, N_k$$

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Now a simple strategy would be we consider the response descriptor which is function of the P1 N system parameters, and for the experimental model I will expand the response descriptor around the initial guesses that we make with delta 1, delta 2, delta N being the corrections, so a Taylor's expansion would lead to various terms and these gradients are to be evaluated at the, from the mathematical model with the initial guesses on the system parameters. Now this delta gamma K suppose the first term on this side I take it to the left side this will be gamma K, PD1, PD2, PDN - gamma K PU1, PU2, PUN, the first term is that respond descriptor that we have determined from experiments this is the respond descriptor that we have predicted from the mathematical model before we have taken the measurements, so the difference between the two is a known quantity in our work, so they can be related to these other terms in the Taylor's expansion through this terms.

In a first order method we omit this quadratic and other higher order terms, and I approximate delta gamma K as a linear function of delta I, so in this method we need to evaluate the gradients of response descriptors with respect to the system parameters, if there are NK number of system parameters and N number of, NK number of system response characteristics and N number of system parameters this will be NK/N matrix. The second order method clearly also indicates the quadratic terms in the expansion.

Consider the first order method

$$\Delta\Gamma_k = \sum_{i=1}^{n} \frac{\partial \Gamma_k}{\partial p_{ui}} \Delta_i ; k = 1,2,\cdots,N_k$$

$$\{\Delta\Gamma\} = [S]\{\Delta\} \text{ with } S_{ki} = \frac{\partial \Gamma_k}{\partial p_{ui}}$$

Knowns: $\Delta\Gamma_k, k = 1,2,\cdots,N_k$

Unknowns: $\Delta_i, i = 1,2,\cdots,n$

$[S] = N_k \times n$ matrix to be determined from the postulated FE model.

$\Rightarrow$

$$\{\Delta\} = [S]^{+}\{\Delta\Gamma\}$$

$[S]^{+}$ = Matrix pseudoinverse

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So let us consider the first order method, so we have this and this equation delta gamma K equal to summation of this gradient into the increments, that needs to be written for K running from 1 to NK, so I can cast this set of equations as a matrix equation with the vector delta gamma is equal to matrix S into delta, where this element of SKI is this gradient, delta gamma K divided by Dou/dou PUI of gamma K, the knowns here as I already mentioned the left-hand side is known here, the unknowns are the system parameters, corrections to the system parameters delta, so these are the unknowns.

Now S is an NK x N matrix to be determined from the postulated finite element model, so this would be known to us, now these constitute a set of typically over determined set of equations and this will be a rectangular matrix therefore we cannot directly in work that so we use the pseudo inverse theory and write delta S pseudo inverse of delta gamma, so this is a matrix pseudo inverse we discussed about this when we talked about substructuring methods, so the same you know theory is applicable here also.

Remarks

- The equation $\{\Delta\Gamma\} = [S]\{\Delta\}$ is used in design sensitivity analysis (forward problem) to determine (the unknown) changes in response characteristics $\Delta\Gamma$ caused by (known) small changes in system parameters Δ.
- In problem of model updating the role of knowns and unknowns is reversed and we call the problem of determining Δ from known values of $\Delta\Gamma$ as inverse senstivity problem.
- Typically, the number of knowns (N_k) and unknowns (n) do not match and the matrix S is often not well conditioned.
- In evaluating $[S]$ the Taylor expansion has been carried out around the initial guess p_u. The reference state about which the Taylor expansion is done could be updated once an estimate of p_d is obtained by linearization around p_u.

⇒ Iterative strategy

$$\{\Delta\}^{\bar{k}+1} = [S]^{\bar{k}}\{\Delta\Gamma\}^{\bar{k}}, \bar{k} = 0,1,2,\cdots$$

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Now we can make few remarks here if you consider this equation that is delta gamma = S into delta, and you are looking in this direction that means given changes that are made to system parameters I want to know what would be the change in response characteristics, this is a problem in design sensitivity analysis, for example you may like to change the stiffness of some element in a structure and you want to know what would be the change in natural frequencies and so on and so forth, so this is a forward problem, and that is a design sensitivity problem, so here we determine the unknown changes in response characteristics delta gamma caused by known small changes in the system parameter. Now on the problem, on hand the situation is quite the opposite, here in problem of model updating the role of knowns and unknowns is reversed and we call the problem of determining delta from known values of delta gamma as the inverse sensitivity problem, the word inverse sensitivity is associated with this description. Typically the number of knowns that is, typically the number of knowns which are system characteristics, and the unknowns which are the system parameters do not match and the matrix S is often not well conditioned, I will talk about condition number of a matrix slightly later in the lecture it is actually ratio of the highest singular value of S to the lowest singular value of S, I need to introduce those terms, we will come to that shortly.

In evaluating matrix S the Taylor expansion has been carried out around the initial guess PU, now the reference state about which the Taylor expansion is done could be updated once an estimate of PD is obtained by linearization around PU, so that would mean we can set a global iteration strategy over and above this formulation where we start with the initial guess and solve this problem and get an improved estimate for delta and that we feedback and use that as initial guess and then reiterate on this and we will get the, that is the S matrix is now linearized around an updated value of delta and this is solved system in a iterative way.

Further discussions

- Determination of the sensitivity matrix S.
- Solution of the resulting equations
  - Pseudo-inverse
  - Regularization
  - Global iterations
  - Second order sensitivity analysis

Present discussion based on
S Venkatesha, 2007, Inverse sensitivity methods in linear structural damage detection using vibration data, MSc(Engg) thesis, IISc, Bangalore.

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Now to implement this method clearly we need to discuss how to determine the sensitivity matrix S, and how do you solve the resulting set of equations, there are various issues like pseudo inverse, regularization, global iteration, and we may like to include second-order sensitivity terms in our analysis, and how to proceed if we do that, so to present discussion is based on MA synthesis this is cited here by Mr. S. Venkatesh.

Classically damped systems

1	Equilibrium equation employed for determination of the modal properties	$M\ddot{x} + Kx = 0$
2	Assumed solution	$x(t) = \Phi e^{i\omega t}$
3	Eigenvalue problem	$K\Phi = \omega^2 M\Phi$
4	Eigensolution	$\omega_1^2 \le \omega_2^2 \le \cdots \le \omega_N^2$, $\Phi = [\Phi_1 \Phi_2 \cdots \Phi_N]$
5	Orthogonality relations	$\Phi^t M \Phi = I, \Phi^t K \Phi = \Lambda$, $\Lambda = diag[\omega_i^2], i = 1, 2, \ldots N$
6	Generalized structural matrices	$\bar{M} = \Phi^t M \Phi = I$, $\bar{K} = \Phi^t K \Phi = \Lambda$, $\bar{C} = diag[2\eta_i\omega_i], i = 1, 2, \ldots N$
7	Receptance matrix	$\alpha(\omega) = [-\omega^2 M + i\omega C + K]^{-1}$ $= \Phi^t[-\omega^2 I + i\omega\bar{C} + \Lambda]^{-1}\Phi$ $\alpha_{jk}(\omega) = \sum_{r=1}^{N} \frac{\Phi_{jr}\Phi_{kr}}{\omega_r^2 - \omega^2 + i2\eta_r\omega\omega_r}$
8	Mobility matrix	$Z(\omega) = i\omega[-\omega^2 M + i\omega C + K]^{-1}$ $Z_{jk}(\omega) = \sum_{r=1}^{N} \frac{(i\omega)\Phi_{jr}\Phi_{kr}}{\omega_r^2 - \omega^2 + i2\eta_r\omega\omega_r}$
9	Accelerance matrix	$a(\omega) = -\omega^2[-\omega^2 M + i\omega C + K]^{-1}$ $a_{jk}(\omega) = \sum_{r=1}^{N} \frac{(-\omega^2)\Phi_{jr}\Phi_{kr}}{\omega_r^2 - \omega^2 + i2\eta_r\omega\omega_r}$
10	Impulse response	$h_{jk}(t) = \sum_{n=1}^{N} \frac{\Phi_{jn}\Phi_{kn}}{\bar{\omega}_n} e^{-\eta_n\omega_n t}\sin(\bar{\omega}_n t)$
11	Forced response in time domain	$M\ddot{x} + C\dot{x} + Kx = F(t)$ $x(0) = x_0, \dot{x}(0) = \dot{x}_0$ $x = \Phi z$ $\{p_n(t)\} = \Phi^t F(t)$ $x_k(t) = \sum_{n=1}^{N} \Phi_{kn} \int_0^t h_n(t-\tau) p_n(\tau) d\tau$ $p_n(t) = \sum_{k=1}^{N} \Phi_{kn} F_k(t)$ $h_n(t) = \frac{1}{\bar{\omega}_n} e^{-\eta_n\omega_n t}\sin(\bar{\omega}_n t)$

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So we will first, to start with we will quickly summarize the main results from linear vibration theory, suppose if you consider the undamped free vibration response of a multi degree freedom system MX double dot + KX = 0, we assume all points on the structure vibrate harmonically at the same frequency and we formulate this eigenvalue problem, this leads to set of N real valued natural frequencies and a set of N real valued eigenvectors and they have this orthogonality properties, and using these matrices we diagonalize this I mean uncouple the equation of motion and we determine various frequency response functions like receptance, mobility, accelerance, this we have considered in earlier lectures you can recall, and we can construct as well the impulse response function for the system in terms of the system model disk responses, and if you want force response in time domain using modal decomposition we uncouple the equation of motion and use Duhamel integral theory and get the impulse response function, so this is we are quite familiar with what these issues are.

Similarly for non-classically damped system if you recall we rewrote the equation of motion in this form AY dot + BY = F(t), so that A and B were symmetric matrices, and we did free vibration analysis and obtain a set of 2N complex valued natural frequencies and 2N complex valued eigenvectors, and we showed that they appear as conjugate pairs, the eigenvalues and eigenvectors and the structure of the modal matrix we delineate it, and which was of this form, and they R matrix which is the modal matrix in this case satisfy these orthogonality relations and using this we have derived the receptance, mobility, and accelerants matrix, and impulse response functions and response to the force response and so on and so forth.

Inverse eigensensitivity analysis : undamped systems

$KX = \omega^2 MX = \lambda MX$

Let $F_i = K - \lambda_i M$

$\Rightarrow F_i X_i = 0$

$\Rightarrow X_i^T F_i X_i = 0$

$$\Rightarrow \frac{\partial X_i^T}{\partial p_j} F_i X_i + X_i^T \frac{\partial F_i}{\partial p_j} X_i + X_i^T F_i \frac{\partial X_i}{\partial p_j} = 0$$

Note: $F_i X_i = 0$ & $X_i^T F_i^T = X_i^T F_i = 0 \Rightarrow X_i^T \frac{\partial F_i}{\partial p_j} X_i = 0$

Noting that $F_i = K - \lambda_i M$ we get $X_i^T \left[\frac{\partial K}{\partial p_j} - \frac{\partial \lambda_i}{\partial p_j} M - \lambda_i \frac{\partial M}{\partial p_j} \right] X_i = 0$

For mass normalized modal matrix, we get $\frac{\partial \lambda_i}{\partial p_j} = X_i^T \left[\frac{\partial K}{\partial p_j} - \lambda_i \frac{\partial M}{\partial p_j} \right] X_i$

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So this is available, we’ve already done I have summarized in one place all the main results, so we'll start with study of undamped systems that means assuming that we are going to invoke classical damping models and use these information for uncoupling the equation of motion, so the eigenvalue problem to be solved I can write it as KX = omega squared MX, for omega square I will write as lambda, and write it as lambda MX, so I will write this as K for i-th eigenvalue I’ll write it as K - lambda IM, and therefore this equation is equivalent to writing FI XI = 0, I can pre multiply by XI transpose and write this equation in this form.
Now what is my objective? I would like to derive the K and M matrices, will have our system parameters P1, P2, P3, PN and I would like to know the gradient dou lambda/dou PI, for I = 1 to N, and similarly dou XJ, that is j-th eigenvector by dou PI for I-th running from 1 to N, so that is the objective, now with that in mind I differentiate this equation now, so using a chain rule we get this equation, okay, so this is dou XI transpose dou PJ, FI, XI remains as it is, and I differentiate the second term and so on and so forth.
Now since FI, XI = 0 it means the transpose of this is also 0, XI transpose, FI transpose is 0 and XI transpose FI is also 0, because FI transpose is FI you know it's a symmetric matrix FI, so that would mean the accepting the middle term other two term drop-off and I get this equation. Now FI is basically K - lambda MI and dou FI/dou PJ will now involve to find dou FI/dou PJ, I have to differentiate this that will be dou K/dou PJ - dou lambda I/dou PJ into M - lambda I into dou M/dou PJ = 0, so now I have the required quantity dou lambda I/dou PJ here and I take it on the other side I have the required expression for the sensitivity I-th eigenvalue this is known, this is from the mathematical model, this is known and this we can differentiate and find out. Suppose PJ is stiffness of, suppose in a beam frame structure suppose P, one of the PJ is EI of

one of the elements, so when you assemble like global stiffness matrix K you will be able to identify which element is associated with that parameter and you will be able to arrive at this differentials.

Model updating with information on natural frequencies

$$\frac{\partial \lambda_i}{\partial p_j} = X_i^T \left[\frac{\partial K}{\partial p_j} - \lambda_i \frac{\partial M}{\partial p_j} \right] X_i$$

$$\lambda_i = \lambda_i[p_1 + \Delta p_1, p_2 + \Delta p_2, \cdots, p_n + \Delta p_n] = \lambda_i[p_1, p_2, \cdots, p_n] + \Delta\lambda_i;$$

$$\Delta\lambda_i = \sum_{j=1}^{n} \frac{\partial \lambda_i}{\partial p_j} \Delta p_j + \text{hot}, i = 1, 2, \cdots, \bar{N}; \bar{N} \le dof$$

$$\Rightarrow$$

$$\begin{Bmatrix} \Delta\lambda_1 \\ \Delta\lambda_2 \\ \vdots \\ \Delta\lambda_{\bar{N}} \end{Bmatrix}_{\bar{N}\times 1} = \begin{bmatrix} \frac{\partial \lambda_1}{\partial p_1} & \frac{\partial \lambda_1}{\partial p_2} & \cdots & \frac{\partial \lambda_1}{\partial p_n} \\ \frac{\partial \lambda_2}{\partial p_1} & \frac{\partial \lambda_2}{\partial p_2} & \cdots & \frac{\partial \lambda_2}{\partial p_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial \lambda_{\bar{N}}}{\partial p_1} & \frac{\partial \lambda_{\bar{N}}}{\partial p_2} & \cdots & \frac{\partial \lambda_{\bar{N}}}{\partial p_n} \end{bmatrix}_{\bar{N}\times n} \begin{Bmatrix} \Delta p_1 \\ \Delta p_2 \\ \vdots \\ \Delta p_n \end{Bmatrix}_{n\times 1}$$

$$\Rightarrow \{\Delta\lambda\} = \left[\frac{\partial \lambda}{\partial p}\right] \{\Delta p\} \Rightarrow \{\Delta p\} = \left[\frac{\partial \lambda}{\partial p}\right]^{+} \{\Delta\lambda\}$$

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Now this is sensitivity analysis, suppose, how do I do the model updating with information on natural frequencies, for example I have measured natural frequencies and I have initial postulate for the natural frequencies, so from the mathematical, this is the equation that we have just derived, so the lambda I, I will write it as lambda I(P1 + delta P1 P2 + delta P2) etcetera, PN + delta PN, so P1, P2, PN are the initial guesses I have dropped the subscript U and delta P are the increment that we need to find out, so this delta lambda I is given by this, so now we have derived this expression for dou lambda I/dou PJ so this I have to use for I, for the first, second, and N bar eigenvalues, you may use first three, first ten or whatever, it will be typically be less than the number of DOF of this system, we can assemble all these matrices in this form and we get this equation, so each term in this matrix need to be evaluated using this formulation, so this is the equation for the unknown increment in delta P in terms of the measured changes in natural frequencies, so this is known, this is the mathematical model so I will use the pseudo inverse and find delta PS dou lambda/dou P lambda into delta lambda, so this is a formulation which uses only the information on natural frequencies, so you see here a term called HOT these are higher order terms, HOT stands for higher order terms.

Inverse sensitivity analysis using information on mode shapes

$$X_i^T M X_s = \delta_{is}$$

$$\Rightarrow \frac{\partial X_i^T}{\partial p_j} M X_s + X_i^T \frac{\partial M}{\partial p_j} X_s + X_i^T M \frac{\partial X_s}{\partial p_j} = 0$$

Note that $X_i^T M \frac{\partial X_s}{\partial p_j}$ & $\frac{\partial X_i}{\partial p_j} M X_s$ are scalars

$$\Rightarrow X_s^T M \frac{\partial X_i}{\partial p_j} + X_i^T M \frac{\partial X_s}{\partial p_j} = -X_i^T \frac{\partial M}{\partial p_j} X_s$$

Similarly we have

$$X_i^T K X_s = \lambda_i \delta_{is}$$

$$\Rightarrow X_s^T K \frac{\partial X_i}{\partial p_j} + X_i^T K \frac{\partial X_s}{\partial p_j} = \frac{\partial \lambda_i}{\partial p_j} \delta_{is} - X_i^T \frac{\partial K}{\partial p_j} X_s$$

Furthermore $F_i X_i = 0$

$$\Rightarrow F_i \frac{\partial X_i}{\partial p_j} = -\frac{\partial F_i}{\partial p_j} X_i$$

$$F_s \frac{\partial X_s}{\partial p_j} = -\frac{\partial F_s}{\partial p_j} X_s$$

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Now how about mode shapes? Suppose if I have been able to measure mode shapes and I am able to predict mode shape from my postulated FE model I would then know the difference between the measured mode shape and the predicted mode shapes, so how do I get sensitivity with respect to mode shapes? So to be able to do that we will assume that mode shapes are mass normalized, so I will consider a pair of Eigen solutions with indices I and S, so I have XI transpose MXS is delta S, delta S is the Kronecker delta function, now I will differentiate this with respect to PJ I get this equation.

Now note, by noting that these quantities XI transpose M and dou XS/dou PJ and these are scalars, so I can reverse the way in which they are written and I can rewrite this equation in this form, the quantity that I am looking for are this dou XI/dou PJ and dou XS/dou PJ that we have to segregate and find out.

Now similarly we have other orthogonality relation XI transpose KXS is lambda I, delta S, so again I will differentiate this and we get one more equation. Now we also have the basic equation FI, XI = 0, this is a eigenvalue problem from this I get the, if I differentiate with respect to PJ I get XI and XS I get two equations as shown here. So now I have four equations,

$$\begin{bmatrix} F_i & 0 \\ 0 & F_s \\ X_s^T M & X_i^T M \\ X_s^T K & X_i^T K \end{bmatrix} \begin{Bmatrix} \dfrac{\partial X_i}{\partial \delta_j} \\ \dfrac{\partial X_s}{\partial \delta_j} \end{Bmatrix} = \begin{bmatrix} -\dfrac{\partial F_i}{\partial \delta} X_i \\ -\dfrac{\partial F_s}{\partial \delta} X_s \\ -X_i^T \dfrac{\partial M}{\partial p_j} X_s \\ \dfrac{\partial \lambda_i}{\partial \delta_j} \delta_{is} - X_i^T \dfrac{\partial K}{\partial \delta_j} X_s \end{bmatrix}$$

Remark

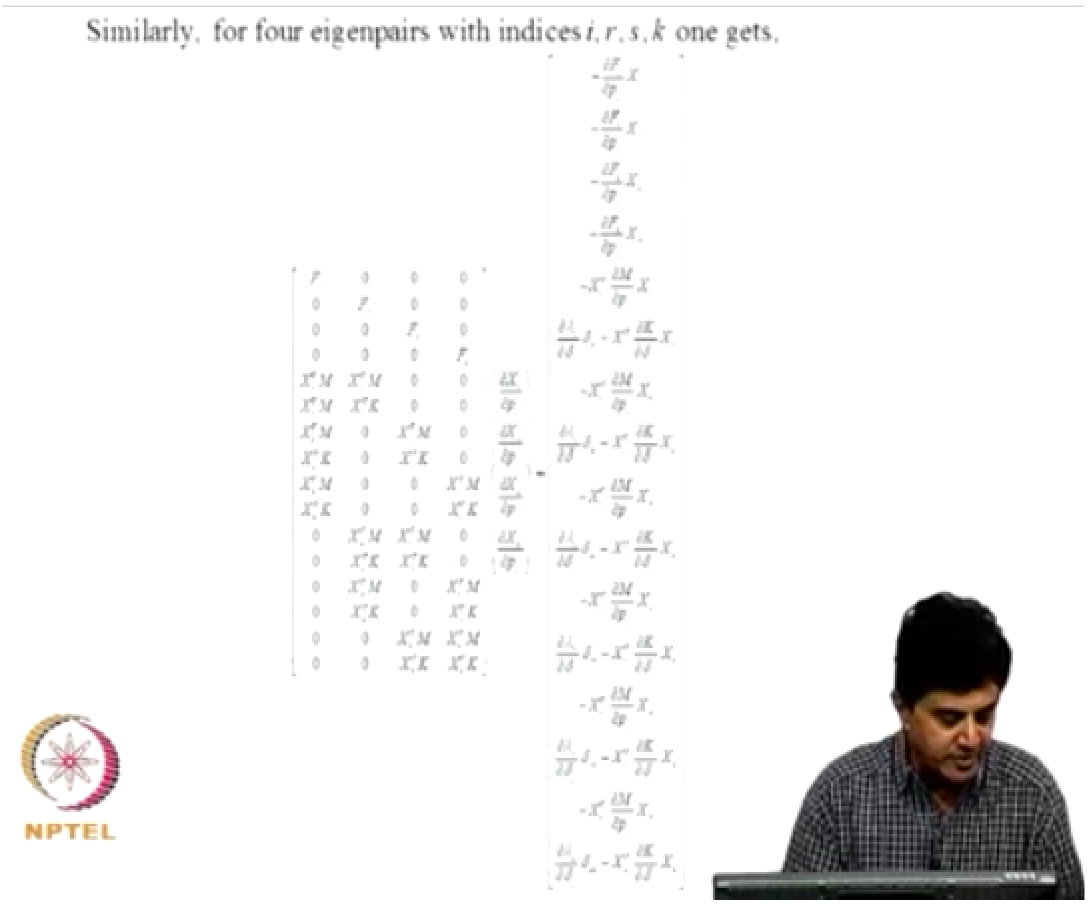
This equation has been derived based on considering two eigen pairs (λ_i, X_i) & (λ_s, X_s). The formulation can be generalized by considering more eigensolutions.

I can club all of them, so what I have used is the basic statement of eigenvalue problem and the two orthogonality relations, so I have I-th eigenvalue and S-th eigenvalue therefore I have two equilibrium equation one for I, and one for S, from that I have got these two equations but I also know XI and XS have certain orthogonality property, namely it is XI's are orthogonal to mass matrix and XI's are orthogonal to stiffness matrix, right so I have used these four equations basically to derive the gradient, so this I cast it in this form, so that is this equation has been derived based on considering two Eigen pairs here and this is dou XI/ dou delta J, Dou XS/dou delta J is equal to this.

For three eigen pairs $(\lambda_i, X_i), (\lambda_r, X_r) \& (\lambda_s, X_s)$ one gets

$$\begin{bmatrix} F_i & 0 & 0 \\ 0 & F_r & 0 \\ 0 & 0 & F_s \\ X_r^T M & X_i^T M & 0 \\ X_r^T K & X_i^T K & 0 \\ X_s^T M & 0 & X_i^T M \\ X_s^T K & 0 & X_i^T K \\ 0 & X_s^T M & X_r^T M \\ 0 & X_s^T K & X_r^T K \end{bmatrix} \begin{Bmatrix} \frac{\partial X_i}{\partial p_j} \\ \frac{\partial X_r}{\partial p_j} \\ \frac{\partial X_s}{\partial p_j} \end{Bmatrix} = \begin{bmatrix} -\frac{\partial F_i}{\partial p_j} X_i \\ -\frac{\partial F_r}{\partial p_j} X_r \\ -\frac{\partial F_s}{\partial p_j} X_s \\ -X_i^T \frac{\partial M}{\partial p_j} X_r \\ \frac{\partial \lambda_r}{\partial \delta_j} \delta_{ir} - X_i^T \frac{\partial K}{\partial \delta_j} X_r \\ -X_i^T \frac{\partial M}{\partial p_j} X_s \\ \frac{\partial \lambda_s}{\partial \delta_j} \delta_{is} - X_i^T \frac{\partial K}{\partial \delta_j} X_s \\ -X_r^T \frac{\partial M}{\partial p_j} X_s \\ \frac{\partial \lambda_s}{\partial \delta_j} \delta_{rs} - X_r^T \frac{\partial K}{\partial \delta_j} X_s \end{bmatrix}$$

Now why to consider only two Eigen solutions, we can consider three Eigen solutions so we can repeat the whole story, I can get while considering three Eigen solutions I will write the equilibrium equation for the three, pertaining to three Eigen solutions and there will be orthogonality relations with respect to mass and stiffness between three Eigen solutions, so if I combine all that I get a larger set of equations. So here I am considering I-th at R-th and S-th Eigen pairs so this can continue, so I can consider I, R, S, K and get a much larger set of



equations, so somewhere you have to stop with this formulation and that becomes one of the algorithmic parameters in implementing the method, so now therefore based on this I have now equations for, using this of course I can again consider the changes in mode shapes at various coordinates and for various modes, and again get an equation of the form this, where now this delta gamma will consist of changes in the values of mode shapes, between the what is measured and what is predicted, and S is the this matrix consisting of these gradients which have to be determined using one of these formulations, and delta P is the change in parameter that we wish to do, so we have now changes in natural frequencies and changes in mode shapes so all of them can be clubbed.

By combining information on natural frequencies and mode shapes one gets

$$\begin{Bmatrix} \Delta\lambda_1 \\ \Delta\lambda_2 \\ \vdots \\ \Delta\lambda_{\bar{N}} \\ \Delta X_1 \\ \Delta X_2 \\ \vdots \\ \Delta X_{\bar{N}} \end{Bmatrix} = \begin{bmatrix} \left[\frac{\partial \lambda_1}{\partial p}\right] \\ \left[\frac{\partial \lambda_2}{\partial p}\right] \\ \vdots \\ \left[\frac{\partial \lambda_{\bar{N}}}{\partial p}\right] \\ \left[\frac{\partial X_1}{\partial p}\right] \\ \left[\frac{\partial X_2}{\partial p}\right] \\ \vdots \\ \left[\frac{\partial X_{\bar{N}}}{\partial p}\right] \end{bmatrix} \{\Delta p\}$$

$$\Rightarrow \{\Delta\Gamma\} = [S]\{\Delta p\}$$

Sizes

$\bar{R}$=number of mode shapes included

$\bar{S}$=number of spatial points where the mode shapes are evaluated

$\Delta\Gamma : (\bar{N} + \bar{R}\bar{S}) \times 1,$

$S : (\bar{N} + \bar{R}\bar{S}) \times n$

$\Delta p : (n \times 1)$

And we can write now if we consider N bar number of modes I have results on N bar natural frequencies and N bar mode shapes, and those mode shapes themselves will be measured at several points, so that also has to be understood, so there will be issues about sizes of this equation, suppose if I have R bar number of mode shapes included, and S bar number of spatial points where the mode shapes are evaluated this delta gamma will be of this size, S will be of this size, and delta P will be of this size, so the final equation to be solved is this and again I get delta PS S pseudo-inverse delta gamma, so in this approach what we have done is therefore we are considering only natural frequencies, then eigenvectors also, and then while formulating sensitivity we may consider two Eigen pairs at a time, three Eigen pairs at a time, four Eigen pairs at a time so on and so forth.

Inverse eigensenstivity analysis : damped systems

$$\begin{bmatrix} 0 & M \\ M & C \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \dot{x} \end{Bmatrix} + \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix} \begin{Bmatrix} \dot{x} \\ x \end{Bmatrix} = \begin{Bmatrix} 0 \\ f(t) \end{Bmatrix}$$

$$\Rightarrow A\dot{y} + By = F(t); y(0) = y_0$$

Free vibration solutions

$$\Omega AR = -BR$$

$$\{\Omega\} = \{\Omega_1, \ \Omega_2, \ \cdots, \ \Omega_N, \Omega_1^*, \ \Omega_2^*, \ \cdots, \ \Omega_N^*\}^T_{2N\times 1}$$

$$[R] = \begin{bmatrix} \Omega_1 R_1 & \Omega_2 R_2 & \cdots & \Omega_N R_n & \Omega_1^* R_1^* & \Omega_2^* R_2^* & \cdots & \Omega_N^* R_n^* \\ R_1 & R_2 & \cdots & R_n & R_1^* & R_2^* & \cdots & R_n^* \end{bmatrix}_{2N\times 2N}$$

such that

$$R^T AR = I$$

$$R^T BR = \begin{bmatrix} -\Omega_1 & 0 & \cdots & 0 \\ 0 & -\Omega_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\Omega^*_N \end{bmatrix} \text{ or } R^t BR = -\begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda^* \end{bmatrix}$$

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Now this analysis was for undamped natural frequencies, but in an experimental work what you measure will be often invariably the damped natural frequencies and damped normal modes, there is no way you can you know eliminate damping in an experimental work, so what we measure in a laboratory is always even for free vibration characteristics it is always damped natural frequencies and damped normal modes, if you recall we have discussed the nature of these solutions, we have shown that the natural frequencies and mode shapes will be complex valued and they appear as complex conjugates, so if we consider equation of motion in this form we write this equation AY dot + BY = F(t), now A and B are the new structural matrices they no longer have the direct interpretation of being either mass or damping matrices, A has in fact M and C, B has M and K and so on and so forth.

Now the free vibration solution if you want to consider it will be AY dot + BY = 0, that is again a set of constant coefficients, linear, homogeneous, differential equations, therefore an exponential solution would be acceptable and that leads to this eigenvalue problem, so it will lead to a set of 2N Eigen values where the Eigen values appear in complex pair and Eigen solutions also appear in complex pairs and we have this relation, and these R, modal matrix R has these two orthogonality relations.

$$A\Omega_r R_r = -BR_r$$
$$F_r = A\Omega_r + B \Rightarrow F_r R_r = 0$$
$$\Rightarrow R_r^T F_r R_r = 0$$
$$\Rightarrow \frac{\partial R_r^T}{\partial p_j} F_r R_r + R_r^T \frac{\partial F_r}{\partial p_j} R_r + R_r^T F_r \frac{\partial R_r}{\partial p_j} = 0$$
$$\text{Since } F_r R_r = 0 \Rightarrow R_r^T F_r^T = R_r^T F_r = 0$$
$$\Rightarrow R_r^T \frac{\partial F_r}{\partial p_j} R_r = 0$$
$$F_r = A\Omega_r + B \Rightarrow R_r^T \left[\frac{\partial A}{\partial p_j}\Omega_r + \frac{\partial \Omega_r}{\partial p_j} A + \frac{\partial B}{\partial p_j} \right] R_r = 0$$
$$\Rightarrow -R_r^T \left[\frac{\partial A}{\partial p_j}\Omega_r + \frac{\partial B}{\partial p_j} \right] R_r = \frac{\partial \Omega_r}{\partial p_j} R_r^T A R_r$$
$$\text{We have } R_r^T A R_r = 1 \Rightarrow \frac{\partial \Omega_r}{\partial \delta_j} = -R_r^T \left[\Omega_r \frac{\partial A}{\partial \delta_j} + \frac{\partial B}{\partial \delta_j} \right] R_r$$

Now again we can consider sensitivity of only the eigenvalues, suppose you consider the R-th eigenvalue with the governing equation will be A omega R, RR is – BR, so I will write FR as A omega R + B that is FR that leads to FR into RR = 0, so I will pre multiply by RR transpose and write this, I differentiate this with respect to PJ and use the fact that this is 0 therefore these are also 0 and I get this equation, so the formulation proceeds on exactly the same lines as we did for undamped system, but of course these quantities are complex valued.

Now since FR is A omega R + B, dou FR/dou PJ can be evaluated from this I get this, now again using the fact that some of these are scalars etc I will be able to write this.

$$\Rightarrow \begin{Bmatrix} \Delta\Omega_1 \\ \Delta\Omega_2 \\ \vdots \\ \Delta\Omega_{\wp} \end{Bmatrix} = \begin{bmatrix} \frac{\partial\Omega_1}{\partial p_1} & \frac{\partial\Omega_1}{\partial p_2} & \cdots & \frac{\partial\Omega_1}{\partial p_n} \\ \frac{\partial\Omega_2}{\partial p_1} & \frac{\partial\Omega_2}{\partial p_2} & \cdots & \frac{\partial\Omega_2}{\partial p_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial\Omega_{\wp}}{\partial p_1} & \frac{\partial\Omega_{\wp}}{\partial p_2} & \cdots & \frac{\partial\Omega_{\wp}}{\partial p_n} \end{bmatrix} \begin{Bmatrix} \Delta p_1 \\ \Delta p_2 \\ \vdots \\ \Delta p_n \end{Bmatrix}$$

Or more compactly as

$$\{\Delta\Omega\} = \left[\frac{\partial\Omega}{\partial p}\right]\{\Delta p\}$$

Note that $\left[\frac{\partial\Omega}{\partial p}\right]$ is complex valued $\Rightarrow \left[\frac{\partial\Omega}{\partial p}\right] = U + iV$ & $[\Delta\Omega] = W + iZ$

$$\begin{bmatrix} W \\ Z \end{bmatrix} = \begin{Bmatrix} U \\ V \end{Bmatrix}\{\Delta p\} \Rightarrow \{\Delta p\} = \begin{bmatrix} U \\ V \end{bmatrix}^{+} \begin{Bmatrix} W \\ Z \end{Bmatrix}$$

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Now since RR transpose ARR is 1, I get dou omega R/dou delta J as this, this is the gradient that we are looking for. So if you are going to use only changes in natural frequencies as for updating you get this updating equation, these are changes in delta P, this is changes in the natural frequencies complex valued, and this is a gradient matrix that you have to evaluate from your FE model, more compactly I can write it as delta gamma as this. Now this is a complex valued quantity therefore I can separate the real and imaginary part I will write U + IV and delta, omega itself at W + IZ, so substitute here and separate real and imaginary parts, I'll put it here and I get this questions, from this I can get delta P as this equation, okay so this is the updating equation that we have to use.

Inverse sensitivity analysis using information on mode shapes

$$R_i^T A R_s = \delta_{is} \Rightarrow \frac{\partial R_i^T}{\partial p_j} A R_s + R_i^T \frac{\partial A}{\partial p_j} R_s + R_i^T A \frac{\partial R_s}{\partial p_j} = 0$$

Noting that $R_i^T A \frac{\partial R_s}{\partial p_j}$ & $\frac{\partial R_i}{\partial p_j} A R_s$ are scalars we get

$$\frac{\partial R_i^T}{\partial p_j} A R_s + R_i^T \frac{\partial A}{\partial p_j} R_s + R_i^T A \frac{\partial R_s}{\partial p_j} = 0$$

Consider next

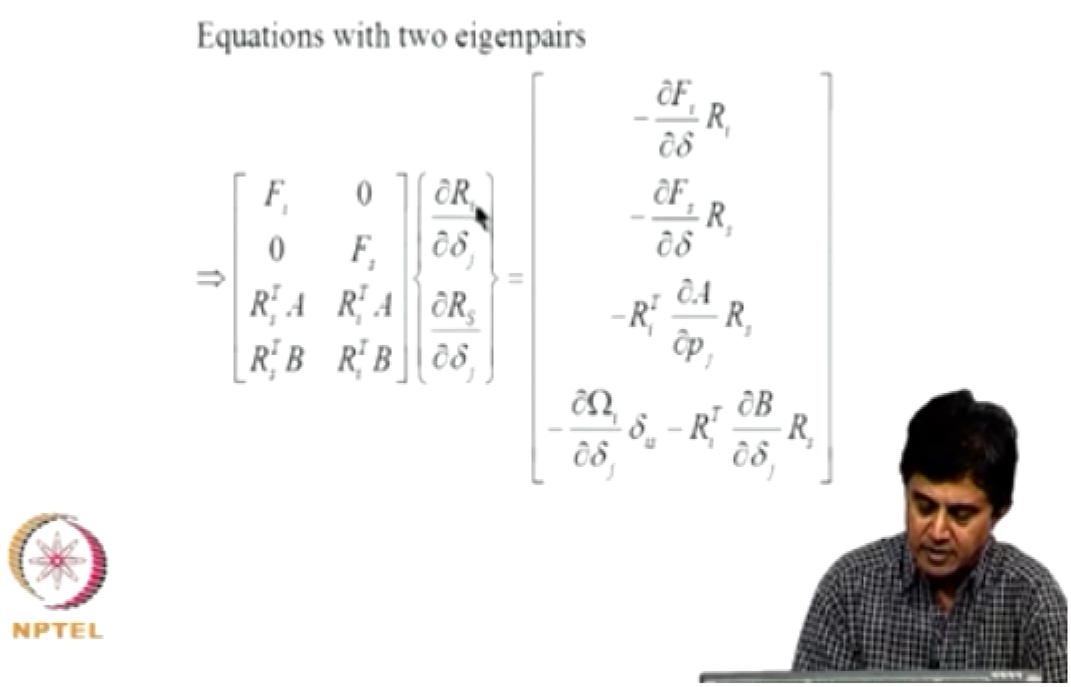
$$R_i^T B R_s = -\Omega_i \delta_{is} \Rightarrow R_s^T B \frac{\partial R_i}{\partial p_j} + R_i^T B \frac{\partial R_s}{\partial p_j} = -\frac{\partial \Omega_i}{\partial p_j} \delta_{is} - R_i^T \frac{\partial B}{\partial p_j} R_s$$

Also, using $F_i R_i = 0$ we get

$$F_i \frac{\partial R_i}{\partial p_j} = -\frac{\partial F_i}{\partial p_j} R_i$$

$$F_s \frac{\partial R_s}{\partial p_j} = -\frac{\partial F_s}{\partial p_j} R_s$$

Now if you want to include mode shapes, the story is same, again you can consider 2 Eigen pairs, 3 Eigen pairs, 4 Eigen pairs so on and so forth, so again just for illustration we will consider one instance RI transpose ARS is delta IS, from this I get this equation by differentiating, and noting that these are true we simplify this and I get this equation. Then similar equation I get with respect to the other orthogonality relations and this is that equation, now I have statements of Eigen value problem for I-th mode and S-th mode starting from that I get other 2 equation, so the 2 equations for I-th and S-th mode emanate from the statement of the Eigen value problem, and other two from the 2 orthogonality relation, so these four I will



club and obtain a whole determined set of equations for gradients of Eigen vectors with respect to delta J, so I can again do a pseudo inverse and find this point.

Equations with three eigenpairs

$$
\begin{bmatrix}
F_1 & 0 & 0 \\
0 & F_2 & 0 \\
0 & 0 & F_3 \\
R_2^T A & R_1^T A & 0 \\
R_2^T B & R_1^T B & 0 \\
R_3^T A & 0 & R_1^T A \\
R_3^T B & 0 & R_1^T B \\
0 & R_3^T A & R_2^T A \\
0 & R_3^T B & R_2^T B
\end{bmatrix}
\begin{Bmatrix}
\frac{\partial R_1}{\partial p_j} \\
\frac{\partial R_2}{\partial p_j} \\
\frac{\partial R_3}{\partial p_j}
\end{Bmatrix}
=
\begin{bmatrix}
-\frac{\partial F_1}{\partial p_j} R_1 \\
-\frac{\partial F_2}{\partial p_j} R_2 \\
-\frac{\partial F_3}{\partial p_j} R_3 \\
-R_1^T \frac{\partial A}{\partial p_j} R_2 \\
-\frac{\partial \Omega_1}{\partial \delta_j} \delta_{12} - R_1^T \frac{\partial B}{\partial \delta_j} R_2 \\
-R_1^T \frac{\partial A}{\partial p_j} R_3 \\
-\frac{\partial \Omega_1}{\partial \delta_j} \delta_{13} - R_1^T \frac{\partial B}{\partial \delta_j} R_3 \\
-R_2^T \frac{\partial A}{\partial p_j} R_3 \\
-\frac{\partial \Omega_2}{\partial \delta_j} \delta_{23} - R_2^T \frac{\partial B}{\partial \delta_j} R_3
\end{bmatrix}
$$

So if you consider 3 Eigen pairs, this is it, so you will write 3 equations for a eigenvalue problem and 3 Eigen orthogonality relations, among the 3 Eigen pairs with respect to A and B matrices, so all those equations if you club you get this equation. 4 Eigen pairs the equation

becomes more complicated, but all these terms will be available to you, you can do this, again if you combine all the equations for obtaining the final updating equations if you have data on N bar natural frequencies and N bar mode shapes evaluated at some S bar number of points and

Equations by combining natural frequency and mode shapes

$$\begin{Bmatrix} \Delta\Omega_1 \\ \Delta\Omega_2 \\ \vdots \\ \Delta\Omega_{\varsigma} \\ \Delta R_1 \\ \Delta R_2 \\ \vdots \\ \Delta R_{\varsigma} \end{Bmatrix} = \begin{bmatrix} \left[\dfrac{\partial\Omega_1}{\partial p}\right] \\ \left[\dfrac{\partial\Omega_2}{\partial p}\right] \\ \vdots \\ \left[\dfrac{\partial\Omega_{\varsigma}}{\partial p}\right] \\ \left[\dfrac{\partial R_1}{\partial p}\right] \\ \left[\dfrac{\partial R_2}{\partial p}\right] \\ \vdots \\ \left[\dfrac{\partial R_{\varsigma}}{\partial p}\right] \end{bmatrix} \{\Delta p\}$$

$$\Delta\Gamma = S\Delta p$$

Sizes

$\bar{R}$=number of mode shapes included

$\bar{S}$=number of spatial points where the mode shapes are evaluated

$\Delta\Gamma : \left(2\bar{N} + \bar{R}\bar{S}\right)\times 1,$

$S : \left(2\bar{N} + \bar{R}\bar{S}\right)\times n$

$\Delta p : (n\times 1)$

so on and so forth, if the final equations can be written as delta gamma as S delta P, and various sizes of these quantities as before are eliminated here, and you should notice that these are state space form therefore dimensions will be 2 into N bar and what about it was earlier, so we get the final equation in this form, again delta P is S pseudo-inverse of delta gamma, so since they

$$\begin{Bmatrix} \Delta\Omega_1 \\ \Delta\Omega_2 \\ \vdots \\ \Delta R_{\varsigma} \end{Bmatrix} = W + iZ \;\&\; \begin{bmatrix} \left[\dfrac{\partial \Omega_1}{\partial p}\right] \\ \left[\dfrac{\partial \Omega_2}{\partial p}\right] \\ \vdots \\ \left[\dfrac{\partial R_{\varsigma}}{\partial p}\right] \end{bmatrix} = U + iV$$

$$\Rightarrow \begin{bmatrix} W \\ Z \end{bmatrix} = \begin{Bmatrix} U \\ V \end{Bmatrix} \{\Delta p\}$$

$$\Rightarrow \{\Delta p\} = \begin{bmatrix} U \\ V \end{bmatrix}^{+} \begin{Bmatrix} W \\ Z \end{Bmatrix}$$

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are complex valued to facilitate computation I can separate real and imaginary parts, rewrite the equations and I get the final updating equation to be this, this is with respect to natural

Inverse sensitivity analysis of frequency response functions

Receptance: $\alpha(\omega)=\left[-\omega^2 M+i\omega C+K\right]^{-1}$

Mobility: $z(\omega)=i\omega\left[-\omega^2 M+i\omega C+K\right]^{-1}$

Accelerance: $a(\omega)=-\omega^2\left[-\omega^2 M+i\omega C+K\right]^{-1}$

For the purpose of illustration, we consider receptance $\alpha(\omega)$.

$$\Rightarrow\left[\alpha(\omega)\right]\left[D(\omega)\right]=I \text{ with } \left[D(\omega)\right]=\left[-\omega^2 M+i\omega C+K\right]$$

$$\Rightarrow\frac{\partial\alpha}{\partial p_j}D+\alpha\frac{\partial D}{\partial p_j}=0$$

$$\frac{\partial\alpha}{\partial p_j}=-\alpha\frac{\partial D}{\partial p_j}D^{-1}=-\alpha\frac{\partial D}{\partial p_j}\alpha$$

$$\frac{\partial\alpha}{\partial p_j}=-\alpha\left[-\omega^2\frac{\partial M}{\partial p_j}+i\omega\frac{\partial C}{\partial p_j}+\frac{\partial K}{\partial p_j}\right]\alpha$$

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frequencies in mode shapes which could be real valued or complex valued and you can include as many number of modes, as many number of orthogonality relations as you wish and develop these methods.

Now if you look into the experimental model analysis literature the primary quantity that we measure in an experimental work often happens to be either the impulse response function or the frequency response function, from the given frequency response function a matrix of frequency response function we extract the natural frequencies and mode shapes, so if you don't want to do that extraction you want to deal directly with what has been measured, I mean even that FRF processed from what we measure but that is more relatively a more primary quantity than the secondary quantities like natural frequencies and mode shapes and damping and so on and so forth, so if you want to now perform a sensitivity analysis on frequency response functions itself so you can predict the frequency response function from your postulated mathematical model and compare directly it with the measured frequency response function, so you will get certain differences and that you can now study to determine peace. So how do we do that? So we have now several descriptors of frequency response function this we have again seen in earlier lectures, we have receptance, mobility, accelerance, we have described all this earlier, these are all complex valued quantities.

Now let us consider for purpose of illustration receptance matrix, so let's assume to start with that it is a square matrix, now I can write alpha(omega) into D omega = I, where D is the inverse of the receptance matrix. Now I want to differentiate, what I want to find is dou alpha IJ with respect to dou PK, some IJ-th element evaluated at some frequency omega with respect to some PK-th parameter, so I'm basically I am interested in this gradient, so differentiate with respect to PJ I get dou alpha/dou PJ into D + alpha into dou D/dou PJ = 0, so if you solve for dou alpha/dou PJ I get this equation, and this is - alpha dou D/dou PJ into D inverse, D inverse is nothing but alpha, D is, we can now differentiate D and find out this quantity.

In the identification process we take that $\left[\alpha_{rs}(\omega)\right]$ for $r = 1, 2, \cdots, N_r$ & $s = 1, 2, \cdots, N_s$ are measured.

$$\Rightarrow \begin{Bmatrix} \Delta\alpha_{11}(\omega_1) \\ \Delta\alpha_{12}(\omega_1) \\ \vdots \\ \Delta\alpha_{N_rN_s}(\omega_1) \\ \Delta\alpha_{11}(\omega_2) \\ \Delta\alpha_{12}(\omega_2) \\ \vdots \\ \Delta\alpha_{N_rN_s}(\omega_2) \\ \vdots \\ \Delta\alpha_{11}(\omega_{N_\omega}) \\ \Delta\alpha_{12}(\omega_{N_\omega}) \\ \vdots \\ \Delta\alpha_{N_rN_s}(\omega_{N_\omega}) \end{Bmatrix}_{N_rN_sN_\omega\times 1} = \begin{bmatrix} \left[\frac{\partial\alpha_{11}(\omega_1)}{\partial p}\right] \\ \left[\frac{\partial\alpha_{12}(\omega_1)}{\partial p}\right] \\ \vdots \\ \left[\frac{\partial\alpha_{N_rN_s}(\omega_1)}{\partial p}\right] \\ \left[\frac{\partial\alpha_{11}(\omega_2)}{\partial p}\right] \\ \left[\frac{\partial\alpha_{12}(\omega_2)}{\partial p}\right] \\ \vdots \\ \left[\frac{\partial\alpha_{N_rN_s}(\omega_2)}{\partial p}\right] \\ \vdots \\ \left[\frac{\partial\alpha_{11}(\omega_{N_\omega})}{\partial p}\right] \\ \left[\frac{\partial\alpha_{12}(\omega_{N_\omega})}{\partial p}\right] \\ \vdots \\ \left[\frac{\partial\alpha_{N_rN_s}(\omega_{N_\omega})}{\partial p}\right] \end{bmatrix}_{N_rN_sN_\omega\times n} \{\Delta p\}_{n\times 1}$$

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Now therefore I have got dou alpha/dou PJ as given by this, now suppose if we have measured alpha RS(omega) for say R = 1 to NR, and S = 1 to NS we will get a NR/NS matrix and this omega could again vary from 1 to some N omega number of frequencies, so if I assemble all the observed changes they will be related to the unknown changes in system parameter through this matrix, so the sizes are spelt out here and you can easily imagine these matrices will be now very large sized, because for every frequency omega and for every alpha RS(omega) you are writing one equation, so the number of equations to be solved can be excessively large in relation to number of system parameters to be determined and this can pose considerable computational difficulties. Again we can separate the real and imaginary parts because we are

$$\begin{Bmatrix} \Delta\alpha_{11}(\omega_1) \\ \Delta\alpha_{12}(\omega_1) \\ \vdots \\ \Delta\alpha_{N_s N_t}(\omega_{N\omega}) \end{Bmatrix} = W + iZ \;\&\; \begin{bmatrix} \left[\dfrac{\partial\alpha_{11}(\omega_1)}{\partial p}\right] \\ \left[\dfrac{\partial\alpha_{12}(\omega_1)}{\partial p}\right] \\ \vdots \\ \left[\dfrac{\partial\alpha(\omega_{n\omega})_{N_s N_t}}{\partial p}\right] \end{bmatrix} = U + iV$$

$$\Rightarrow \begin{bmatrix} W \\ Z \end{bmatrix} = \begin{Bmatrix} U \\ V \end{Bmatrix} \{\Delta p\}$$

$$\Rightarrow \{\Delta p\} = \begin{bmatrix} U \\ V \end{bmatrix}^{+} \begin{Bmatrix} W \\ Z \end{Bmatrix}$$

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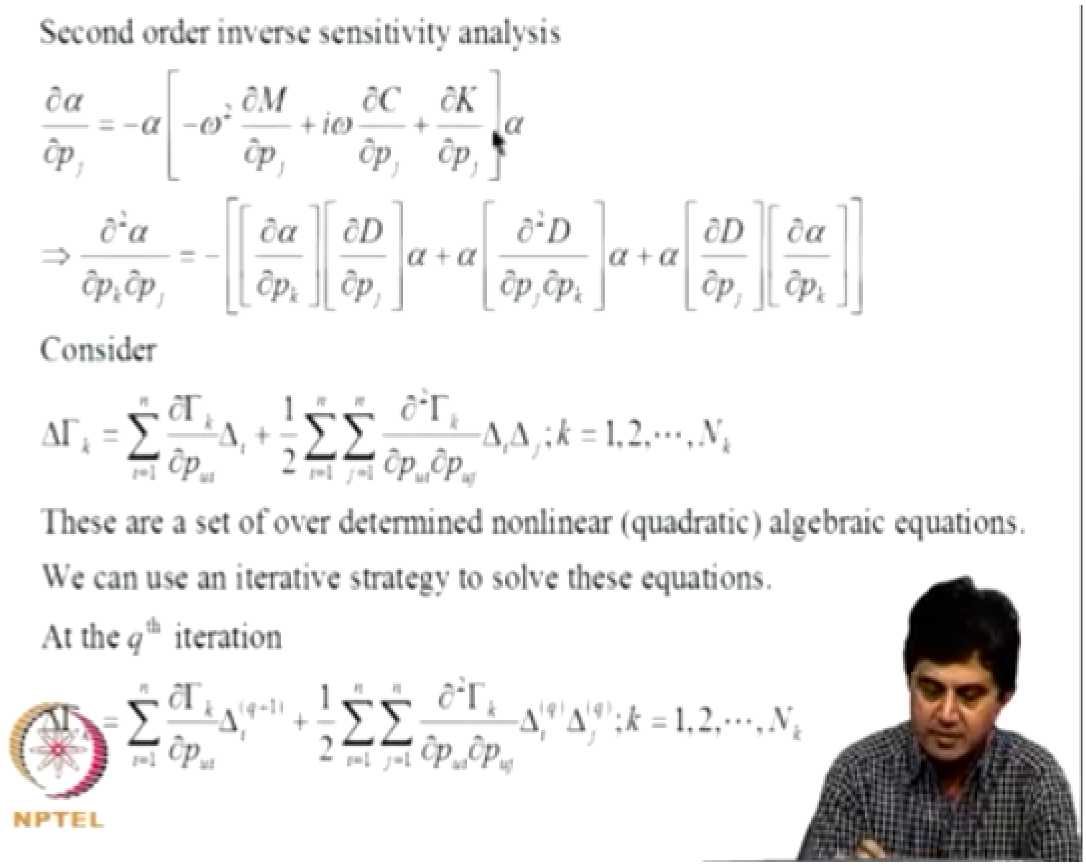
dealing with frequency response function, therefore they will have real and imaginary part so we have to separate and we could get a similar equation as we got earlier.



Now let's quickly foray into what happens if we consider second-order terms in our Taylor's expansion, so it is easy to explain that with respect to FRF sensitivity so we got this first-order sensitivity this is exact, and if I now differentiate with respect to PJ, PK, I have done it for PJ, now if I differentiate with respect to PK I need to simply differentiate these terms so I get this, so this is straightforward to be evaluated, so evaluation of this first-order and second-order sensitivity for FRF's presents no difficulties it can be done in a straightforward manner.

So let's now consider change in observed parameter using first order terms and second order terms, these are a set of over determined nonlinear in this case it is quadratic, algebraic equation, we can use an iterative strategy to solve this equation, so what we will do is we will start a iteration count Q, and at the Q-th iteration step for the second order terms I will use the, here I will use Q + 1, here I'll use Q, so we can iterate this and find out the solutions.

$$\Delta\Gamma_k = \sum_{i=1}^{n} \frac{\partial \Gamma_k}{\partial p_{ui}} \Delta_i^{(q+1)} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 \Gamma_k}{\partial p_{ui} \partial p_{uj}} \Delta_i^{(q)} \Delta_j^{(q)}; k = 1, 2, \cdots, N_k$$

This can be written as

$$[S]\{\Delta\}^{(q+1)} = \{\Delta\Gamma\} - \{S_{II}\}^q$$

with

$$\{S_{II}\}_{N_k \times 1}^q = \left\{ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 \Gamma_k}{\partial p_{ui} \partial p_{uj}} \Delta_i^{(q)} \Delta_j^{(q)} \right\}$$

$$\Rightarrow \{\Delta\}^{(q+1)} = [S]^{-} \left[\{\Delta\Gamma\} - \{S_{II}\}^q \right]; q = 0, 1, 2, \cdots, N_q$$

To begin the iteration it can be assumed that $\{\Delta\}^{(0)} = [S]^{-} \{\Delta\Gamma\}$

Remark

This approach is likely to lead to large number of equations in few unknowns and this may pose numerical difficulties.

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So we can write this as, this equation matrix form S delta Q + 1 is delta gamma - S2 Q, so where S2 Q is the second order gradients evaluated the previous step of iteration, so to start the iteration we can use first order analysis, so you can do this and then start the second order iteration, so this approach is likely to lead to a large number of equations few unknowns and this may pose numerical difficulties as before, but the advantage of second-order sensitivity is if your initial guess is far away from what is the true value it gives you a greater margin of a, you know error between what your prediction and the true value, and because you are including quadratic terms in your expansion, so this is an advantage of this method, later on with through some numerical examples we will be able to see this.

Singular value decomposition of the FRF matrix and complex mode indicator function (CMIF)

Preliminaries

Let A be a $n \times n$ nonsingular matrix and consider the eigenvalue problem

$Ax = \lambda x$

Let $\{\lambda_i\}_{i=1}^{n}; [\Phi]_{n \times n}$ be the eigensolutions such that $\Phi^T \Phi = I$.

$$\Rightarrow \Phi^T A \Phi = \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$\Rightarrow A\Phi = \Phi\Lambda$

$\Rightarrow A = \Phi\Lambda\Phi^T$

Question: what happens if A is a rectangular matrix?

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Now we have looked at now natural frequencies, mode shapes FRF's, are there any other descriptors that we can look at? Now FRF matrices in experimental work are often rectangular, what happens if we perform a singular decomposition of those FRF matrices? What are the quantities that we get? We have seen that the equation for FRF's if we use there will be very large number of equations to be solved for the operating system parameters, so can we you know effectively do some kind of data reduction by instead of considering all the FRF's can we simply condense the data by considering singular values and singular vectors and so on and so forth, that is a type of question.

We need to start with some preliminaries we discussed what is known as complex mode indicator function, so before that as a precursor to that we need to discuss what is singular value decomposition, so I will quickly run through this, so let A be a N x N nonsingular matrix and consider the eigenvalue problem AX = lambda X, so we will consider the situations where I get N natural eigenvalues and N/N eigenvector matrix and I will normalize the modal matrix so that phi transpose phi is I, and phi transpose A phi is this diagonal matrix of eigenvalues, so the eigenvalues and eigenvector therefore satisfy the relation A phi is phi lambda, and now pre multiplying, post multiplying by phi transpose I get A to be phi lambda phi transpose, so what I am doing is I am decomposing A in terms of an orthogonal matrix phi and a diagonal matrix lambda, so this representation is very useful in evaluating for example functions of A and so on and so forth.

Now the question we ask is what happens if A is a rectangular matrix, can we get a similar type of decomposition for a rectangular matrix? Obviously we cannot talk about eigenvalues and

Let A be a $m \times n$ matrix. Introduce

$$[B]_{m \times m} = [A]_{m \times n} [A^T]_{n \times m}$$

$$[B^T]_{n \times n} = [A^T]_{n \times m} [A]_{m \times n}$$

Consider eigenvalue problems associated with matrices B and B^t.

Let Q_1 be the $m \times m$ eigenvector matrix of B such that $Q_1^T Q_1 = I$ and

let Q_2 be the $n \times n$ eigenvector matrix of B^t such that $Q_2^T Q_2 = I$.

The non-zero eigenvalues of B and B^t would be identical.

$\Rightarrow$ We can write

$$A = Q_1 \Sigma Q_2^T$$

where Σ is the $n \times m$ diagonal matrix of square root of the nonzero eigenvalues of B and B^t. It can be verified that

$$AA^T = Q_1 \Sigma Q_2^T Q_2 \Sigma^T Q_1 = Q_1 \Sigma \Sigma^T Q_1$$

$$A^T A = Q_2 \Sigma^T Q_1^T Q_1 \Sigma Q_2^T = Q_2 \Sigma^T \Sigma Q_2$$

Q_1 = Matrix of left singular vectors

Q_2 = Matrix of right singular vectors

Nonzero elements of Σ=singular values of A.

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eigenvectors of A directly, so what we do is we define 2 matrices B is A into A transpose, suppose A is M x N, B would be, A transpose will be M x M, and B transpose will be N x N, so we can do eigenvalue analysis on B and B transpose, okay and we can find the M x M modal matrix for matrix B and that will have this orthogonality relation. Similarly Q2 be the N x N eigenvector matrix of B transpose so that Q2 transpose Q2 is I, we can show that the nonzero eigenvalues of B and B transpose will be identical, see B will have M eigenvalues, B transpose will have N eigenvalues, but there will be certain rank deficiencies associated with these matrices, the nonzero eigenvalues of B and B transpose can be shown to coincide, in fact we'll be able to write the DA is Q1 some sigma Q2 transpose, where sigma is a N x M diagonal matrix of square root of the nonzero eigenvalues of B and B transpose, we can verify that for example A is Q1, sigma Q2 transpose, suppose I am post multiply by A transpose and use these definitions I will be able to show that AA transpose will be Q1 sigma sigma transpose Q1, so this is the kind of decomposition for B that we have just now discussed for a square matrix. Similarly A transpose A which is N x N, I will get a similar decomposition.

Now what is of interest is A itself can be decomposed like this, this is known as singular value decomposition of matrix A, and this Q1 and Q2 are known as singular vectors, Q1 is a left singular vector and Q2 is the right singular vector, and sigma is the singular values of A, we can see a quickly an example suppose A is a 4 x 2 matrix we can find out the sigma matrix that

Example-1

$$A=\begin{bmatrix} 0.5377 & 0.3188 \\ 1.8339 & -1.3077 \\ -2.2588 & -0.4336 \\ 0.8622 & 0.3426 \end{bmatrix}$$

$$\Sigma=\begin{bmatrix} 3.1014 & 0 \\ 0 & 1.4129 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$Q_1=\begin{bmatrix} -0.1590 & 0.2717 & 0.8669 & -0.3864 \\ -0.6397 & -0.7548 & 0.1386 & 0.0435 \\ 0.7049 & -0.5057 & 0.4120 & 0.2786 \\ -0.2619 & 0.3174 & 0.2439 & 0.8782 \end{bmatrix} \Rightarrow Q_1'Q_1 = I$$

$$Q_2=\begin{bmatrix} -0.9920 & 0.1259 \\ 0.1259 & 0.9920 \end{bmatrix} \Rightarrow Q_2'Q_2 = I$$

$$Q_1\Sigma Q_2' = \begin{bmatrix} 0.5377 & 0.3188 \\ 1.8339 & -1.3077 \\ -2.2588 & -0.4336 \\ 0.8622 & 0.3426 \end{bmatrix}$$

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will be this, and Q1 and Q2 will be this, and you can verify that Q1 transpose Q1 is I, and Q2 transpose Q2 is I, and if you multiply now Q1, Q2 transpose we will get this matrix which is nothing but A, you can verify that. This is just an illustration of what I am telling.

SVD of the FRF matrix and the complex mode indicator function

Consider $N_r \times N_s$ FRF matrix α.

Define

$B = \alpha\alpha^H$

$Q = \alpha^H \alpha$

The superscript H here denotes conjugate transpose.

$B : N_r \times N_r$

$Q : N_s \times N_s$

B & Q are real, symmetric with real eigenvalues.

The spectrum of these eigenvalues are called the complex mode indicator functions (CMIF-s).

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We will now consider the question, what will happen if I now perform singular value decomposition of the FRF matrix itself? So that leads to what is known as complex mode indicator function or CMIF, so let us consider NR x NS FRF matrix alpha its rectangular, we will define B as alpha into alpha H, and Q as alpha H into alpha where the H is the conjugate transpose.

Now B will be NR x NR, Q is NS x NS, so B and Q are real symmetric with real eigenvalues, so the spectrum of these eigenvalues are called complex mode indicator functions, what they are? To understand that we will consider a simple example, I'll consider a 7 degree of freedom

Example-2 Consider a 7-dofs system with

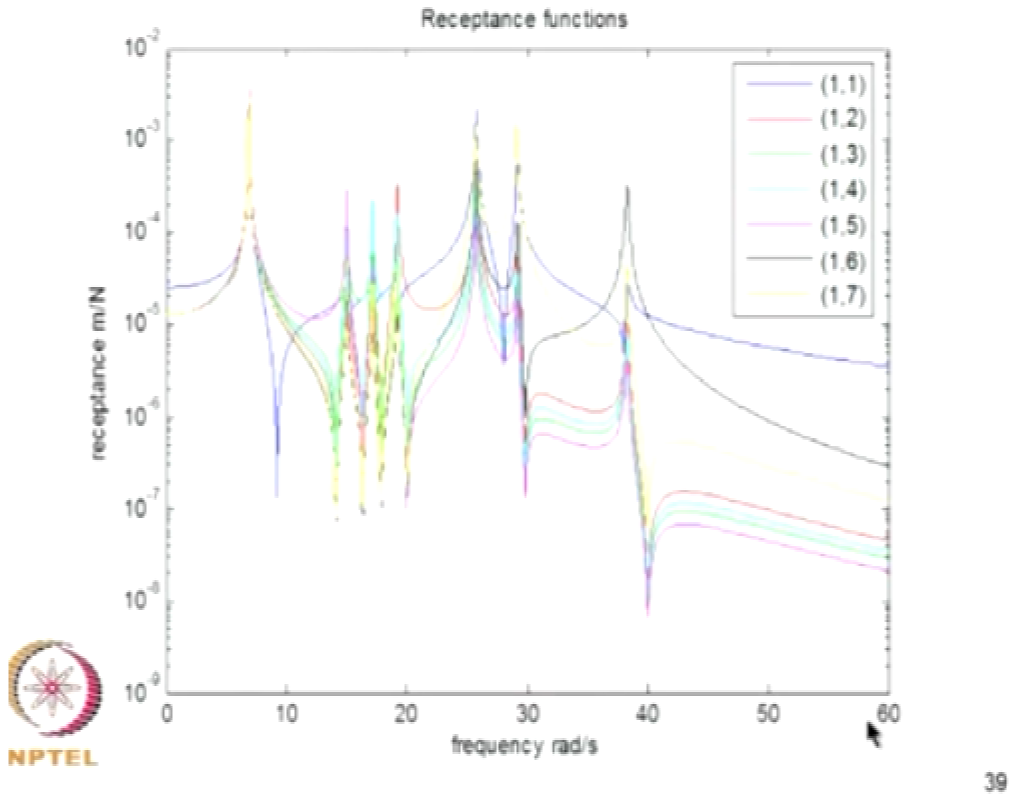
$$M = \begin{bmatrix} 100 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 200 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 300 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 250 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 400 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 50 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 90 \end{bmatrix}$$

$$K = \begin{bmatrix} 70000 & -10000 & -10000 & -10000 & -10000 & -10000 & -10000 \\ -10000 & 70000 & -10000 & -10000 & -10000 & -10000 & -10000 \\ -10000 & -10000 & 70000 & -10000 & -10000 & -10000 & -10000 \\ -10000 & -10000 & -10000 & 70000 & -10000 & -10000 & -10000 \\ -10000 & -10000 & -10000 & -10000 & 70000 & -10000 & -10000 \\ -10000 & -10000 & -10000 & -10000 & -10000 & 70000 & -10000 \\ -10000 & -10000 & -10000 & -10000 & -10000 & -10000 & 70000 \end{bmatrix}$$

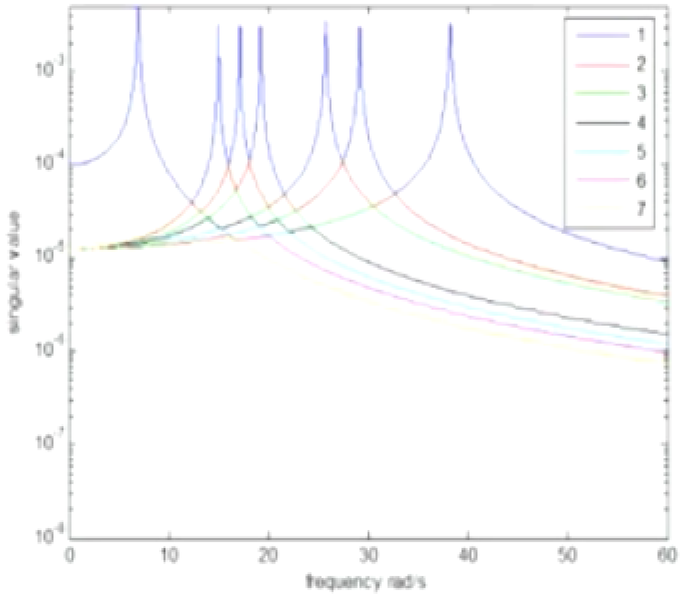
$\{\omega\} = (6.9283 \quad 15.0033 \quad 17.1302 \quad 19.2332 \quad 25.7690 \quad 29.1782 \quad 38.2654)$ rad/s.

All modes viscously damped with 2% damping

system with mass matrix as this, and stiffness matrix as this. Now if you compute the natural frequencies you will be able to get these natural frequencies you will see that all these 7 natural frequencies are distinct.



Now if I plot the CMIF, this is FRF's, suppose for one row of receptance functions I show these have 7 peaks corresponding to 7 natural frequencies. Now if you plot the spectrum of singular



Spectrum of singular values for the system
Seven distinct peaks in the first singular values
No repeated natural frequencies.

values for the system if you see the blue line, the blue line is the spectrum for the first singular value, and you clearly see 7 peaks which correspond to the 7 natural frequencies of the system, so no problem here.

Example-3 Consider a 7-dofs system with

$$M = \begin{bmatrix} 100 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 100 \end{bmatrix}$$

$$K = \begin{bmatrix} 70000 & -10000 & -10000 & -10000 & -10000 & -10000 & -10000 \\ -10000 & 70000 & -10000 & -10000 & -10000 & -10000 & -10000 \\ -10000 & -10000 & 70000 & -10000 & -10000 & -10000 & -10000 \\ -10000 & -10000 & -10000 & 70000 & -10000 & -10000 & -10000 \\ -10000 & -10000 & -10000 & -10000 & 70000 & -10000 & -10000 \\ -10000 & -10000 & -10000 & -10000 & -10000 & 70000 & -10000 \\ -10000 & -10000 & -10000 & -10000 & -10000 & -10000 & 70000 \end{bmatrix}$$

$$\omega_n = (10.0000 \quad 28.2843 \quad 28.2843 \quad 28.2843 \quad 28.2843 \quad 28.2843 \quad 28.2843) \text{ rad/s}$$

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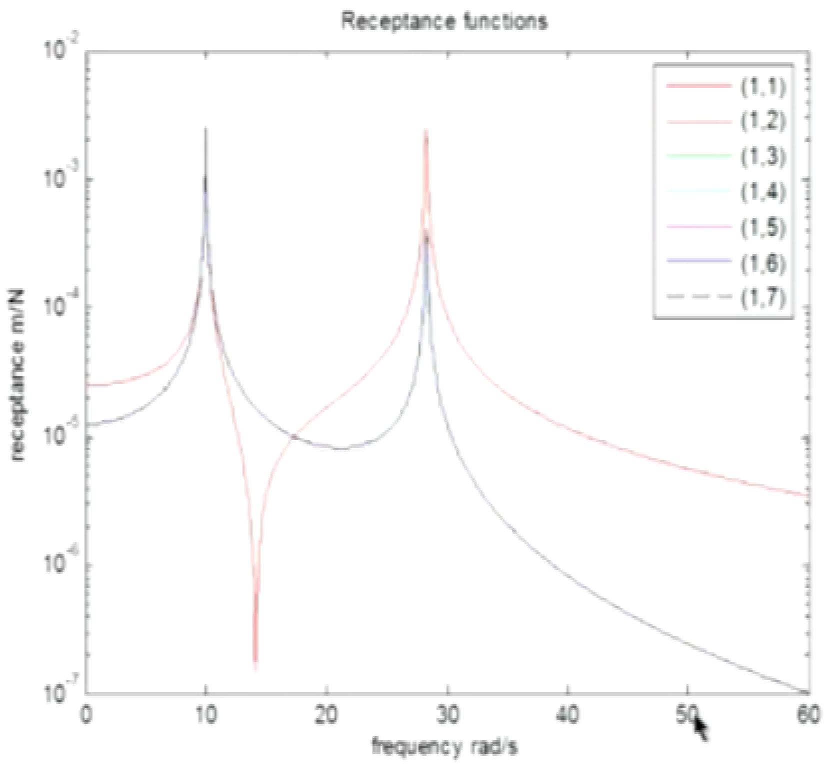
All modes viscously damped with 2% damping

Now we will change the system slightly, we will alter the mass and stiffness matrix, now I have a very peculiar system in which there are again 7 natural frequencies, the first natural frequency is 10, but all remaining 6 natural frequencies are 28.2843, that that means the remaining natural frequencies repeat 6 times, so now if you compute the frequency response function you will see



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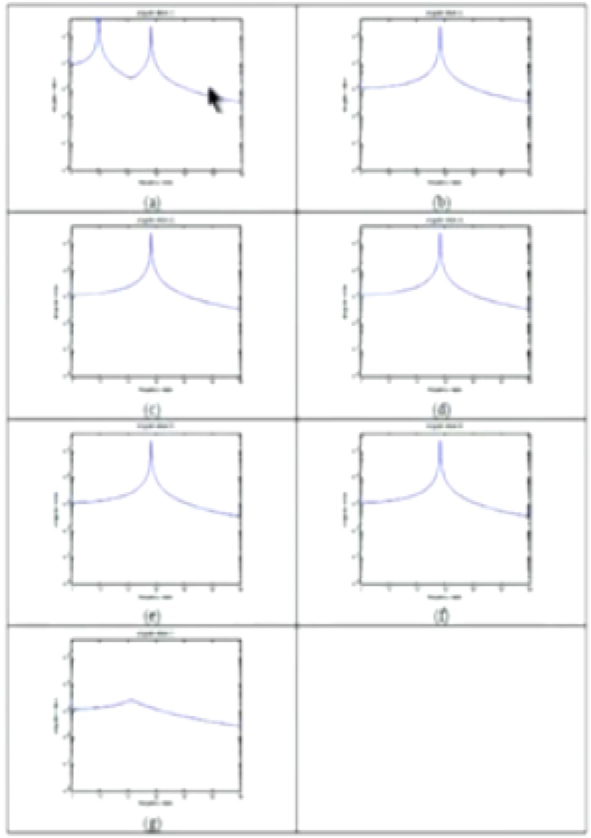
only 2 peaks, it appears as though you are dealing with a 2 degree freedom system, so this FRF matrix will not show, FRF plot will not show that some eigenvalues are repeated, but on the other hand if you plot the spectrum of singular values you will see that there will be, if you plot

Spectrum of singular values
(a) singular value 1;
(b) singular value 2;
(c) singular value 3;
(d) singular value 4;
(e) singular value 5;
(f) singular value 6;
(g) singular value 7

FRF-s have only two Peaks.
Spectrum of singular Values show seven distinct peaks at 28.2843 rad/s.
Repeated natural frequencies implied.



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the singular values, for those plots of 7 singular values you will see that there will be 7 places where this peak, and I obtain peaks for the second, third, fourth, fifth and sixth at the frequency 28 thereby indicating that the frequency 28.2843 etcetera is repeating 6 times, so this is used in industrial experimental works to characterize repeated natural frequencies are closely spaced natural frequencies, so this is a very useful tool.

Inverse sensitivity of singular values of FRF matrix

Consider $N_r \times N_i$ FRF matrix α.

Define

$B = \alpha\alpha^H$

$Q = \alpha^H\alpha$

The superscript H here denotes conjugate transpose.

$B : N_r \times N_r$

$Q : N_i \times N_i$

$B \& Q$ are real, symmetric with real eigenvalues.

Consider the eigenvalue problem

$BX = \mu X$

$$X^T X = I \;\&\; X^T BX = \begin{bmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_N \end{bmatrix}$$

Note: this analysis is carried out for different values of the driving frequency, $\omega = \{\omega_i\}_{i=1}^{N\omega}$.

Now motivated by this we can consider problems of inverse sensitivity of a singular values of FRF matrix, we can do a single inverse sensitivity analysis with respect to CMIF itself for example, so we will again consider this we have introduced these notations, now B and Q are these matrices I will consider the eigenvalues, BX, with respect to B and I have these relations and you must notice that when I talk about FRF it is at this value, there is a frequency driving frequency parameter implied in this, so all these analysis has to be done for every frequency, and so the driving frequency is now fixed and there could be N omega number of driving frequencies that has to be borne in mind.

Sensitivity analysis

$$BX_i = \mu_i I X_i$$

$$F_i = B - \mu_i I$$

$$\Rightarrow F_i X_i = 0$$

$$\Rightarrow X_i^T F_i X_i = 0$$

$$\Rightarrow \frac{\partial X_i^T}{\partial p_j} F_i X_i + X_i^T \frac{\partial F_i}{\partial p_j} X_i + X_i^T F_i \frac{\partial X_i}{\partial p_j} = 0$$

$$F_i X_i = 0 \Rightarrow X_i^T F_i^T = X_i^T F_i = 0$$

$$\Rightarrow X_i^T \frac{\partial F_i}{\partial p_j} X_i = 0$$

$$\Rightarrow X_i^T \left[\frac{\partial B}{\partial p_j} - \frac{\partial \mu_i}{\partial p_j} I \right] X_i = 0 \Rightarrow \frac{\partial \mu_i}{\partial p_j} = X_i^T \left[\frac{\partial B}{\partial p_j} \right] X_i$$

$$B = \alpha \alpha^H \Rightarrow \frac{\partial B}{\partial p_j} = \frac{\partial \alpha}{\partial p_j} \alpha^H + \alpha \frac{\partial \alpha^H}{\partial p_j}$$

$$\Rightarrow \frac{\partial \mu_i}{\partial p_j} = X_i^T \left[\frac{\partial \alpha}{\partial p_j} \alpha^H + \alpha \frac{\partial \alpha^H}{\partial p_j} \right] X_i$$

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Now I can do the, now for B matrix and this Q matrix I can do the Eigen sensitivity analysis whatever I did for natural frequencies, mode shapes etcetera, so I get, I will not run into these steps we get by analyzing B matrix I get certain equations with eigenvalues alone, eigenvectors

$$\frac{\partial \mu_i}{\partial p_j} = X_i^T \left[\frac{\partial \alpha}{\partial p_j} \alpha^H + \alpha \frac{\partial \alpha^H}{\partial p_j} \right] X_i$$

$$\Rightarrow$$

$$\begin{Bmatrix} \Delta\mu_1 \\ \Delta\mu_2 \\ \vdots \\ \Delta\mu_{N_s} \end{Bmatrix} = \begin{bmatrix} \frac{\partial \mu_1}{\partial p_1} & \frac{\partial \mu_1}{\partial p_2} & \cdots & \frac{\partial \mu_1}{\partial p_n} \\ \frac{\partial \mu_2}{\partial p_1} & \frac{\partial \mu_2}{\partial p_2} & \cdots & \frac{\partial \mu_2}{\partial p_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial \mu_{N_s}}{\partial p_1} & \frac{\partial \mu_{N_s}}{\partial p_2} & \cdots & \frac{\partial \mu_{N_s}}{\partial p_n} \end{bmatrix} \begin{Bmatrix} \Delta p_1 \\ \Delta p_2 \\ \vdots \\ \Delta p_n \end{Bmatrix}$$

also, this eigenvalue equation with only eigenvalues we can focus only on singular values, we will not include singular vectors in our discussion, so this is an equation.

Writing this equation for $\omega = \{\omega_i\}_{i=1}^{N\omega}$, we get

$$\{\Delta\mu\} = \left[\frac{\partial\mu}{\partial p}\right]\{\Delta p\}$$

$$\Rightarrow \{\Delta p\} = \left[\frac{\partial\mu}{\partial p}\right]^{+}\{\Delta\mu\}$$

$$\begin{Bmatrix} \Delta\mu_1(\omega_1) \\ \Delta\mu_2(\omega_1) \\ \vdots \\ \Delta\mu_{N_c}(\omega_1) \\ \Delta\mu_1(\omega_2) \\ \Delta\mu_2(\omega_2) \\ \vdots \\ \Delta\mu_{N_c}(\omega_{N_\omega}) \\ \vdots \\ \Delta\mu_1(\omega_{N_\omega}) \\ \Delta\mu_2(\omega_{N_\omega}) \\ \vdots \\ \Delta\mu_{N_c}(\omega_{N_\omega}) \end{Bmatrix}_{NN_\omega\times 1} = \begin{bmatrix} \left[\frac{\partial\mu_1(\omega_1)}{\partial p}\right] \\ \left[\frac{\partial\mu_2(\omega_1)}{\partial p}\right] \\ \vdots \\ \left[\frac{\partial\mu_{N_c}(\omega_1)}{\partial p}\right] \\ \left[\frac{\partial\mu_1(\omega_2)}{\partial p}\right] \\ \left[\frac{\partial\mu_2(\omega_2)}{\partial p}\right] \\ \vdots \\ \left[\frac{\partial\mu_{N_c}(\omega_2)}{\partial p}\right] \\ \vdots \\ \left[\frac{\partial\mu_1(\omega_{N_\omega})}{\partial p}\right] \\ \left[\frac{\partial\mu_2(\omega_{N_\omega})}{\partial p}\right] \\ \vdots \\ \left[\frac{\partial\mu_{N_c}(\omega_{N_\omega})}{\partial p}\right] \end{bmatrix}_{NN_\omega\times m} \{\Delta p\}_{m\times 1}$$

Now if we write this equation for N omega number of driving frequencies I get a set of, large set of equations as shown here and these are the updating equations that can be used okay, so we will see that this helps us to deal with repeated natural frequencies when I consider the derivation of the Eigen sensitivities the question of possibility of eigenvalues repeating was not addressed, so if you have a system with certain symmetries so eigenvalues could repeat, so in that case how do you do updating, because gradients of natural frequencies for frequencies which repeat involve certain additional considerations.

Tikhonov regularization

Consider the set of equations $[A]\{x\}=\{B\}$

$A: m \times n$

$B: m \times 1$

$x: n \times 1$

Condition number of A is defined as the ratio of the largest singular value of A and the smallest singular value of A.

Consider the situation in which A is ill-conditioned (that is, has large condition number).

Instead of considering $[A]\{x\}=\{B\}$, we consider

$$\left[A^T A+\xi I\right]\{x\}=A^T B$$

ξ = scalar paramter known as the regularization paramter, to be selected such that $\left[A^T A+\xi I\right]$ is better conditioned.

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Now what we have done is the generic form of, no matter which respond descriptor you use, the generic form of the equation has been delta P = some S + delta gamma. Now is this solution strategy always workable, is the next question we have to consider. So actually it turns out that it is advantageous to refine this solution strategy by using what is known as regularization that is Tikhonov regularization, what it means is what I am going to explain now. Consider this set of equation AX = B, A is a square matrix, M x N, B is M x 1, X is N x 1.

Example

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$Cond(A)$ is not defined.

A → A+small perturbation

$$A = \begin{bmatrix} 1.0092 & 1.0004 & 1.0076 & 1.0017 & 1.0005 & 1.0032 \\ 1.0079 & 1.0085 & 1.0074 & 1.0071 & 1.0010 & 1.0095 \\ 1.0096 & 1.0093 & 1.0039 & 1.0003 & 1.0082 & 1.0003 \\ 1.0066 & 1.0068 & 1.0066 & 1.0028 & 1.0069 & 1.0044 \end{bmatrix}$$

$Cond(A) = 1629.4$

A is ill-conditioned

Clearly, $Cond(I) = 1$.

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Now we define condition number of A as the ratio of the largest singular value of A and the smallest singular value of A, now before I proceed I can take a simple example, suppose if I take A to be a matrix 4 x 6 matrix of ones, the condition number A is not defined because the lowest condition number is 0 it is ranked deficient so there will be a problem, now what I do is I add small perturbations to this, so this is A matrix now, okay and condition number becomes 1629.4, so if in your analysis if this is a matrix that you have to deal with but because of perturbation errors and so on and so forth you observe this, then if you attempt to invert or solve these problems, find pseudo-inverse etcetera you are dealing with a highly ill-conditioned matrix, so I will show some more issues related to this as we go along. Clearly condition number of identity matrix is 1, this you have to, now what we do is instead of considering AX = B, we consider a modified version of this, for example how we proceed to find pseudo-inverse? I pre multiply by A transpose, and A transpose A is a square matrix that I will invert, that is what we have been doing, that is I have AX = B, I can pre multiply by this, so this is N x M, this is M x N, this is N x 1, and this is M x 1, and A transpose is N x M so I get N x 1 equation and I can invert this matrix and find X that is what is our definition of pseudo inverse is.

Tikhonov regularization

Consider the set of equations $[A]\{x\} = \{B\}$

$A : m \times n$

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$x : n \times 1$

Condition number of A is defined as the ratio of the largest singular value of A and the smallest singular value of A.

Consider the situation in which A is ill-conditioned (that is, has large condition number).

Instead of considering $[A]\{x\} = \{B\}$, we consider

$$\left[A^T A + \xi I\right]\{x\} = A^T B$$

ξ = scalar paramter known as the regularization paramter, to be selected such that $\left[A^T A + \xi I\right]$ is better conditioned.

$Ax = B$

$A^t A x = A^t B$

$n\times m \; m\times n \; n\times 1 \quad n\times m \; m\times 1$

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Now I don't want to do that, what I will do is I will introduce a additional term XI into I, this XI is a scalar parameter, now instead of inverting A transpose A I will invert this matrix, okay, this is known as regularization parameter to be selected such that we improve upon the condition number of this A transpose A matrix, okay, now what that means? Suppose from this I get X as

$$\left[A^T A + \xi I\right]\{x\} = A^T B \Rightarrow \{x\} = \left[A^T A + \xi I\right]^{-1} A^T B$$

This can be shown to be equivalent to minimizing the quantity $\|Ax - B\| + \xi \|x\|$

$\|Ax - B\|$: error norm

$\|x\|$: measure of smoothness of the solution

If ξ becomes arbitrarily large, it would alter the basic physics of the problem.

If ξ=0, we return to the problem of having to deal with ill-conditioned A.

$\Rightarrow$

In selecting ξ a trade-off on the values of $\|Ax - B\|$ & $\|x\|$ becomes relevant.

This is best illustrated through an example.

Hansen P C, 1994, Regularization tools: A Matlab package for analysis and solution of discrete ill-posed problems, Numerical Algorithms, 6, 1–35.

A transpose A + XII inverse A transpose B, we can show that this solution is equivalent to minimizing the quantity AX - B modulus + norm + XI into norm(x). Now what is norm(AX-B), it is a error norm, okay, for a given value of AX - B must be equal to 0 but you are not getting that, so this norm is actually error norm.

On the other hand this norm(x) is a measure of smoothness of the solution, if there are two alternate solutions one which is smooth is what I prefer that means if elements of X oscillate too much that is highly a non-smooth type of solution, whereas all elements are close to each other then the norm of that matrix will be less. Now if XI becomes arbitrarily large, how to select XI is still the question that we have to answer, see we cannot go on increasing XI indefinitely, then you will fiddling with the physics of the problem, you will be altering that, that is not acceptable. On the other hand if you put XI = 0, then you are back to the problem of inverting a ill-conditioned matrix, so there is obviously trade off in selecting XI, between the values of this norm and this norm, so what is done is we consider what is known as L curve, and there is a useful reference I have given here you can see that, this is available on the web.

Example

$$A = \begin{bmatrix} 0.1600 & 0.1000 \\ 0.1700 & 0.1100 \\ 2.0200 & 1.2900 \end{bmatrix}$$

$$A = U\Sigma V =$$

$$= \begin{bmatrix} -0.0782 & 0.8336 & -0.5468 \\ -0.0839 & -0.5520 & -0.8296 \\ -0.9934 & -0.0190 & 0.1131 \end{bmatrix} \begin{bmatrix} 2.4127 & 0 \\ 0 & 0.0022 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -0.8428 & 0.5382 \\ -0.5382 & -0.8428 \end{bmatrix}$$

$cond(A) = 1097.5$

Consider $Ax = B$ with B= = $[0.2600 \;\; 0.2800 \;\; 3.3100]^t \Rightarrow x = A^{+}B = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

Let us now add some noise to B & let

$\bar{B} = [0.2700 \; 0.2500 \; 3.3300]^t$

$\Rightarrow \{x_b\} = [A]^{+}\{\bar{B}\} = [7.0089 \;\; -8.3957]^t$

This happens because of large condition number of A.

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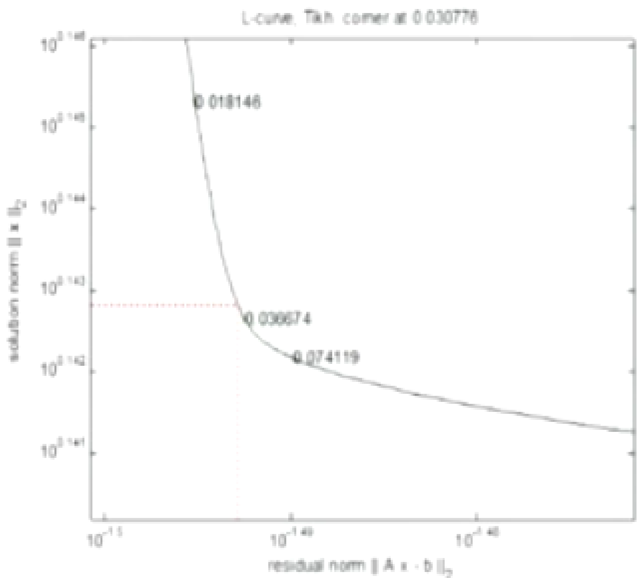
Let us consider a simple example A is this matrix, now I will do a singular value decomposition of this and I get these 3, this is a left single vector, this is the singular value, and this is a right matrix of right singular vectors. We can show that condition number of A is about 1097.5, now you consider AX = B where B is given by this, now if you find A pseudo inverse B, I get answer as 1 1, which is a nice solution.

Now let's consider now, we will add a slight noise to B, B is 0.26, 0.28, 3.3, I will make it 0.27. 0.25, 3.33, this is quite conceivable in a experimental work such type of noise is quite possible, now if you now find using the same formulation X will becomes, true answer is 1 1 it becomes 7 and – 8, that means in this type of calculations, this type of calculations are unforgiving as for as noise is concerned, a slight noise can distort the answer, this is a very eloquent illustration of that, this happens because the condition A matrix which we are trying to invert A transpose A matrix has a very large condition number, right so what we do is we now, instead of solving

Consider now solving the probleem using Tikhnov's regularization.

$$\left[A^T A + \xi I\right]\{x\} = A^T B$$



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that problem I consider A transpose A+ XI(I), X = 0. Now how this XI is selected is we plot the two you know scalars AX - B norm and norm(x) for different values of XI, and the point which is closest to the origin which happens to this typically turns out to be L shape curve, and at the bend the point is taken as an optimal value, so this can be programmed and we get the Tikhonov regularization parameter as 0.03.

Consider now solving the probleem using Tikhnov's regularization.

$$D = \left[A^T A + \xi I \right] = \begin{bmatrix} 4.1657 & 2.6405 \\ 2.6405 & 1.7170 \end{bmatrix}$$

$$= \begin{bmatrix} -0.8428 & -0.5382 \\ -0.5382 & 0.8428 \end{bmatrix} \begin{bmatrix} 5.8519 & 0 \\ 0 & 0.0308 \end{bmatrix} \begin{bmatrix} -0.8428 & -0.5382 \\ -0.5382 & 0.8428 \end{bmatrix}$$

$$Cond(D) = 190.11$$

$$x = [1.1997 \quad 0.7008]^t$$

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If I use that now, I get now the D matrix, A transpose A + XII is this, and if I do a singular value decomposition I see that condition number is now 190, it has dropped from nearly 1097 to 190, the solution I get is 1.19 and 0.70 so this is lot more acceptable than 7 and – 8, so what we have to do is every step where we are solving were determined set of equations we have to do a regularization that is always helpful.

Summary

- $\{\Delta\Gamma\} = [S]\{\Delta p\}$
- $\left[S^t S + \xi I\right]\{\Delta p\} = \{\Delta\Gamma\}$

Select ξ by using the L-curve

$\{\Delta p\} = \left[S^t S + \xi I\right]^{-1}\{\Delta\Gamma\}$

- Impose a global iteration
- Refinement: introduce second order terms and perform an iterative solution.

$\{\Delta\Gamma\}$

- Undamped natural frequencies and mode shapes
- Damped natural frequencies and mode shapes
- FRF-s
- Singular values of FRF matrix
- Singular values and singular vectors of the FRF matrix

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So in summary now what we have done is, where the updating equations have this form delta gamma is S into delta P, and we use regularization and find delta P, and we select XI by using the L curve approach and delta P leads to this, on this we will impose a global iteration that means I'll start with the initial guess on P and I will evaluate this S matrix at that value of P, I will solve this and find the increment to P, and I will now revise my S matrix instead of evaluating at the original value, I will evaluate the upgraded value, so this iteration I will continue till some norm on delta P converges.

Now a refinement on this would be to introduce a second order terms in the Taylor's expansion, so again we can retain all these ingredients, regularization, global iteration, all these steps can be introduced, this delta gamma as we have seen we've used undamped natural frequencies and mode shapes, damped natural frequencies and mode shapes FRF's, singular values of FRF matrix, you can of course include singular values and singular vectors of the FRF matrix, and the question on at what frequencies you would like to include this arises that can be handled, there are some few issues associated with that, so we can close this discussion by making few

Remarks

Effect of noise is mitigated to some extent due to averaging process involved in measuring FRF-s

$\{\Delta\Gamma\}$

- Undamped natural frequencies and mode shapes (Large data gets compressed)
- Damped natural frequencies and mode shapes (Large data gets compressed)
- FRF-s (One has to deal with large amount of data)
- Singular values of FRF matrix (Large data gets compressed)
- Singular values and singular vectors of the FRF matrix (Large data gets compressed)

  Inverse eigensolution method can be obtained as special cases of this method.

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observations, what happened to measurement noise in this? There is something interesting here in the sense when we use FRF's, FRF's are typically obtained by averaging across several measurements, so to some extent the measurement noises, effect of measurement noise is mitigated when you first use average FRF's, okay, say noise is eliminated by averaging, so that is one place where we explicitly handle presence of noise, but as far as imperfections in the mathematical model itself is concerned and there is no explicit model for the imperfections, so the answers we get on delta P are deterministic in this approach, this delta gamma if you use undamped natural frequencies and mode shapes large data gets compressed, large data set comprising of FRF gets compressed to few scalar numbers and few functions that is the mode shapes, and few natural frequencies.

Similarly this is also true if you are dealing with damped natural frequencies and mode shapes, again there is a data compression, but on the other hand if you are using FRF's you have to deal with very large amount of data. Similarly singular values of FRF matrix, there is a data compression, again singular values and singular vectors of FRF matrix if you use again there is a compression of large data.

Now actually if you use singular values and singular vectors of FRF matrix you can show that the inverse Eigen solution method will be a special case of this approach, I am not sure if we will be able to get into all the details, but I am just pointing out you can explore that fact if that is true by your own methods.

Now in the next class what we will do is, we will consider a few examples and illustrate the updating method that we have discussed in this lecture. So at this point we will close this lecture.

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