Indian Institute of Science
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NP-TEL
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Course Title
Finite element method for structural dynamic
And stability analyses
Lecture – 34
Fe modeling of vehicle structure interactions

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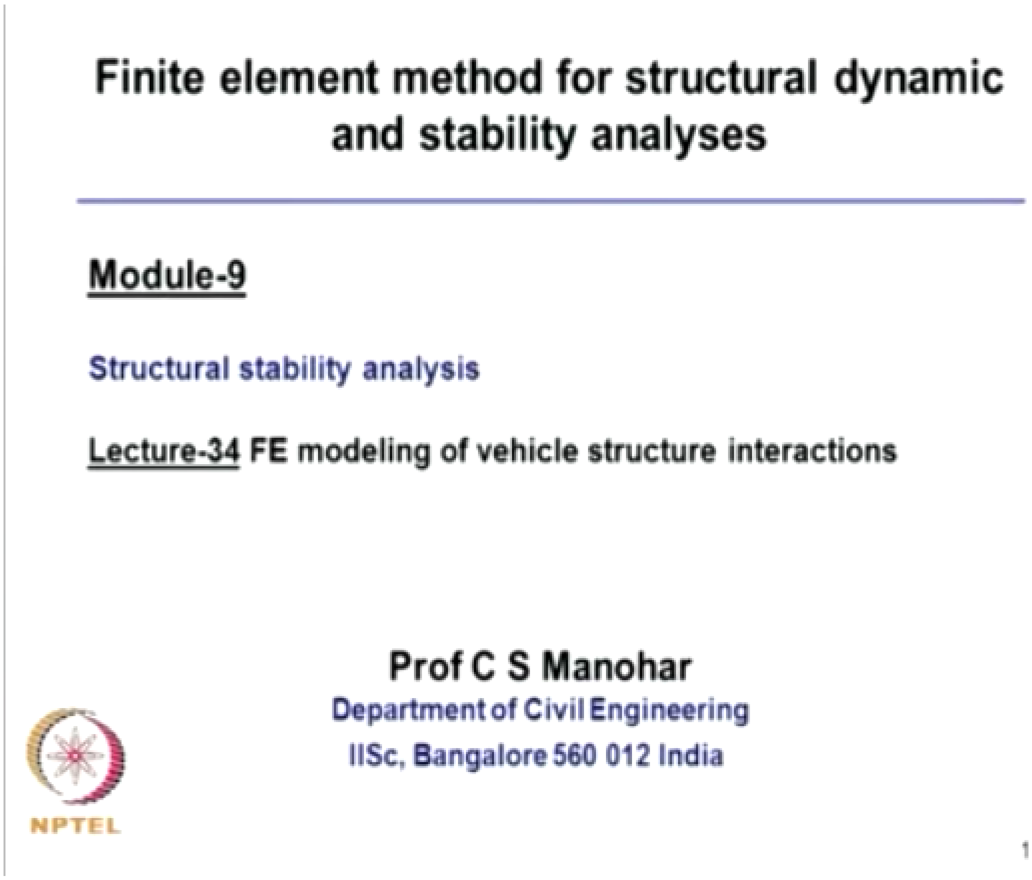

We are considering problem of modeling vehicle structure interactions using finite element method, this will be the concluding lecture on structural stability analysis we will begin a new topic towards the end of this lecture on finite element model updating.

Recall

- Floquet's theory for periodically time varying systems
- Need for dynamic analysis in understanding follower force systems
- Governing PDE-s for beam-moving oscillator systems
- Weighted integral and weak formulations to develop FE models starting from governing PDE-s

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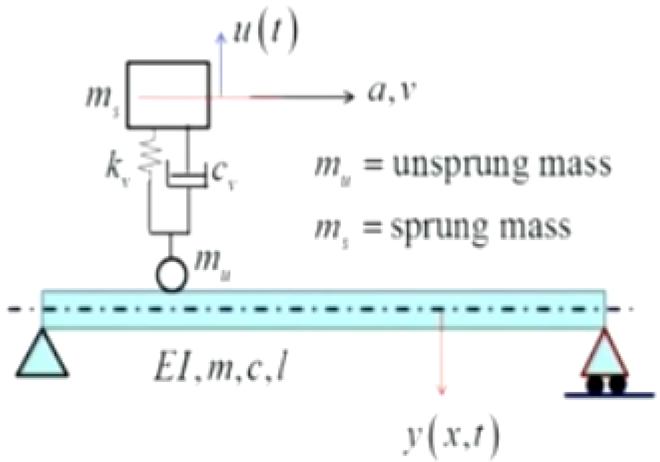
So what we have done in the study of time varying systems and stability is we considered Floquet's theory for periodical time varying systems, these coefficients help us to determine boundaries of stability for response of dynamical systems with period time varying, periodic coefficients.

In certain class of problems that is statically loaded structures in certain class of problems a dynamic analysis is needed to resolve the questions on stability, and this we discussed in the previous lecture, and we have derived the governing partial differential equation for beam moving oscillator systems, and we started discussing weighted integral and weak formulations to develop FE models starting from governing partial differential equation.

FE analysis of vehicle-structure interactions



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High speed trains
Mobile launchers
Numerically controlled machning

So we will continue with this and now consider the finite element analysis of vehicle structural interactions, so this quickly recall the supporting structure is modeled as the Euler-Bernoulli beam with these parameters, and the moving vehicle is modeled as a single degree freedom oscillator, MU is unsprung mass, MS is the sprung mass, KV and CV are the vehicle spring and damping coefficients, this type of problems are currently gaining importance, study of this type of problems because of development of high-speed trains, and mobile, aircraft launchers, and in even a numerically controlled machines where the tools move on the job at a fairly high speed to enhance productivity, so when we talk about vehicle structure interactions the scope of the problem need not really be confined to only highway bridges or railway bridges it can encompass a broader class of problems and the kind of formulation that we are going to discuss is in principle applicable to this wide range of problems.

for $0 < t < t_{exit}$

$$m_s\ddot{u} + c_v\left\{\dot{u} - \frac{D}{Dt}y\left[x(t),t\right]\right\} + k_v\left\{u - y\left[x(t),t\right]\right\} = 0$$

$$EIy^{iv} + m\ddot{y} + c\dot{y} = f(x,t)\delta\left(x - vt - \frac{1}{2}at^2\right)$$

$$f(x,t) = (m_u + m_s)g + k_v\left\{u - y\left[x(t),t\right]\right\} + c_v\left\{\dot{u} - \frac{D}{Dt}y\left[x(t),t\right]\right\}$$

$$-m_u\frac{D^2}{Dt^2}y\left[x(t),t\right]$$

$f(x,t) =$ wheel force

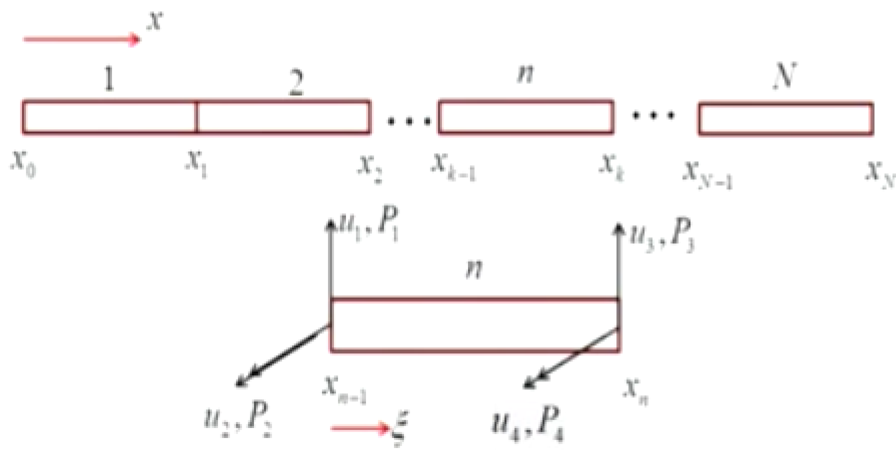
for $t > t_{exit}$

$$EIy^{iv} + m\ddot{y} + c\dot{y} = 0$$

with conditions at t_{exit} obtained from equations valid for $0 < t < t_{exit}$

Approach: integral and weak formulation

This equation we have derived in the previous lectures, so the vehicle enters the bridge span at T = 0 and exits at T exit it moves with an acceleration A and velocity V which remain time which do not change when the vehicle is on the bridge, and we have one equation for the vehicle a single degree freedom oscillator it has terms containing the bridge response and the beam response itself is governed by a Euler-Bernoulli beam equation, and the wheel force has contribution from weight of the vehicle and the vehicle spring stiffness, damper and the inertial effects due to the unsprung mass, we are using total derivatives here because the wheel moves on a deflected profile of the beam and it will induce Coriolis forces, and once the vehicle exists the bridge we consider only the free vibration of the beam assuming that our interest is basically on the beam and not so much on the vehicle, so we discretize the beam into set of



Let the beam be divided into N elements.

Consider the n^{th} element:

l_n=length of the element

t_{n-1}=time at which the vehicle enters the n^{th} element

t_n= time at which the vehicle leaves the n^{th} element

capital N elements, and for an Nth element the coordinates are XN – 1 and XN, and the degrees of freedom we model this element with 2 noded Euler-Bernoulli beam element with 2 degrees of freedom per node U1, U2, U3, U4 are displacements, P1, P2, P3, P4 are the corresponding stress resultants.

Now LN is the length of the beam, and TN - 1 is a time at which the vehicle enters the nth element, and at TN it exits the vehicle, sorry the vehicle exits the nth element, so we are now considering what happens during the time period when vehicle enters this element and leave this element.

The equations of motion while the vehicle is on the n^{th} element are as follows:

$$m_s\ddot{u} + c_v\dot{u} + k_v u - I\left[t_{n-1} < t < t_n\right]\left\{c_v \frac{Dy_0}{Dt} + k_v y_0\right\} = 0$$

$$EIy^{iv} + m\ddot{y} + c\dot{y} = f(x,t)\,\delta(x - x_0)$$

$$f(x,t) = I\left[t_{n-1} < t < t_n\right]\left\{(m_u + m_s)g + k_v\{u - y_0\} + c_v\left(\dot{u} - \frac{Dy_0}{Dt}\right)\right\} - m_u \frac{D^2 y_0}{Dt^2}$$

$I\left[t_{n-1} < t < t_n\right] = 1$ if $t_{n-1} < t < t_n$, otherwise zero (indicator function)

$$y_0(t) = y(x_0, t)$$

$$x_0 = vt + \frac{1}{2}at^2$$

Note: x and x_0 are calculated from the left end of the beam.

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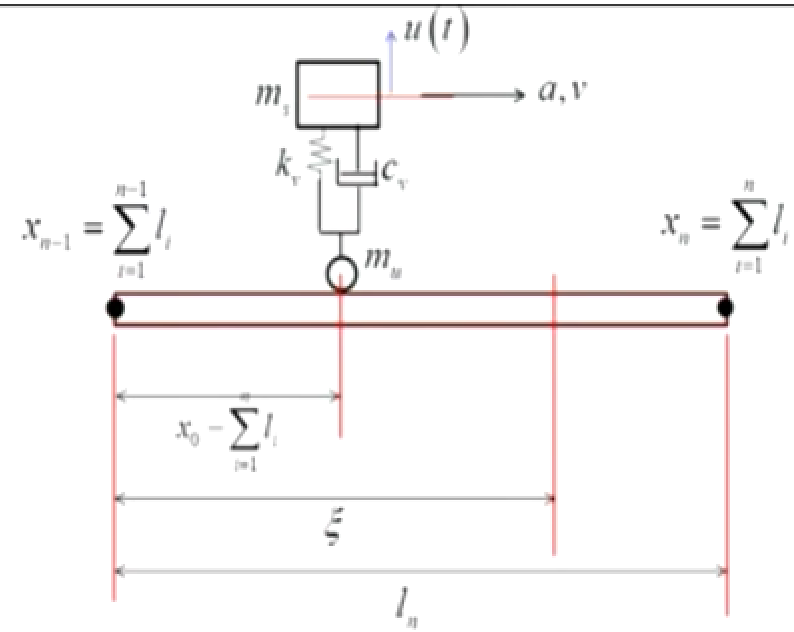
So the equation of motion while the vehicle is on the nth element are as follows, this is oscillator equation, so we use an indicator function to indicate that this force is valid only when the element is carrying the load so the vehicle exists on the element over the time period TN – to TN an indicator function takes a value of 1 when T is in this range otherwise it is 0, and this is a wheel force again it is multiplied by an indicator function and these are the terms that we already discussed.

Now we denote by Y naught (t) the deflection of the supporting structure under the wheel load which is Y(x naught, t) where X naught is VT + 1/2 AT square, we must note that X and X naught are calculated from the left end of the beam.



$$\xi = x - \sum_{i=1}^{n-1} l_i \Rightarrow 0 < \xi < l_n$$

$$\xi_0 = x_0 - \sum_{i=1}^{n-1} l_i = \text{position of the wheel in local coordinate system}$$

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Now we make a coordinate transformation, this is the nth element of length LN we introduce a coordinate XI as defined through this X – actually, XN - 1 and by this arrangement the XI would lie between 0 to LN. Now sai naught is the position of the wheel in the local coordinate system, so this is XN - 1 is some addition of all the length elements up to this point, and this XN is the length that includes this LN as well.

The equation of motion while the vehicle is on the n^{th} element are as follows:

$$m_s\ddot{u} + c_v\dot{u} + k_v u - I\left[t_{n-1} < t < t_n\right]\left\{c\frac{Dy_0}{Dt} + k_v y_0\right\} = 0$$

$$EIy^{iv}(\xi,t) + m\ddot{y}(\xi,t) + c\dot{y}(\xi,t) = f(\xi,t)\delta(\xi - \xi_0)$$

$$f(\xi_0,t) = I\left[t_{n-1} < t < t_n\right]\left\{(m_u + m_s)g + k_v\{u - y_0\} + c_v\left(\dot{u} - \frac{Dy_0}{Dt}\right) - m_u\frac{D^2 y_0}{Dt^2}\right\}$$

$$I\left[t_{n-1} < t < t_n\right] = 1 \text{ if } t_{n-1} < t < t_n, \text{ otherwise zero (indicator function)}$$

$$y_0(t) = y(\xi_0,t)$$

$$y' = \frac{\partial y}{\partial \xi}$$

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Now the equation of motion while the vehicle is on the nth element we have already written, now this prime here is now with respect to XI, so this is the only difference we have introduced the XI coordinate system which is specific to the nth element.

Weighted residual form of the beam equation

$$\int_0^l w(\xi)\left\{EIy^{iv}(\xi,t)+m\ddot{y}(\xi,t)+c\dot{y}(\xi,t)-f(\xi,t)\delta(\xi-\xi_0)\right\}d\xi = 0$$

$w(\xi)$ = weight function

$$\Rightarrow \left\{w(\xi)EIy'''(\xi,t)\right\}_0^{l_n} - \int_0^l w'(\xi)EIy'''(\xi,t)d\xi +$$

$$\int_0^{l_n} w(\xi)\left\{m\ddot{y}(\xi,t)+c\dot{y}(\xi,t)-f(\xi,t)\delta(\xi-\xi_0)\right\}d\xi = 0$$

$$\Rightarrow \left\{w(\xi)EIy'''(\xi,t)\right\}_0^{l_n} - \left\{w'(\xi)EIy''(\xi,t)\right\}_0^{l_n} + \int_0^l w''(\xi)EIy''(\xi,t)d\xi +$$

$$\int_0^{l_n} w(\xi)\left\{m\ddot{y}(\xi,t)+c\dot{y}(\xi,t)-f(\xi,t)\delta(\xi-\xi_0)\right\}d\xi = 0$$

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Now we write the weighted residual form of the beam equation, so we take all the forcing functions to the left side and multiply by a weight function and integrate over 0 to LN = 0, so W(xi) is the weight function, now to get the weak form we carry out integration by parts so that the demands on continuity on trial functions and weight function can be equally distributed, so this we have seen in the previous lecture so this leads to this equation for the first upon doing integration by parts twice I get this equation. Now this is our weak form of the, weak statement of the equation.

Let $V(\xi,t) = EIy'''(\xi,t)$ & $M(\xi,t) = EIy''(\xi,t)$

$V_{n-1} = V(0,t); V_n = V(l_n,t); M_{n-1} = M(0,t); M_n = M(l_n,t)$

Weak form

$$w(l_n)V_n - w(0)V_{n-1} - w'(l_n)M_n + w'(0)M_{n-1} +$$

$$\int_0^l w''(\xi)EIy''(\xi,t)d\xi + \int_0^{l_n} w(\xi)\{m\ddot{y}(\xi,t) + c\dot{y}(\xi,t) - f(\xi,t)\delta(\xi-\xi_0)\}d\xi = 0$$

$$y(\xi,t) = \sum_{i=1}^{4} u_i(t)\phi_i(\xi)$$

$$\left.\begin{array}{l} \phi_1(\xi) = 1 - 3\frac{\xi^2}{l^2} + 2\frac{\xi^3}{l^3}; \phi_2(\xi) = \xi - 2\frac{\xi^2}{l} + \frac{\xi^3}{l^2}; \\ \phi_3(\xi) = 3\frac{\xi^2}{l^2} - 2\frac{\xi^3}{l^3}; \phi_4(\xi) = -\frac{\xi^2}{l} + \frac{\xi^3}{l^2} \end{array}\right\}$$

We obtain equations for $u_i(t), i = 1,2,3,4$ by using $w(\xi) = \phi_i(\xi), i = 1,2,3,4$

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Now we denote by V, the shear force and the bending moment EIY ripple prime and EIY double prime and VN - 1 is the value of shear force at the left end, and VN is the shear force at the right end, similarly we define MN – 1, MN as the bending moments at XI = 0 and LN, the weak form using these notations now take this you know they will be represented by this equation. Now our job is to approximate the field variable in terms of the nodal degrees of freedom and suitable interpolation functions, so this is the representation that we use, there are 2 nodes and 2 degrees of freedom per node therefore we need 4 generalized coordinates, and as before we take cubic polynomials and we obtain equations, we substitute into this and we use the weight functions phi I(xi) for 1, 2, 3, 4 and to get a set of four equations which lead to the required equation for the nodal degrees of freedom UI(t).

$$\phi_1(\xi)=1-3\frac{\xi^2}{l^2}+2\frac{\xi^3}{l^3};\phi_2(\xi)=\xi-2\frac{\xi^2}{l}+\frac{\xi^3}{l^2};$$

$$\phi_3(\xi)=3\frac{\xi^2}{l^2}-2\frac{\xi^3}{l^3};\phi_4(\xi)=-\frac{\xi^2}{l}+\frac{\xi^3}{l^2}$$

$$\Rightarrow$$

$$\Phi=\left[\phi_1(\xi)\quad\phi_2(\xi)\quad\phi_3(\xi)\quad\phi_4(\xi)\right]$$

$$\Phi(\xi=0)=\Phi_0=\left[1\quad 0\quad 0\quad 0\right]$$

$$\Phi(\xi=l_n)=\Phi_{l_n}=\left[0\quad 0\quad 1\quad 0\right]$$

$$\Phi'(\xi=0)=\Phi'_0=\left[0\quad 1\quad 0\quad 0\right]$$

$$\Phi'(\xi=l_n)=\Phi'_{l_n}=\left[0\quad 0\quad 0\quad 1\right]$$

The terms $w(l_n)V_n-w(0)V_{n-1}-w'(l_n)M_n+w'(0)M_{n-1}$

lead to the force vector $\left\{-V_{n-1}\quad M_{n-1}\quad V_n\quad -M_n\right\}^t$

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Now I write, I use capital Phi to denote this trial functions, it is 1 row and 4 columns here, and at XI = 0 it is given by this, and at XI = Ln this is given by this, and similarly phi dash of, phi dash at XI = 0, and XI = LN are given by this, now using these relations we can take care of the terms at the boundary, and we can show that the terms that is this boundary terms lead to force vector – V N – 1, M N-1, VN and – MN transpose because of these you know with properties of the trial function.

Now we can examine each term one by one and see what is the contribution they make, so this

The terms $\int_0^{l_n} mw(\xi)\ddot{y}(\xi,t)d\xi$ & $\int_0^{l} w''(\xi)EIy''(\xi,t)d\xi$ lead respectively to the beam element mass and stiffness matrices given by $(\text{with } l = l_n)$

$$M = \frac{ml}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \& K = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

Similarly $\int_0^{l_n} cw(\xi)\dot{y}(\xi,t)d\xi$ leads to the beam element damping matrix.

The evaluation of $\int_0^{l_n} w(\xi)f(\xi,t)\delta(\xi-\xi_0)d\xi = w(\xi_0)f(\xi_0,t)$ requires elaboration.

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first term MW(sai) Y double dot(sai, t) and the second term W double prime EIY double prime lead respectively to a beam, mass, and stiffness matrices given by this, where L is LN is given by this, so this we have seen in the last class. Similarly the third term involving damping terms leads to the beam element damping matrix. Now the evaluation of the other term which is W(sai) F, that the wheel force term which is actually since is a direct delta function we can quickly do the integration it is given by this, so this requires elaboration.

Denote $\alpha(t)=\{u_1(t) \quad u_2(t) \quad u_3(t) \quad u_4(t)\}^t$

We have

$$f(\xi_0,t)=I\left[t_{n-1}<t<t_n\right]\left\{(m_u+m_s)g+k_v\{u-y_0\}+c_v\left(\dot{u}-\frac{Dy_0}{Dt}\right)-m_u\frac{D^2y_0}{Dt^2}\right\}$$

$$y_0(t)=y(\xi_0,t)=\Phi(\xi_0)\alpha$$

$$\frac{D}{Dt}=(v+at)\frac{\partial}{\partial\xi}+\frac{\partial}{\partial t}$$

$$\frac{Dy_0}{Dt}=\left\{(v+at)\frac{\partial}{\partial\xi_0}+\frac{\partial}{\partial t}\right\}\Phi(\xi_0)\alpha(t)$$

$$=(v+at)\Phi'(\xi_0)\alpha(t)+\Phi(\xi_0)\dot{\alpha}(t)$$

$$\frac{D^2y_0}{Dt^2}=\left\{(v+at)\frac{\partial}{\partial\xi_0}+\frac{\partial}{\partial t}\right\}\left\{(v+at)\Phi'(\xi_0)\alpha(t)+\Phi(\xi_0)\dot{\alpha}(t)\right\}$$

$$=(v+at)^2\Phi''(\xi_0)\alpha(t)+(v+at)\Phi'(\xi_0)\dot{\alpha}(t)+a\Phi'(\xi_0)\alpha(t)$$

$$+(v+at)\Phi'(\xi_0)\dot{\alpha}(t)+\Phi(\xi_0)\ddot{\alpha}(t)$$

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Now what I will do is I will use alpha(t) to denote the nodal degrees of freedom U1, U2, U3, U4 transpose, so it is a 4 x 1 vector, so now F(xi naught, t) is given by using the definitions that we have introduced given by, this is given by this, and now Y naught (t) itself is Y(xi naught, t) which is phi(xi naught) alpha, that is the representation we are using, so this is 1 x 4, this is 4 x 1, therefore this is scalar representation.

Denote $\alpha(t)=\{u_1(t) \quad u_2(t) \quad u_3(t) \quad u_4(t)\}^t$

$$\frac{D^2 y_0}{Dt^2}=\left\{(v+at)\frac{\partial}{\partial \xi_0}+\frac{\partial}{\partial t}\right\}\{(v+at)\Phi'(\xi_0)\alpha(t)+\Phi(\xi_0)\dot{\alpha}(t)\}$$

$$=(v+at)^2\Phi''(\xi_0)\alpha(t)+(v+at)\Phi'(\xi_0)\dot{\alpha}(t)+a\Phi'(\xi_0)\alpha(t)$$

$$+(v+at)\Phi'(\xi_0)\dot{\alpha}(t)+\Phi(\xi_0)\ddot{\alpha}(t)$$

$$=\Phi(\xi_0)\ddot{\alpha}(t)+\left[\Phi(\xi_0)+2(v+at)\Phi'(\xi_0)\right]\dot{\alpha}(t)+$$

$$\left[(v+at)\Phi'(\xi_0)+(v+at)^2\Phi''(\xi_0)+a\Phi'(\xi_0)\right]\alpha(t)$$

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Now I have to take care of this terms D/DT of Y naught, D square/DT square of Y naught, so we can write now D/DT as this is operator nu + AT dou/dou sai + dou/dou T, operating on Y naught it produces this equation and we get this equation, that is this, this acts on Y naught and we get this equation, and the second derivative accordingly can also be obtained by repeating this product operation on this, so we get the terms this and this are computed.

$$\phi_1(\xi)=1-3\frac{\xi^2}{l^2}+2\frac{\xi^3}{l^3};\phi_2(\xi)=\xi-2\frac{\xi^2}{l}+\frac{\xi^3}{l^2};$$

$$\phi_3(\xi)=3\frac{\xi^2}{l^2}-2\frac{\xi^3}{l^3};\phi_4(\xi)=-\frac{\xi^2}{l}+\frac{\xi^3}{l^2}$$

$$\Rightarrow$$

$$\Phi=\begin{bmatrix}\phi_1(\xi) & \phi_2(\xi) & \phi_3(\xi) & \phi_4(\xi)\end{bmatrix}$$

$$\Phi'^{t}=\begin{Bmatrix}-\frac{6\xi}{l^2}+6\frac{\xi^2}{l^3}\\ 1-4\frac{\xi}{l}+3\frac{\xi^2}{l^2}\\ 6\frac{\xi}{l^2}-6\frac{\xi^2}{l^3}\\ -2\frac{\xi}{l}+3\frac{\xi^2}{l^2}\end{Bmatrix}\&\Phi''^{t}=\begin{Bmatrix}-\frac{6}{l^2}+12\frac{\xi}{l^3}\\ \frac{4}{l}+6\frac{\xi}{l^2}\\ \frac{6}{l^2}-12\frac{\xi}{l^3}\\ -\frac{2}{l}+6\frac{\xi}{l^2}\end{Bmatrix}$$

Now, so we have D square Y naught/DT square using this notation we get this representation, so we can rearrange the terms and now see what happens, so this is the phi sai transpose and phi double sai transpose that is required in defining these functions, that is elaborated here.

$$\left\{w(\xi)EIy'''(\xi,t)\right\}_0^{l_e}-\left\{w'(\xi)EIy''(\xi,t)\right\}_0^{l_e}+\int_0^{l}w''(\xi)EIy''(\xi,t)\,d\xi\;+$$

$$\int_0^{l_e}w(\xi)\left\{m\ddot{y}(\xi,t)+c\dot{y}(\xi,t)-f(\xi,t)\delta(\xi-\xi_0)\right\}d\xi=0$$

$$\Rightarrow$$

$$\begin{Bmatrix}-V_{n-1}\\M_{n-1}\\V_n\\-M_n\end{Bmatrix}+K\alpha+M\ddot{\alpha}+C\dot{\alpha}=$$

$$I\left[t_{n-1}<t<t_n\right]\Phi^t(\xi_0)\left\{(m_u+m_s)g+k_vu+c_v\dot{u}\right\}$$

$$-I\left[t_{n-1}<t<t_n\right]k_v\Phi^t(\xi_0)\Phi(\xi_0)\alpha$$

$$-I\left[t_{n-1}<t<t_n\right]c_v\Phi^t(\xi_0)\left[(v+at)\Phi'(\xi_0)\alpha(t)+\Phi(\xi_0)\dot{\alpha}(t)\right]$$

$$-I\left[t_{n-1}<t<t_n\right]m_u\Phi^t(\xi_0)[(v+at)^2\Phi''(\xi_0)\alpha(t)+(v+at)\Phi'(\xi_0)\dot{\alpha}(t)$$

$$+a\Phi'(\xi_0)\alpha(t)+(v+at)\Phi'(\xi_0)\dot{\alpha}(t)+\Phi(\xi_0)\ddot{\alpha}(t)]$$

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Now we can now write the resulting equations, so what we have done is the weight function is taken to be one of the trial functions, there are four trial functions and we get four equations, and these equations are assembled in the matrix form, so the terms involving flexural rigidity will lead to K alpha, inertia will lead to M alpha double dot and damping is C alpha dot, this is a wheel four term, so here of course the indicator function multiplies all the terms so that, it is clear that these terms are valid only when the vehicle is on the element under consideration, so these details can be absorbed but what we should notice is this has terms involving alpha, alpha dot, and as well as alpha double dot, so these terms can be transferred to the left hand side, so if

$$\begin{Bmatrix} -V_{n-1} \\ M_{n-1} \\ V_n \\ -M_n \end{Bmatrix} + \left[M + I\left[t_{n-1} < t < t_n\right] m_u \Phi^t(\xi_0)\Phi(\xi_0) \right] \ddot{\alpha}(t) +$$

$$\left[C + I\left[t_{n-1} < t < t_n\right] \Phi^t(\xi_0)\left[c_v \Phi(\xi_0) + 2m_u (v + at)\Phi'(\xi_0) \right] \right] \dot{\alpha}(t) +$$

$$[K + I\left[t_{n-1} < t < t_n\right] \Phi^t(\xi_0)$$

$$\left[k_v \Phi(\xi_0) + c_v (v + at)\Phi'(\xi_0) + m_u (v + at)^2 \Phi''(\xi_0) + m_u a \Phi'(\xi_0) \right]] \alpha(t)$$

$$= I\left[t_{n-1} < t < t_n\right] \Phi^t(\xi_0) \left\{ (m_u + m_s) g + k_v u + c_v \dot{u} \right\}$$

$$m_s \ddot{u} + c_v \dot{u} + k_v u = I\left[t_{n-1} < t < t_n\right] \left\{ c_v \frac{Dy_0}{Dt} + k_v y_0 \right\}$$

$$= I\left[t_{n-1} < t < t_n\right] c_v \left[(v + at)\Phi'(\xi_0)\alpha(t) + \Phi(\xi_0)\dot{\alpha}(t) \right]$$

$$+ I\left[t_{n-1} < t < t_n\right] k_v \Phi(\xi_0)\alpha(t)$$

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I transfer all the terms that multiply alpha double dot to the left hand side the mass matrix will have, the mass matrix of the beam element plus the contribution from the wheel force, the damping matrix that of the beam element plus the contribution from the wheel force, and similarly stiffness of the beam element plus the contribution from the wheel force, this must be equal to the contribution due to the cell fate of the structure and the load transferred from the vehicle.

$$\begin{Bmatrix} -V_{n-1} \\ M_{n-1} \\ V_n \\ -M_n \end{Bmatrix} + \left[M + \tilde{M}(t)\right]\ddot{\alpha}(t) + \left[C + \tilde{C}(t)\right]\dot{\alpha}(t)[K + \tilde{K}(t)]\alpha(t)$$

$$-I\left[t_{n-1} < t < t_n\right]\Phi^t(\xi_0)\{k_v u + c_v \dot{u}\} = I\left[t_{n-1} < t < t_n\right]\Phi^t(\xi_0)(m_u + m_s)g$$

$$\tilde{M}(t) = I\left[t_{n-1} < t < t_n\right]m_u \Phi^t(\xi_0)\Phi(\xi_0)$$

$$\tilde{C}(t) = I\left[t_{n-1} < t < t_n\right]\Phi^t(\xi_0)\left[c_v \Phi(\xi_0) + 2m_u(v + at)\Phi'(\xi_0)\right]$$

$$\tilde{K}(t) = I\left[t_{n-1} < t < t_n\right]\Phi^t(\xi_0)$$

$$\left[k_v \Phi(\xi_0) + c_v(v + at)\Phi'(\xi_0) + m_u(v + at)^2 \Phi''(\xi_0) + m_u a \Phi'(\xi_0)\right]$$

$$m_s \ddot{u} + c_v \dot{u} + k_v u - I\left[t_{n-1} < t < t_n\right]\bar{\psi}_1(\xi_0)\alpha(t) - I\left[t_{n-1} < t < t_n\right]\bar{\psi}_2(\xi_0)\dot{\alpha}(t) = 0$$

$$\bar{\psi}_1(\xi_0) = c_v(v + at)\Phi'(\xi_0) + k_v \Phi(\xi_0)$$

$$\bar{\psi}_2(\xi_0) = c_v \Phi(\xi_0)$$

Now the vehicle a degree of freedom itself can be written in this form, there is a Y naught here and DY naught/DT I will use the representation that we have derived earlier this is what we get here, so I can write now the mass matrix as M + M tilde, damping matrix as C + C tilde, and stiffness as K + K tilde, this related quantities are functions of time and they are described here, so XI naught itself is a position of the wheel which changes with time, so it is the VT + 1/2 AT square so there is a time dependency here also, so all these terms are functions of time, so we get the governing equation for the beam and for the vehicle.

Element level equation of motion

$$\text{Let } d_n = \begin{Bmatrix} \alpha(t) \\ u(t) \end{Bmatrix} \ \& \ F_n = \begin{Bmatrix} -V_{n-1} \\ M_{n-1} \\ V_n \\ -M_n \end{Bmatrix}$$

$$\begin{bmatrix} M+\tilde{M}(t) & 0 \\ 0 & m_s \end{bmatrix} \begin{Bmatrix} \ddot{\alpha}(t) \\ \ddot{u}(t) \end{Bmatrix} +$$

$$\begin{bmatrix} C+\tilde{C}(t) & I[t_{n-1}<t<t_n]\Phi^t(\xi_0)c_v \\ -I[t_{n-1}<t<t_n]\bar{\psi}_1(\xi_0) & c_v \end{bmatrix} \begin{Bmatrix} \dot{\alpha}(t) \\ \dot{u}(t) \end{Bmatrix}$$

$$+\begin{bmatrix} K+\tilde{K}(t) & -I[t_{n-1}<t<t_n]\Phi^t(\xi_0)k_v \\ -I[t_{n-1}<t<t_n]\bar{\psi}_1(\xi_0) & k_v \end{bmatrix} \begin{Bmatrix} \alpha(t) \\ u(t) \end{Bmatrix}$$

$$\begin{Bmatrix} I[t_{n-1}<t<t_n]\Phi^t(\xi_0)(m_u+m_s)g \\ 0 \end{Bmatrix} + \begin{Bmatrix} -F_n \\ 0 \end{Bmatrix}$$

Further steps: assembly, imposition of BCs

Now I can now combine the beam degree of freedom with the vehicle degree of freedom and define a combined displacement vector call it as DN, that is alpha(t) + and U(t), FN I write it as the nodal forces as the consisting of these quantity as FN, so with these notations the equation now become some M(t) into DN double dot + C(t) into DN dot + some K(t) into D dot is equal to some forcing function plus the nodal forces, so this you know is of the form that we have been talking, so this is like so we have now the time-varying, mass, stiffness, and damping terms, so this is the main feature of this problem, this is the equation at the element level, now further steps involves assembly and imposition of boundary conditions this I am not going to elaborate because we have done this several times for various problems so we need not revisit this issue again.

Now what is to be noted here is that because of the time dependency of the structural matrices these equations although they are linear the concept of natural coordinates etcetera are not applicable here, so you can’t talk about natural frequency and mode shapes for this type of systems, and the only way to tackle these problems quantitatively is to integrate them using one of the integration schemes that we discuss like Newmark-Beta method or any other method that we discussed earlier. Of course if a series of loads pass on this and if there is some periodicity associated with these functions then one can use Floquet’s theory and determine stability conditions for the vehicle, for the bridge vehicle system that is of course one possibility if we are interested in qualitative behavior, if only one vehicle passes on the bridge these time dependency is of a transient nature and there is a questions about stability will not be you know steady state, stability of steady state solutions are not discussed for this type of problems.

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Now as I already said what we have idealized the supporting structure as Euler-Bernoulli beam and the moving oscillator as a, moving vehicle as a single degree freedom system. Now in practice however the supporting structure can be very complex, this is one of the simpler forms of a bridge, the bridge structures are lot more complicated than this, and this is a moving train you know it is a series of elastics you know systems having mass, and stiffness, and damping characteristics, the thing is when do we need to do a vehicle structure interaction problem, that is a question that we can consider. Typically if the mass of the moving vehicle is comparable to the mass of the supporting structure, and if the speed of the vehicle is high, and if the natural frequency of the vehicle and the bridge are comparable, there are various conditions under which one can think of expecting that vehicle section interactions would be important. In this type of problems for example in railway bridge, a plate girder bridge, a single span plate girder bridge typically could weigh about 15 to 20 tons, whereas a locomotive like this can weigh a more than 100 tons, 110 to 120 tons, when this type of vehicle passes on bridge the mass of the vehicle becomes predominantly higher than that of the supporting structure and vehicle structure interaction would be significantly important.

Laboratory model on beam-oscillator system

S Abhinav, PhD student, IISc

Beam parameters

A simply supported beam
length of the beam: L = 5 m
mass per unit length: m = 8 kg/m
thickness of beam = 0.05 m
E = 2.0E+11 N/m2

Oscillator parameters

A single degree of freedom oscillator having the following parameters moves over the beam with uniform acceleration, starting at the left end at t=0, and exiting the beam at t=T
sprung mass m1 = 4mL kg
unsprung mass m2 = mL kg
initial velocity v_i = 10 m/s
acceleration = 5 m/s2
damping coefficient of the oscillator suspension c_o = 4 Ns/m
stiffness coefficient of the oscillator suspension k_o = 1000 N/m

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Now to illustrate the formulation that we have developed a simple exercise has been done this one of our students has developed the software based on this formulation, and this is a data for the beam structure a single span beam, this beam structure is more like a laboratory model than

Finite element discretization

number of elements nE = 12;

length of each element l = L/nE

Newmark-Beta implicit scheme

The parameters used are

beta = 0.5

gamma = 0.5

Time parameters

time step dt = 1e-5 s

Total time T = 0.4495 s

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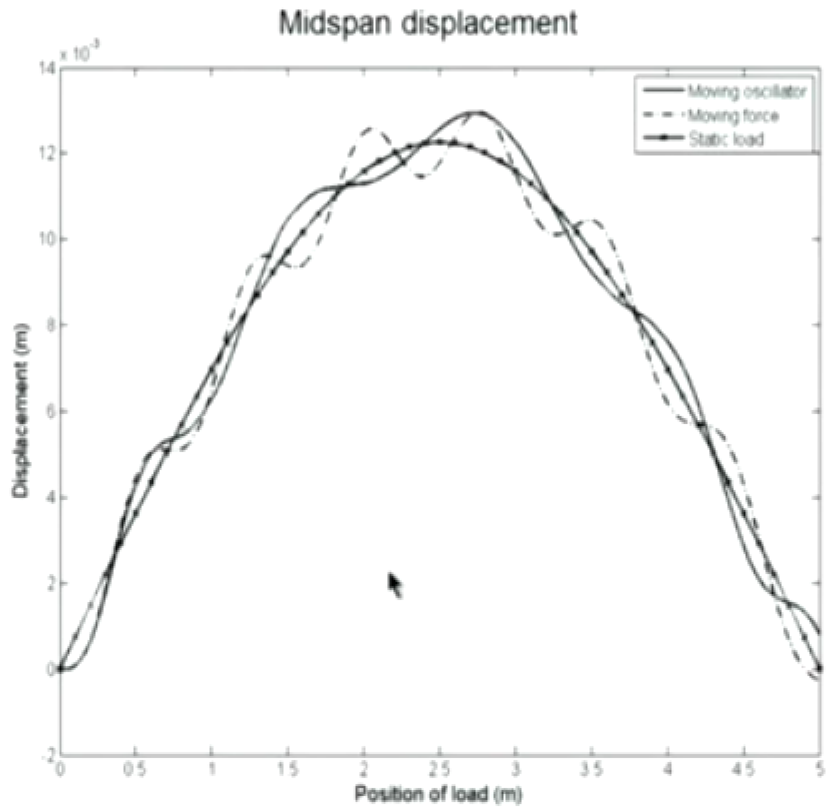
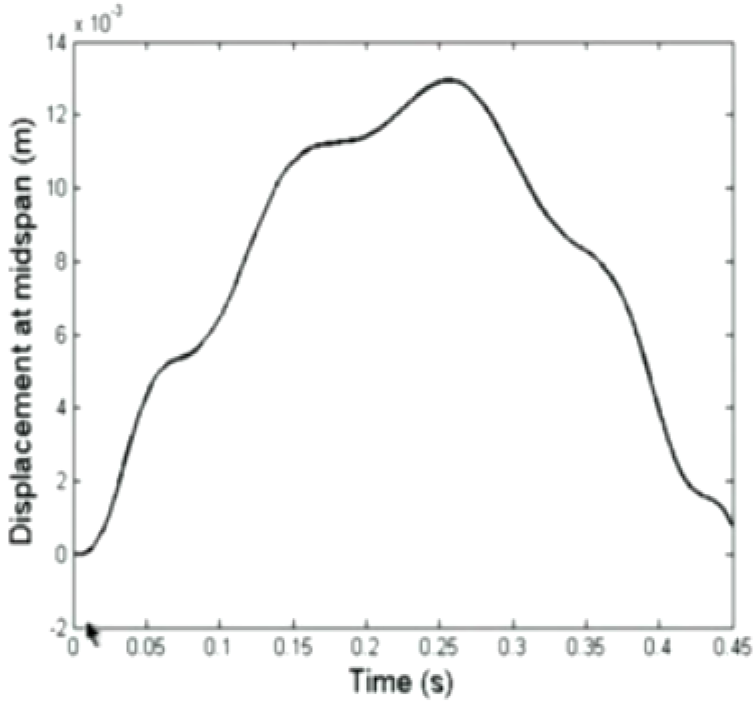
a realistic real life structure, so this structure has been discretized with 12 elements and there are 25 degrees of freedom in this model, and then the Newmark-Beta implicit scheme has been used to integrate with these details of integration and some of the results are shown here, so this



is mid span deflection as a function of the position of the load, if you see here there is one result that corresponds to analysis of structure under static loads, that is the traditional influence line type of studies that we would do.

The second graph that is the moving force diagram takes into account, the weight of the structure travelling, weight of the vehicle moving on the bridge system, the full line is a complete coupled dynamic analysis and this results are specifically designed to show the reasonableness of the algorithm developed and the calculations perform, so this is actually the



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mid-span displacement as a function of the position of the load, and this is actually the same response now plotted as a function of time. So this is just an illustration of the formulation that is developed.

References

- L Fryba, 1999, Vibration of solids and structures under moving loads, Thomas Telford Publishing, Dordrecht

- Y B Yang, J D Yau, and Y S Wu, 2004, Vehicle-bridge interaction dynamics, World Scientific, New Jersey

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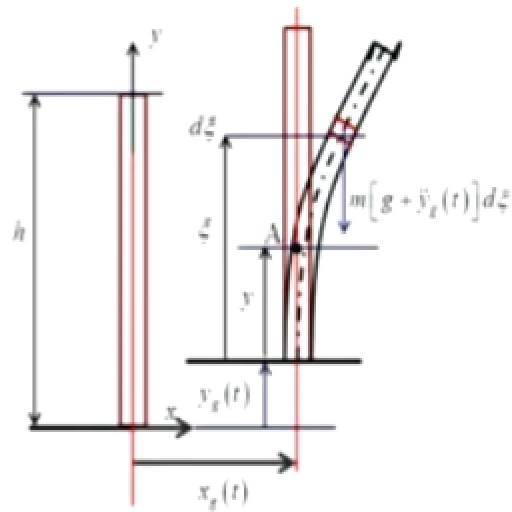
Now there are several studies on structures under moving vehicles, the book by Fryba is one of the classical books in this field, there is another book recent book by Yang and others on vehicle bridge interaction dynamics, these two books contain quite useful information for those who wish to study this subject further.

Exercise: develop FE model for the Stack under bi-axial support motions using weak formulations



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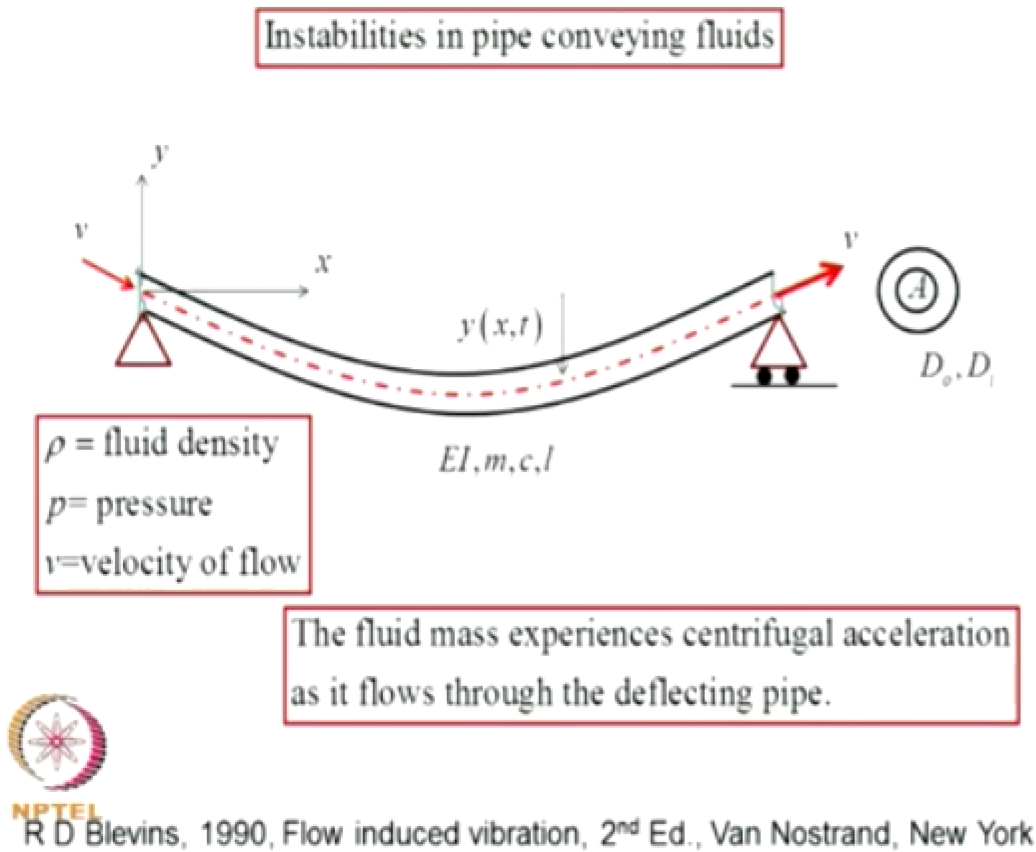
Now as an exercise I would suggest that the problem of a stack under biaxial earthquake ground motion we have formulated in the previous lecture, the exercise is to develop a finite element model for this structure using the weak formulation, so that means you start with a given partial differential equation include the parametric excitation terms and of course the time-dependent boundary support motion and then discretize it using say Euler-Bernoulli beam formulation and just as you have done for vehicle structure interaction problem the exercise is to develop a model for finite element model for this system, here again you will see that the structural matrices would be time dependent.



Now before we leave the subject of structural stability analysis there are couple of instances where stability questions arise, I just would like to mention because they have you know considerable you know interest both from application point of view and from the point of view of understanding the collage of different kinds of problems that arise in stability analysis, so this is a problem of vibration of a pipe that is conveying fluid, the fluid is moving in this pipe with a velocity V, this is a pipe cross section, area of flow is A and these are the diameters, outer and inner diameters the fluid density is rho and P is the pressure under which it is flowing, and V is the velocity of the flow.

Now the thing is that what we should model here is the fluid mass experiences centrifugal acceleration as it flows through the deflecting pipe, so this is the issue that we have to additionally handle other than the stiffness and inertia of the pipe and mass of the fluid that is flowing through, so this book by Blevins on flow induced vibration contain many examples of fluid structure interaction problems especially vibration problems of piping and other systems, so this illustration is picked up from details provided in this book.

$$EIy^{iv} + \rho A v y'' + 2\rho A v \dot{y}' + (m + \rho A)\ddot{y} = 0$$

$$y(0,t) = 0,\ y(l,t) = 0,\ y''(0,t) = 0,\ y''(l,t) = 0$$

$\rho A v y''$ = Centrifugal force due to acceleration of fluid through the curvature of the deformed pipe. Produces axial compression. Reduces natural frequencies. Causes buckling.

$2\rho A v \dot{y}'$ = Force required to rotate the fluid mass with local pipe rotation. Coriolis force. Notice the mixed derivative. Causes an asymmetric distortion of the classical mode shapes

$M = (m + \rho A)$

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Without getting into the fluid mechanics arguments I will present the governing field equation and explain what the terms mean, so we have EIY4 rho A VY double dot 2 rho AV Y dot prime and M + rho A Y double dot = 0, so this is a boundary condition for a simply supported beam condition, this is a flexural rigidity, terms coming from flexure of the beam this is the inertial term, M is the mass per unit length of the beam and this is mass per unit length of the fluid, so these 2 terms are the new terms, if there was no flow we would write this equation I mean omit these 2 terms and write the equation.

Now what is this rho A, VY double prime, prime is DY/DX dou/dou X, this is a centrifugal force due to the acceleration of fluid through the curvature of the deformed pipe, it produces axial compression and as you have seen presence of axial load reduces the natural frequencies and it causes buckling, so this is the effect that we can expect from this term, this is also an interesting term, this is a force required to rotate the fluid mass with local pipe rotation, this is a Coriolis force, you should notice the mixed derivative here Y dot prime is actually this is dou square Y/dou X dou T, so this is a mixed derivative term which creates some interesting features in obtaining the solution it causes an asymmetric distortion of the classical mode shapes, so sine N pi X/L will not you know as we will shortly see sine N pi X/L will not be the mode shape for this, exact mode shape for this. Capital M is mass per unit length and that includes mass per unit length of the pipe and the flowing fluid.

$$EIy'''' + \rho A v y'' + 2\rho A v \dot{y}' + (m + \rho A)\ddot{y} = 0$$

If we take $y(x,t) = \phi(x)\sin\omega t$, the coriolis force would lead to $\cos\omega t$ terms.

We can take $y(x,t) = \phi(x)(a\sin\omega t + b\cos\omega t)$

Similarly, if we take $\phi(x) = \sin\frac{n\pi x}{l}$ the coriolis terms produce spatially asymmetric terms for symmetric mode shapes (n=1,3,5,$\cdots$), and spatially symmetric terms for asymmetric mode shpapes (n=2,4,6,$\cdots$).

$\Rightarrow$ Assume

$$y(x,t) = \sum_{n=1,3,5,\cdots}^{\infty} a_n \sin\frac{n\pi x}{l}\sin\omega t + \sum_{n=2,4,6,\cdots}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\omega t$$

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Now we want to now construct a solution for this, and for example estimate natural frequency and see how it depends on the flow velocity that is the question. Are there any stability related issues in handling this problem? Now if we start by taking that phi(x) is sine omega T and when you substitute here this Y dot prime term will lead to cosine terms, so you will have problem in dealing with that. Similarly if you take Y(x,t) as phi(x) into A sine omega T + B cos omega T says, now to allow for presence of cos omega T term I can include cos omega T terms, so we can avoid this problem by doing this, but if we take phi(x) as sine N pi X/L, this prime Y dot prime will produce cosine functions, so again that will be problematic in handling because it will be a special asymmetric terms, it will induce spatially asymmetric terms for symmetric mode shapes for 1, 3, 5, and spatially symmetric terms for asymmetric mode shapes, so what we do is therefore a mode shape, a single mode shape is assumed in a Fourier series like this, it has sine N pi X/L term with odd indices multiplying sine omega T and again sine N pi X/L with even number of terms multiplying cos omega T.

Mind you this is not a modal summation this is a series representation for a single mode, okay that should be understood, this summation is not over omega N.

$$EIy^{iv} + \rho A v y'' + 2\rho A v \dot{y}' + (m + \rho A)\ddot{y} = 0$$

$$y(x,t) = \sum_{n=1,3,5,\cdots}^{\infty} a_n \sin\frac{n\pi x}{l}\sin\omega t + \sum_{n=2,4,6,\cdots}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\omega t$$

$$\dot{y}'(x,t) = \sum_{n=1,3,5,\cdots}^{\infty} a_n \frac{n\pi}{l}\cos\frac{n\pi x}{l}\omega\cos\omega t - \sum_{n=2,4,6,\cdots}^{\infty} b_n \frac{n\pi}{l}\cos\frac{n\pi x}{l}\omega\sin\omega t$$

The treatment of $\cos\frac{n\pi x}{l}$ terms poses difficulties.

$$\cos\frac{n\pi x}{l} = \sum_{p=1,2,3,\cdots}^{\infty} b_{np}\sin\frac{p\pi x}{l}, n = 1,2,3\cdots, 0 \le x \le l$$

$$b_{np} = \begin{cases} 0 & n+p \text{ even} \\ \dfrac{4p}{\pi\left(p^2 - n^2\right)} & n+p \text{ odd} \end{cases}$$

Note: this is an approximation. The representation would not be good near supports. This however brings in only sine terms to represent spatial variations.

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Now we substitute this into the governing equation we compute Y, Y dot prime and we will have problem here because we have sine N pi X/L and when I compute the mixed derivative there will be cos N pi X/L term, this term will be difficult to handle, so what we do is this cos N pi X/L term itself, we will expand in a Fourier series containing sign P pi X/L, okay now so this can be done, these terms can be evaluated but the problem is that the cos N pi X/L term will not satisfy the boundary conditions, although it may represent the behavior away from the boundaries, at the boundaries it will not satisfy the prescribed boundary condition, but we will ignore that effect and we'll go ahead because it brings in only sign terms to represent spatial variations and that is helpful for us.

$$a_n\left[EI\left(\frac{n\pi}{l}\right)^4-\rho A v^2\left(\frac{n\pi}{l}\right)^2-M\omega^2\right]=\frac{8\rho A v\omega}{l}\sum_{p=2,4,6,\cdots}^{\infty}a_p\frac{pn}{n^2-p^2}, n=1,3,5,\cdots$$

$$a_n\left[EI\left(\frac{n\pi}{l}\right)^4-\rho A v^2\left(\frac{n\pi}{l}\right)^2-M\omega^2\right]=-\frac{8\rho A v\omega}{l}\sum_{p=1,3,5,\cdots}^{\infty}a_p\frac{pn}{n^2-p^2}, n=2,4,6,\cdots$$

Truncate the series at N terms.

$$\bar{a}=\begin{bmatrix}a_1 & a_2 & \cdots & a_N\end{bmatrix}^T$$

$$k_{np}=\begin{cases}EI\left(\frac{n\pi}{l}\right)^4-\rho A v^2\left(\frac{n\pi}{l}\right)^2 & n=p\\ -\frac{8\rho A v\omega}{l}\frac{pn}{n^2-p^2} & n=\text{odd}, n+p=\text{odd}\\ \frac{8\rho A v\omega}{l}\frac{pn}{n^2-p^2} & n=\text{even}, n+p=\text{odd}\\ 0 & n\neq p, n+p=\text{even}\end{cases}$$

Eigenvalue problem

$$K\bar{a}-\omega^2 MI\bar{a}=0$$

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So with all this we can go ahead and obtain these equations, for N odd I get this equation, for N even I get this equation, these are infinite number of equations, they are infinite number of terms, so for practical computation we truncate this at capital N number of terms, suppose A bar is A1, A2, AN then I get KNP as this, and we can write the eigenvalue problem in terms of KNP as KA bar - omega square MIA = A bar, I is identity matrix, M is the total mass.

$K\bar{a} - \omega^2 M I \bar{a} = 0$

For nontrivial solutions, $\left|K - \omega^2 M I\right| = 0$

Illustration: retain only 2 terms.

$\Rightarrow$

$$\left[1-\left(\frac{v}{v_c}\right)^2-\left(\frac{\omega}{\omega_N}\right)^2\right]\left[16-4\left(\frac{v}{v_c}\right)^2\left(\frac{\omega}{\omega_N}\right)^2\right]-\frac{256}{9\pi^2}\left(\frac{v}{v_c}\right)^2\left(\frac{\rho A}{M}\right)\left(\frac{\omega}{\omega_N}\right)^2=0$$

$\omega_N = \dfrac{\pi^2}{l^2}\sqrt{\dfrac{EI}{M}} =$ fundamental natural frequency of the pipe without fluid flow

$v_c = \dfrac{\pi}{l}\sqrt{\dfrac{EI}{\rho A}} =$ critical velocity for buckling

$$\left(\frac{\omega}{\omega_N}\right)=\alpha\pm\left\{\alpha^2-4\left[1-\left(\frac{v}{v_c}\right)^2\right]\left[4-\left(\frac{v}{v_c}\right)^2\right]\right\}^{0.5}$$

$$\alpha=\frac{17}{2}-\left(\frac{v}{v_c}\right)^2\left[\frac{5}{2}-\left(\frac{128}{9\pi^2}\right)\left(\frac{\rho A}{M}\right)\right]$$

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Now for non-trivial solutions the determinant of this equation should be 0, now for illustration what we will do is, we will retain only 2 terms so that we can get some simple explanation terms and examine how the solution behaves, so we will carry out this as details are not provided here, so we introduce 2 parameters omega capital N, it is not the capital N natural frequency but instead it is the fundamental natural frequency of the pipe in absence of fluid flow, and VC is the critical velocity for buckling, as we saw this term is like a compressive force so we can compute the Euler buckling load corresponding to that and that gives rise to this VC, and this is this. Now in terms of that I get this equation, so this is an characteristic equation for omega I can solve for it, I get this equation.

$$\left(\frac{\omega}{\omega_N}\right) = \alpha \pm \left\{\alpha^2 - 4\left[1-\left(\frac{v}{v_c}\right)^2\right]\left[4-\left(\frac{v}{v_c}\right)^2\right]\right\}^{0.5}; \alpha = \frac{17}{2} - \left(\frac{v}{v_c}\right)^2\left[\frac{5}{2} - \left(\frac{128}{9\pi^2}\right)\left(\frac{\rho A}{M}\right)\right]$$

Remarks

- $\omega_1 \& \omega_2$ are real for $\frac{v}{v_c} \le 1$
- $\frac{\omega_1}{\omega_N} \approx \left[1-\left(\frac{v}{v_c}\right)^2\right]^{0.5}$
- At critical velocity of flow, the pipe buckles.
- $\omega = \omega_1 \Rightarrow \frac{a_2}{a_1} = \frac{8}{3\pi^2}\left(\frac{\omega_1 l}{v_c}\right)\left(\frac{v}{v_c}\right)\left[16 - 4\left(\frac{v}{v_c}\right)^2 - \left(\frac{\omega_1}{\omega_N}\right)^2\right]^{-1}$

For $\frac{v}{v_c} \le 1, \Rightarrow \left|\frac{a_2}{a_1}\right| < 0.094 \Rightarrow$ mode shape is predominantly sinusoidal

Exercise: Develop a FE model and by performing an eigenvalue analysis that includes coriolis terms, estimate the critical velocity

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Now if we examine this we see that there are 2 roots because we written 2 terms, and omega 1 and omega 2 real whenever velocity is less than the critical velocity, and we can approximate this by ignoring the terms involving mass, and this is actually we can simplify this and as a first approximation we can get the natural frequency as given by this, so at critical velocity of flow the pipe buckles, because the natural frequency goes to 0, so at that time omega is a, natural frequency would be 0.

Now how about the mode shapes? We can put omega = omega 1, and evaluate the ratio of A2/A1 and we can show this, and whenever this velocity flow is less than critical velocity you can show that this ratio A2/A1 is less than about the small number this indicates that mode shape is predominantly sinusoidal, although there is an asymmetric distortion this you know seems to produce marginal effect and results from this formulation have been compared in the Blevins book with experimentally observed data and reasonable comparisons have been found.

Now I would like to set an exercise we will consider this problem, and starting with this equation the problem is to develop a finite element model by performing an eigenvalue analysis that introduce Coriolis terms estimate the critical velocity, that means we will not adopt this representation in solving the problem, that is this representation we will discretize deal with a generalized eigenvalue problem and examine the relationship between complex valued natural frequencies mode shapes and the flow velocity, and infer without introducing any ad hoc assumptions or you know selecting functions which don’t satisfy boundary conditions and so on and so forth, we can examine what the finite element model teaches us, so this is left as an exercise.

Problems of transient parametric excitations

Recall

$$\ddot{x} + \omega^2 x = \frac{1}{m} U(t); x(0) = 0, \dot{x}(0) = 0$$

$$x(t) = A\cos\omega t + B\sin\omega t + \frac{1}{m\omega^2}$$

$$x(0) = 0 \Rightarrow A = -\frac{1}{m\omega^2}$$

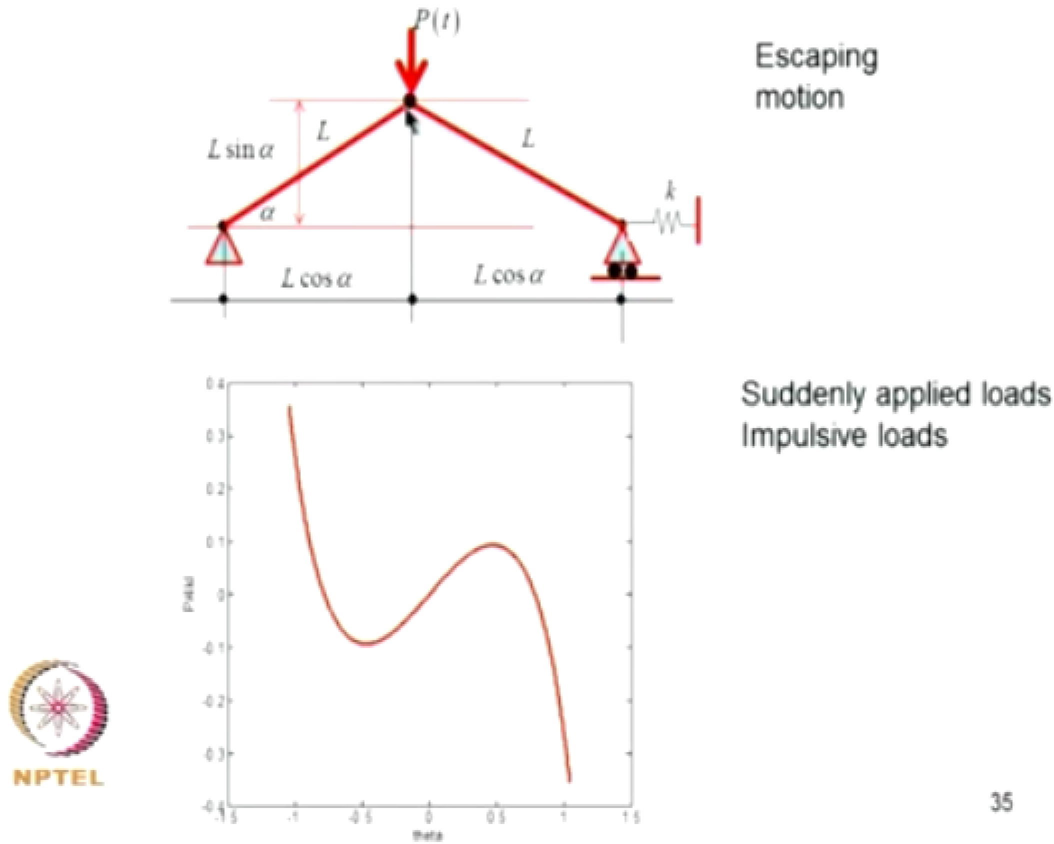
$$\dot{x}(0) = 0 \Rightarrow B = 0$$

$$x(t) = \frac{1}{k}(1 - \cos\omega t)$$

$$\text{Dynamic amplification} = \frac{\max|x(t)|}{x_{st}} = 2$$

L N Virgin, 2007, Vibration of axially loaded structures, Cambridge University Press, NY
G J Simitses and D H Hodges, 2006, Fundamentals of structural stability, Elsevier, Amsterdam.

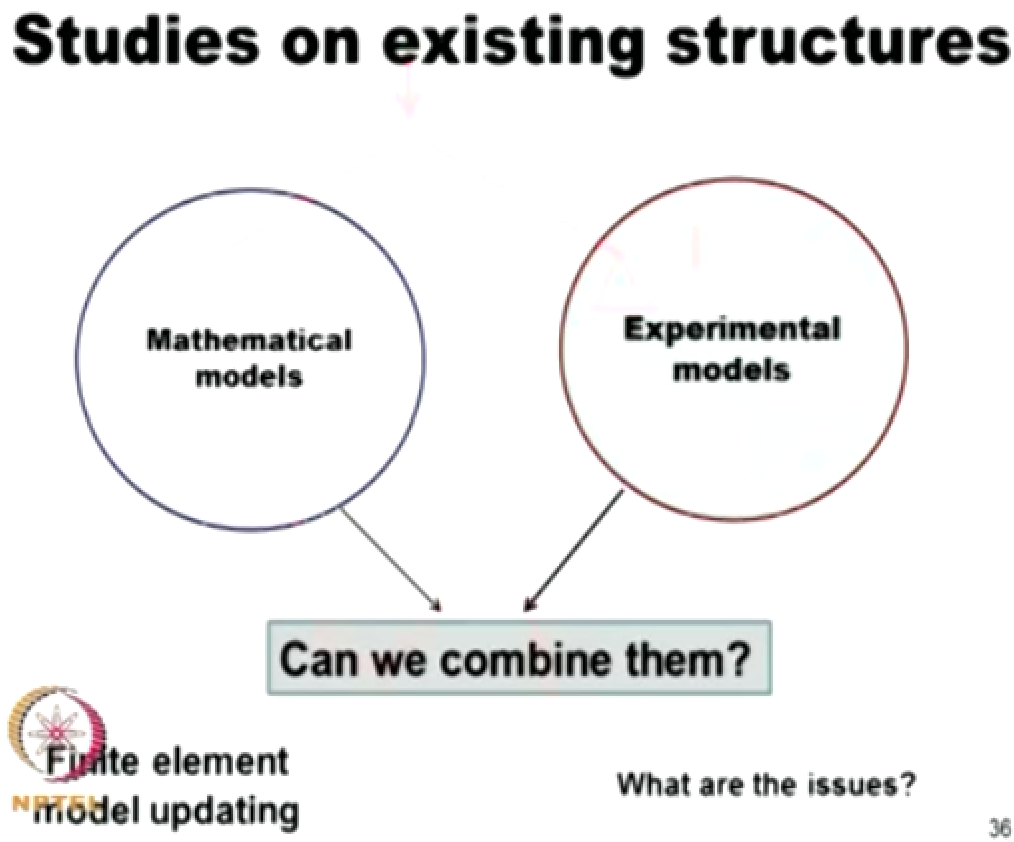
Now there is one more topic which has been quite widely studied in existing literature, can there be parametric instabilities if the parametric excitations are transient in nature, for example if we have suddenly applied loads or impulsive loads, impulsive axial loads or suddenly applied axial loads how does the structure behave? Now we can quickly recall a simple model that is undamped single degree freedom system which is subjected to N suddenly applied load, so we can examine how the system respond, so you can see U(t) the step function, that means a constant load is applied suddenly at T = 0, and let us assume that initial conditions system is at rest when this happens, and this is a complementary function plus particular integral, and using this prescribed initial conditions we can show that the total solution is given by 1/K (1 - cos omega T) 1/K is actually the static response, if the load were to be not suddenly applied, so the dynamic amplification because we have applied the load suddenly for the undamped case is about 2, okay, so this tells you that for statically loaded systems actually loaded, structures which are actually loaded by static forces, if the force were to be applied suddenly there could be instabilities, so this has been studied I have given two references here which have useful



information, see for example we have studied this problem of snap through under constant load, so when P(t) was P we found that this has a load deflection path and there is a snap through and things like that.

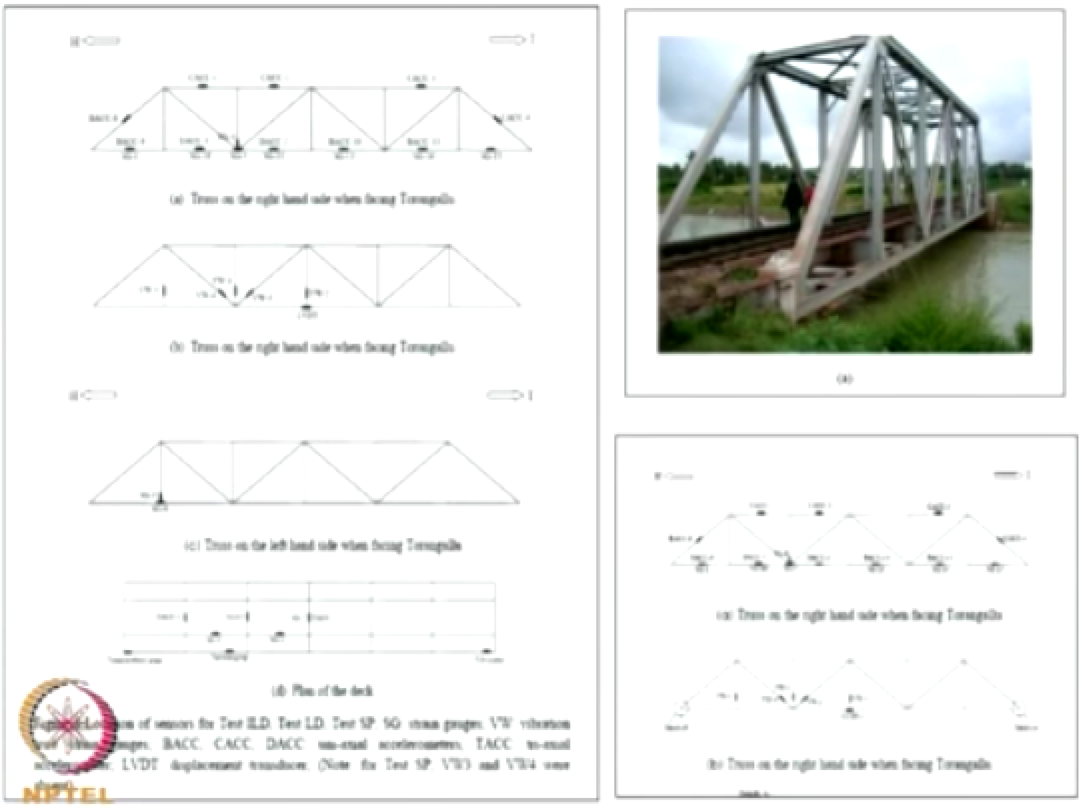
Now suppose snap through occurs at some P = P critical say, now if this P, instead of applying slowly if I apply suddenly then you can easily imagine that there is room for dynamic amplification of the response because load is applied suddenly then the response can shoot up, so even for value of P which is less than P critical the structure can lose stability and it can, you know escape and it can snap and start oscillating somewhere else, so when that happens the load that you are applying may be less than the static critical load value, because the load has been applied dynamic suddenly that happen, so this type of problems have been studied I am not going to discuss this in detail, suddenly applied loads and even impulsive loads that is load acting for a short time you know so they will, they can also cause buckling.

So in general the shell type of structures and other structures if you have impulsive type of loading there will be membrane forces that will be set up due to that transient membrane forces and during that period they can interact with flexural responses and create instability conditions, so that intuitively one can expect that might happen, but how to characterize this etcetera is this matter that it has to be studied and as I said already there are a couple of references that I have suggested for that. So with this we conclude our discussion on stability of structures.



So we move on to now a new module that is, that addresses studies on existing structures, so before a structure comes into existence the only way we can investigate how the structure might behave is through developing mathematical models, that is what we often do when we design structures and at the same time of design the structure would not have come into existence, but once the structure comes into existence then you have the opportunity to measure the response of the structure and you get an experimental approach to study the structure you know as an option. Now even after a structure comes into existence we can continue to use a mathematical modeling, now if we conduct a measurement on the performance of the structure under a given set of loading and observe the structural response, and if we were to predict the same structural response for the similar type of loading using a mathematical model it is quite likely that the two results would not match.

Now the basic question is can we combine these two models, okay, what are the issues that arise when we try to answer this question? So this type of questions are studied in a subject known as finite element model updating, so the question is you make a finite element model for an existing system you have a set of measurements that are available to you on this structure and you can mimic the situations under which you have got these measurements through a mathematical model, and the prediction from mathematical model and the observed performance of the structure may or may not match, and if there is difference how do you deal with that? Can we, for example improve upon the finite element model so that the results from finite element model are reconciled with prediction from measurements from experiment



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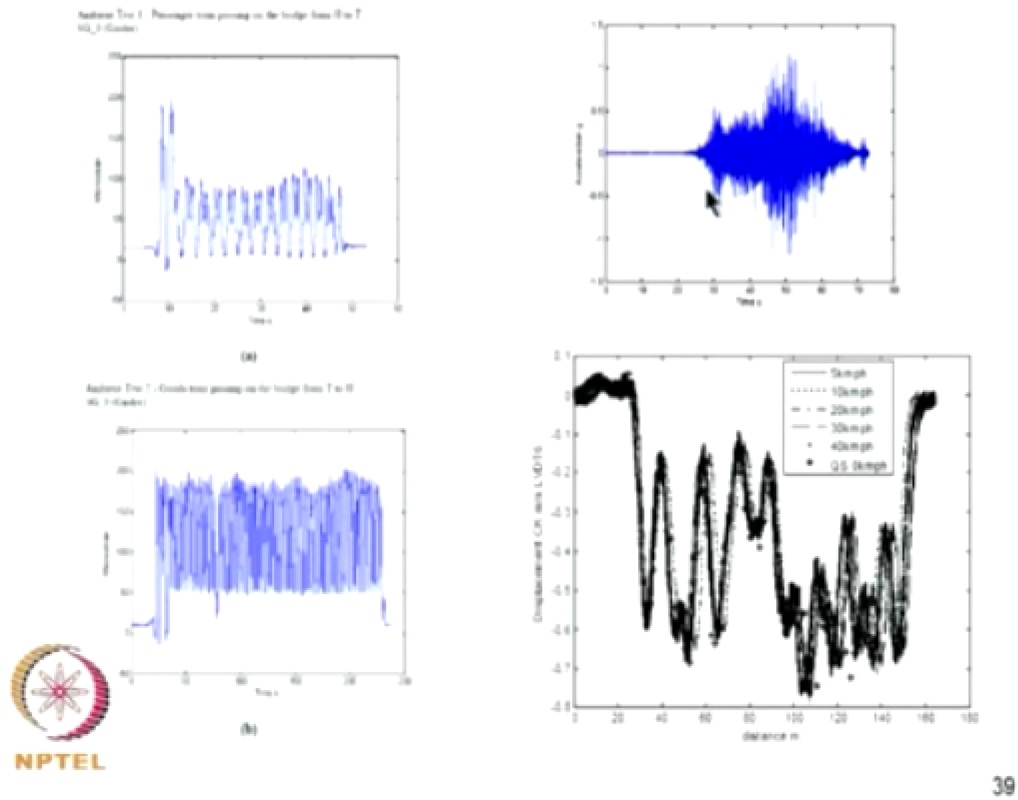
model, to address this we have to understand that given the advent of sensing technology it has become possible now to instrument structures, so this is a railway bridge structure in which we did a field work where this was instrumented through nearly about 50 to 100 sensors, we had strain gauges, we had a LV DT’s that measured displacement, we had vibrations wire strain gauges that again there is a strain gauge, accelerometers, uniaxial, biaxial etcetera, so this bridge structure was instrumented and its performance was measured under operating loads, static, dynamic, diagnostic loads where we knew what loads we were applying, and ambient loads which was due to the prevailing traffic on this bridge line, and so on and so forth.

Now the question is after obtaining this data from this existing bridge, and suppose if I were to make the finite element model for this and conduct this test numerically on my finite element model I will still be able to predict what should be the readings from these sensors, but those readings will not, those predictions will not match with what exactly we observed in the field,

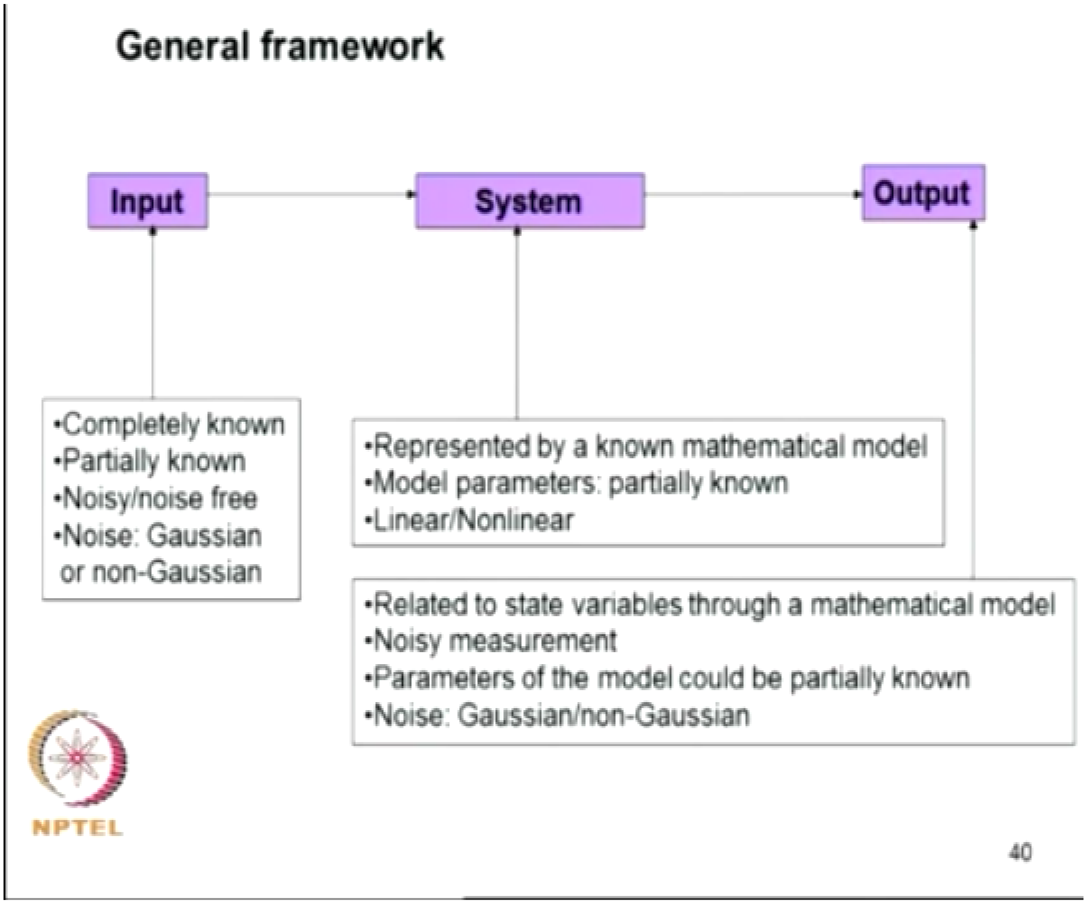
Loads

- Formations
  - **Specially configured**
  - **Ambient**
- Speeds
- Exigencies: brake binding, movement from rest & braking
- Instrumented sledge hammer

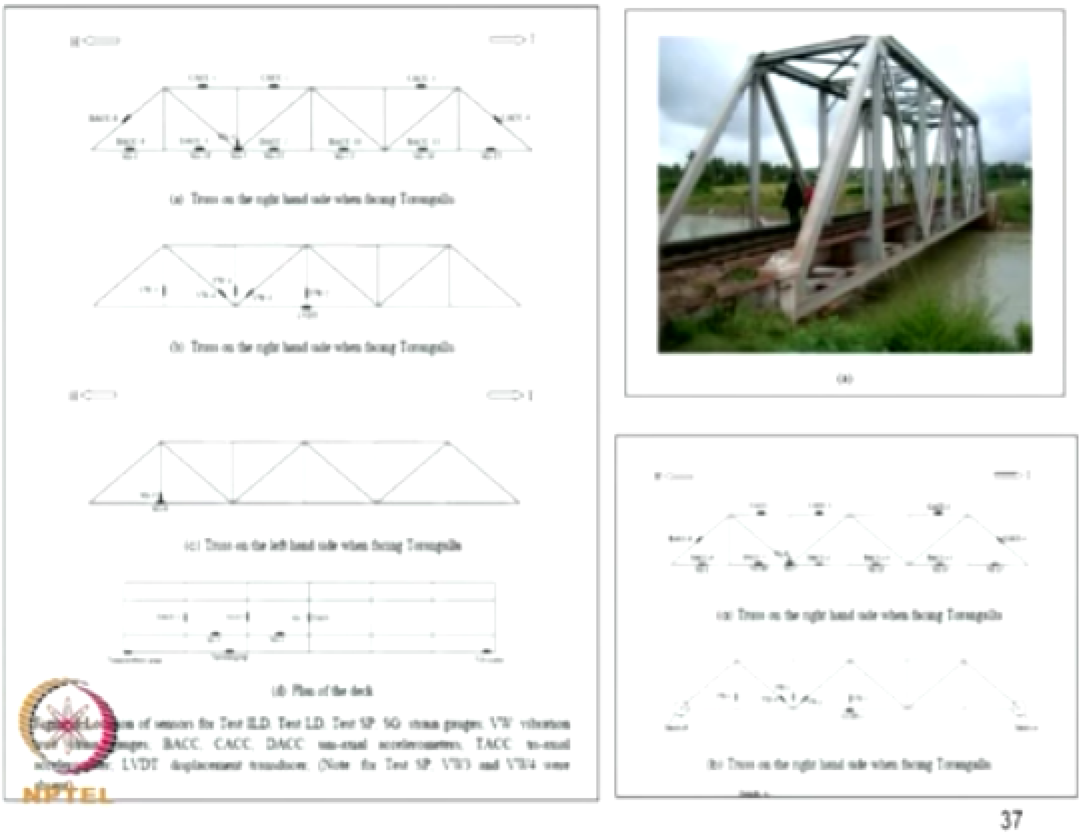
so we can measure several things, this is a test that is done to measure impulse response functions from which we can extract the frequency response function of the system, there are various tests that can be done on the structure, like a bridge structure, and these are the typical



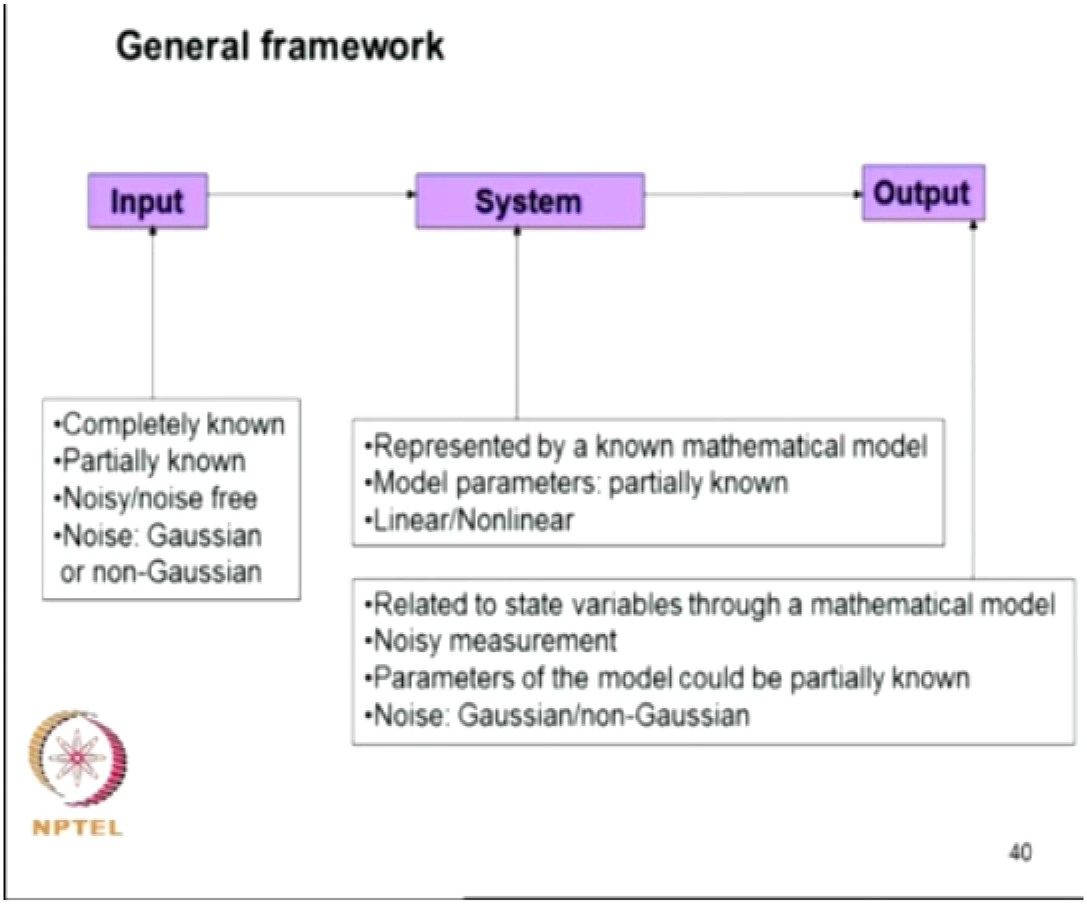
you know readings that we get, this is a strain gauge reading, this is an accelerometer reading, and this is readings from LV DT'S, so this type of data can emanate from existing structures.



Now we need another question that we try to answer is having had these measurements can we identify the parameters of the finite element model that we postulate for the given structure, the need for such exercise for example in the context of these bridges arise for example these bridges might have been designed to carry a specific level of axial loads, the user may like to



enhance the axial loads for future use, or have trains that move faster than what they were enraged at a time of design heavier vehicles, faster movement, longer trains, so on and so forth, so can these bridges cope up with those increased demands or should we repair and retrofit these retrofit these bridges to carry higher you know level of loads and so on and so forth. To be able to answer that one way is to really apply those enhanced loads and see whether the bridge can take it or not, but in life bridges that is not possible, we can’t really apply you know loads that might cause destruction to these bridges, so the best way is to apply the loads that are permitted on these bridges and then take measurements of the kind that I mentioned and make a finite element model whose predictions are reconciled with the measurements made by updating the model parameters or details of modeling, and then on the updated finite element model we can apply enhance loads, make the loads move with faster speeds and so on and so forth, and predict the performance of the structure.



Now so the general framework for this type of problems can be you know classified as shown here, so we have a system input and output, suppose if we know inputs completely and if the system is represented by a known mathematical model completely, and if we want to predict the output that is a problem of response analysis, but it is not always that we deal with that tag of problems, inputs could be partially known, it could be noisy, the noise itself could be having certain mathematical features the system could be represented by a known mathematical model, but the model parameters could be partially known, and the system behavior could be linear or non-linear, the output could be related to state variables where a mathematical model for example if you measure strain and in the mathematical model you have displacement as the state variable the relationship between measured strain and displacement is through a strain displacement relation which is, could be linear or non-linear that is a call that modular has to take, so there is a mathematical modeling issues involved in relating what we measure through the system states of the mathematical model. The measurements would be noisy, and the parameters of the model could be partially known, and again the noise characteristic would have complicating features.

PROBLEMS IN VIBRATION ENGINEERING

Input → System → Output

Input	System	Response	Remark
Given	Given	To be determined	Response analysis
Given	To be determined	Given (partially)	System identification
To be determined	To be determined	Given (partially)	Blind system identification
To de determined	Given	Given	Measurement Force identification
Given	To be selected	Bounds on response given	Design

Response analysis (forward problem): SIMPLER
Other problems: More difficult.

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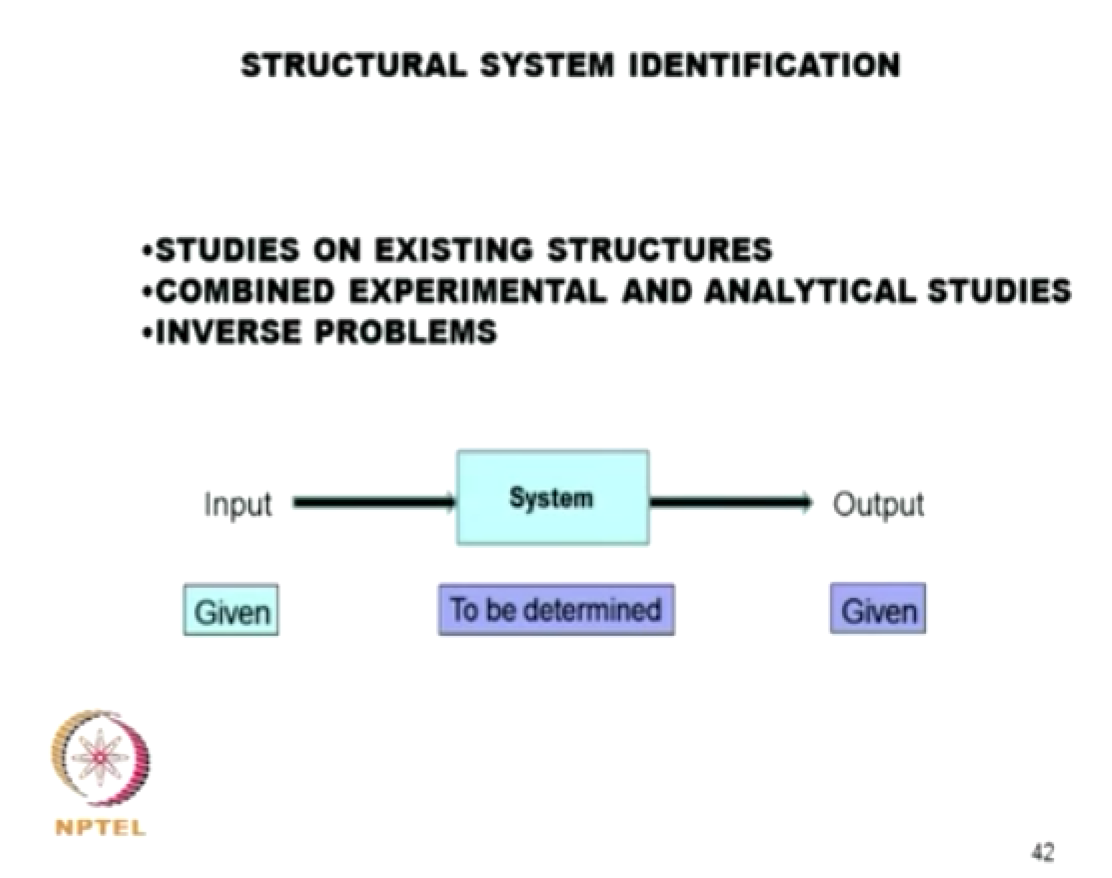
So with this, within this framework we can classify problems in vibration engineering into several categories, for example if input is given and system is given, and if response has to be determined there is a problem in response analysis, this is a forward problem, on the other hand if I know the input and if I know the output may be partially, and our task is to determine the system, that means we have to determine the mathematical model parameters of the system then this problem is known as system identification, there is various levels of making system identification it could be parametric or nonparametric as well, but it pertains to a mathematical model for the system behavior.

Similarly if the input is to be determined and system is to be determined and response is given, this is what is known as blind system identification, you only know the response of the system you have not even measure the input nor the, nor you know many things about the system. Now if input is not given and if system is known and the response is known, the unknown will be the applied input, so this is a problem of force identification or it is also a problem of measurement this is a principle on which a sensor would work, in a sensor we know the system characteristics and the response of the sensor, but we will not know what is causing the sensor to produce an output.

Now we can generalize this kind of classification, for example if input is given and system has to be selected and on the response I have prescribed bounds, then how do you select parameter, system parameters, that is a problem of design. Now response analysis is a forward problem it is a fairly simple problem that is what we have been discussing so far in the course, other problems are more difficult, problem of system identification is an inverse problem, it has its own set of difficulties and some of that hopefully we will be able to see as we go along.



So in structural system identification basically these are studies on existing structures, and these studies represent combined experimental and analytical studies, and these are inverse problems, as I already said given the input and the output our problem is to determine the system parameter.

Sources of errors in finite element models

- **Those which are inherent in FEM**
  - Discretization error: interpolation
  - Solution errors: integration, round off, eigen extraction, modal truncation, and matrix inverse calculations
  - Modeling errors: damping
- **Those which are introduced by the analyst**
  - Choice of elements to represent a given geometry
  - Omission of unimportant details
  - Modeling of boundary conditions
  - Modeling of joints
  - Choice of constitutive laws
  - Numerical values assigned to the model parameters

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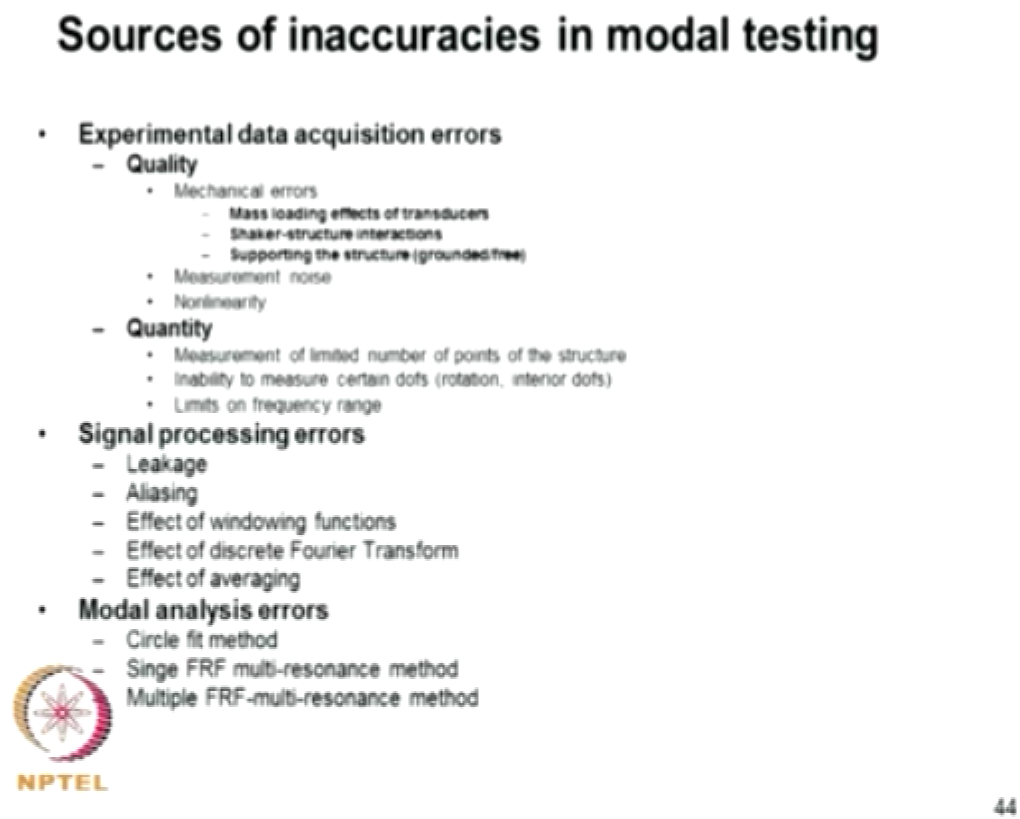
Now if I make a finite element model for an existing structure what could be sources of errors, those which are inherent in finite element method, for example there will be discretization error, and a question of interpolation, then there are solution errors you carry out integrations, there is the round off on a computer you will adopt certain algorithms for eigenvalue extraction, you expand response in you know mode shapes and you may truncate the modes at a given value and there are many computational issues like inverting matrices and so on and so forth. The other issue is modeling of damping, so damping is a very contentious issue whether damping is linear or non-linear, viscous, or structural, proportional or non-proportional, or a mixture of all of them, there are many issues there.

Now these are some of the things that are inherent in any finite element, whenever you use finite element method you have to make choices on all these parameters related to this so that you get acceptable results, but whenever an analyst use this finite element method for the same problem not analyst will come out with the same model, so one has to choose elements to represent the given geometry then you may omit unimportant details, then modeling of boundary conditions requires careful consideration, most of the important decisions that a modeler takes in context of modern finite element method, application of final element method to structural engineering problems relate to questions on boundary conditions, is the boundary condition on displacement fixed or hinged or sliding or whatever.

Then modeling of joints, so if there is partial fixity or the joints are flexible, for example in a transmission line tower the joints may not be pinned, there may be partial transmission of moments, and similarly at joints where we think there is a fixity a condition it may not be truly fixed there could be some flexibility. Then choice of constitutive laws, we often make assumptions of homogeneity, isotropy, and so on and so forth, how far they are valid? Then numerical values assigned to the model parameters, these are the decisions that analyst makes, analyst has options and to make more and more refined models to make you know to deal with each of these issues, but in any given situation some choice will be made.



How about in experimental work? So there is data acquisition errors, so there are mass loading effects of the transducers, then the exciter and structure interactions could be there, and how do you support the structure in a modal testing? These are in relation to model testing, and there is measurement noise, and how do you know the structure is behaving linearly when you are conducting a test aimed at finding natural frequencies and mode shapes? Then there is again in experimental work also the number of sensors is limited, so measurement of limited number of points on the structure and certain degrees of freedom may not be possible to measure like rotation and interior degrees of freedom, the sensing technology is improving with you know many things are becoming possible, but still there are limitations.

Then limits on frequency range, the signal processing errors, there is leakage, aliasing, windowing, and effect of using discrete Fourier transform, and when you measure frequency response function you will have to do averaging across a fixed number of samples that has, that introduces sampling fluctuations, and then after you measure frequency response functions to extract the natural frequencies and mode shapes that you have to use certain algorithm, a simple one is circle fit method, you can use a single FRF or multiple FRF method so on and so forth, so we are not discussed these issues but I am summarizing the possible sources when we do model testing, model testing are aimed at finding natural frequencies mode shapes participation factors and modal damping through experimental methods.

Updating:

FE Model: studies on an idealized mathematical model
Experimental model: studies on actual structure

It is generally believed that experimental results are more trustworthy than the results based on numerical modeling

General philosophy of updating
Results from numerical models + corrections=Results from experiments

The Two philosophies
•Local Corrections
- •Error localization
- •Error Correction

It is assumed that the mathematical model consists of discrete locatable errors associated with a physical meaning.

•Global Corrections
Corrections are made in a curve-fitting sense. The corrections would not have any specific physical meaning

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Now the problem of updating you know the issue is that finite element model are aimed at studies on an idealized mathematical model, and experimental model are studies on actual structure, so it is generally believed that experimental results are more trustworthy than the result based on numerical modeling. The general philosophy of updating is that results from numerical models plus corrections lead to results from experiments I mean this is oversimplified statement but this is a general philosophy.

Now we can do, there are two things, we can do local corrections, that means we can find out where exactly within the finite element model there are errors, we localize the errors and correct the errors, it is assumed that the mathematical model consists of discrete locatable errors associated with the physical meaning. Then global corrections, corrections are made in a curve fitting sense the corrections would not have any specific physical meaning simply the mathematical model is forced to reconcile with the prediction from experiments.

Why experimental models are more acceptable?

- No compromise on
  - constitutive laws,
  - boundary conditions,
  - joint behavior,
  - dissipation characteristics,
  - stiffness and mass distributions
  - presence of residual stresses

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Why experimental models are more acceptable? There are no compromises on constitutive laws, boundary conditions, joint behavior, damping characteristics, and stiffness and mass distributions, and any presence of residual stresses, so we don't make any assumption, they are what they are in an experiment, but whereas in a mathematical model we need to make a model for each one of this, it can be as refined as you wish but finally a choice has to be still made.

Difficulties associated with locating errors in theoretical model

- Insufficient experimental modes
- Insufficient experimental coordinates
- Size and mesh incompatibility of the experimental and FE models
- Experimental random and systematic errors
- Absence of damping in FE normal modes and presence of damping in experimental normal modes

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Now difficulties associated with locating errors in the theoretical model, we may have insufficient experimental modes, insufficient experimental coordinates, and size and mesh incompatibility of the experiment and finite element models, then experimental random and systematic errors, and absence of damping in FE normal modes and presence of damping in experimental normal modes, so when you measure natural frequencies using experimental methods it will be always damped natural frequencies, and damped normal modes there is no way you can switch off damping in an experimental work, whereas in finite element model traditionally we use undamped natural frequencies and undamped normal modes, and derive natural coordinates from them.

Scope of the discussion

- Discuss mathematical framework for carrying out FE model updating
  - Inverse sensitivity methods
- Tools for comparing two models
  - Model correlation

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So this is just a quick overview of some of the issues that arise when we think of reconciling finite element model predictions with experimental, so what I aim to do in next 1 or 2 lectures is to give a glimpse of basic issues related to finite element model updating, the discussion will be focused on mathematical framework for carrying out a FE model updating, we specifically discuss what is known as inverse sensitivity methods. And then what are the tools for comparing 2 models, suppose one experimental model you have and another mathematical model you have, how do we compare the two models? So that takes us into a discussion on model correlation, so some of the metrics used for this we will discuss. So these two topics will try to cover in the following lectures, at this stage we will close this lecture.

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