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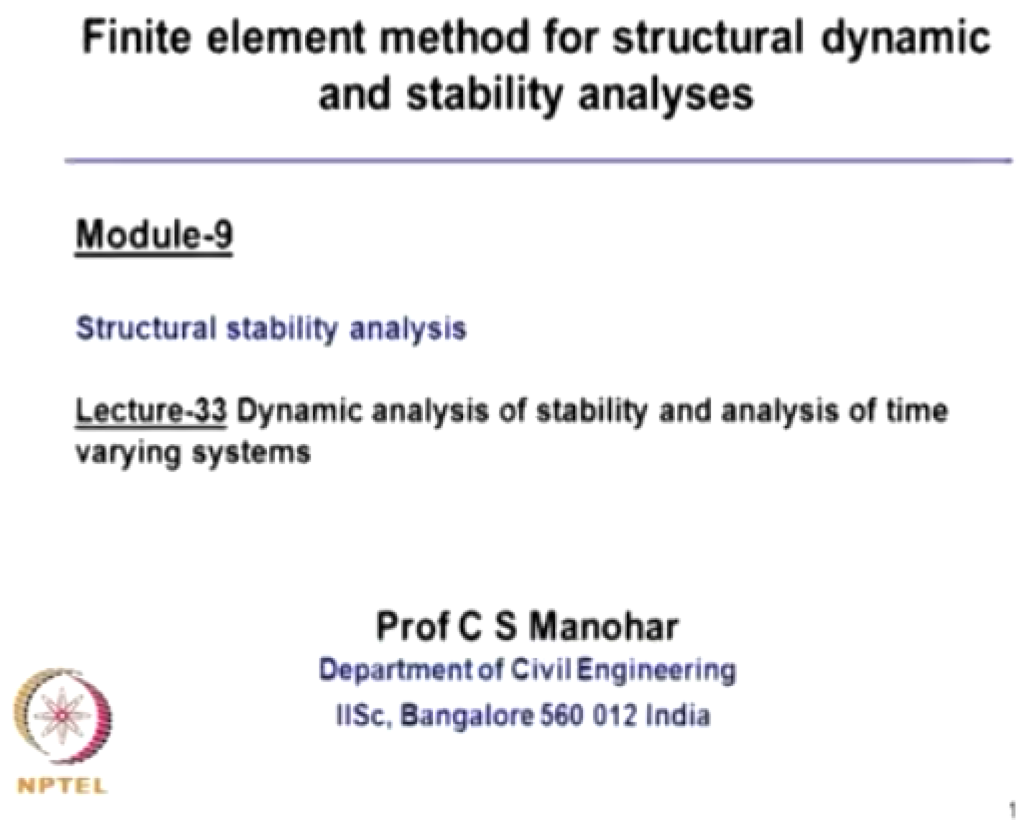
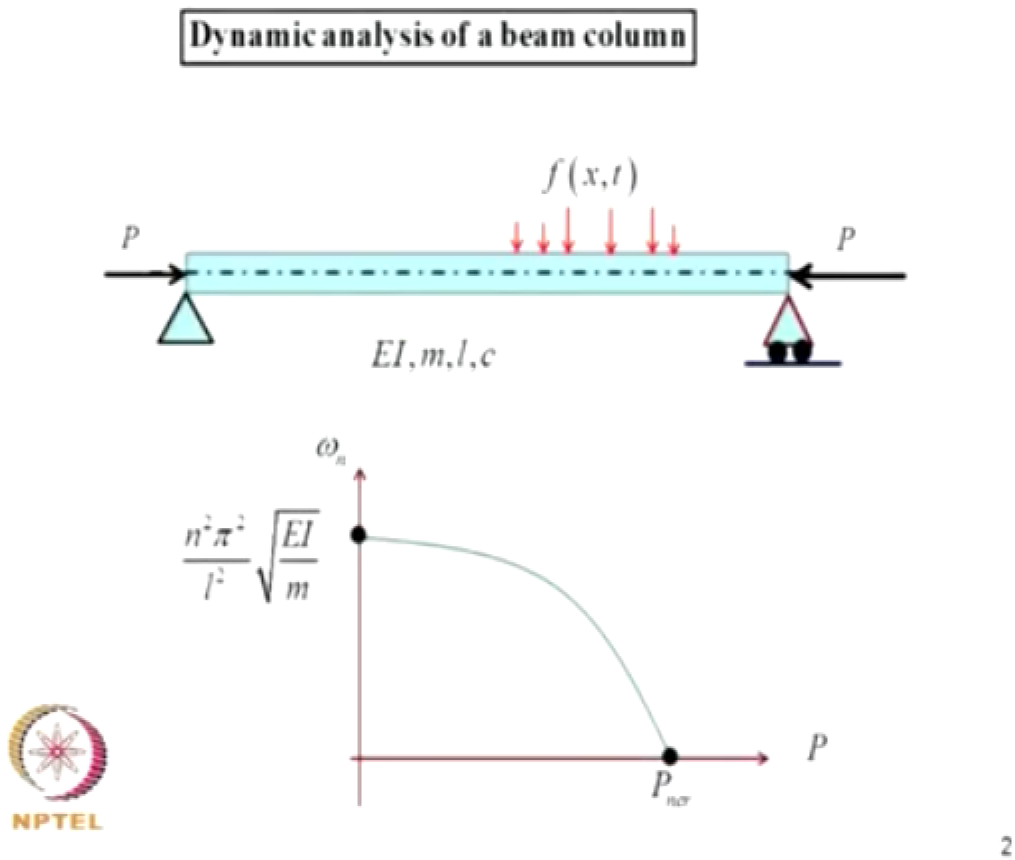
Course Title
Finite element method for structural dynamic
And stability analyses
Lecture – 33
Dynamic analysis of stability and
Analysis of time varying systems
(continued)

By
Prof. CS Manohar
Professor
Department of Civil Engineering
Indian Institute of Science,
Bangalore-560 012
India

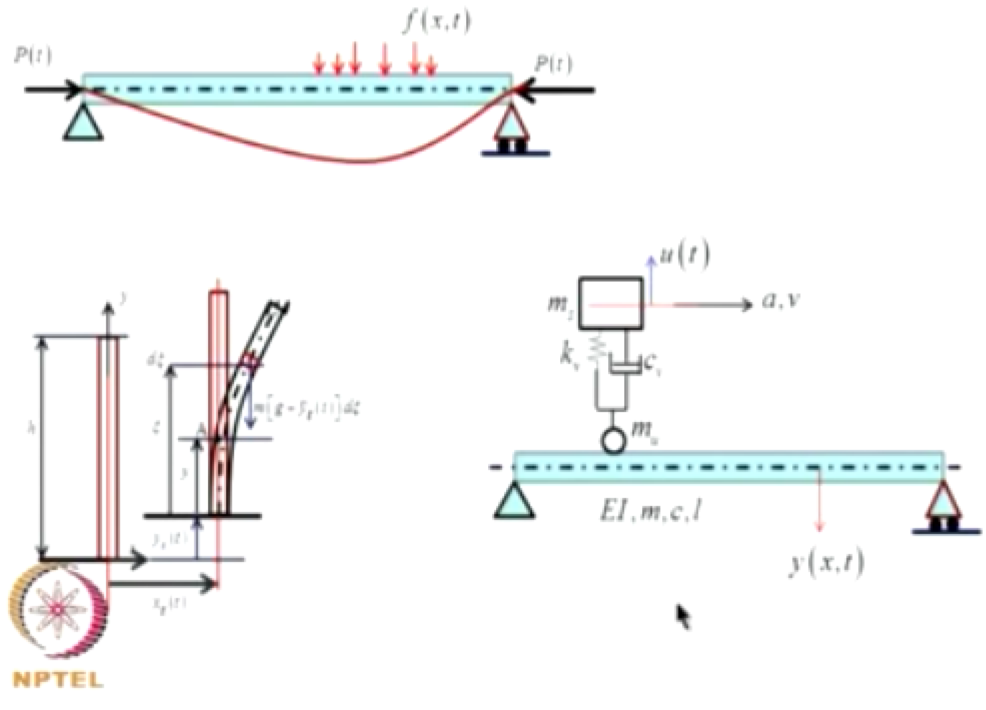


In the last class we started discussing topics related to dynamic analysis of stability and analysis of time varying systems, so we started by discussing the effect of presence of an axial load on

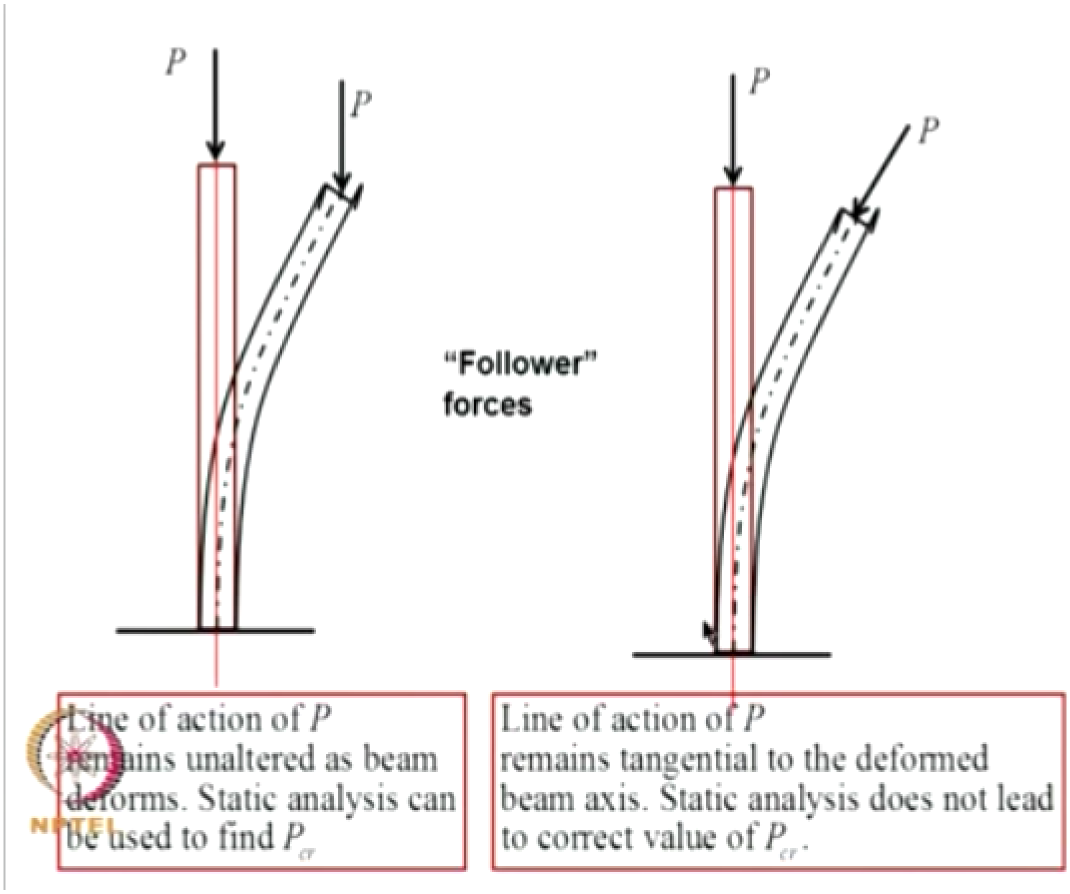


natural frequency of a beam like this, and we showed that when P approaches the critical value the natural frequency drops to 0 and the solutions will start growing in time.

Parametrically excited systems



The next question we considered was what happens if the axial loads are time dependent, we considered few situations, for example straight forward case is when a beam like this is a subjected to axial load P(t) or a stacked like this subjected to bi-axial earthquake ground motions, so there will be a vertical component we saw that the presence of vertical component appears as a parameter in the governing differential equation and we call this type of systems as parametrically excited systems. Here if you write the equation for the beam oscillations P(t) will appear as a parameter in the equation of motion, in the sense it multiplies a term involving dependent variable therefore it is called parametrically excited systems.
Another important application in civil engineering problems is problems of vehicle structure interaction that is encountered typically in bridge engineering problems, so here as a vehicle traverses supporting structure like this if one writes the combined equation for dynamics of vehicle and the beam we saw that the governing partial differential equation had time varying terms.



So we will continue our discussion with this and we posed a few problems one of the problem that we need to we've agreed to address is that of problems of follower force, so here suppose if you consider a cantilever beam which is loaded actually like this and as the structure deforms if the direction of the application of the load remains unaltered we already analyze this problem, suppose if you imagine a situation where the line of action of the load remains tangential to the deformed middle axis of the beam, then we need to revisit this problem and we need to find out what is the critical value of P, so we will consider this problem in today's class and we will show that if you perform a static analysis for this problem it will not yield a satisfactory solution, but on the other hand if you perform a dynamic analysis it gives a different answer and that seems to be logically agreeable.

Problem 1

How to characterize resonances in systems governed by equations of the form

$$M(t)\ddot{X} + C(t)\dot{X} + K(t)X = 0; X(0) = X_0; \dot{X}(0) = \dot{X}_0$$

when the parametric excitations are periodic.

Problem 2

How to arrive at FE models for PDE-s with time varying coefficients?

Problem 3

Are there any situations in statically loaded systems, wherein one needs to use dynamic analysis to infer stability conditions?

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So we will consider three problems, one is how to characterize resonances in systems governed by equations of the form M(t) X double dot, C(t) X dot + K(t) X = 0. Now the resonances occur because there are time varying terms in the mass, stiffness, and damping matrices, and if these terms are periodic in nature and then the question of resonance is appropriate because it's excitations have a steady state character, so we could expect steady state responses from the system as well and see whether the responses remain bounded or grow in time.

Next we'll consider another problem where we start with partial differential equations with time varying coefficients, we saw that in the previous lecture that this type of systems are governed by partial differential equations which have time varying terms, and we will ask the question how to make finite element models for such systems.

Next we will consider if there are any situations in statically loaded systems wherein one needs to use dynamic analysis to infer stability conditions, so these are the some of the questions that is on our agenda, we'll consider this in this and the following lecture.

Qualitative analysis of parametrically excited systems

$$\ddot{u}(t)+p_1(t)\dot{u}(t)+p_2(t)u(t)=0$$

$$u(0)=u_0;\dot{u}(0)=\dot{u}_0$$

$$p_i(t+T)=p_i(t), i=1,2$$

The governing equation is a linear second order ODE with time varying coefficients. It admits two fundamental solutions.

So we'll begin by considering qualitative analysis of parametrically excited systems for purpose of illustrating the basic concepts, we will consider a single degree freedom system and that is U(t) is the scalar variable, a scalar function of time, and it is governed by a second order differential equation of the form U double dot P1(t) U dot + P2(t) UT = 0.
Now let us assume that there are some nonzero initial conditions, and we are assuming that P1 and P2 are periodic with period capital T, now the governing equation is a linear second order ordinary differential equation with time varying coefficients, it admits 2 fundamental solutions

$$\ddot{u}(t)+p_1(t)\dot{u}(t)+p_2(t)u(t)=0$$

Let $u_1(t)$ and $u_2(t)$ be the fundamental solutions of this equation.

$$\Rightarrow u(t)=c_1u_1(t)+c_2u_2(t)$$

Consider the governing equation at $t+T$

$$\ddot{u}(t+T)+p_1(t+T)\dot{u}(t+T)+p_2(t+T)u(t+T)=0$$

Since $p_i(t+T)=p_i(t), i=1,2,$ we get

$$\ddot{u}(t+T)+p_1(t)\dot{u}(t+T)+p_2(t)u(t+T)=0$$

$\Rightarrow$ If $u(t)$ is a solution $\Rightarrow u(t+T)$ is also a solution.

$\Rightarrow$

$$u_1(t+T)=a_{11}u_1(t)+a_{12}u_2(t)$$

$$u_2(t+T)=a_{21}u_1(t)+a_{22}u_2(t)$$

$$\Rightarrow \{u(t+T)\}=[A]\{u(t)\}$$

We are interested in nature of the solution as $t\to\infty$.

so let U1 and U2 be the fundamental solutions of this equation, then any solution U(t) can be written as linear superposition of the fundamental solutions therefore I get C1 U1(t) + C2 U2(t), now we will consider the governing equation at T = t + T, so if I write this it will be U double dot(t + T) + P1(t+ T) etcetera equal to 0, now since the coefficients are periodic with period capital T, I can write for P1(t + T) and P2(t + T) I cannot P1(t) and P2(t), so what this means is if U(t) is a solution then U(t + T) also is a solution to the governing equation, because if U(t) satisfies this equation U(t + T) also satisfies this equation, so this enables us to write U1, that is the two fundamental solutions at t + T is another solution in terms of U1 and U2, so I get U1(t + T) as this and U2(t + T) this, and in a matrix form I get this equation, so we are interested in nature of the solution as T tends to infinity, so this we briefly touched upon towards the end of the previous lecture, so we will formulate this problem in greater detail as we go along.

$$\{u(t+T)\} = [A]\{u(t)\}$$

$$\lim_{t \to \infty} u(t) \to ?$$

This is equivalent to asking $\lim_{n \to \infty} u(t + nT) \to ?$

$$u(t+T) = Au(t)$$

$$u(t+2T) = Au(t+T) = A^2 u(t)$$

$$\vdots$$

$$u(t+nT) = Au(t+(n-1)T) = A^n u(t)$$

$\Rightarrow$ The behavior of $\lim_{n \to \infty} u(t+nT)$ is controlled by the behavior of $\lim_{n \to \infty} A^n$.

Intutively, one can see that this, in turn, depends upon the nature of eigenvalues of A.

Now we are interested in finding limit T tending to infinity what happens to U(t), this is equivalent to asking I will write U(t) as (t + NT) and allow N to infinity, where N is an integer 1, 2, 3, 4, 5, 6 etcetera, now U(t + T) is AU(t), therefore U(t + 2T) is AU(t + T) which is A square U(t) and so on and so forth, so if I write U(t + NT) it will be A to the power of N U(t), so the behavior limit of N tending to infinity of U(t + NT) is controlled by the behavior of this behavior of A to the power of N as N tends to infinity, so intuitively one can see that this in turn depends upon the nature of eigenvalues of A.

$u(t+T) = Au(t)$

Introduce the transformation $u(t) = Qv(t)$

$u(t+T) = Au(t) \Rightarrow Qv(t+T) = AQv(t)$

Pre-multiply by Q^t

$Q^t Qv(t+T) = Q^t AQv(t)$

Select Q such that A is diagonalized.

That is, we wish to find Q such that $Q^t Q$ & $Q^t AQ$ are diagonal.

Consider the eigenvalue problem: $A\phi = \lambda\phi$

Select Q to be the matrix of eigenvectors of A.

$Q = [\Phi_1 \quad \Phi_2]$

$$\Rightarrow v(t+T) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} v(t)$$

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9

Now so what we do to analyze this problem, we consider this equation U(t + T) = AUT, then we introduce the transformation U(t) is Q into V(t) with the aim to diagonalize A matrix, so U(t + T) is AUT that means QV(t + T) is AQV(t) that means in this equation I'm substituting for U according to this transformation. Now I will pre multiply by Q transpose so I get this equation, now the objective of making this transformation is that we wish to select Q such that A is diagonalized that is we wish to find Q such that Q transpose Q and Q transpose AQ are diagonal, so we can start by considering the eigenvalue problem associated with matrix A, A phi is lambda phi. Now mind you A is not a symmetric, need not be a symmetric matrix it is real valued but not necessarily symmetric.

Now we select Q to be the matrix of eigenvectors of A, that is phi 1 phi 2, and we can show that V(t + T) is lambda 1 V(t) because the eigenvectors will be satisfying the conditions phi T phi will be I, I can normalize them such that this is true and then I can Q phi will be some diagonal matrix lambda, which is the matrix of eigenvalues of A as shown here.

$$\Rightarrow v_i(t+T) = \lambda_i v_i(t), i = 1,2$$

$$\Rightarrow v_i(t+nT) = \lambda_i^n v_i(t), i = 1,2$$

We have

- $\lim_{t\to\infty} v_i(t) = \lim_{n\to\infty} v_i(t+nT) \to 0$ if $|\lambda_i| < 1, i = 1,2$
- $\lim_{t\to\infty} v_i(t) = \lim_{n\to\infty} v_i(t+nT) \to \infty$ if $|\lambda_i| > 1, i = 1,2$
- $v_i(t)$ is periodic with period T if $\lambda_i = 1, i = 1,2$
- $v_i(t)$ is periodic with period $2T$ if $\lambda_i = -1, i = 1,2$

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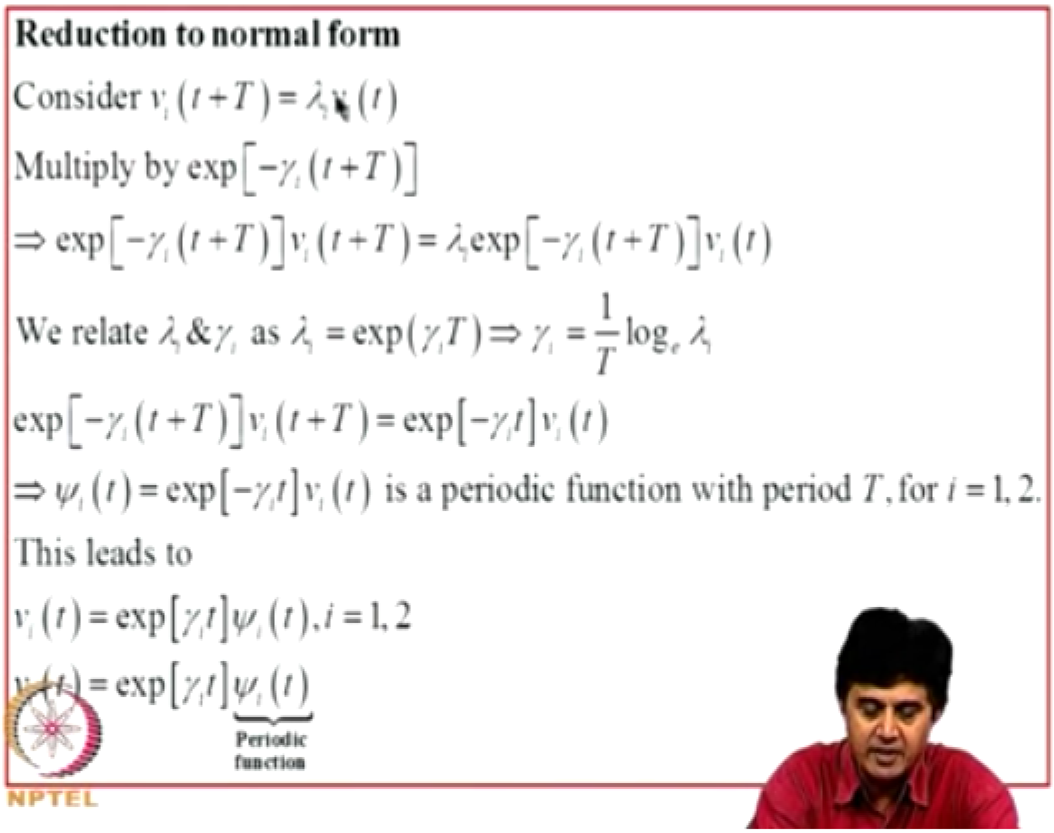
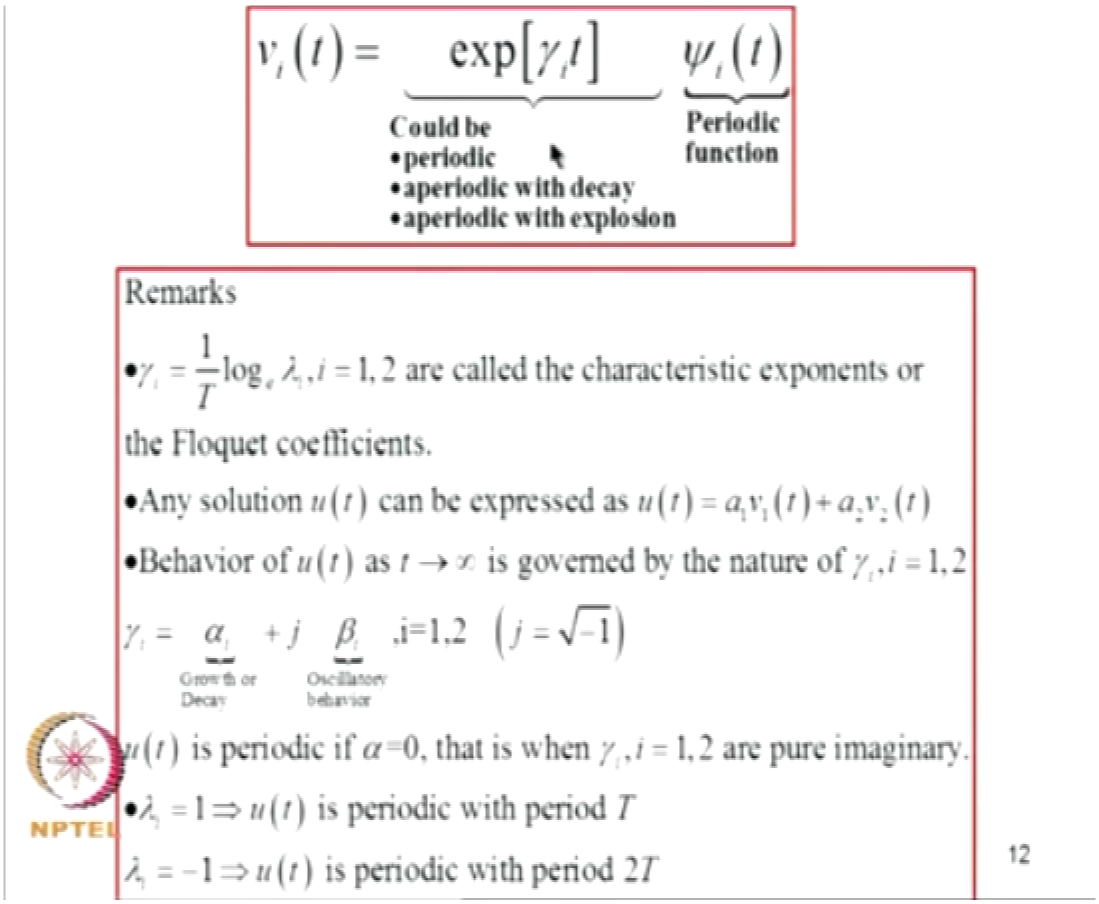
Now that would mean VI(t + T) is lambda VI((t) for I equal to 1 to 2, now since the matrix is diagonalized I can do that, now therefore VI(t + NT) is lambda I to the power of N, VI(t) this, now we want to now investigate what happens to these solutions as N tends to infinity, the original function U(t) can be constructed by superposing V1 and V2, so therefore behavior of U is controlled by behavior of V1 and V2 as N tends to infinity, so we can see that as T tends to infinity that means N tending to infinity of VI(t + NT) it goes to 0 if modulus of lambda is less than 1, because a scalar equation this we saw in the last class this is a condition, because every time you multiply this number starts shrinking and it goes to 0 as N tends to infinity. On the other hand if modulus of lambda is greater than 1 this solution will blow off, if lambda is equal to 1 then it is periodic with period capital T, on the other hand if it is - 1 then after every 2T it returns to the original state, the first time you operate it will become minus of the quantity and then square of that quantity which is again returns to the original state, so the period will be 2T, so depending on the nature of eigenvalues there are 4 different behaviors possible either the solution will go to 0 as T tends to infinity or it will shoot to infinity as T tends to infinity, or it will be periodic with period capital T, or it will be periodic with period capital 2T.



Now we will try to reduce these results into a standard form, so what we will do is we will consider this equation V(t + T) is lambda I VI(t), now I introduce a new variable gamma I and I multiply the both side of this equation by this quantity exponential - gamma I (t + T) as I do this lambda and gamma are related by this equation that you can see that and therefore gamma I is nothing but 1/T log E lambda I.

Now therefore exponential of - gamma T and (t + T) is VI (t + T) which is exponential gamma IT VI(t), now if I call this quantity exponential - gamma IT into VIT as sai(t) it is clear from this expression that sai(t) is periodic function with period capital T, so therefore I can write VI(t) as exponential gamma I(t) into a periodic functions sai(t), so this is a periodic function, so



what we have done now? We have expressed the solution as a periodic function multiplied by an exponent and this exponent could be periodic, aperiodic with decay or aperiodic with explosion depending on nature of gamma I, so we will make some remarks, gamma I 1/T log E lambda I, these are called as the characteristic exponents or Floquet coefficients. Any solution U(t) can be expressed as linear superposition of V1 and V2, behavior of U(t) as T tends to infinity is governed by the nature of the exponent, suppose gamma IT is, we can expect it to be complex value so I write gamma as alpha I + J beta I, J is now square root – 1, I, I am using for index, so the complex number is now expressed in terms of J, so if the real part of this you know controls the growth or decay of the solution and this contributes to the oscillatory behavior so U(t) is periodic if alpha = 0, that means if this is pure imaginary this will be E raise to pure imaginary exponent this is sine and cosine terms will emerge, and therefore VI(t) will be periodic, so when gamma I is pure imaginary U(t) is periodic, if lambda = 1, U(t) is periodic with period capital T, that means this exponent you have to find out and related to gamma I, if lambda is – 1, U(t) is periodic with period 2T.

Determination of the characteristic exponents

$$\ddot{u}(t)+p_1(t)\dot{u}(t)+p_2(t)u(t)=0$$

Let $u_1(t)\,\&\,u_2(t)$ be two solutions of this equation with

$u_1(0)=1, \dot{u}_1(0)=0$ and

$u_2(0)=0, \dot{u}_2(0)=1$

We have

$$u_1(t+T)=a_{11}u_1(t)+a_{12}u_2(t)$$

$$u_2(t+T)=a_{21}u_1(t)+a_{22}u_2(t)$$

$$u_1(0)=1\Rightarrow u_1(T)=a_{11}u_1(0)+a_{12}u_2(0)=a_{11}$$

$$\dot{u}_1(0)=0\Rightarrow \dot{u}_1(T)=a_{11}\dot{u}_1(0)+a_{12}\dot{u}_2(0)=a_{12}$$

$$u_2(0)=1\Rightarrow u_2(T)=a_{21}u_1(0)+a_{22}u_2(0)=a_{21}$$

$$\dot{u}_2(0)=0\Rightarrow \dot{u}_2(T)=a_{21}\dot{u}_1(0)+a_{22}\dot{u}_2(0)=a_{22}$$

$$A=\begin{bmatrix}u_1(T) & \dot{u}_1(T)\\ u_2(T) & \dot{u}_2(T)\end{bmatrix}$$

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13

Now this is a well-known classical result in study of parametrically excited systems, so the Floquet's coefficients help us to determine boundaries of stable solutions. Now in numerical work, how do we determine Floquet's constants? So what we can do is we revisit this equation U double dot(t) + P1(t) U dot(t) + P2(t) U(t) = 0, now let U1 and U2 be two solution of this equation with the following initial condition, U1(0) is 1, U1 dot(0) is 0, and U2(0) is 0, and U2 dot(0) is 1 that means I am selecting two solutions which start from a set of linearly independent initial conditions, so I can write U1(t + T) therefore I can write in terms of A11 U1(t) + A12 U2(t), and similarly U2(t) is written like this. Now if you find, our objective is to find A matrix you see for a given system, how do you find A matrix? Moment you find A matrix you perform eigenvalue analysis and look at the nature of the eigenvalues that answers our question on stability, so it's as simple as that, so U1(0) you see it is U1(t) which is A11 U1(0) + A12 U2(0) which is nothing but A11, similarly U1 dot(0) is U1 dot(T) which is A12, similarly I can find out A11, A12, A21, A22 so therefore elements of A matrix are determined by solving this equation with two linearly independent solution over one period of excitation, that is all the numerical work that we need to do, moment we do that we will be able to form this matrix and after we can perform an eigenvalue analysis of this and then examine the nature of eigenvalues, and we will be able to infer whether the solution is periodic, decays to 0 or grows and it is periodic with period capital T or periodic with period capital 2T.

$$A = \begin{bmatrix} u_1(T) & \dot{u}_1(T) \\ u_2(T) & \dot{u}_2(T) \end{bmatrix}$$

Find eigenvalues of A

Infer nature of solutions by using the criteria

- $\lim_{t\to\infty} v_i(t) = \lim_{n\to\infty} v_i(t+nT) \to 0$ if $|\lambda_i| < 1, i = 1$ and 2
- $\lim_{t\to\infty} v_i(t) = \lim_{n\to\infty} v_i(t+nT) \to \infty$ if $|\lambda_i| > 1, i = 1,$ or 2
- $v_i(t)$ is periodic with period T if $\lambda_i = 1, i = 1, 2$
- $v_i(t)$ is periodic with period $2T$ if $\lambda_i = -1, i = 1, 2$

So we find eigenvalues of A and infer the nature of solutions by using the following criteria this we have discussed, so if modulus of lambda I is less than 1 for I = 1 to 2 then the solution will decay, and if it is greater than 1 for any of I's, if any one of the eigenvalues has satisfy this property solution will go to infinity, and this is periodic with this, if both lambdas are equal to 1, and it is periodic with period 2T both lambda's are -1, so this completes the you know a calculation of Floquet's coefficient by simple integral, so you can use if you formulate a finite element model you could use a numeric beta or any other method that we have discussed earlier and integrate this equation over one period of the parametric excitation.

If the condition $|\lambda_i| > 1$ occurs, we say that the system has got into parametric resonance.

Here the motion grows exponetially with time.

Presence of damping does not limit the amplitude of oscillations.

Amplitudes could get limited due to nonlinear effects.

This is contrast with resonance in externally driven systems:

$$\ddot{x} + \omega^2 x = P\cos\lambda t; x(0) = 0; \dot{x}(0) = 0 \Rightarrow x(t) = \frac{P}{\lambda^2 - \omega^2}\left[\cos\omega t - \cos\lambda t\right]$$

$$\lim_{\lambda\to\omega} x(t) = \lim_{\lambda\to\omega} \frac{P}{\lambda^2 - \omega^2}\left[\cos\omega t - \cos\lambda t\right] = \frac{Pt}{2\lambda}\sin\lambda t$$

$$\Rightarrow \lim_{\lambda\to\omega}\lim_{t\to\infty} x(t) \to \infty$$

Resonance response amplitudes are limited by damping.

Nonlinearity would also become important as response grows.

15

If the condition lambda, modulus of lambda greater than one occurs we say that the system has got into parametric resonance, so here the motion grows exponentially with time, okay you recall how, if that is here if you recall how a harmonically driven single degree freedom system gets into resonance for that if you consider X double dot + omega square X = P cos lambda T with say 0 initial conditions you can show that solution is given by this. Now as lambda tends to omega we get the solution as PT/2 lambda sine lambda T, so as lambda goes to omega and T tends to infinity, X(t) goes to infinity and the growth is linear in time, so this is when external excitation creates resonance in the system, but if the system gets into parametric resonance the growth is exponential.

Now in the resonance in external excitations, the resonance response amplitudes are limited by damping and non-linearity of course would also be important as response grows, but in parametrically excited systems damping has no role, if the system gets into resonance presence of damping does not limit the amplitude of the response that is you can see here already I have a U dot term, so presence of U dot term is not changing the nature of the solution.

So in this case if your damping treatment won't solve problems of parametric resonances, whereas that will solve the problem of resonance under external excitations, now here of course as amplitudes become large for a linear system amplitude tends to infinity but once amplitude cross certain limits the system nonlinearities kick in and the behavior will be altered due to presence of nonlinear terms.

Extension to MDOF systems

Consider n-dof system

$$\ddot{U}(t)+[P(t)]\dot{U}(t)+[Q(t)]U(t)=0$$

with

$$[P(t+T)]=[P(t)]_{n\times n}$$

$$[Q(t+T)]=[Q(t)]_{n\times n}$$

$\Rightarrow$

$$\{U(t+T)\}=[A]\{U(t)\}$$

Recipe

Generate a set of n independent solutions of the governing equation by using a set of n linearly independent ics.

Form the A matrix

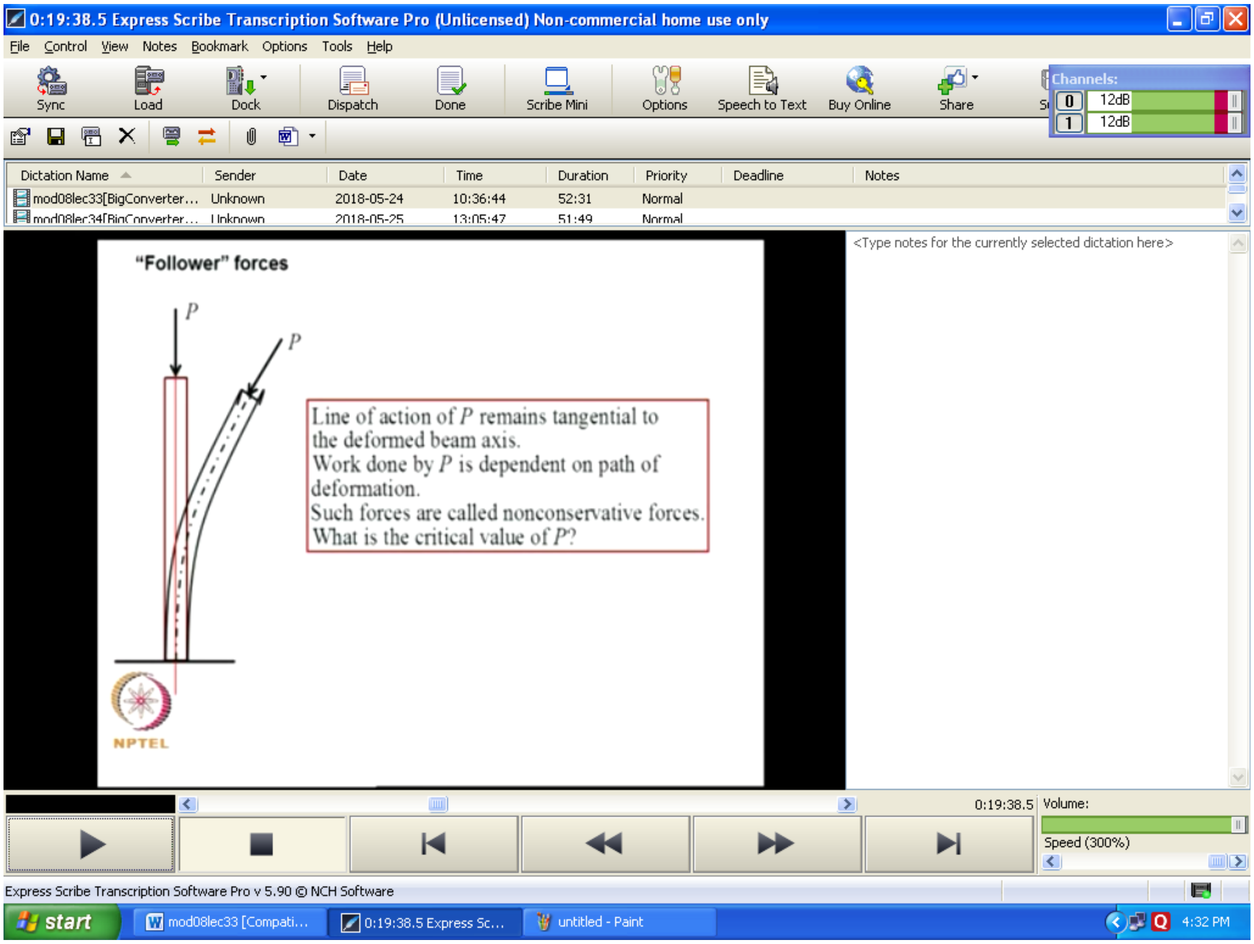
Find eigenvalues of matrix A.

Infer the nature of the solution by examining the nature of the eigenvalues.

16

How do you deal with multi degree freedom systems? So we can consider equation of this form U double dot P(t), U dot + Q(t) U, again let us assume P and Q are periodic with period capital T, so in the equation that we got say M(t) X double dot + CX dot etcetera that is in the model that we got we had M(t) X double dot + C(t) X dot + K(t) X = 0, so I can reduce this equation to this form by multiplying by M inverse, so it reduces to this form.

Now again by using the logic that we have discussed U(t + T) can be written as A into U(t), now A will be, size of A will be equal to the, suppose U is N x 1, A will be N x N, so how do you find A? We solve this equation over one period of excitation with a set of linearly independent initial condition, we have to select a set of N linearly independent initial conditions and integrate the solutions over one period, and by examining the response at capital T will be able to construct A matrix, again we can examine the eigenvalues of A and infer whether the solution grows in time, or decays in time, or becomes periodic, etcetera, so this is how a qualitative analysis of a time varying system can be performed.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"Follower" forces
P
P
Line of action of P remains tangential to the deformed beam axis.
Work done by P is dependent on path of deformation.
Such forces are called nonconservative forces.
What is the critical value of P?
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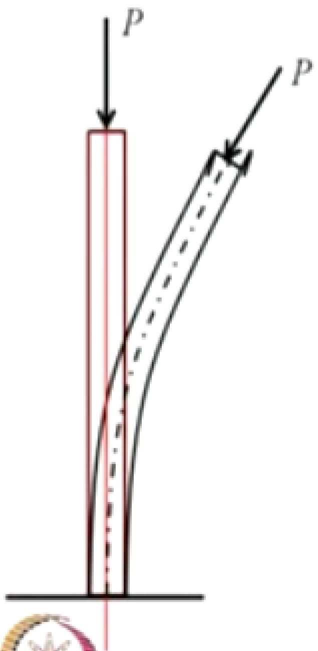
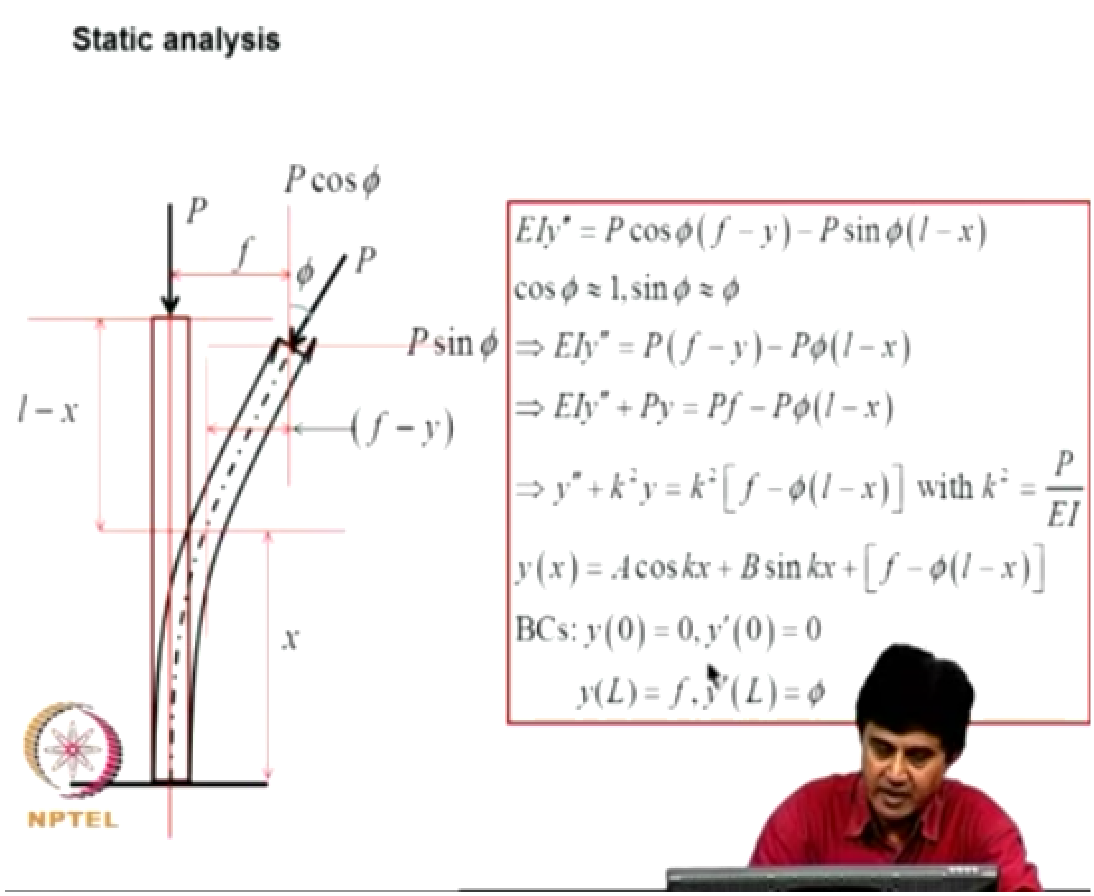
“Follower” forces



Line of action of P remains tangential to the deformed beam axis.
Work done by P is dependent on path of deformation.
Such forces are called nonconservative forces.
What is the critical value of P?

Now the next problem that I mentioned was that of so-called follower force problems, so here the line of action of P remains tangential to the deform beam axis, so work done by P is dependent on path of the deformation, so such forces are called non conservative forces. So now what is the critical value of P that is the question we are asking?



So what we can do is we can perform a static analysis, see when we performed stability analysis of a beam where load was, direction of the load was unaltered what was a conceptual framework within which we did that, we started with small values of P and incremented P in steps, at every increment, at every step of incrementing P we gave a slight perturbation to the system and the system gets into oscillations, and presence of damping ensures that the system either returns to its original state our assumes an neighboring equilibrium position, so there was an element of dynamics even in this, so but if you were to perform a dynamic analysis of that problem and examine the nature of fixed points associated with the governing equations we would reach the same conclusion as we would do by simply performing a static analysis, we have discussed the issue about fixed points etcetera in one of the earlier lectures.

Now inspired by that what we will do is we will formulate this problem using purely a static consideration, so we can resolve this load as P sine phi and P cos phi, and when I write the equation for bending moment at any section I will include bending moment due to the horizontal component and due to the vertical component, so I will set up a coordinate system X is measured from this and I want to write bending moment at X, so it will be P cos phi into this lever arm, which will be F – Y, and P sine phi will be a lever arm will be this, okay, so that is sine of phi L – X.

Now if we assume that phi is small we can assume cos phi as 1 and sine phi as phi, so the governing equation becomes by simplifying this I get EIY double prime + PY = PF - P phi L-X), F is the displacement here and phi is the rotation. So what are the unknowns in this problem? F and phi are unknowns here, we don't know what they are, because they have to be determined by analyzing the problem, now I'll divide by EI and introduce K square as P/EI and

I will get this equation, and this is a linear equation and I can write the complementary function and particular integral.
Now what are the boundary conditions? At X = 0, Y(0) is 0, Y prime (0) is 0, and what are these F and phi? F is Y(l) and phi is Y prime(l), that is how we have introduced that. Now

$$y(x) = A\cos kx + B\sin kx + \left[f - \phi(l-x)\right]$$

$$y'(x) = -Ak\sin kx + Bk\cos kx + \phi$$

$$\text{BCs: } y(0)=0,\ y'(0)=0$$

$$y(l) = f,\ y'(L) = \phi$$

$$y(0) = 0 \Rightarrow A + f - \phi l = 0$$

$$y'(0) = 0 \Rightarrow Bk + \phi = 0$$

$$y(l) = f \Rightarrow f = A\cos kl + B\sin kl + f$$

$$y'(L) = \phi \Rightarrow \phi = -Ak\sin kl + Bk\cos kl + \phi$$

$$\begin{bmatrix} 1 & 0 & 1 & -l \\ 0 & k & 0 & 1 \\ \cos kl & \sin kl & 0 & 0 \\ -k\sin kl & k\cos kl & 0 & 0 \end{bmatrix} \begin{Bmatrix} A \\ B \\ f \\ \phi \end{Bmatrix} = 0$$

19

therefore there are now 4 constants A, B, F and phi which are unknowns, and there are four conditions, what are these four conditions? These are the four conditions, so I can now impose that so Y(x) is this and Y prime(x) I can find out by differentiating this, and by imposing the four boundary condition Y(0) is 0, Y prime(0) is 0, Y(l) is F, Y prime(l) = phi, I get a set of four equations, and the unknowns are A, B, F and phi, so I can cast it in a matrix form and write in this form, so for non-trivial solution the determinant of this equation must be 0, so if I

For non trivial solutions

$$\begin{vmatrix} 1 & 0 & 1 & -l \\ 0 & k & 0 & 1 \\ \cos kl & \sin kl & 0 & 0 \\ -k\sin kl & k\cos kl & 0 & 0 \end{vmatrix} = 0$$

$$\text{We get } \Delta = \begin{vmatrix} 1 & 0 & 1 & -l \\ 0 & k & 0 & 1 \\ \cos kl & \sin kl & 0 & 0 \\ -k\sin kl & k\cos kl & 0 & 0 \end{vmatrix} = -1$$

This means that only trivial solution is possible for all values of k.

Structure's state of rest $(y = 0)$ is always stable for all values of P.

This defies expectations.

Did we miss something?

20

impose that we get the determinant of this matrix will be – 1 if we expand that, and it is independent of the applied load, so that means determinant of this matrix is not 0, so the only possible solution is a trivial solution so what it means that only trivial solution is possible for all values of K, that means no matter what is a load P, the equilibrium position is always stable, so that is structure state of rest is always stable for all values of P, this defies expectations, so did we miss anything in doing this problem?

Idea

The loss of structural stability is accompanied by oscillations whose amplitude grow in time.

Therefore, include inertial effects in considering stability of equillibrium state.

Consider the case when P was applied in a conservative manner.

When dynamic analysis was performed, when $P = P_{cr}$,

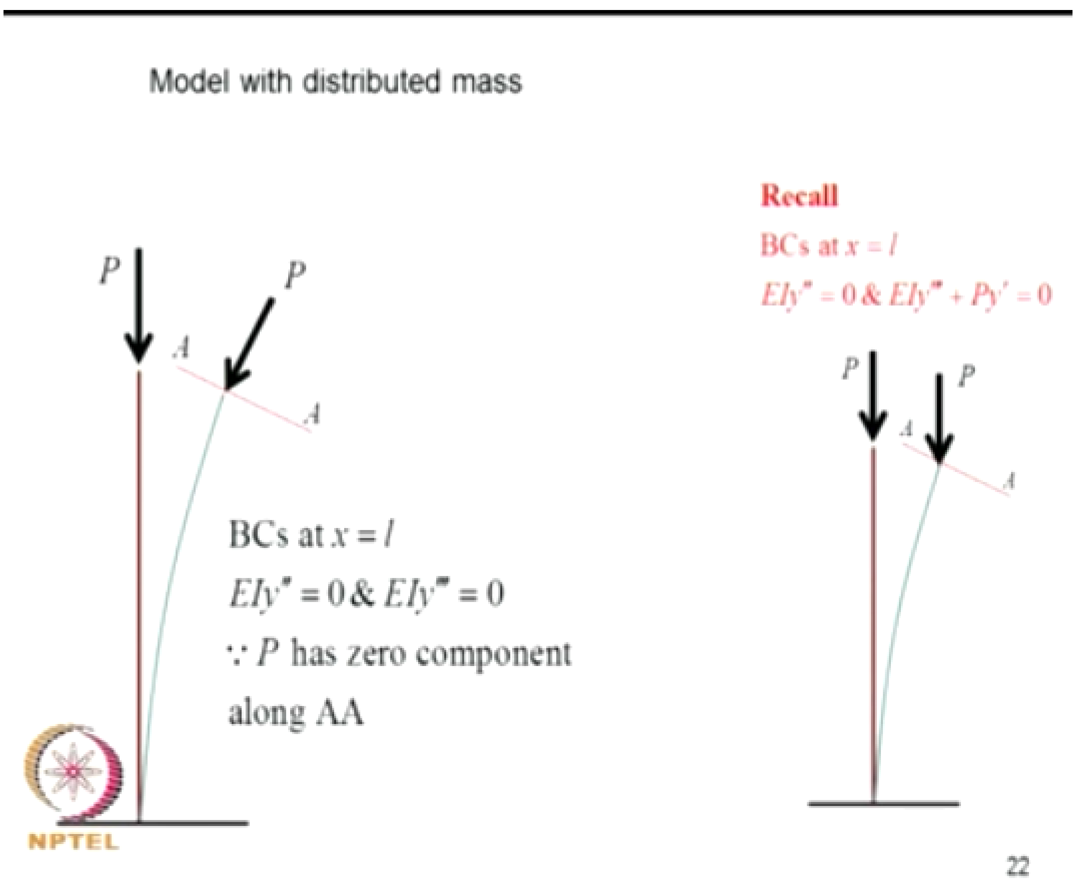
the response grew linearly in time with natural frequency=0.

The results from static and dynamic analysis coincided.

NPTEL

21

So the thing that we have missed is a, we have done a static analysis. So the idea is if we say that loss of structural stability is accompanied by oscillations whose amplitude grows in time, whose amplitude grow in time if we notice that then we can include inertial effects in considering the stability of the equilibrium state, suppose if you do that there is nothing wrong in trying that out, so let us consider the case when P was applied in a conservative manner that means direction of P would remain unaltered due to deformation of the structure, so when dynamic analysis was performed and when P = P critical the response grew linearly in time when natural frequency was 0, the result from static and dynamic analysis coincided, okay, so this we can recall in this context.



Now what we will do is we will reformulate this problem, now let's again assume that at some stage in the deformation the beam occupies this position, neighboring equilibrium position and P remains tangential to the deform axis of the beam, now what are the boundary conditions at X = L for the deformed configuration? The bending moment and shear force must be equal to 0, this is because P has 0 components along A, okay so now therefore the boundary conditions at, whereas when we consider this problem the boundary conditions at X = L was EIY double prime = 0 + EIY triple prime + PY prime = 0, so P appeared in specification of boundary condition here, but whereas here since it appears tangent in load remains tangential to the axis of the beam, the boundary conditions involving shear force P won't appear.

$$EIy^{iv} + Py'' + m\ddot{y} = 0$$

$$\text{BCs: } y(0,t) = 0, y'(0,t) = 0, EIy''(l,t) = 0, EIy'''(l,t) = 0$$

$$y(x,t) = \phi(x)\exp(i\omega t)$$

$$EI\phi^{iv} + P\phi'' - m\omega^2\phi = 0$$

$$\phi^{iv} + k^2\phi'' - a\omega^2\phi = 0$$

$$\phi(x) = \phi_0 \exp(sx)$$

$$s^4 + k^2 s^2 - a\omega^2 = 0$$

$$s^2 = \lambda \Rightarrow \lambda^2 + k^2\lambda - a\omega^2 = 0$$

$$\lambda_1 = -\frac{k^2}{2} + \sqrt{a\omega^2 + \frac{k^4}{4}} \; \& \; \lambda_2 = -\frac{k^2}{2} - \sqrt{a\omega^2 + \frac{k^4}{4}}$$

$$\Rightarrow \phi(x) = A\cosh\lambda_1 x + B\sinh\lambda_1 x + C\cos\lambda_2 x + D\sin\lambda_2 x$$

$$\phi(0) = 0, \phi'(0) = 0, \phi''(l) = 0, \phi'''(l) = 0$$

23

Now equipped with this now I will write this equation EIY4 + PY double prime + MY double dot = 0, so the boundary conditions are at X = 0, Y and Y prime are 0, and X = L bending moment and shear force are 0, so I will again assume a solution where all points on the structure oscillate harmonically at the same frequency and I get this eigenvalue problem, so I divide this equation by EI and introduce K Square and a parameter A, which is square root M/EI, so here this is a linear equation with constant coefficients therefore exponential must satisfy this equation, so I get phi(x) as phi naught E raise to SX and this is a characteristic equation, so this is biquadratic equation so I can get the roots by solving this quadratic equation and these are the roots, so based on this I will be able to write the solution phi(x) is A cos lambda 1X + B sine H lambda 1X + C cos lambda 2X + D sine lambda 2X, this is negative therefore square root of lambda 2 will be imaginary therefore we will get sine and cosine terms, whereas this will be positive therefore we will get sine and cosine term, sine H and cos H terms. So now I have four boundary conditions and I can do that I am skipping those steps, the

$$\phi(x) = A\cosh\lambda_1 x + B\sinh\lambda_1 x + C\cos\lambda_2 x + D\sin\lambda_2 x$$

$$\phi(0) = 0, \phi'(0) = 0, \phi''(l) = 0, \phi'''(l) = 0$$

Condition for nontrivial solution

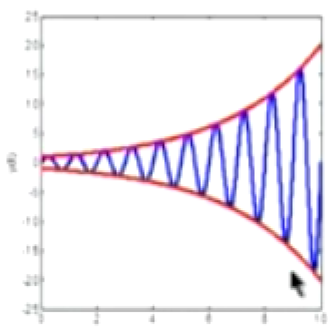
$$2a\omega^2 + k^4 + 2a\omega^4 \cos\lambda_1 l + k^2\sqrt{a\omega^2}\sinh\lambda_1 l \sin\lambda_2 l = 0$$

This leads to the relation between P and ω.

$$y(x,t) = \phi(x)\exp(i\omega t)$$

Write $\omega = a + ib \Rightarrow y(x,t) = \phi(x)\exp\left[(ia - b)t\right]$

Instability when $b < 0 \Rightarrow P_{cr} = 19.739\frac{EI}{l^2}$



Response for $P > P_{cr}$

condition for non-trivial solution we can obtain in this form by you know writing the four equations, forming the coefficient matrix and demanding that the determinant of the coefficient matrix is 0 I get this equation, now this actually is the characteristic equation and it relates P and omega, so this leads to the relation between P and omega and we have this Y(x,t) is exponential I omega T, this omega need not be real in this case, so if I assume omega to be A + IB then Y(x,t) will be exponential IA - B into T, so for B less than 0 there will be instability, because the real part of this exponent will be positive and as T tends to infinity the solution grows and that helps us to determine the critical load which is 19.739 EI/L square, so this result contradicts this analysis which showed that for all values of P the response is stable.

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- V V Bolotin, 1963, Nonconservative problems of the theory of elastic stability, Pergammon, Oxford.
- M A Langthejm and Y Sugiyama, 2000, Dynamic stability of columns subjected to follower loads: a survey, Journal of Sound and Vibration, 238(5), 809-851.

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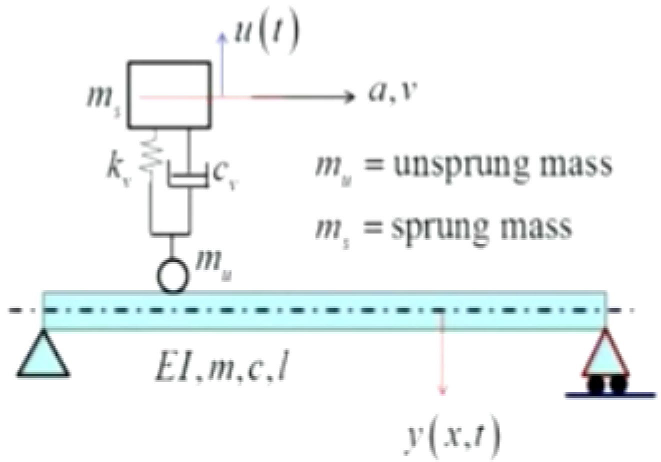
Now if P were to be greater than P critical and this is how the solution would grow, it will be oscillatory and it grows, so this is called flutter and things like that, now there are two books, which this book by Bolotin gives a classical book which discusses these problems, and in the existing literature there has been criticism of this model and the basic question that has been asked is are there any situations where we can apply static loads which obey this hypothesis that they remain tangential to the deformed axis, so apparently no experiment till today has been done to characterize this that is a claim, but there is a review paper in which these issues are discussed so if you wish to understand the issues related to this discussion in the existing literature you can read this reference.

FE analysis of vehicle-structure interactions



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Now we will now move on to the next item on our agenda on how to make finite element analysis for systems which are governed by partial differential equations with time varying coefficients, so we will consider this problem we have considered in the previous class, where I have explained all the basic terminologies of this problem and we have got the governing differential equation, so the coordinate system is the origin is here, X is measured along this axis, Y is the displacement measured from the neutral axis here and vehicle is taken to enter the bridge at T = 0 and it leaves the bridge at T exit, and the time that it spends on the bridge is governed by the its motion parameters, acceleration and velocity we assume that these two are constant the time the vehicle is on the bridge, the vehicle itself is characterized in terms of an unsprung mass and a spring mass, and stiffness and damping characteristics of the isolation.

for $0 < t < t_{exit}$

$$m_s \ddot{u} + c_v \left\{ \dot{u} - \frac{D}{Dt} y\left[x(t), t\right] \right\} + k_v \left\{ u - y\left[x(t), t\right] \right\} = 0$$

$$EIy^{iv} + m\ddot{y} + c\dot{y} = f(x,t)\,\delta\left(x - vt - \frac{1}{2}at^2 \right)$$

$$f(x,t) = (m_u + m_s)g + k_v \left\{ u - y\left[x(t), t\right] \right\} + c_v \left\{ \dot{u} - \frac{D}{Dt} y\left[x(t), t\right] \right\} - m_u \frac{D^2}{Dt^2} y\left[x(t), t\right]$$

$f(x,t)$ = wheel force

for $t > t_{exit}$

$$EIy^{iv} + m\ddot{y} + c\dot{y} = 0$$

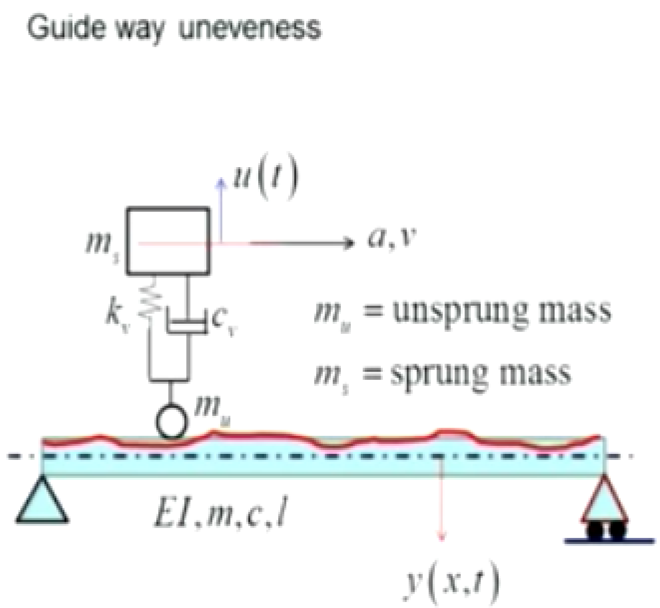
with conditions at t_{exit} obtained from equations valid for $0 < t < t_{exit}$

Approach: integral and weak formulation

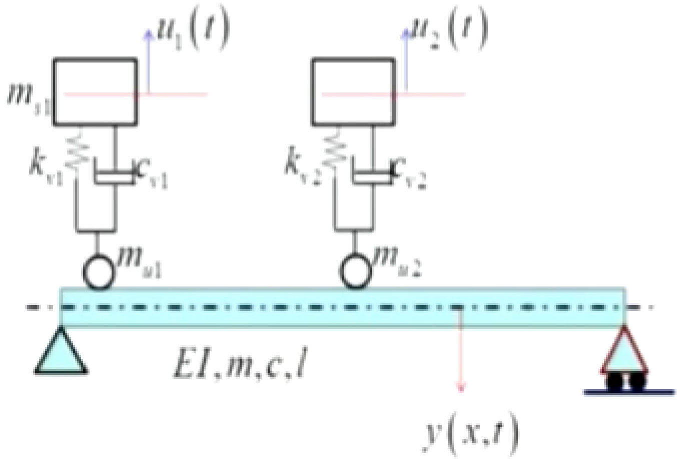
So there will be now a degree of freedom associated with the vehicle and dependent variable Y(x,t) associated with deformation of the beam, so these are the governing differential equations, the equation for U will obviously have the displacement of the beam because the force in the spring and the force in the damper depends on the relative displacement and velocity between this point, between this point and this point, so this point itself is deforming so we will get these terms, and also I pointed out that as the structure deforms and this mass rolls on the bridge, this wheel rolls on the bridge it will be rolling on a deflected profile, therefore when I compute velocity and accelerations needed to characterize the spring forces and the inertial forces we need to consider the total derivative, so that is why we are writing capital D/DT of Y(X(t),t) and this is the equation for the vehicle degree of freedom, and this is the equation for the beam oscillations, F(x,t) is the wheel force that consists of weight of the vehicle, the force transferred from the spring, and the force transferred from the damper, and the inertial force of the unsprung mass, and this is a concentrated force which point of application changes with time as the vehicle moves and that is depicted through this direct delta function, so F(x,t) is the wheel force, as the vehicle exits the bridge the bridge undergoes small oscillations and since our interest is primarily on the bridge we will not write the corresponding equation for the vehicle.

So at T exit to solve this equation if you wish to model how the free vibration decay takes place after the vehicle lose the bridge you should, this is valid from T exit and the state of the bridge at T exit should be computed by analyzing this problem, so that means as the vehicle leaves the bridge the displacement and velocity field of the bridge must be captured through the model that is applicable for this time origin, so what remains as the snapshot of the bridge response at T exit will serve as initial conditions to solve this problem.

So what we will do is we will try to develop a finite element model for this pair of equation, ordinary differential equation and partial differential equation by using what is known as an integral and weak formulation. In the development of finite element method so far in the course we have started with the variational principle, we didn't start from the governing differential equation when we started discussing approximate method we saw that Rayleigh-Ritz method the way we apply Rayleigh Ritz and Galerkin method we're somewhat different, Galerkin method we applied on a governing differential equation whereas Rayleigh-Ritz was on a variational formulation, so here we will assume that the starting point for discussion is a partial differential equation, okay, and how do we analyze this? So we need to prepare some basics for



that, before we get into this we can consider some more aspects of this problem, see there are certain idealizations made in arriving at this model for example we are assuming that the bridge deck is smooth, but in reality the bridge deck could be, there could be guide way unevenness and the vehicle will actually be moving on a bumpy bridge, one the deformation of the bridge along with the roughness of the bridge surface contributes to the input to the vehicle, for example if this bridge was undeforming but it is, that means if this vehicle was running on a rigid pavement but with undulations it will still feel the oscillations, not here the vehicle is running on a deforming elastic medium, but it has certain undulations, so this can also be included in our model.



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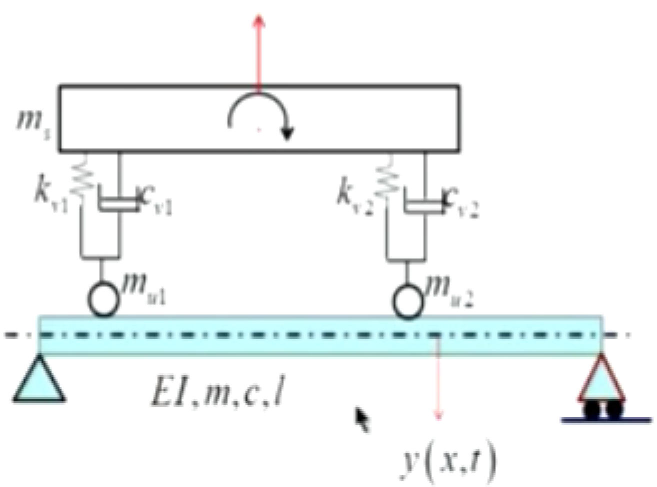
Vehicles and the beam interact

29

And there could be series of loads crossing the bridge, so again principle of superposition won't be valid here, by that I mean if you find the bridge response bridge vehicle system response due to passage of one vehicle and similar response analysis for passage of another vehicle that cannot be superposed to find response when both the vehicles pass the bridge that principle of superposition is not acceptable, we can improve upon the vehicle model, we can include



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Vehicles and the beam interact

30

translation and pitching of the vehicle, we can model for the vehicle in greater details, and this type of models have also been studied in the literature.

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31

In practical situation of course the supporting structure itself will be a 3, you know 3 dimensional structure lattice girder bridge like this and the moving vehicle itself will be a fairly complicated engineering system, so this should be borne in mind when we analyze this type of models, so in reality to explore the, to realize the full potential of finite element method this type of problems need to be analyzed in the framework of a supporting structure being modeled as a, using as a finite element model and the moving system also using another finite element model, and these two models move relative to each other, it is that problem that we should eventually be able to solve as we start with such an ideal situations such as this.

Prelude : Integral and weak formulations for modeling beam vibrations

We consider situations in which the system to be analyzed is described in terms of a governing differential equation. This is in contrast to our studies so far wherein we started with Hamilton's principle in formulating the problem.

Consider

$$\left(EIy''\right)'' + m(x)\ddot{y} = f(x,t)$$

$$y(0,t) = 0, y'(0,t) = 0, EIy''(l,t) = M_0(t), \left(EIy''\right)'(l,t) = 0$$

$$y(x,0) =, \dot{y}(x,0) = 0$$

We aim to find an approximate solution to this equation in the form

$$y(x,t) = \sum_{n=1}^{N} a_n(t)\phi_n(x) + \phi_0(x)$$

As we have seen earlier, the substitution of the assumed solution into the governing equation leads to a residue. 32

Now as a prelude to for developing finite element model starting from a partial differential equation we will revisit some you know basic notions about how to do this by considering a simpler problem, so what we do is we consider situations in which the system to be analyzed described in terms of a governing differential equation, this is in contrast over study so far wherein we started with Hamilton's principle in formulating the problem, so what we do is we consider the equilibrium equation which is a partial differential equation say in homogeneous beam EIY double prime double prime + M(x) Y double dot = F(x,t) and let's assume that the beam initial condition there is a time varying term at the boundary, for example it's a fixed at the left end and it is free at the other end but it carries a time varying moment, and it's assumed that it starts from rest.

Now our aim is to find an approximate solution to this equation in the form with certain trial function phi N(x) and generalized coordinates AN(t), and we have seen earlier the substitution of the assumed solution into the governing equation leads to a residue, so what we do is we

⇒ Weighted residual statement

$$\int_0^l w(x)\left[\left(EIy''\right)'' + m(x)\ddot{y} - f(x,t)\right]dx = 0$$

•If $y(x,t)$ is the exact solution, $\left[\left(EIy''\right)'' + m(x)\ddot{y} - f(x,t)\right] = 0.$

•If $y(x,t)$ is replaced by its approximation,

$$y(x,t) = \sum_{n=1}^{N} a_n(t)\phi_n(x) + \phi_0(x) \Rightarrow \left[\left(EIy''\right)'' + m(x)\ddot{y} - f(x,t)\right] \neq 0.$$

•The above statement implies that the error of representation is zero in a weighted integral sense.

•By choosing N independent weight functions, we get N independent equations for the unknowns $a_n(t), n = 1, 2, \cdots, N$

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write what is known as a weighted residual statement for the problem, what we do is we take all the terms which are on the right hand side to the left hand side, and we get on the right hand side 0, and multiply that by a weight function W(x) and integrate over the domain of the problem, so we get this equation, this equation is the statement of the, is known as a weighted residual statement, where W(x) is a weight function, if Y(x,t) is the exact solution the term inside the bracket would be 0, so this will be automatically satisfied.

On the other hand if Y(x,t) is replaced by an approximation, this term inside the bracket won't be 0 and this can be used to obtain equation for AN(t), so how we'll do this? We select a set of, suppose there is a capital N generalized coordinates, we select a capital N set of weighting functions and write this equation for those capital N number of weight functions, and that leads to the equation for AN(t), so the above statement implies that the error of representation is 0 in the weighted integral sense by choosing N independent weight functions, we get N independent equation for the unknowns AN(t), N = 1 to N, and that is how we formulate the problem.

Now here if you see here when I am representing, here we have to choose two things now, one the trial function and the weight function, now if we now look at the demands on continuity of this trial function and the weight function here you will see that the trial function need to be differentiable up to fourth order, whereas weight function need not be differentiable also it should be simply integral okay, so the demand on the weight function and the trial function is not even, if that is not a restriction we can simply go ahead and select a capital N number of weight function and solve the problem, but obviously that is not a fair situation to construct the trial functions which are differentiable up to fourth order requires fairly elaborate representation for phi N(x) and that leads to increased computational burden, so we want to reduce the demand

on differentiability on phi N(x), so what we do? So the observation is that continuity requirements and W(x) and phi N(x) are different, the requirements on phi N(x) are more stringent, the weighted integral statement is equivalent to the governing field equation and does not take into the boundary conditions, because boundary condition issue has not yet come up, the unknowns AN(t) can be determine by considering N weight functions as shown here, so you write this equation for a set of N weight functions, and we get the required N equation.
So to proceed further with the solution we need to select the trial functions phi N(x), N = 1 to N which possess fourth order derivatives and satisfy the prescribed boundary conditions, but there is no such stringent requirements on the weight functions, so this looks a bit unfair, because we have to select both of them.

Consider $\int_0^l w(x)\left[(EIy'')'' + m(x)\ddot{y} - f(x,t)\right]dx$ and integrate the first term by parts:

$$\Rightarrow \left[w(x)(EIy'')'\right]_0^l - \int_0^l w'(x)(EIy'')'dx + \int_0^l w(x)\left[m(x)\ddot{y} - f(x,t)\right]dx = 0$$

$$\Rightarrow \left[w(x)(EIy'')'\right]_0^l - \left[w'(x)(EIy'')\right]_0^l + \int_0^l w''(x)(EIy'')dx$$

$$+\int_0^l w(x)\left[m(x)\ddot{y} - f(x,t)\right]dx = 0$$

This is known as the weak form.

Notice: differentiability requirement on $y(x)$ [and hence on trial functions $\phi_n(x), n = 1,2,\cdots N$] has come down to 2 and the requirement on $w(x)$ has gone up to 2.

The integration by parts has enabled us to trade the differentiability requirements between trial functions and the weight functions.

35

So now what we do is we integrate the term by parts, by integrating by parts first time, the first term integrated by part, other terms we can retain as it is, so it becomes now the fourth derivative becomes third derivative, and here we get the first derivative on the weight function, the other terms remains the same, so integrate once again then this term is following from this, and from this term I get W prime(x) EIY double prime, so the derivative on this, this derivative has now vanished, and in the integrand I get now W double prime(x) and Y double prime(x) DX + this term as it is, now this statement is known as a weak form.
Now if you decide that we will work with this form of the equation then the original weighted residual form statement, then we see that the trial function now need to be only differentiable up to second order, but now the weight function need to be differential up to second order, so the demand on the trial function and the weight function as far as differentiability goes is now even, so by integrating by parts we've achieved the trade off from on differentiability

requirement on trial functions and weight function, so we'll make these comments, so the differentiability requirement on Y(x) and hence on the trial functions has come down to 2, and the requirement on W(x) has gone up to 2.

The integration by parts has enabled us to trade the differentiability requirements between trial functions and the weight functions.

Consider the terms $\left[w(x)(EIy'')'\right]_0^l$ & $\left[w'(x)(EIy'')\right]_0^l$

We can identify two types of BCs: natural and essential.

We call coefficients of the weight function and its derivatives in the above terms as secondary variables.

Thus, (EIy'') & $(EIy'')'$ are secondary variables.

Specification of secondary variables on the boundaries constitute the natural (or force) BCs.

The dependent variables expressed in the same form as the weight function as appearing in the boundary terms are called the primary variables.

Thus $y(x,t)$ & $y'(x,t)$ are the primary variables.

Specification of the primary variables on the boundaries constitutes the essential (or geometric) boundary conditions.

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Now consider now we have 2 more terms because of integrating by parts, now let us consider those 2 terms. Now based on this we can identify 2 types of boundary condition, these terms are associated with what happens at the boundaries X = 0, and X = L, so these are clearly associated boundary conditions, so based on this we can identify 2 types of boundary conditions, one set known as natural, and the other one known as essential. Now the rule for this division is as follows, we call coefficients of the weight function and its derivatives in the above terms as secondary variables, for example weight function W(x), the coefficient is EIY double prime prime, so EIY double prime prime is a secondary variable, similarly here weight function derivative is multiplied by EIY double prime, so EIY double prime is called the secondary variable. Specification of the secondary variables on the boundaries constitute the natural or force boundary conditions, the dependent variables expressed in the same form as a weight function, as appearing the boundary terms are called the primary variables, thus we have W(x), therefore Y(x,t) and we have W prime(x) and thus Y prime(x,t) are the primary variables. Specification of the primary variables on the boundaries constitutes the essential or geometric boundary condition, so we divide, in summary we are dividing the boundary conditions into natural and essential, and we have now developed a prescription for classifying these boundary conditions, the variables as being primary and secondary and the boundary condition being natural or geometric.

Remarks

- The number of primary and secondary variables would be equal.
- The SVs have direct physical meaning.

(EIy''): bending moment

$(EIy'')'$: shear force.

Each PV is associated with a corresponding SV

Secondary variable	Primary variable
Bending moment : (EIy'')	Slope $y'(x,t)$
Shear force: $(EIy'')'$	Displacement $y(x,t)$

Essential BCs invovle specifying displacement & slope at the boundaries.

Natural BCs involve specifying BM and SF at boundaries

Now in this type of formulation if you are dealing with even order differential equations which we are always doing in many of the structural engineering problems, the number of primary and secondary variables will be equal, the secondary variables have direct physical meaning as far as the problem is concerned, for example here EIY double prime is a bending moment, EIY double prime is the shear force, so each primary variable is associated with a corresponding secondary variable, so there is a natural pairing of primary and secondary variables, for example if you have secondary variable which is bending moment, the primary variable is the slope, if the secondary variable is the shear force the primary variable is a displacement, so essential boundary conditions involve specifying displacement and slope at the boundaries, and natural boundary conditions involves specifying bending moment and shear force at the boundaries for the beam problem.

Remarks (Continued)

- On the boundary either a pv can be specified or the corresponding sv can be specified. A given pair of sv and pv cannot be specified simultaneously at the same boundary.

Thus, at a free end we can specify BM to be zero and slope remains unspecified; similarly, SF can be specified to be zero and displacement remains unspecified.

By denoting: $(EIy'')' = V \,\&\, (EIy'') = M$, we write

$$\left[w(x)V(x)\right]_0^l - \left[w'(x)M(x)\right]_0^l + \int_0^l w''(x)(EIy'')dx + \int_0^l w(x)\left[m(x)\ddot{y} - f(x,t)\right]dx = 0$$

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Now on the boundary either a primary variable can be specified or the corresponding secondary variable can be specified, a given pair of SV and PV cannot be specified simultaneously the same boundary, thus for example at a free end of a beam we can specify bending moment to be 0 but the slope remains unspecified, similarly shear force can be specified to be 0 in a simple supported end and displacement remains, no in the free end shear force can be specified to be 0 but the displacement remains unspecified, okay, so now we use some notation EIY double prime prime as V which is shear force, EIY double prime as M we write the weak form in this way, so the weak form statement is this.

$$\left[w(x)V(x)\right]_0^l - \left[w'(x)M(x)\right]_0^l + \int_0^l w''(x)(EIy'')dx +$$

$$\int_0^l w(x)\left[m(x)\ddot{y} - f(x,t)\right]dx = 0$$

We now require the weight functions to satisfy the essential BCs of the problem.

Recall: the BCs we are considering are

$$y(0,t)=0, y'(0,t)=0, EIy''(l,t)=M_0(t), (EIy'')'(l,t)=0$$

Accordingly, we demand $w(0)=0, w'(0)=0$

Thus we have

$$\left[w(x)V(x)\right]_0^l = w(l)V(l) - w'(l)M(l) = -w'(l)M_0(t)$$

The weak form thus reads

$$-w'(l)M_0(t) + \int_0^l w''(x)(EIy'')dx + \int_0^l w(x)\left[m(x)\ddot{y} - f(x,t)\right]dx = 0$$

Now we now require the weight functions to satisfy the essential boundary conditions of the problem, recall the boundary conditions we are considering is Y at X = 0, Y and Y prime is 0, and at X = L, we have M naught applied bending moment and shear force which are 0. Now we demand that at X = 0, W(0) and W prime(0) is 0 so this is similar to the virtual displacement concept where virtual displacement conforms to the prescribed boundary condition, so this weight functions must also conform to the prescribed boundary condition, thus we have W(x) V(x), 0 to L will be simply this, which is this, and the weak form with this understanding that the weight function satisfy the geometric boundary conditions we get as this, this can be now used to proceed further with the problem.

$$-w'(l)M_0(t)+\int_0^l w''(x)(EIy'')dx+\int_0^l w(x)\left[m(x)\ddot{y}-f(x,t)\right]dx=0$$

This is equivalent to the original differential equation and the natural BCs.

Recall that we have $y(x,t)=\sum_{n=1}^{N}a_n(t)\phi_n(x)+\phi_0(x)$

with $a_n(t), n=1,2,\cdots,N$ to be determined. $\Rightarrow$

$$-w'(l)M_0(t)+\int_0^l w''(x)\left(EI\left\{\sum_{n=1}^{N}a_n(t)\phi_n''(x)+\phi_0''(x)\right\}\right)dx+$$

$$\int_0^l w(x)\left[m(x)\left\{\sum_{n=1}^{N}\ddot{a}_n(t)\phi_n''(x)\right\}-f(x,t)\right]dx=0$$

We can use $w(x)=\phi_n(x), 1,2,\cdots,N$ and obtain equations for $a_n(t), n=1,2,\cdots,N$.

So this is equivalent to the original differential equation and the natural boundary conditions. Now what we did was we started with this assumed solution, so what we will do now is here AN(t) needs to be determined, so now I will substitute this into the weak form I get this, and I will now use in one way of proceeding further W(x) to be equal to the trial function themselves, and we obtain a set of N equations for AN(t) and we can proceed further with the analysis.

Remarks (continued)

- The method leads to symmetric coefficient matrices.
- The natural boundary conditions are included in the weak form and the approximate solution needs to satisfy only the essential boundary conditions.

J N Reddy, 2006, An introduction to the finite element method, 3rd Edition, Tata McGraw-Hill, New Delhi

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This you know approach leads to symmetric coefficient matrices when you formulate the problem, and the natural boundary conditions are included in the weak form and the approximate solutions need to satisfy only the essential boundary conditions, so a discussion on this is available in the textbook by JN Reddy you can see this also, so what is weak here is the

Remarks

- The number of primary and secondary variables would be equal.
- The SVs have direct physical meaning.

(EIy''): bending moment

$(EIy'')'$: shear force.

Each PV is associated with a corresponding SV

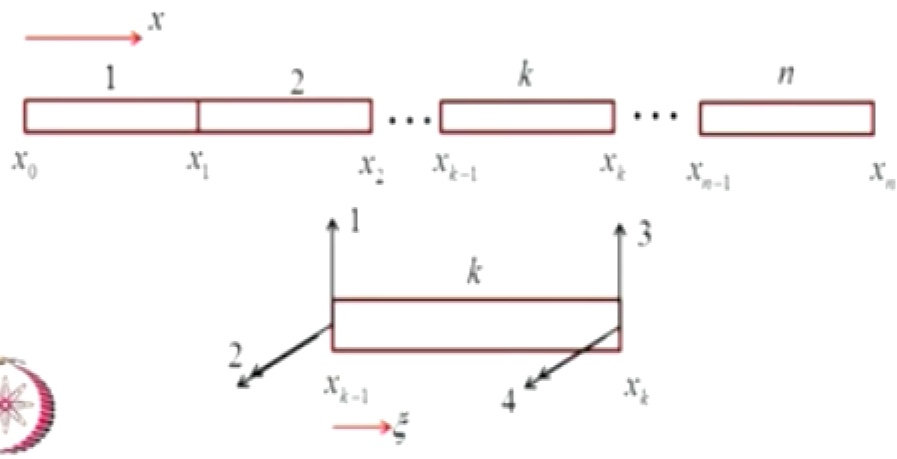
Secondary variable	Primary variable
Bending moment : (EIy'')	Slope $y'(x,t)$
Shear force: $(EIy'')'$	Displacement $y(x,t)$

Essential BCs invovle specifying displacement & slope at the boundaries.

Natural BCs involve specifying BM and SF at boundaries

requirement on differentiability of the trial function has been weakened, it is in that sense the formulation is called a weak formulation.

$$(EIy'')'' + m(x)\ddot{y} = f(x,t)$$
$$y(0,t)=0, y'(0,t)=0, y(l,t)=0, EIy''(l,t)=0$$
$$y(x,0)=, \dot{y}(x,0)=0$$



42

How do we develop finite element solutions for this? Phi N(x) where globally valued shape functions in the previous formulation, now if you want to apply finite element method we divide the domain into say N elements, so and if you consider a typically a K-th element with

Consider the k^{th} element

Let $\xi = x - \sum_{i=1}^{n-1} l_i$ where $l_i = x_{i+1} - x_i$

$x_{k-1} < x < x_k \Rightarrow 0 < \xi < l_k$

For the k^{th} element we have $\left(y' = \frac{\partial y}{\partial \xi} \right)$

$$(EIy'')'' + m(x)\ddot{y} = f(x,t)$$

$$y(0,t) = u_1(t), y'(0,t) = u_2(t), y(l_k,t) = u_3(t), y(l_k,t) = u_4(t)$$

$$EIy'''(0,t) = F_1(t), EIy''(0,t) = F_2(t), EIy'''(l_e,t) = F_3(t), EIy''(l_e,t) = F_4(t)$$

Weighted residual statement

$$\int_0^{l} w(\xi)\left[(EIy'')'' + m(\xi)\ddot{y} - f(\xi,t) \right] d\xi = 0$$

coordinate XK - 1 on the left and XK on the right, and suppose each node has 2 degrees of freedom, so what we can do is we can consider the K-th element and introduce a local coordinate system XI as X - I = N - 1 LI, where LI is XI + 1 - XI which is a length of the element, so as X varies from XK - 1 to XK, XI varies from 0 to LK, so for the K-th element using the notation prime, Y prime as dou Y/dou sai I can write the equation in this form.

Weak form

$$\left[w(\xi)(EIy'')'\right]_0^{l_k} - \left[w'(\xi)(EIy'')\right]_0^{l_k} + \int_0^{l_k} w''(\xi)(EIy'')d\xi$$

$$+\int_0^{l_k} w(\xi)\left[m(\xi)\ddot{y} - f(\xi,t)\right]d\xi = 0$$

$$\Rightarrow \int_0^{l_k} w''(\xi)(EIy'')d\xi + \int_0^{l_k} w(\xi)\left[m(\xi)\ddot{y} - f(\xi,t)\right]d\xi$$

$$-F_1 w(0) - w(l_k)F_3 - F_2\left\{-w'(0)\right\} - F_4\left\{-w'(l_k)\right\} = 0$$

$$F_1 = (EIy'')'(0)$$

$$F_2 = (EIy'')(0)$$

$$F_3 = -(EIy'')'(l_k)$$

$$F_4 = -(EIy'')(l_k)$$

44

So now the boundary conditions are here at XI = 0, the displacements are U1 and U2, and XI = 1, displacements are U3 and U4, similarly there will be a shear force and bending-moment here a shear force and bending moment here, so these are the set of 8 boundary conditions that we need to you know consider. So the weighted residual statement is given by this, the weak form including the required boundary conditions, that means again we differentiate integrate by parts twice and the order of differentiability on Y and W becomes equal, and we get this weak form. Now we have this function F1, F2, F3, F4 which appear on the boundary I write in this form, so I have this as a weak form for the K-th element.

$$\int_0^{l_k} w''(\xi)(EIy'')d\xi + \int_0^{l_k} w(\xi)\left[m(\xi)\ddot{y} - f(\xi,t)\right]d\xi$$

$$-F_1 w(0) - w(l_k)F_3 - F_2\{-w'(0)\} - F_4\{-w'(l_k)\} = 0$$

$$y(x,t) = \sum_{i=1}^{4} u_i(t)\phi_i(\xi)$$

and by selecting $w(\xi) = \phi_i(\xi), i = 1,2,3,4$ we get

$$\int_0^{l_k} \phi_j''(\xi) EI \sum_{i=1}^{4} u_i(t)\phi_i''(\xi) d\xi + \int_0^{l_k} \phi_j(\xi)\left[m(\xi)\sum_{i=1}^{4} \ddot{u}_i(t)\phi_i(\xi) - f(\xi,t)\right]d\xi$$

$$-F_1\phi_j(0) - \phi_j(l_k)F_3 - F_2\{-w'\phi_j'(0)\} - F_4\{-\phi_j'(l_k)\} = 0$$

$$\Rightarrow \sum_{i=1}^{4} K_{ij}u_i(t) + \sum_{i=1}^{4} M_{ij}\ddot{u}_i(t) - P_j(t) = 0, j = 1,2,3,4$$

$$K_{ij} = \int_0^{l_k} EI\phi_j''(\xi)\phi_i''(\xi)d\xi ; M_{ij} = \int_0^{l_k} m(\xi)\phi_j(\xi)\phi_i(\xi)d\xi$$

$$P_j(t) = \int_0^{l_k} \phi_j(\xi) f(\xi,t)d\xi + F_j$$

45

Now this is a weak form and now I assume the solution Y(x,t) is I = 1 to 4, UI(t) phi (XI) and we can now select phi(XI) to be the cubic polynomials, I don't want to discuss the issues on how to select the polynomials here we have seen that, so we can select them and we select W(sai) to be the trial functions themselves, and I can get I can now perform these integrations and the terms involving EI will lead to the stiffness matrix KIJ as shown here, and terms involving mass will lead to the inertial the mass matrix as shown here, and the remaining terms contribute to the equivalent nodal forces as shown here, this is for K-th element, so these are

$$\phi_1(\xi) = 1 - 3\frac{\xi^2}{l^2} + 2\frac{\xi^3}{l^3};$$

$$\phi_2(\xi) = \xi - 2\frac{\xi^2}{l} + \frac{\xi^3}{l^2};$$

$$\phi_3(\xi) = 3\frac{\xi^2}{l^2} - 2\frac{\xi^3}{l^3};$$

$$\phi_4(\xi) = -\frac{\xi^2}{l} + \frac{\xi^3}{l^2}$$

$$\Rightarrow$$

$$M = \frac{ml}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \& K = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

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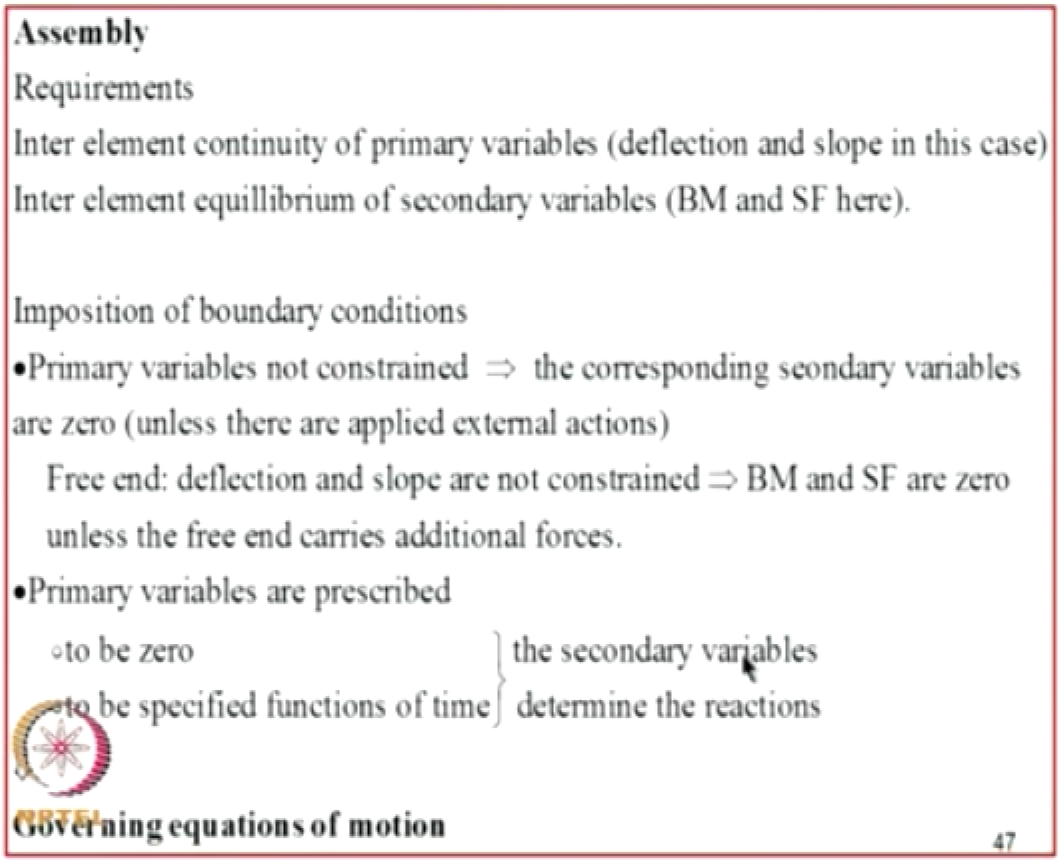
like cubic polynomials I select and the mass matrix and stiffness matrix we have derived earlier in our studies and this is same, there's a same consistent mass matrix and the elastic stiffness matrix.



Now we need to assemble. Assembling, this is a formulation up to K-th element, now the requirements for assembling is the inter element continuity of primary variables into deflection and slope in this case, and inter element equilibrium of secondary variable that is bending moment and shear force, so that is how, I mean this we need not discuss in greater detail because we've already seen how to do these things, then imposition of boundary conditions primary variables are not constrained, the corresponding secondary variables are 0, if primary variables are not constrained the secondary variables will be 0, for example at a free end the translation and rotation are not constrained, therefore this shear force and bending-moment would be 0, unless you have externally applied actions at the free end, that's what I'm saying. Then primary variables are prescribed, for example they are prescribed to be 0 or they can be prescribed to specified functions of time as in the earthquake ground motion or any other support motion problem, so in that case the corresponding secondary variables determine the reactions, so this leads to the governing equation of motion. So this is a framework where we start with the governing partial differential equation and develop the finite element model for the problem.

for $0 < t < t_{exit}$

$$m_s \ddot{u} + c_v \left\{ \dot{u} - \frac{D}{Dt} y\left[x(t), t\right] \right\} + k_v \left\{ u - y\left[x(t), t\right] \right\} = 0$$

$$EIy^{iv} + m\ddot{y} + c\dot{y} = f(x,t)\,\delta\left(x - vt - \frac{1}{2}at^2 \right)$$

$$f(x,t) = (m_u + m_s)g + k_v \left\{ u - y\left[x(t), t\right] \right\} + c_v \left\{ \dot{u} - \frac{D}{Dt} y\left[x(t), t\right] \right\} - m_u \frac{D^2}{Dt^2} y\left[x(t), t\right]$$

$f(x,t)$ = wheel force

for $t > t_{exit}$

$$EIy^{iv} + m\ddot{y} + c\dot{y} = 0$$

with conditions at t_{exit} obtained from equations valid for $0 < t < t_{exit}$

Approach: integral and weak formulation

So in the next class what we will do is we will consider this framework and consider the problem of vehicle structure interaction, this is a partial differential equation and the ordinary differential equation that we have derived, and we will develop the finite element model for this case starting from the integral and weak formulation. So we will close the lecture at this, this lecture at this stage, and we will pick up on this in the next lecture.

Programme Assistance
Guruprakash P
Dipali K Salokhe
Technical Supervision
B K A N Singh
Gururaj Kadloor
Indian Institute of Science
Bangalore